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Avner Friedman

Chaocheng Huang
Wright State University - Main Campus, chaocheng.huang@wright.edu

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AVERAGED MOTION OF CHARGED PARTICLES IN A CURVED STRIP*

AVNER FRIEDMAN† AND CHAOCHENG HUANG‡

Abstract. This paper is concerned with the motion of electrically charged particles in a "curved" infinite strip. The system is $\Delta\varphi = -P$, $\partial^2\psi/\partial t^2 = -\nabla\varphi(\psi, t)$, and $P = P_0(\psi^{-1})J(\psi^{-1})$ with initial and boundary conditions, where φ is the electric potential, ψ is the particle trajectory, P is the charge distribution, and J is the Jacobian. The lower boundary Γ_0 of the strip is grounded. Therefore, once a particle reaches Γ_0, it loses its charge and becomes immobile. We prove existence and uniqueness of solutions for small time. For nearly axially symmetric initial data, we also show that the solution can be extended until the time when all the particles have migrated to Γ_0. A numerical simulation based on our model is implemented. The results indicate that particles tend to accumulate less around the convex parts of Γ_0 and more around the concave parts.

Key words. charged particles, two-phase materials, homogenization, dynamical system, conservation law

AMS subject classifications. Primary, 70F99, 78A35; Secondary, 35A05, 35B60, 45G15

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1. The model. We consider a simplified model of motion of electrically charged spherical particles, with uniform charge and mass, in a "curved" infinite strip in R^n :

$$\Omega = \{x = (x', x_n) \in R^n : \; g(x') < x_n < 1\}.$$

Let $P(x, t)$ denote the mass (or charge) distribution of the particles at a point x in Ω and time $t \geq 0$, and $\varphi(x, t)$ the electric potential. By Maxwell's equation,

$$\Delta\varphi = -P \text{ in } \Omega. \tag{1.1}$$

A voltage difference M is maintained between the upper boundary $\Gamma_\infty = \{x_n = 1\}$ and the lower boundary

$$\Gamma_0 = \{(x', x_n) : \; x_n = g(x'), \; x' \in R^{n-1}\};$$

thus,

$$\varphi = M \text{ on } \Gamma_\infty \tag{1.2}$$

and

$$\varphi = 0 \text{ on } \Gamma_0. \tag{1.3}$$

The particles may collide, but collisions are assumed to be "soft" in the sense that nothing mechanically happens when two particles move into the same spot. We also

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†Institute for Mathematics and Its Applications, University of Minnesota, Minneapolis, MN 55455 (friedman@ima.umn.edu). The research of this author was supported in part by National Science Foundation grant DMS 94-01251.
‡Department of Mathematics and Statistics, Wright State University, Dayton, OH 45435 (chuang@math.wright.edu).

neglect gravity. Then the only force acting on the particles is the electric field $-\nabla\varphi$. If we denote by $\psi(x,t)$ the particle trajectory, then, by Newton's law,

$$\frac{d^2\psi(x,t)}{dt^2} = -\nabla\varphi(\psi(x,t),t). \tag{1.4}$$

By conservation of mass we also have

$$P(x,t) = P_0\left(\psi^{-1}(x,t)\right) J\left(\psi^{-1}(x,t)\right),\ P(x,0) = P_0(x), \tag{1.5}$$

where $P_0(x)$ is the initial distribution, J is the Jacobian, and ψ^{-1} is the inverse of the mapping $x \mapsto \psi(x,t)$ that we require to be 1-1. Finally, we prescribe the initial conditions

$$\psi(x,0) = x, \quad \psi_t(x,0) = \psi^0(x). \tag{1.6}$$

The model (1.1)–(1.6) was developed and studied by the authors [3] in case Ω is the entire space R^n. It was proved that a unique solution exists for a small time interval $0 \le t \le T$ but, in general, not for all time (since $P(x,t)$ may blow up in finite time). For a class of initial data, however, the solution was proved to exist for all time t [3].

We note that if we introduce the Eulerian variables $z = \psi(x,t), \vec{v}(z,t) = \psi_t(x,t)$, then (1.4) becomes

$$\vec{v}_t(z,t) + (\vec{v}\cdot\nabla_z)\,\vec{v}(z,t) = -\nabla_x\varphi(z,t). \tag{1.7}$$

Differentiating in t the relation

$$J(\psi(x,t))\,P(z,t) = P_0(x)$$

(which is another way of writing (1.5)) and using the well-known formula

$$\frac{\partial}{\partial t}J(\psi(x,t)) = (\nabla\cdot\vec{v})\,J(\psi(x,t)),$$

we obtain

$$P_t + \nabla\cdot(P\vec{v}) = 0, \tag{1.8}$$

which is the standard form of conservation of mass. Although equations (1.7), (1.8) together with (1.1) look similar to the Euler–Poisson system [5], it is difficult to work with this form of the model, because we cannot very well account for the particles that leave the domain Ω. We shall therefore stick to the Lagrangian variable x and the formulation (1.1)–(1.6).

The geometry of the domain Ω and the boundary conditions (1.2), (1.3) are motivated by a problem in electrostatic spray painting [1] [2, Chap. 4]. The surface of the sprayer is located at $\Gamma_\infty = \{x_n = 1\}$ and the workpiece (which is being painted) is Γ_0. The workpiece is grounded ($\varphi = 0$), whereas a potential M is maintained at Γ_∞. The potential M is chosen appropriately large in order to force the particles to move toward Γ_0 (see Remark 3.1 for the dependence of M on the initial data and Γ_0; see also (4.18) and (4.19) for a special case). The electrostatically charged paint particles stream through Ω from Γ_∞ toward Γ_0 and, as they reach the workpiece Γ_0, they stick to it to form a paint layer. In the present model we assume that, once reaching Γ_0, the particles lose their electrostatic charge. We also assume that the thickness of the layer formed by the particles that accumulate on the boundary Γ_0 is small compared with the thickness of the strip Ω, so that the domain Ω may be assumed to be fixed in time. Finally, and most importantly, we consider here only the submodel whereby no new particles are injected from Γ_∞ into Ω at times $t > 0$; i.e., all the charged particles are already in Ω at time $t = 0$.

We define “finishing time” as the time after which a portion of the workpiece does not get any more paint, and “global-finishing time” as the time when all the paint has migrated to the workpiece.

In sections 2 and 3 we shall establish local existence and uniqueness for the system (1.1)–(1.6) by an adaptation of the method in [3]. We shall show that the solutions can be extended to $t < T \approx C_0/\sqrt{M}$. Under some conditions we shall also prove that the solutions exist roughly up to the finishing time.

In section 4 we shall consider the case of axially symmetric data

$$g\left(x'\right) = 0,\ P_0\left(x\right) = P_0\left(x_n\right),\ \psi^0\left(x\right) = \left(0, bx_n\right) \tag{1.9}$$

and establish a global solution; i.e., the solution exists for $0 < t < T$, where T is the global finishing. It is also shown that $T\sqrt{M} \to \sqrt{2}$ as $M \to \infty$.

In section 5 we shall consider a perturbation problem with the initial data

$$g(x') = \varepsilon h(x'), \quad P_0(x) = p_0, \quad \psi^0(x) = 0,$$

where p_0 is a positive constant and ε is a small parameter. Using the method of section 3 and deriving some a priori estimates, we show that there is a unique solution for $0 \leq t \leq T_\varepsilon$, where $T_\varepsilon \to T_0$ as $\varepsilon \to 0$ and $T_0 \approx \sqrt{2}/\sqrt{M}$ (for large M) is the global-finishing time for the problem with $\varepsilon = 0$.

In section 6 we consider for simplicity the case $n = 2$ and establish the asymptotic expansion

$$\psi_\varepsilon = \psi_0 + \varepsilon\psi_1 + O\left(\varepsilon^2\right),\ f_\varepsilon = f_0 + \varepsilon f_1 + O\left(\varepsilon^2\right),\ \varphi_\varepsilon = \varphi_0 + \varepsilon\varphi_1 + O\left(\varepsilon^2\right), \tag{1.10}$$

where (φ_0, ψ_0, f_0) is the solution of an axially symmetric system and (φ_1, ψ_1, f_1) is the solution of the linearized problem about (φ_0, ψ_0, f_0).

We are interested in the thickness $W_\varepsilon\left(x'\right)$ (along the normal direction to Γ_0) of the layer of the particles that have accumulated at a point (x', x_n) of the workpiece Γ_0 during the time interval $0 \leq t \leq T_\varepsilon$. Using (1.10), we can write

$$W_\varepsilon\left(x'\right) = W_0 + \varepsilon W_1\left(x'\right) + O\left(\varepsilon^2\right), \tag{1.11}$$

where W_0 is a constant. In section 7 we compute numerically the solution of a linearized problem and, in particular, the term $W_1\left(x'\right)$. The numerical results explore the relation of $W_1\left(x'\right)$ to the geometry of the workpiece.

2. A general existence theorem. We anticipate that the particles will move downward toward the workpiece Γ_0. Denote by Γ_t the surface consisting of all points x in Ω such that the trajectory $s \mapsto \psi\left(x, s\right)$ hits Γ_0 at time t and write

$$\Gamma_t = \{(x', f\left(x', t\right)) :\ x' \in R^{n-1}\};$$

i.e., $f\left(x', t\right)$ is the function such that its graph is Γ_t. Note that $g\left(x'\right) \leq f\left(x', t\right) \leq 1$ and $f\left(x', 0\right) = g\left(x'\right)$. We assume that as soon as the trajectory $s \mapsto \psi\left(x, s\right)$ hits Γ_0, it becomes inactive (i.e., immobile and with zero charge). Therefore, we need to consider the function $\psi\left(x, t\right)$ only for x in the closure of the domain

$$\Omega_t = \{(x', x_n) :\ f\left(x', t\right) < x_n < 1\}.$$

Setting

$$\Omega_t^\psi = \psi\left(\Omega_t, t\right) = \{\psi(x,t) :\ x \in \Omega_t\},$$

we note that the lower boundary of Ω_t^ψ is Γ_0. Denote by $\mathcal{X}_K$ the characteristic function of a set K. We can now reformulate problem (1.1)–(1.6) more precisely: find functions $\varphi(x,t), P(x,t), \psi(x,t)$, and $f(x',t)$ satisfying

$$\Delta\varphi\left(x,t\right) = -P\left(x,t\right)\mathcal{X}_{\Omega_t^\psi} \quad \text{in} \quad \Omega, \tag{2.1}$$

$$\psi_{tt}\left(x,t\right) = -\nabla\varphi\left(\psi\left(x,t\right),t\right) \quad \text{if} \quad f\left(x',t\right) < x_n < 1, \tag{2.2}$$

$$P\left(x,t\right) = P_0\left(\psi^{-1}\left(x,t\right)\right) J\left(\psi^{-1}\left(x,t\right)\right) \quad \text{in} \quad \Omega_t^\psi, \tag{2.3}$$

$$\psi_n\left(x', f\left(x',t\right), t\right) = g\left(\psi_1\left(x', f\left(x',t\right),t\right), ..., \psi_{n-1}\left(x', f\left(x',t\right),t\right)\right), \tag{2.4}$$

$$g\left(x'\right) \le f\left(x',t\right) \le 1, \tag{2.5}$$

where $\psi = (\psi_1, \psi_2, ..., \psi_n)$, together with the boundary conditions

$$\varphi = 0 \quad \text{on} \quad \Gamma_0, \quad \varphi = M \quad \text{on} \quad \Gamma_\infty, \quad \varphi \text{ is bounded in } \Omega, \tag{2.6}$$

and the initial conditions

$$\psi\left(x,0\right) = x,\ \psi_t\left(x,0\right) = \psi^0\left(x\right),\ f\left(x',0\right) = g\left(x'\right). \tag{2.7}$$

In (2.1)–(2.7) we have implicitly assumed that $\psi\left(x,t\right)$ is defined only in the closure of Ω_t, ψ^{-1} and P in (2.3) are defined in Ω_t^ψ, P in (2.1) is understood to be zero in $\Omega\backslash\Omega_t^\psi$, and

$$\Omega_t^\psi \subset \Omega, \quad \psi\left(\cdot,t\right) :\ \ \Omega_t \longmapsto \Omega_t^\psi \ \text{ is invertible.} \tag{2.8}$$

We shall further require that

$$f_t\left(x',t\right) > 0. \tag{2.9}$$

This condition is motivated by the physical assumptions that the particles in Ω move downward and that no new particles are injected from Γ_∞ at time $t > 0$.

To prove existence and uniqueness we introduce the space $C^{m+\alpha}\left(\Omega\right)$ of functions in Ω for which the first m derivatives are α-Hölder continuous; the norm will be denoted by $\|\cdot\|_{C^{m+\alpha}(\Omega)}$. For any function $\varphi\left(x,t\right)$ defined in $\Omega \times [0,T]$, we shall briefly write $\|\varphi\|_{C^{m+\alpha}}$ instead of $\|\varphi\left(\cdot,t\right)\|_{C^{m+\alpha}(\Omega)}$, and $\nabla\varphi$ for the spatial gradient. Throughout the paper, we make the following assumptions:

$$g \in C^{2+\alpha}\left(R^{n-1}\right),\ g \le 1 - 2\delta_0\ (\delta_0 > 0),\ P_0 \in C^\alpha\left(\bar{\Omega}\right),\ P_0 \ge 0, \tag{2.10}$$

$$\psi^0 \in C^{1+\alpha}\left(\bar{\Omega}\right),\ \psi_n^0\left(x\right) \le 0,\ \vec{n}\left(y\right) \cdot \psi^0\left(x\right) \le 0 \ \text{ for } x \in\ \Omega,\ y \in \Gamma_0, \tag{2.11}$$

where $\vec{n}$ is the inward normal vector to Γ_0. In this and the next section we also assume, for simplicity, that

$$\psi^0 = 0 \text{ and } P_0 = 0 \ \text{ for } |x| \text{ large.} \tag{2.12}$$

It is sometimes inconvenient to work directly with the function $\psi\left(x,t\right)$ because the domain Ω_t (of its spatial variable x) varies with time t. We therefore introduce a new function Ψ, defined in $\Omega \times [0,T]$ by

$$\Psi\left(x', x_n, t\right) = \psi\left(x', z_n, t\right) \quad \text{or} \quad \psi\left(x', x_n, t\right) = \Psi\left(x', w_n, t\right), \tag{2.13}$$

where

$$z_n = f(x',t) + \frac{1-f(x',t)}{1-g(x')}(x_n - g(x')), \tag{2.14}$$

$$w_n = g(x') + \frac{1-g(x')}{1-f(x',t)}(x_n - f(x',t)). \tag{2.15}$$

Note that at $t=0$, $f_t = (1, -\nabla g)\cdot\psi^0$, and

$$\nabla\Psi = \nabla\psi \cdot \begin{bmatrix} I_{n-1} & 0 \\ \nabla_{x'} z_n & \dfrac{1-f}{1-g} \end{bmatrix}, \quad \nabla\psi = \nabla\Psi \cdot \begin{bmatrix} I_{n-1} & 0 \\ \nabla_{x'} w_n & \dfrac{1-g}{1-f} \end{bmatrix}, \tag{2.16}$$

and

$$\Psi_t = \psi_t + D_t z_n D_n \psi, \quad \psi_t = \Psi_t + D_t w_n D_n \Psi, \tag{2.17}$$

where D_i means differentiating with respect to the ith spatial variable, D_t is the derivative in t, and I_m is the identity matrix in R^m. A solution of (2.1)–(2.7) for $0 \le t \le T$ is said to be classical if (i) f and ∇f are continuous in $R^{n-1}\times[0,T]$, (ii) P is continuous in $\cup_{0\le t\le T}(\overline{\Omega_t^{\psi}}\times\{t\})$, (iii) ψ, $\nabla\psi$, ψ_{tt} are continuous in $\cup_{0\le t\le T}(\overline{\Omega_t}\times\{t\})$, (iv) φ is continuous in $\bar\Omega\times[0,T]$, and (v) $\Delta\varphi$ is bounded in $\bar\Omega\times[0,T]$. In the sequel, all solutions are understood to be classical solutions. We further require that, for any $0\le t\le T$,

$$f(\cdot,t),\ \Psi(\cdot,t)\in C^{1+\alpha} \text{ and } f_t(\cdot,t),\ \Psi_t(\cdot,t)\in C^{1+\alpha} \tag{2.18}$$

in their respective domains. We shall often denote a solution simply by (ψ, f) or (Ψ, f).

THEOREM 2.1. *If* (2.10)–(2.12) *hold, then there exist two positive constants* M_0 *and* C_0 *such that for* $M \ge M_0$ *there exists a unique solution of* (2.1)–(2.7) *for* $0 \le t \le T = C_0/\sqrt{M}$.

Note that in the above, M_0 and C_0 depend only on the initial data and Γ_0 (see Remark 3.1). The proof is given in the next section.

3. Proof of Theorem 2.1. We begin with local existence.

LEMMA 3.1. *Suppose* (2.10)–(2.12) *hold. Then there exists a constant* $M_1 \ge 0$ *such that for* $M \ge M_1$ *there exists a unique solution to* (2.1)–(2.7) *for* $0\le t\le T$ *for some* $T>0$.

Proof. For any $\eta\ge1$, $1\ge T>0$, denote by $K(\eta,T)$ the set of all functions (Ψ,f) which satisfy the following conditions:

(i) $\Psi(x,t)$ is defined in $\bar\Omega\times[0,T]$ with values in $\bar\Omega$, and $f(x',t)$ is defined in $R^{n-1}\times[0,T]$ with values in R^1;

(ii) for any $0\le t\le T$, (2.18) holds and, denoting by $\mathrm{id}(x)=x$,

$$\|\Psi(\cdot,t)-\mathrm{id}\|_{C^{1+\alpha}},\ \left\|\nabla\Psi^{-1}\right\|_{L^\infty}\le\eta,$$
$$\|f(\cdot,t)\|_{C^{1+\alpha}},\ \|f_t(\cdot,t)\|_{C^\alpha},\ \|\Psi_t(\cdot,t)\|_{C^\alpha}\le\eta;$$

(iii) for any $0\le t\le T$, $x=(x',x_n)\in R^n$, and δ_0 defined in (2.10),

$$f_t(x',t)>0,\ f(x',t)\le1-\delta_0,\ f(x',0)=g(x'),$$
$$\Psi(x,0)=x,\ \Psi_t(x,0)=\psi^0(x)-\begin{pmatrix}0\\ D_t z_n(x,0)\end{pmatrix},$$

where z_n is defined in (2.14), and

$$\Psi_n(x',g(x'),t)=g(\Psi_1(x',g(x'),t),...,\Psi_{n-1}(x',g(x'),t)).$$

Note that in (ii) the assumption on $(\Psi - \mathrm{id})$, instead of Ψ, is to avoid technical difficulties in handling unbounded function Ψ, since $\Psi(x,0) = x$.

For any $(\Psi, f) \in K(\eta, T)$, we shall define a mapping $\mathcal{A}(\Psi, f) = (\tilde{\Psi}, \tilde{f})$ in such a way that (Ψ, f) is a solution of (2.1)–(2.7) if and only if it is a fixed point of $\mathcal{A}$. We construct the mapping $\mathcal{A}$ in several steps.

Step 1. Given (Ψ, f), define ψ by (2.13), P by (2.3), and φ by (2.1), (2.6).

Step 2. Set $\varphi = \varphi_1 + M\varphi_2$, where φ_1, φ_2 are bounded functions satisfying

$$\begin{aligned} \Delta\varphi_1 &= -P\chi_{\Omega_t^\psi} \quad \text{in } \Omega, \\ \varphi_1 &= 0 \quad \text{on } \Gamma_0 \cup \Gamma_\infty \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} \Delta\varphi_2 &= 0 \quad \text{in } \Omega, \\ \varphi_2 &= 0 \quad \text{on } \Gamma_0, \\ \varphi_2 &= 1 \quad \text{on } \Gamma_\infty. \end{aligned} \tag{3.2}$$

From (ii), $J\left(\Psi^{-1}\right) > 0$ is uniformly bounded (independent of $\eta \geq 1$). It follows from (2.10), (2.16), and (ii) that $P = P_0 J\left(\psi^{-1}\right) = P_0 \; J\left(\Psi^{-1}\right)(1-g)(1-f)^{-1} > 0$ is uniformly bounded. By the gradient estimates and the maximum principle, we readily find that

$$|\nabla\varphi_1| \leq C_1 \text{ in } \Omega, \; \frac{\partial\varphi_1}{\partial\vec{n}} \geq 0 \text{ on } \Gamma_0, \tag{3.3}$$

where C_1 depends on $\|P_0\|_{L^\infty}$ and $\|g\|_{C^{1+\alpha}}$, and that

$$\frac{\partial\varphi_2}{\partial x_n} \geq C_2 > 0 \text{ in } \Omega, \; \frac{\partial\varphi_2}{\partial\vec{n}} \geq C_2 \text{ on } \Gamma_0, \tag{3.4}$$

where C_2 depends only on $\|g\|_{C^{1+\alpha}}$. We now choose M_1 such that $C_3 = M_1C_2 - C_1 > 0$. It follows that

$$\frac{\partial\varphi}{\partial x_n} \geq C_3 \text{ in } \Omega, \; \frac{\partial\varphi}{\partial\vec{n}} \geq C_3 > 0 \text{ on } \Omega(\Gamma_0), \tag{3.5}$$

provided that $M \geq M_1$, where $\Omega(\Gamma_0) \subset \Omega$ is a small neighborhood of Γ_0. By the method in [3] we can show that φ_1 is a $C^{2+\alpha}$ function and therefore so is φ, and $\|\varphi\|_{C^{2+\alpha}(\Omega)} \leq C(\eta, M)$. We extend φ to the whole space R^n by a function $\hat{\varphi}$ such that

$$\|\hat{\varphi}\|_{C^1(R^n)} \leq C_0 \|\varphi\|_{C^1(\Omega)}, \; \|\hat{\varphi}\|_{C^{2+\alpha}(R^n)} \leq C_0 \|\varphi\|_{C^{2+\alpha}(\Omega)},$$

where C_0 depends only on $\|g\|_{C^{2+\alpha}}$.

Step 3. Define $\tilde{\psi}(x,t)$, for any $x \in \Omega$, as the unique solution of

$$\tilde{\psi}(x,t) = x + t\psi^0(x) - \int_0^t \int_0^s \nabla\hat{\varphi}\left(\tilde{\psi}(x,\tau), \tau\right) d\tau ds; \tag{3.6}$$

the method of successive iterations shows that indeed there exists a unique solution to (3.6). Set

$$F(x', x_n, t) = \tilde{\psi}_n(x,t) - g\left(\tilde{\psi}_1(x,t), ..., \tilde{\psi}_{n-1}(x,t)\right). \tag{3.7}$$

Then

$$D_nF(x,t) = 1 + tD_n\psi_n^0 - \int_0^t\int_0^s D_n\left(D_n\hat{\varphi}\left(\tilde{\psi}(x,\tau),\tau\right)\right)d\tau ds$$
$$-\nabla_{x'}g\cdot\left[tD_n\left(\psi^0\right)' - \int_0^t\int_0^s D_n\left(\nabla_{x'}\hat{\varphi}\left(\tilde{\psi}(x,\tau),\tau\right)\right)d\tau ds\right],$$

where $\left(\psi^0\right)' = \left(\psi_1^0,...,\psi_{n-1}^0\right)$. Hence, for small $T > 0$ (dependent on η and M),

$$D_nF(x,t) \geq \frac{1}{2} \text{ if } 0 \leq t \leq T. \tag{3.8}$$

From the implicit function theorem it then follows that there exists a unique function $x_n = \tilde{f}(x',t)$ of

$$F\left(x', \tilde{f}(x,t), t\right) \equiv 0. \tag{3.9}$$

We now use (2.13) to define $\tilde{\Psi}$ corresponding to $\tilde{\psi}$, $\tilde{f}$ and then define $\mathcal{A}(\Psi, f) = (\tilde{\Psi}, \tilde{f})$.

Step 4. We claim that

(a) $g(x') \leq \tilde{f}(x',t) \leq 1 - \delta_0$, $\tilde{f}_t(x',t) > 0$, and $\tilde{f}(x',0) = g(x')$;
(b) $\tilde{\psi}(x,t) \in \Omega$ for $\tilde{f}(x',t) < x_n < 1$;
(c) $(\tilde{\psi}, \tilde{f})$ (hence $(\tilde{\Psi}, \tilde{f})$) is independent of the particular extension $\hat{\varphi}$ of φ.

To prove (a), we first observe that from (3.7) and (3.9) at $t = 0$ it follows that $\tilde{f}(x',0) = g(x')$. Next we differentiate (3.7) in t to obtain

$$F_t(x,t) = D_t\tilde{\psi}_n - \nabla_{x'}g\cdot D_t\tilde{\psi}' = \sqrt{1 + |\nabla_{x'}g|^2}\,\vec{n}\cdot D_t\tilde{\psi}(x,t), \tag{3.10}$$

where $\vec{n}$ is the inward normal to Γ_0 at $(\tilde{\psi}'(x,t), g(\tilde{\psi}'(x,t)))$. Note that by (3.3) and (3.6), $|\tilde{\psi}(x,s) - x| \leq TC$ (depending on C_1 in (3.3) and M). Hence for small T, $\tilde{\psi}(x,s) \in \Omega(\Gamma_0)$. Differentiating (3.6) in t and taking the scalar product with the normal $\vec{n}$ at $x_n = \tilde{f}(x',t)$, we get

$$\vec{n}\cdot D_t\tilde{\psi}(x,t) = \vec{n}\cdot\psi^0(x) - \int_0^t \vec{n}\cdot\nabla\hat{\varphi}\left(\tilde{\psi}(x,s), s\right)ds < 0$$

at $x_n = \tilde{f}(x',t)$, where we used (3.5), (2.11); consequently, $F_t(x', \tilde{f}(x',t), t) < 0$. Since, by (3.9),

$$(D_nF)\tilde{f}_t(x',t) + F_t\left(x', \tilde{f}(x',t), t\right) = 0, \tag{3.11}$$

it follows (using also (3.8)) that $\tilde{f}_t > 0$. Finally, since $\tilde{f}(x',0) = g(x') \leq 1 - 2\delta_0$, we obtain $g(x') \leq \tilde{f}(x',t) \leq 1 - \delta_0$.

The assertion (b) can be verified as follows. Since $F(x', \tilde{f}(x',t), t) \equiv 0$ and $F(x', x_n, t)$ is strictly monotone increasing in x_n, it follows that $F(x', x_n, t) > 0$ if $x_n > \tilde{f}(x',t)$. Hence, by (3.7),

$$\tilde{\psi}_n(x,t) > g\left(\tilde{\psi}'\right) \text{ if } x_n > \tilde{f}(x',t).$$

It remains to show that $\tilde{\psi}_n(x,t) \leq 1$ in the set $\tilde{f}(x',t) < x_n < 1$. Suppose this is not true. Then we can find a point $\left(x^0, t^0\right)$ such that $\tilde{f}\left(x^{0\prime}, t^0\right) \leq x_n^0 < 1$, $\tilde{\psi}_n\left(x^0, t^0\right) = 1$, and $\tilde{\psi}_n\left(x^0, t\right) \leq 1$ for $t \leq t^0$. From (3.6),

$$1 = \tilde{\psi}_n\left(x^0, t^0\right) = x_n^0 + t\psi_n^0\left(x^0\right) - \int_0^{t^0}\int_0^s D_n\hat{\varphi}\left(\tilde{\psi}\left(x^0, \tau\right), \tau\right) d\tau ds$$
$$= x_n^0 + t\psi_n^0\left(x^0\right) - \int_0^{t^0}\int_0^s D_n\varphi\left(\tilde{\psi}\left(x^0, \tau\right), \tau\right) d\tau ds \leq x_n^0 < 1 \quad \text{(by (3.5))},$$

a contradiction.

We now show (c). By (b), $\tilde{\psi}(x,t) \in \Omega$ if $\tilde{f}(x', t) \leq x_n \leq 1$, and, therefore, $\hat{\varphi}(\tilde{\psi}(x,t),t) = \varphi(\tilde{\psi}(x,t),t)$. Hence for any $x \in \Omega, \tilde{\psi}(x,t)$ is a solution of (2.2) for $t < \min\left(\tilde{t}(x), T\right)$, where $\tilde{t}(x)$ is the largest number less that T such that $\tilde{f}\left(x', \tilde{t}(x)\right) \leq x_n$. Suppose that $\check{\varphi}$ is another extension of φ. Then by the above procedure we can define $\check{\psi}, \check{f}$, and $\check{F}$ corresponding to $\check{\varphi}$. By (b), we know that $\check{\psi}(x,t)$ solves (2.2) for $t < \min\left(\check{t}(x), T\right)$. By uniqueness of the solution to (2.2) (for fixed x), we then obtain $\tilde{\psi}(x,t) = \check{\psi}(x,t)$ for $t < \min\left(\tilde{t}(x), \check{t}(x)\right)$. It follows (from (3.7)) that $F\left(x', x_n, t\right) = \check{F}\left(x', x_n, t\right)$ for $t \leq \min\left(\tilde{t}(x), \check{t}(x)\right)$ and, since the solution to $F\left(x', x_n, t\right) = 0$ is unique, $\tilde{f}(x', t) = \check{f}(x', t)$. We thus conclude that $\tilde{\psi}$ is independent of the extension of φ.

Step 5. To prove that $\mathcal{A}$ maps $K(\eta, T)$ into itself (for small T and large η), we need to estimate the $C^{1+\alpha}$-norms of $\tilde{\psi}$ and $\tilde{f}$. Using the method in [3], we can show that

$$\left\|\tilde{\psi}(\cdot, t)\right\|_{C^{1+\alpha}} + \left\|\tilde{\psi}_t(\cdot, t)\right\|_{C^{1+\alpha}} \leq \left\|\psi^0\right\|_{C^{1+\alpha}} + tC(\eta, M) \tag{3.12}$$

for $0 \leq t \leq T$, where $C(\eta, M)$ depends on the initial data and on η, M. By (3.6), (3.10), (3.11), and $C^{1+\alpha}$ estimates of $\tilde{\psi}_t$ in (3.12), we also have

$$\left\|\nabla\tilde{\psi}(\cdot, t) - I_n\right\|_{L^\infty} \leq tC(\eta, M), \quad \left\|\tilde{f}_t(\cdot, t)\right\|_{C^\alpha} \leq C_0 + tC(\eta, M),$$

where the constant C_0 depends only on the initial data. The first inequality implies $\|\nabla\tilde{\psi}^{-1}(\cdot,t)\|_{L^\infty} \leq tC(\eta, M)(1-tC)^{-1}$. Since also $\psi(x,0) = x$, if we choose η large enough and T sufficient small (depending on η and M), then $(\tilde{\Psi}, \tilde{f}) \in K(\eta, T)$.

Having constructed the mapping $\mathcal{A}$, we now define successively

$$\left(\Psi^{(m+1)}, f^{(m+1)}\right) = \mathcal{A}\left(\Psi^{(m)}, f^{(m)}\right)$$

and let

$$\delta^{(m)}(t) = \sup_{x\in\Omega}\left|\Psi^{(m+1)}(x,t) - \Psi^{(m)}(x,t)\right| + \sup_{x'\in R^{n-1}}\left|f^{(m+1)}(x',t) - f^{(m)}(x',t)\right|.$$

As in [3], one can show that

$$\delta^{(m)}(t) \leq C^m t^m \left|\log t\right|^m. \tag{3.13}$$

Therefore, for $0 \leq t \leq T$ (T small), the sequence $\Psi^{(m)}(x,t)$ is uniformly convergent as $m \to \infty$. It follows that the mapping $\mathcal{A}$ admits a fixed point. As in [3], one can also establish uniqueness by slightly modifying the proof of the estimates (3.13). Clearly, a fixed point of $\mathcal{A}$ is a solution to (2.1)–(2.7). Conversely, any solution to (2.1)–(2.7) must belong to $K(\eta, T)$ for some η and T and is a fixed point of $\mathcal{A}$. This concludes the proof of Lemma 3.1. □

Proof of Theorem 2.1. For simplicity, we consider only the case $\psi^0 = 0$ and $\delta_0 \leq 1/2$ (δ_0 as in (2.10)). Lemma 3.1 guarantees a unique solution for small time if $M \geq M_1$. Suppose that (ψ, f) is a solution. Set

$$k(t) = \|\psi - \mathrm{id}\|_{C^{1+\alpha}} + \left\|\nabla\psi^{-1}\right\|_{C^\alpha} + \|f\|_{C^{1+\alpha}} + \left\|(1-f)^{-1}\right\|_{C^0},$$

where $\mathrm{id}(x) = x$. By slightly modifying the proof of Lemma 3.1, one sees that the solution can be extended as long as $k(t)$ is a priori bounded. We shall proceed to estimate $k(t)$.

Notice that $\left\|J\left(\psi^{-1}\right)\right\|_{C^\alpha} \leq C_1 \left\|\nabla\psi^{-1}\right\|_{C^1}^{4+\alpha} \|\nabla\psi\|_{C^{1+\alpha}}$, where C_1 is a generic constant (depending on Ω), and Ω_t^ψ contains a strip of width $\|\nabla\psi^{-1}\|_{C^0}^{-1}\|(1-f)^{-1}\|_{C^0}^{-1} \geq k(t)^{-2}$. From (3.2), it is easy to see that $\|\varphi_2\|_{C^{2+\alpha}} \leq C_1$. To estimate φ_1, we introduce the scaled functions $\tilde{\varphi}_1(x,t) = \varphi_1(x/k^2, t)$, $\tilde{P}(x,t) = P(x/k^2, t)$ (for fixed t). Equation (3.1) then becomes $\Delta\tilde{\varphi}_1 = k^{-4}\tilde{P}\chi_{\left(k^2\Omega_t^\psi\right)}$, where the region $k^2\Omega_t^\psi$ contains a strip of width 1. By an estimate analogous to (3.12) (by using [3, Theorem 7.1]), we find that

$$\|\tilde{\varphi}_1\|_{C^{2+\alpha}} \leq C_1 \left(\left\|k^{-4}\tilde{P}\right\|_{C^\alpha\left(k^2\Omega_t^\psi\right)} + \left\|k^2\nabla\psi\right\|_{C^\alpha}\right) \ln\left(2 + \left\|k^2\nabla\psi\right\|_{C^\alpha}\right),$$

since $k^2\Omega_t^\psi$ is bounded by the curves $k^2\Gamma_0$ and $\left\{k^2\psi(x', 1, t) : x' \in R^{n-1}\right\}$. It follows that

$$\|\varphi_1\|_{C^{2+\alpha}} \leq k^{4+2\alpha} \|\tilde{\varphi}_1\|_{C^{2+\alpha}} \leq C_1 k^8(\tau) \ln(2 + k(\tau)).$$

Since the solution (ψ, f) is the fixed point of the mapping $\mathcal{A}$ defined in the proof of Lemma 3.1, one sees from (3.6) that

$$\psi(x,t) = x - \int_0^t \int_0^s \left(M\nabla\varphi_2(\psi,\tau) + \nabla\varphi_1(\psi,\tau)\right) d\tau ds. \tag{3.14}$$

Hence

$$\nabla\psi = I - \int_0^t \int_0^s \left(MD^2\varphi_2(\psi,\tau) + D^2\varphi_1(\psi,\tau)\right) \nabla\psi \, d\tau ds. \tag{3.15}$$

Analogously, since $F(x', f(x',t), t) = 0$, where F is defined in (3.7), we have

$$f(x',t) = g(\psi') + \int_0^t \left(\int_0^s MD_n\varphi_2(\psi,\tau) + D_n\varphi_1(\psi,\tau)\right) d\tau ds, \tag{3.16}$$

where we used (2.4) and (3.14). Note that

$$\left\|D^2\varphi_2(\psi,\tau)\right\|_{C^\alpha} \leq C_1 \left\|D^2\varphi_2\right\|_{C^\alpha} \|\nabla\psi\|_{C^0} \leq C_1 k(\tau)$$

and

$$\begin{aligned}\left\|D^2\varphi_1(\psi,\tau)\right\|_{C^\alpha} &\leq \left\|D^2\varphi_1\right\|_{C^\alpha} \|\nabla\psi\|_{C^0} \\ &\leq C_1 k^9(\tau) \ln(2 + k(\tau)) \leq C_1 k^{10}(\tau).\end{aligned}$$

By differentiating (3.16) in x' and using (3.15) and the last two estimates, we get

$$\|\psi - \mathrm{id}\|_{C^{1+\alpha}} + \|f\|_{C^{1+\alpha}} \le C_1 + C_1 \int_0^t \int_0^s \left(Mk^2(\tau) + k^{11}(\tau)\right) d\tau ds, \tag{3.17}$$

$$\|\nabla\psi - I\|_{C^0} \le C_1 \int_0^t \int_0^s \left(Mk(\tau) + k^{10}(\tau)\right) d\tau ds. \tag{3.18}$$

Using (3.16) and the fact that $1 - g \ge 2\delta_0$ (see (2.10)), we also have

$$1 - f(x', t) \ge 2\delta_0 - C_1 \int_0^t \int_0^s \left(Mk(\tau) + k^{10}(\tau)\right) d\tau ds. \tag{3.19}$$

From (3.18), (3.19), and the inequality $\left\|\nabla\psi^{-1}\right\|_{C^0} \le C_1 \left(1 - \|\nabla\psi - I\|_{C^0}\right)^{-1}$, it follows that $\left\|\nabla\psi^{-1}\right\|_{C^0} + \left(1 - f(x', t)\right)^{-1} \le 2C_1 + 1/\delta_0$, provided that

$$C_1 \int_0^t \int_0^s \left(Mk(\tau) + k^{10}(\tau)\right) d\tau ds \le \delta_0. \tag{3.20}$$

Combining these inequalities with (3.17) and recalling the definition of $k(t)$, we obtain the inequality

$$k(t) \le C_1 + C_1 \int_0^t \int_0^s Mk^{11}(\tau)\, d\tau ds \equiv h(t), \tag{3.21}$$

as long as (3.20) is satisfied. Clearly, $h'' \le MC_1h^{11}$ or $h' \le MC_1th^{11}$. Solving this differential inequality, we find that

$$k(t) \le h(t) \le \frac{C_1}{\left(1 - 5MC_1^{11}t^2\right)^{1/10}} \le 2C_1, \tag{3.22}$$

provided

$$t^2 M \le \frac{1}{10C_1^{11}}. \tag{3.23}$$

Substituting (3.22) into (3.20), it is easy to see that (3.20) and (3.23) remain true as long as $t^2 \le C_0/M$, provided $M \ge M_0 = \max(M_1, 1)$, where M_1 is the same constant as in Lemma 3.1, and

$$C_0 = 2^{-10}C_1^{-11}\delta_0.$$

A continuity argument then completes the proof of Theorem 2.1.

Remark 3.1. One can see from the proof that M_1 in Lemma 3.1 can be any number large than $C_1C_2^{-1}$, where C_1 and C_2 are obtained from certain elliptic a priori estimates and depend also on the geometry; the dependence can be made more specific by looking carefully into the proofs of these a priori estimates. In the case that $\psi_n^0(x) \le -\sigma$ and $\vec{n} \cdot \psi^0 \le -\sigma$, $\sigma > 0$, the proof simplifies and we can choose $M_1 = 0$ (since (3.5) is then not needed). The constant M_0 in Theorem 2.1 can be any positive number larger than or equal to M_1.

DEFINITION 3.1. *We call T the* finishing time *if a solution of* (2.1)–(2.7) *exists for $t < T$ and* $\limsup_{t\to T} \|f(\cdot, t)\|_{C^0} = 1$.

Physically, the finishing time is the moment when some particles situated initially on the top boundary of the region have travelled all the way to the workpiece. No more paint is deposited on some parts of the workpiece after that time.

Remark 3.2. In general a solution may not exist for long time. For systems in the whole space, there are examples (see [3, Theorem 7.1]) of initial data for which the solutions do not exist beyond a finite time T_0, due to the fact that Jacobian $J(\psi)$ degenerate as $t \to T_0$. For the special geometry of the present paper, nonexistence may result also from the loss of the $C^{1+\alpha}$ regularity of f. In addition, near the finishing time T, the thickness of Ω_t^{ψ} may vary sharply, and this would further speed up development of singularities through the deterioration of $C^{2+\alpha}$-norm of the potential φ_1 in (3.1). It is not clear whether a singularity would occur before the finishing time. In some special cases, existence of solutions until "very close" to the finishing time can be proved, as will be seen in the next three sections. For general initial data, we have the following estimates for the finishing time for large M.

COROLLARY 3.2. *Suppose that*

$$\frac{\left(\cosh^{-1} 2\right)^2}{2\left(1-\|g\|_{C^0}\right)}\lambda_1 > \lambda_0, \tag{3.24}$$

where

$$\lambda_0 = \left\|D^2\varphi_2\right\|_{C^0},\ \lambda_1 = \inf_{x\in\Omega} D_n\varphi_2,$$

and φ_2 *is the solution of* (3.2). *Let* $T(M)$ *be the maximum time of existence for the solution. Then, for any* $\delta > 0$, *there exists an* $M_1(\delta)$ *such that if* $M \ge M_1(\delta)$, *then*

$$\limsup_{t\to T(M)} \|f(x,t)\|_{C^0} > 1-\delta. \tag{3.25}$$

Corollary 3.2 implies that for large M, the maximum time of existence for the solution is almost the finishing time provided that condition (3.24) holds.

Notice that the function φ_2 from (3.2) depends only on g (which determines the domain Ω). In the case $g = 0, \varphi_2(x) = x_n$ and, consequently, (3.24) holds. We shall see in section 4 that in that case, the solution exists exactly up to the finishing time $T(M) = \sqrt{2/M}$ and $f(T(M)) = 1$. By continuity, (3.24) holds if $\|g\|_{C^{2+\alpha}}$ is small. Notice also that by the strong maximum principle $D_n\varphi_2 > 0$ on the boundary of Ω. Since $D_n\varphi_2$ is harmonic in Ω, it follows again by the strong maximum principle that $\lambda_1 > 0$.

Proof of Corollary 3.2. Without loss of generality, we may assume that $\lambda_0 > 0$ (since otherwise φ_2 is a linear function and, consequently, the assertion follows easily). Choose $\varepsilon > 0$ and x_0' such that

$$\frac{\lambda_1}{2\left(1-g\left(x_0'\right)+\varepsilon\right)}\left(\cosh^{-1}\frac{2(1-\varepsilon)}{1+\varepsilon}\right)^2 > \lambda_0. \tag{3.26}$$

From (3.15), we have

$$\begin{aligned}\nabla\psi - I = &-M\int_0^t\int_0^s D^2\varphi_2(\psi)\,d\tau ds - \int_0^t\int_0^s D^2\varphi_1(\psi)\nabla\psi\,d\tau ds\\ &+M\int_0^t\int_0^s D^2\varphi_2(\psi)(I-\nabla\psi)\,d\tau ds.\end{aligned} \tag{3.27}$$

It follows (analogously to (3.18)) that

$$\|\nabla\psi - I\|_{C^0} \le M\lambda_0\int_0^t\int_0^s \|\nabla\psi - I\|_{C^0}\,d\tau ds + C_1\int_0^t\int_0^s k^{10}\,d\tau ds + \frac{M\lambda_0}{2}t^2. \tag{3.28}$$

It is easy to check that $w(t) = (1+\varepsilon)\cosh\left(\sqrt{\lambda_0 M t^2}\right) - 1$ solves

$$w(\tau) = M\lambda_0 \int_0^t \int_0^s w(\tau)\, d\tau ds + \frac{M\lambda_0}{2} t^2 + \varepsilon. \tag{3.29}$$

If

$$C_1 \int_0^t \int_0^s k^{10}(\tau)\, d\tau ds \le \varepsilon, \tag{3.30}$$

then we can compare w with $\|\nabla\psi - I\|_{C^0}$ by means of the integral equations (3.28) and (3.29) and conclude that

$$\|\nabla\psi - I\|_{C^0} \le w(t) = (1+\varepsilon)\cosh\left(\sqrt{M\lambda_0} t\right) - 1. \tag{3.31}$$

Let

$$C_0 = \lambda_0^{-1}\left(\cosh^{-1}\left(\frac{2(1-\varepsilon)}{1+\varepsilon}\right)\right)^2. \tag{3.32}$$

Suppose for this moment that (3.30) holds. Then for $Mt^2 \le C_0$ we find from (3.31) that $\|\nabla\psi - I\|_{C^0} \le 1 - 2\varepsilon < 1$, and, consequently, $\left\|\nabla\psi^{-1}\right\|_{C^0} \le 1/(2\varepsilon)$. By using once again $\|\nabla\psi - I\|_{C^0} \le 1$, we deduce from (3.27) and (3.16) (analogously to the derivation of (3.17)) that

$$\|\nabla\psi - I\|_{C^\alpha} \le C_1 + C_1 M \int_0^t \int_0^s k(\tau)\, d\tau ds + C_1 \int_0^t \int_0^s k^{11}(\tau)\, d\tau ds \tag{3.33}$$

and

$$\|f\|_{C^{1+\alpha}} \le C_1 + M\lambda_0 \int_0^t \int_0^s k(\tau)\, d\tau ds + C_1 \int_0^t \int_0^s k^{11}(\tau)\, d\tau ds \tag{3.34}$$

for $Mt^2 \le C_0$. Denote by $T(M)$ the maximum time of existence; i.e., solution exists for $t < T(M)$ but not at $t = T(M)$. We claim that for any $\delta > 0$, if $M \ge M_1(\delta)$ (M_1 will be determined later), then $\limsup_{t\to T(M)} \|f\|_{C^0} > 1 - \delta$.

Indeed, proceeding by contradiction, suppose that for $t < T(M)$,

$$\|f\|_{C^0} \le 1 - \delta. \tag{3.35}$$

Then $(1-f) \ge \delta$. For simplicity, we may assume that $\delta \le \varepsilon < 1/4$. It follows from (3.33) and (3.34) that

$$k(t) \le C_1 + \frac{2}{\delta} + C_1 M \int_0^t \int_0^s k(\tau)\, d\tau ds + C_1 \int_0^t \int_0^s k^{11}(\tau)\, d\tau ds \equiv h(t)$$

if $Mt^2 \le C_0$ (with assumption (3.30)). Hence $h'' \le C_1 M h + C_1 h^{11}$ or $h' \le C_1 M t h + C_1 t h^{11}$. It follows that

$$\left(h e^{-C_1 M t^2/2}\right)' \le C_1 t h^{11} e^{-C_1 M t^2/2} \le C_1 t \left(h e^{-C_1 M t^2/2}\right)^{11} e^{5 C_1 M t^2}.$$

Therefore, by integration,

$$k(t) \le h(t) \le 2\left(C_1 + \frac{2}{\delta}\right) \tag{3.36}$$

if $Mt^2 \leq C_0$ and

$$t^2 \leq \frac{\delta}{10C_1\left(2+\delta C_1\right)e^{5C_1C_0}}. \tag{3.37}$$

So far we have established (3.36) under the assumptions (3.30), $Mt^2 \leq C_0$, and (3.37). The same continuity argument as in the proof of Theorem 2.1 deduces that (3.36) holds if $t^2M \leq C_0$ and $M \geq M_1(\delta)$, where

$$M_1(\delta) = 2^{10}\left(C_1 + \frac{2}{\delta}\right)^{11} C_0$$

(which guarantees (3.30) and (3.37)). Consequently, the solution can be extended as long as $t^2M \leq C_0$ so that $T(M) > \sqrt{C_0/M}$. Taking $t = \sqrt{C_0/M} \equiv T_0$, it follows from (3.16), (3.26), and (3.30) that, for $x_1' = \psi^{-1}(x_0', g, T_0)'$ (so that $\psi'(x_1', f, t) = x_0'$),

$$1 - f(x_1', T_0) \leq 1 - g(\psi'(x_1', f, T_0)) - \frac{MT_0^2}{2}\lambda_1 + \varepsilon = 1 - g(x_0') - \frac{C_0}{2}\lambda_1 + \varepsilon \leq 0,$$

a contradiction to (3.35). This completes the proof of Corollary 3.2.

4. Axially symmetric solution. The results in sections 2 and 3 can be extended to the cases where P_0 and ψ^0 do not have compact supports. We shall be interested in global solutions, that is, solutions that exist until time T when all the particles have migrated to Γ_0; i.e., $P(x,T) = 0$. We first restrict our attention to axially symmetric data, i.e.,

$$P_0(x) = P_0(x_n),\ g(x') = 0 \tag{4.1}$$

and

$$\psi^0(x) = (0, ..., 0, b_n x_n),\quad b_n \leq 0; \tag{4.2}$$

in the next section we shall consider a small perturbation of such data.

Set

$$\Omega_0 = \{x \in R^n:\ \ 0 < x_n < 1\}.$$

Since the data are axially symmetric with respect to x_n-axis, we expect that the solution will also be axially symmetric. Thus, we seek a solution in Ω_0 of the form

$$\begin{cases} \varphi(x,t) = \varphi_1(x_n,t) + Mx_n, \\ \psi_i(x,t) = x_i \ \text{ for } \ 1 \leq i \leq n-1, \\ \psi_n(x,t) = \psi_n(x_n,t), \\ f(x',t) = f(t) \end{cases} \tag{4.3}$$

with $P = P(x_n, t)$. From equation (2.1) and the boundary conditions in (2.6),

$$\begin{cases} \Delta\varphi_1 = -P\mathcal{X}_{\Omega_t^\psi} \ \text{ in } \ \Omega_0, \\ \varphi_1 = 0 \ \text{ on } \ \Gamma_0 \cup \Gamma_\infty. \end{cases} \tag{4.4}$$

Hence, for $x_n \leq \psi_n(1,t)$,

$$\begin{aligned} D_n\varphi_1(x_n,t) &= A(t) - \int_0^{x_n} P(\xi,t)\, d\xi \\ &= A(t) - \int_0^{x_n} P_0\left(\psi_n^{-1}(\xi,t)\right) D_n\left(\psi_n^{-1}(\xi,t)\right) d\xi \\ &= A(t) - \int_{f(t)}^{\psi_n^{-1}(x_n,t)} P_0(\xi)\, d\xi, \end{aligned}$$

where we used the fact that $\psi_n(f(t), t) = 0$; here $A(t)$ is a function that will be determined later on. It follows that

$$(4.5)\ D_n\varphi_1(x_n, t) = \begin{cases} A(t) - \int_{f(t)}^{\psi_n^{-1}(x_n,t)} P_0(\xi, t)\, d\xi & \text{for } 0 \le x_n \le \psi_n(1, t), \\ A(t) - \int_{f(t)}^{1} P_0(\xi, t)\, d\xi & \text{for } \psi_n(1, t) \le x_n \le 1. \end{cases}$$

By (2.2)

$$\frac{d^2\psi_n(x_n t)}{dt^2} = -D_n\varphi(\psi_n, t) = -D_n\varphi_1(\psi_n, t) - M.$$

Therefore, as long as the trajectory lies in Ω_0,

$$(4.6)\ \psi_n(x_n, t) = (1 + b_n t)\, x_n - \frac{t^2}{2}M - \int_0^t \int_0^s \left(A(\tau) - \int_{f(\tau)}^{x_n} P_0(\xi)\, d\xi \right) d\tau ds.$$

To determine $A(t)$ we integrate both sides of (4.5) and use the boundary conditions in (4.4):

$$\int_0^{\psi_n(1,t)} \left(A(t) - \int_{f(t)}^{\psi_n^{-1}(x_n,t)} P_0(\xi, t)\, d\xi \right) dx_n + \int_{\psi_n(1,t)}^{1} \left(A(t) - \int_{f(t)}^{1} P_0(\xi, t)\, d\xi \right) dx_n = 0$$

or

$$A(t) - (1 - \psi_n(1, t)) \int_{f(t)}^{1} P_0(\xi, t)\, d\xi = \int_0^{\psi_n(1,t)} \int_{f(t)}^{\psi_n^{-1}(x_n,t)} P_0(\xi, t)\, d\xi dx_n$$

$$(4.7)\quad = \int_{f(t)}^{1} \left(\int_{f(t)}^{z} P_0(\xi)\, d\xi \right) D_n\psi_n(z, t)\, dz \ \ \text{(by substituting } z = \psi_n^{-1}(x_n, t)\text{)}.$$

From (4.6) we have, by differentiation,

$$(4.8)\qquad D_n\psi_n(x_n, t) = (1 + b_n t) + \frac{t^2}{2} P_0(x_n).$$

Substituting this into the right-hand side of (4.7), we find the following expression for $A(t)$:

$$A(t) = (1 - \psi_n(1, t)) \int_{f(t)}^{1} P_0(\xi)\, d\xi$$

$$(4.9)\qquad + \int_{f(t)}^{1} \left(\int_{f(t)}^{z} P_0(\xi)\, d\xi \right) \left(1 + b_n t + \frac{t^2}{2} P_0(z) \right) dz.$$

To compute $\psi_n(1, t)$ we integrate (4.8) (with respect to x_n) and use the fact that $\psi_n(f(t), t) = 0$:

$$(4.10)\qquad \psi_n(1, t) = (1 + b_n t)(1 - f(t)) + \frac{t^2}{2} \int_{f(t)}^{1} P_0(\xi)\, d\xi.$$

Finally, using the relation $\psi_n(f(t), t) = 0$ in (4.6), we arrive at the following formula for f:

$$f(t) = \frac{Mt^2}{2(1+b_n t)} + \frac{1}{1+b_n t}\int_0^t \int_0^s \left(A(\tau) - \int_{f(\tau)}^{f(t)} P_0(\xi)\, d\xi \right) d\tau ds. \tag{4.11}$$

THEOREM 4.1. *Assume that* $b_n \le 0$, $P_0 \ge 0$, *and* P_0 *is continuous. Then there is a constant* $M_0 > 0$ *(depending on* b_n *and* P_0*) such that for any* $M \ge M_0$ *there exists a unique solution to the system* (4.9)–(4.11) *for* $0 \le t \le T(M, b_n)$, *and*

$$0 \le f(t) \le 1,\ f'(t) > 0 \ \ \text{for} \ \ 0 \le t < T(M, b_n),$$

$$f(t) \nearrow 1 \ \ \text{as} \ \ t \nearrow T;$$

moreover, for large M,

$$\frac{\partial T}{\partial M} \le 0, \quad \frac{\partial T}{\partial b_n} \ge 0, \tag{4.12}$$

and

$$T(M, b_n)\sqrt{M} \longrightarrow \sqrt{2} \ \ \text{as} \ \ M \longrightarrow \infty, \tag{4.13}$$

uniformly in b_n, *if* b_n *remains bounded.*

Proof. Consider first this case where $D_n P_0$ is Lipschitz continuous. By extension, we may assume that P_0 is defined for all $x \in R^1$. Multiplying (4.11) by $(1 + b_n t)$ and differentiating in t, we get

$$\begin{aligned}&\left[1 + b_n t + \frac{t^2}{2} P_0(f(t))\right] f'(t)\\ &= Mt - b_n f(t) + \int_0^t \left(A(\tau) - \int_{f(\tau)}^{f(t)} P_0(\xi)\, d\xi \right) d\tau.\end{aligned} \tag{4.14}$$

Differentiating (4.14) once again, we obtain

$$\begin{aligned}&\left[1 + b_n t + \frac{t^2}{2} P_0(f(t))\right] f''(t)\\ &= M + A(t) - \left[2b_n + 2tP_0(f(t)) + \frac{t^2}{2} P_0'(f(t)) f'(t)\right] f'(t).\end{aligned} \tag{4.15}$$

From (4.9) and (4.10), one sees that the right-hand side of (4.15) is Lipschitz continuous in $f(t), f'(t)$, and t. Therefore, by standard ODE theory, equation (4.15) together with the initial conditions $f(0) = f'(0) = 0$ has a unique solution as long as $1 + b_n t > 0$.

Consider first the case that $b_n < 0$. If we substitute (4.10) into (4.9) we find, after some manipulation, that for $t \ge \tau$,

$$\begin{aligned}A(\tau) - \int_{f(\tau)}^{f(t)} P_0(\xi)\, d\xi &= -\frac{\tau^2}{2} \int_{f(\tau)}^1 P_0(\xi) \int_\xi^1 P_0(\eta)\, d\eta\, d\xi\\ &\quad + \int_{f(t)}^1 P_0(\xi)\, d\xi - (1 + b_n \tau) \int_{f(\tau)}^1 \int_\xi^1 P_0(\eta)\, d\eta d\xi.\end{aligned} \tag{4.16}$$

It follows that

$$M + A(\tau) - \int_{f(\tau)}^{f(t)} P_0(\xi)\, d\xi \geq \gamma$$

for any $\gamma > 0$ if

$$M \geq \frac{\tau^2}{2} \left(\int_0^1 P_0(\xi)\, d\xi \right)^2 + \int_0^1 P_0(\xi)\, d\xi + \gamma. \tag{4.17}$$

We choose

$$M_0 = \frac{T_0^2}{2} \left(\int_0^1 P_0(\xi)\, d\xi \right)^2 + \int_0^1 P_0(\xi)\, d\xi + \gamma, \quad T_0 = -\frac{1}{2b_n}, \tag{4.18}$$

where γ is a positive constant that will be determined later on. Then, for any $M \geq M_0$, (4.17) holds for $t < T_0$. Hence from (4.14),

$$f'(t) > 0 \text{ for } 0 \leq t \leq T_0,$$

and from (4.11),

$$f(t) \geq \frac{\gamma t^2}{(1 + b_n t)} \text{ for } 0 \leq t \leq T_0$$

as long as $f(t) \leq 1$. It follows that if

$$\gamma \geq \frac{1 + b_n T_0}{T_0^2} = 2b_n^2,$$

then, for any $M \geq M_0$, $f(t) \nearrow 1$ as $t \nearrow T(M, b_n)$ for some $T(M, b_n) \leq T_0$. The solution $f(t)$ of (4.15) determines the unique solution of (4.9)–(4.11).

In case $b_n = 0$, we choose

$$T_0 = \left(\frac{1}{2} \int_0^1 P_0(\xi)\, d\xi \right)^{-\frac{1}{2}}, \quad \gamma \geq \int_0^1 P_0(\xi)\, d\xi,$$

so that

$$M_0 = 3 \int_0^1 P_0(\xi)\, d\xi. \tag{4.19}$$

As before, if $M \geq M_0$ there exists a $T(M, b_n) \leq T_0$ such that $f'(t) > 0$ for $t < T(M, b_n)$ and

$$f(t) \nearrow 1 \text{ as } t \nearrow T(M, b_n).$$

From now on we fix

$$\gamma = \max\left(2b_n^2, \int_0^1 P_0(\xi)\, d\xi \right) \tag{4.20}$$

and take M_0 to be the maximum of the two number M_0 in (4.18) and (4.19). Then, if $M \geq M_0$ the assertions of the theorem follow.

We now drop the assumption that P_0' is Lipschitz continuous. Let P_m be a sequence of smooth functions that converge uniformly to P_0. For each m, there exists a solution $f_m(t)$ for $t \le T_0$. From (4.14), we know that f_m and f'_m are uniformly bounded. Let $A_m(t)$ denote the function $A(t)$ corresponding to f_m. It is easily seen that

$$|A_l(t) - A_m(t)| \le C|f_l(t) - f_m(t)| + C\|P_l - P_m\|_{L^\infty}$$

and, from (4.11),

$$|f_l(t) - f_m(t)| \le Ct^2\left(\sup_{\tau\le t}|f_l(\tau) - f_m(\tau)| + \|P_l - P_m\|_{L^\infty}\right).$$

Hence for small t, $f_m(t)$ converges uniformly to a function $f(t)$; by (4.14), $f'_m(t)$ also converges uniformly to $f'(t)$. A step-by-step argument now shows that $f_m(t) \to f(t)$ as long as $f(t)$ remains less than 1. By substituting $t = T(M, b_n)$ in (4.11), we find that, since $f(T) = 1$,

$$MT^2 - 2(1 + b_nT) = -2\int_0^T\int_0^s\left(A(\tau) - \int_{f(\tau)}^1 P_0(\xi)\,d\xi\right)d\tau ds. \tag{4.21}$$

From (4.16), $|A(t)| \le C + CT^2$, where C depends on b_n and initial data. Hence, by (4.21), one sees that $T \to 0$ as $M \to \infty$. The limit (4.13) follows by taking $M \to \infty$ in (4.21). Finally, by differentiating (4.21) with respect to M, we find

$$\begin{aligned}&\left(2MT - 2b_n + 2\int_0^T\left(A(\tau) - \int_{f(\tau)}^1 P_0(\xi)\,d\xi\right)d\tau\right)\frac{\partial T}{\partial M}\\ &= -T^2 + \int_0^T\int_0^s\left(\frac{\partial A(\tau)}{\partial M} + \frac{\partial f(\tau)}{\partial M}P_0(f(\tau))\right)d\tau ds.\end{aligned} \tag{4.22}$$

Since $b_n \le 0$ and $|A(t)| \le C + CT^2$, we find (by using (4.13)) that for large M, the coefficient of $\partial T/\partial M$ in (4.22) is positive. By differentiating (4.16) (with $t = T$) in M, we have $|\partial A(\tau)/\partial M| \le C|\partial f(\tau)/\partial M|$. It follows from (4.11) that $\partial f(\tau)/\partial M$ solves an integral equation and, consequently,

$$\left|\frac{\partial A(\tau)}{\partial M}\right| \le C\left|\frac{\partial f(\tau)}{\partial M}\right| \le C\tau^2.$$

The first inequality in (4.12) follows since the left-hand side in (4.22) is negative for large M. The second inequality can be proved analogously. ∎

Remark 4.1. If P_0 is piecewise continuous, then we can extend Theorem 4.1 by establishing existence and uniqueness of a solution f in interval (t_i, t_{i+1}) where $x = f(t_i)$ and $x = f(t_{i+1})$ are two adjacent discontinuities of $P_0(x)$. One can show that $f(x)$ is Lipschitz continuous at $t = t_i$, and $f(t)$ is continuously differentiable with $f'(t) > 0$ if $t \ne t_i$.

We next show that system (4.9)–(4.11) is equivalent to system (2.1)–(2.7) with the data (4.1) and (4.2).

THEOREM 4.2. *Let* $(A(t), f(t))$ *be a solution of* (4.9)–(4.11) *established in Theorem* 4.1. *Then the functions in* (4.3) *with* φ_1 *and* ψ_n *defined by* (4.5) *and* (4.6) *form the unique solution of* (2.1)–(2.7).

Proof. Actually the only thing that remains to be proved is that

$$0 \le \psi_n(x_n, t) \le 1 \;\text{ if }\; f(t_n) \le x_n \le 1,\; 0 \le t \le T \quad (T = T(M, b_n)). \tag{4.23}$$

To prove it, we differentiate (4.10) and use (4.14), (4.20). It follows that $D_t\psi_n(1,t) \leq 0$. Since $\psi_n(1,0) = 1$ (by (4.10)), we conclude that

$$\psi_n(1,t) \leq 1 \text{ for } t \leq T,$$

and, consequently, since $D_n\psi_n(x_n,t) > 0$ (by (4.8)), we have $\psi_n(x_n,t) \leq 1$ and $\psi_n(x_n,t) \geq 0$ if $f(t) \leq x_n \leq 1$ (since $\psi_n(f(t),t) = 0$). This completes the proof of (4.23). □

Example 4.1. Let

$$b_n = 0,\ P_0(x_n) = p_0\mathcal{X}_{(0,1)}(x_n),\ p_0 > 0. \tag{4.24}$$

Then

$$\psi_n(x_n,t) = \left(1 + \frac{p_0t^2}{2}\right)(x_n - f(t)), \tag{4.25}$$

where $f(t)$ solves

$$\left(1 + \frac{p_0t^2}{2}\right)f''(t) + 2p_0tf'(t) = A(t) + M, \quad f(0) = f'(0) = 0, \tag{4.26}$$

and

$$A(t) = \frac{p_0}{2}(1 - f(t))\left[\left(1 + \frac{p_0t^2}{2}\right)f(t) + 1 - \frac{p_0t^2}{2}\right]. \tag{4.27}$$

Note that (4.26) is equivalent to

$$\left(1 + \frac{p_0t^2}{2}\right)f(t) = \frac{Mt^2}{2} + \int_0^t\int_0^s (A(\tau) + p_0f(\tau))\, d\tau ds. \tag{4.28}$$

5. A perturbed system. In the previous section we established existence of a global solution if the domain and the initial data are axially symmetric. We now consider system (2.1)–(2.9) in the case where

$$g(x') = \varepsilon h(x'),\ \psi_t(x,0) = 0,\ P_0(x) = p_0, \tag{5.1}$$

where ε is small, $p_0 > 0$ is a constant, and $h(x')$ is a $C^{2+\alpha}$ function. For simplicity, we assume that $\varepsilon > 0,\ h(x) \geq 0$. By slightly modifying the proof of Theorem 2.1, one can show that there exists a unique solution $(\psi_\varepsilon, f_\varepsilon, \varphi_\varepsilon)$ for $0 \leq t \leq T$ for some $T > 0$ which is independent of ε. We shall use the following notations:

$$\begin{aligned}
\Omega_\varepsilon &= \{(x',x_n) : \ \varepsilon h(x') < x_n < 1,\ x' \in R^{n-1}\},\\
\Omega_{\varepsilon t} &= \left\{(x',x_n) : \ f_\varepsilon(x') < x_n < 1,\ x' \in R^{n-1}\right\},\\
\Gamma_\varepsilon &= \{(x',x_n) : \ x_n = \varepsilon h(x'),\ x' \in R^n\},\\
\Gamma_{\varepsilon t} &= \{(x',x_n) : \ x_n = f_\varepsilon(x'),\ x' \in R^n\};
\end{aligned}$$

the set $\Omega_{\varepsilon t}^{\psi_\varepsilon}$ is defined similarly. In this section Ω_0 and Γ_0 are understood as Ω_ε and Γ_ε, respectively, for $\varepsilon = 0$. To establish global existence, we need the following lemmas.

LEMMA 5.1. *Let* $0 < a < 1,\ g_1(x') \in C^{2+\alpha}\left(R^{n-1}\right),\ g_2(x') \in C^{1+\alpha}\left(R^{n-1}\right),$ *and* $b(x') \in C^{2+\alpha}\left(R^{n-1}\right)$. *For any* ε *small such that* $a + \|\varepsilon g_2\|_{L^\infty} < 1$, *set*

$$\begin{aligned}
G_\varepsilon &= \{(x',x_n) : \ \varepsilon g_1(x') < x_n < a + \varepsilon g_2(x')\},\\
G_1 &= \{(x',x_n) : \ \varepsilon g_1(x') < x_n < 1\},\\
G_0 &= \{(x',x_n) : \ 0 < x_n < a\}.
\end{aligned}$$

Suppose u_ε is a bounded solution of

$$\begin{aligned}\Delta u_\varepsilon &= q\mathcal{X}_{G_\varepsilon} \quad in \quad G_1, \\ u_\varepsilon = 0 \quad on \quad x_n = 1,& \quad u_\varepsilon = b(x') \quad on \quad x_n = \varepsilon g_1(x'),\end{aligned} \tag{5.2}$$

where $q \in C^\alpha(G_1)$. Then there exists a positive constant ε_0 (depending on g_1, g_2, and a) such that $\|\varepsilon_0 g_1\|_{C^{2+\alpha}} + \|\varepsilon_0 g_2\|_{C^{1+\alpha}} \le 1$ and, for $|\varepsilon| \le \varepsilon_0$,

$$\|u_\varepsilon\|_{C^{2+\alpha}(G_\varepsilon)} \le \frac{C}{a^{2+\alpha}} \left(\|q\|_{C^\alpha} + \|b\|_{C^{2+\alpha}}\right), \tag{5.3}$$

where C is a constant independent of a, b, g_i, q, and ε. Moreover, if $g_1 \ge 0$, $g_2 \le 0$, and u_0 is the solution corresponding to $\varepsilon = 0$, then

$$\|u_\varepsilon - u_0\|_{C^{2+\alpha}(G_\varepsilon)} \le \frac{\varepsilon C}{a^{2+\alpha}} \left(\|q\|_{C^\alpha} + \|b\|_{C^{2+\alpha}}\right) \tag{5.4}$$

for $0 \le \varepsilon \le \varepsilon_0$.

The assumptions $\varepsilon \ge 0, g_1 \ge 0$, $g_2 \le 0$ are made only for the sake of simplicity so that $G_\varepsilon \subseteq G_0$. Without these assumptions, however, (5.4) is still true provided u_0 is "properly" extended to G_ε.

Proof. Consider first the case $a = 1/2$. Choose ε_0 such that $|\varepsilon_0 g_i(x')| \le 1/8$. For any $x_0' \in R^{n-1}$, take a cut-off function $r(x')$ that equals 1 if $|x' - x_0'| \le 1/4$ and 0 if $|x' - x_0'| \ge 1/2$. Denote by $B_\rho = B_\rho(x_0')$ the ball in R^n of radius ρ centered at $(x_0', \varepsilon g_2(x_0')))$. Set

$$w(x) = \omega_2 \int_{G_\varepsilon} \frac{q(\xi)\, r(\xi)}{|x - \xi|^{n-2}}\, d\xi,$$

where the constant ω_2 is chosen so that

$$\Delta w = rq\mathcal{X}_{G_\varepsilon} \quad \text{in} \quad G_1.$$

By [3, Lemma 4.2],

$$\|w\|_{C^2\left(G_\varepsilon \cap B_{1/4}\right)} + \|w\|_{C^{2+\alpha}\left(G_\varepsilon \cap B_{1/4}\right)} \le C\left(\|q\|_{C^\alpha} + \|b\|_{C^{2+\alpha}}\right). \tag{5.5}$$

The function $v_\varepsilon = u_\varepsilon - w$ satisfies

$$\Delta v_\varepsilon = 0 \quad \text{in} \quad G_1 \cap B_{1/4}.$$

By the maximum principle, $\|u_\varepsilon\|_{L^\infty} \le C\left(\|q\|_{L^\infty} + \|b\|_{L^\infty}\right)$, and the same estimate then holds for v_ε in $G_\varepsilon \cap B_{1/4}$ (by (5.5)). It follows (by the interior Schauder estimates) that the $C^{2+\alpha}$-norm of v_ε in $G_1 \cap B_{1/8}$ is bounded by $C\left(\|q\|_{C^\alpha} + \|b\|_{C^{2+\alpha}}\right)$. Therefore,

$$\|u_\varepsilon\|_{C^{2+\alpha}\left(G_\varepsilon \cap B_{1/8}\right)} \le C\left(\|q\|_{C^\alpha} + \|b\|_{C^{2+\alpha}}\right). \tag{5.6}$$

By the interior Schauder estimates, the same bound holds in any subsets of G_ε that satisfy $\operatorname{dist}(x, \{x_n = \varepsilon g_2(x')\}) \ge 1/16$, and thus the assertion (5.3) is valid for $a = 1/2$.

To prove (5.4), we first show that

$$\|u_\varepsilon - u_0\|_{L^\infty(G_\varepsilon)} \le \varepsilon C\left(\|q\|_{C^\alpha} + \|b\|_{C^{2+\alpha}}\right). \tag{5.7}$$

Introduce a change of variables

$$\xi' = x', \quad \xi_n = \frac{x_n - \varepsilon g_1(x')}{1 - \varepsilon g_1(x')}, \quad \tilde{u}_\varepsilon(\xi) = u_\varepsilon(x). \tag{5.8}$$

Then $\tilde{u}_\varepsilon$ is defined in Ω_0 and satisfies

$$\begin{aligned}
&\Delta\tilde{u}_\varepsilon = \varepsilon\theta + \tilde{q}\mathcal{X}_{\tilde{G}_\varepsilon} \text{ in } \Omega_0,\\
&\tilde{u}_\varepsilon = 0 \text{ on } x_n = 1,\\
&\tilde{u}_\varepsilon (x', 0) = b(x'),
\end{aligned}$$

where $\tilde{G}_\varepsilon$ is the image of G_ε by the mapping (5.8), θ is a bounded function (by (5.6)), and $\tilde{q}(\xi) \equiv q(x)$ that satisfies $|\tilde{q} - q| \leq C\varepsilon$. It is now easy to deduce (5.7) by L^p estimates for the Poisson equation.

Next, for any fixed x'_0, we choose a smooth function $r(x')$ such that $r(x') = 1$ if $|x' - x'_0| \leq 2$ and $r(x') = 0$ if $|x' - x'_0| \geq 3$, and set $\hat{u}_\varepsilon = ru_\varepsilon,\ \hat{u}_0 = ru_0$. Then

$$\Delta\hat{u}_\varepsilon = q_\varepsilon + rq\mathcal{X}_{G_\varepsilon} \text{ in } \Omega_0, \quad \Delta\hat{u}_0 = q_0 + rq\mathcal{X}_{G_0} \text{ in } \Omega_0,$$

where

$$q_\varepsilon = \tilde{u}_\varepsilon\Delta r + 2\nabla\tilde{u}_\varepsilon \cdot \nabla r, \quad q_0 = u_0\Delta r + 2\nabla u_0 \cdot \nabla r.$$

It is easily seen that

$$\|q_\varepsilon - q_0\|_{C^\alpha(G_\varepsilon)} \leq C\|u_\varepsilon - u_0\|_{C^{1+\alpha}(G_\varepsilon)}. \tag{5.9}$$

We can express both $\hat{u}_\varepsilon$ and $\hat{u}_0$ as Newtonian potentials in the bounded domains plus boundary integrals. By applying the methods in [3, section 12] (to estimate the Newtonian potentials) and [4, Section 3] (to estimate the boundary integrals) we can derive the bound

$$\|\hat{u}_\varepsilon - \hat{u}_0\|_{C^{2+\alpha}(G_\varepsilon\cap B_2)} \leq \varepsilon C\left(\|q\|_{C^\alpha} + \|b\|_{C^{2+\alpha}}\right) + C\|q_\varepsilon - q_0\|_{C^\alpha(G_\varepsilon)}. \tag{5.10}$$

There is actually a difference between the treatments in [3] and in (5.10). In [3], we break the domain of integration G_ε into two parts. One part is a ball tangent to the boundary, whereas the measure of the remaining part is less than εC. Here we take, instead of a ball, the intersection of G_ε with a half space that is tangent to ∂G_ε at a boundary point; then the remaining region is contained in a strip of width less than εC. The rest of the estimates are essentially the same.

From (5.9) and (5.10) it follows that

$$\|u_\varepsilon - u_0\|_{C^{2+\alpha}(G_\varepsilon\cap B_1)} \leq \varepsilon C\left(\|q\|_{C^\alpha} + \|b\|_{C^{2+\alpha}}\right) + C\|u_\varepsilon - u_0\|_{C^{1+\alpha}(G_\varepsilon)}.$$

Using partition of unity, we get

$$\|u_\varepsilon - u_0\|_{C^{2+\alpha}(G_\varepsilon)} \leq \varepsilon C\left(\|q\|_{C^\alpha} + \|b\|_{C^{2+\alpha}}\right) + C\|u_\varepsilon - u_0\|_{C^{1+\alpha}(G_\varepsilon)},$$

and then, by interpolation and (5.7), the estimate (5.4) (in case $a = 1/2$) follows.

For general $a > 0$, we use the scaling $w_\varepsilon(x) = u_\varepsilon(2ax)$. Then

$$\Delta w_\varepsilon = 4a^2\tilde{q}\mathcal{X}_{\tilde{G}_\varepsilon} \text{ in } G_1/2a,$$

$$w_\varepsilon = 0 \text{ on } x_n = \frac{1}{2a}, \quad w_\varepsilon = \tilde{b} \text{ on } x_n = \frac{\varepsilon\tilde{g}_1}{2a},$$

where $\tilde{b}(x') = b(2ax'),\ \tilde{q}(x) = q(2ax),\ \tilde{g}_1(x') = g_1(2ax'),\ \tilde{G}_\varepsilon = G_\varepsilon/2a$. By the maximum principle,

$$\|w_\varepsilon\|_{L^\infty} = \|u_\varepsilon\|_{L^\infty} \leq \|b\|_{L^\infty} + C\|q\|_{L^\infty}.$$

From the proof of the case $a = 1/2$, we see that the estimates on the C^α-norm of the first two derivatives remain true if we replace the domain G_1 by G_1/ρ for $\rho \le 1$, and the constants C do not depend on ρ. Hence (5.3) and (5.4) hold for w_ε. We thus obtain

$$\|w_\varepsilon\|_{C^{2+\alpha}(\tilde{G}_\varepsilon)} \le C\left(a^2\|\tilde{q}\|_{C^\alpha} + \left\|\tilde{b}\right\|_{C^{2+\alpha}} + \|q\|_{L^\infty}\right),$$
$$\|w_\varepsilon - w_0\|_{C^{2+\alpha}(\tilde{G}_\varepsilon)} \le \varepsilon C\left(a^2\|\tilde{q}\|_{C^\alpha} + \left\|\tilde{b}\right\|_{C^{2+\alpha}} + \|q\|_{L^\infty}\right)$$

if $|\varepsilon g_1|, |\varepsilon g_2| \le a/8$ (so that Ω_ε is connected and of class $C^{1+\alpha}$), where $w_0(x) = u_0(2ax)$. Estimates (5.3) and (5.4) now follow by scaling back. ∎

Let (ψ_0, f_0, φ_0) be the global solution corresponding to $\varepsilon = 0$ for $0 \le t \le T_0$, constructed in Example 4.1, with $f_0(T_0) = 1$; the function ψ_0 is understood to be defined for all x in R^n by extending its nth component, as defined in (4.25), to all $x_n \in R$.

LEMMA 5.2. *Suppose the solution $(\psi_\varepsilon, f_\varepsilon, \varphi_\varepsilon)$ of (2.1)–(2.9) exists for $0 \le t \le T$, and $1 - f_\varepsilon(x', t) \ge \delta$ for some $\delta > 0$. Then there exists an $\varepsilon_0 > 0$ and a constant C such that if*

$$\psi_\varepsilon = \psi_0 + \varepsilon\psi_\varepsilon^{(1)}, \quad f_\varepsilon = f_0 + \varepsilon f_\varepsilon^{(1)} \tag{5.11}$$

for $\varepsilon \le \varepsilon_0$, then

$$\left\|\psi_\varepsilon^{(1)}\right\|_{C^{1+\alpha}(\Omega_{\varepsilon t})} + \left\|D_t\psi_\varepsilon^{(1)}\right\|_{C^{1+\alpha}(\Omega_{\varepsilon t})} + \left\|f_\varepsilon^{(1)}\right\|_{C^{1+\alpha}} \le C \tag{5.12}$$

for all $0 \le t \le T$.

Proof. By (5.11), we have, for $f_\varepsilon(x', t) < x_n < 1$,

$$\varepsilon\frac{d^2\psi_\varepsilon^{(1)}}{dt^2} = -\nabla\varphi_\varepsilon(\psi_\varepsilon, t) + \nabla\varphi_0(\psi_0, t). \tag{5.13}$$

We can also write

$$\Delta\varphi_\varepsilon = -p_0 J\left(\psi_0^{-1}\right)\mathcal{X}_{\Omega_{\varepsilon t}}^{\psi_\varepsilon} + \theta_\varepsilon\mathcal{X}_{\Omega_{\varepsilon t}}^{\psi_\varepsilon},$$
$$\Delta\varphi_0 = -p_0 J\left(\psi_0^{-1}\right)\mathcal{X}_{\Omega_{0t}}^{\psi_0},$$

where $\theta_\varepsilon = p_0(J(\psi_0^{-1}) - J(\psi_\varepsilon^{-1}))$. By (5.11),

$$\|\theta_\varepsilon\|_{C^\alpha} \le \varepsilon C\left\|\psi_\varepsilon^{(1)}\right\|_{C^{1+\alpha}(\Omega_{\varepsilon t})}. \tag{5.14}$$

Set $\varphi_\varepsilon = u_\varepsilon + v_\varepsilon$, where u_ε satisfies

$$\Delta u_\varepsilon = -p_0 J\left(\psi_0^{-1}\right)\mathcal{X}_{\Omega_{\varepsilon t}}^{\psi_\varepsilon}$$

and the boundary condition (2.6). By the estimate (5.4) of Lemma 5.1 (with $g_1 = h$, $a = \psi_{0n}(1, t)$, $g_2(x') = \psi_n^{(1)}(x', 1, t)$),

$$\|\nabla u_\varepsilon(\psi_\varepsilon, t) - \nabla\varphi_0(\psi_0, t)\|_{C^{1+\alpha}(\Omega_{\varepsilon t})} \le \varepsilon C\left\|\psi_\varepsilon^{(1)}\right\|_{C^{1+\alpha}(\Omega_{\varepsilon t})}, \tag{5.15}$$

where C depends only on δ. On the other hand, v_ε satisfies

$$\Delta v_\varepsilon = \theta_\varepsilon\mathcal{X}_{\Omega_{\varepsilon t}}^{\psi_\varepsilon}$$

with zero boundary conditions. Again by Lemma 5.1 and (5.14),

$$\left\|\nabla v_{\varepsilon}\left(\psi_{\varepsilon}, t\right)\right\|_{C^{1+\alpha}\left(\Omega_{\varepsilon t}\right)} \leq C\left\|\theta_{\varepsilon}\right\|_{C^{\alpha}} \leq \varepsilon C\left\|\psi_{\varepsilon}^{(1)}\right\|_{C^{1+\alpha}\left(\Omega_{\varepsilon t}\right)}. \tag{5.16}$$

Using (5.15) and (5.16) to estimate the right-hand side of (5.13), we obtain

$$\left\|\frac{d^{2} \psi_{\varepsilon}^{(1)}}{d t^{2}}\right\|_{C^{1+\alpha}\left(\Omega_{\varepsilon t}\right)} \leq C\left\|\psi_{\varepsilon}^{(1)}\right\|_{C^{1+\alpha}\left(\Omega_{\varepsilon t}\right)},$$

so that

$$\left\|\psi_{\varepsilon}^{(1)}\right\|_{C^{1+\alpha}\left(\Omega_{\varepsilon t}\right)}+\left\|D_{t} \psi_{\varepsilon}^{(1)}\right\|_{C^{1+\alpha}\left(\Omega_{\varepsilon t}\right)} \leq C. \tag{5.17}$$

Next, in view of (5.1), (2.4) becomes

$$\psi_{\varepsilon, n}\left(x^{\prime}, f_{\varepsilon}, t\right)=\varepsilon h\left(\psi_{\varepsilon}^{\prime}\left(x^{\prime}, f_{\varepsilon}, t\right)\right).$$

Hence, by (5.11) and (4.25),

$$-\varepsilon\left(1+\frac{p_{0} t^{2}}{2}\right) f_{\varepsilon}^{(1)}=\varepsilon h\left(\psi_{\varepsilon}^{\prime}\left(x^{\prime}, f_{\varepsilon}, t\right)\right)-\varepsilon \psi_{\varepsilon, n}^{(1)}\left(x^{\prime}, f_{\varepsilon}, t\right),$$

so that

$$\left\|f_{\varepsilon}^{(1)}\right\|_{C^{1+\alpha}} \leq C\left\|\psi_{\varepsilon}^{(1)}\right\|_{C^{1+\alpha}\left(\Omega_{\varepsilon t}\right)}+C.$$

Combining this with (5.17), the lemma follows. □

We can now establish "global" existence for the perturbed system.

THEOREM 5.3. *For any $\delta>0$, there exists an $\varepsilon_{0}(\delta)>0$ such that for $\varepsilon \leq \varepsilon_{0}(\delta)$, there exists a unique solution $\left(\psi_{\varepsilon}, f_{\varepsilon}, \varphi_{\varepsilon}\right)$ of (2.1)–(2.9) with (5.1) for $0 \leq t \leq T_{0}-\delta$. Furthermore, $\left|f_{\varepsilon}-f_{0}\right| \leq C_{0} \varepsilon$.*

Proof. By Theorem 2.1, there exists a unique solution for $0 \leq t \leq T$; T is small and independent of ε. To extend the solution beyond $t=T$, we need to verify the following conditions:

$$D_{t} \psi_{\varepsilon, n}(x, T) \leq 0 \text { in } \Omega_{\varepsilon T}, \quad \vec{n} \cdot D_{t} \psi_{\varepsilon}(x, T) \leq 0 \text { on } \Gamma_{0}, \tag{5.18}$$

$$\inf \left\{\left|1-f_{\varepsilon}\left(x^{\prime}, T\right)\right|: x^{\prime} \in R^{n-1}\right\} \geq \omega>0, \tag{5.19}$$

$$D_{n} \psi_{\varepsilon, n}(x, T)-\varepsilon \nabla_{x^{\prime}} h\left(\psi_{\varepsilon}^{\prime}\right) \cdot D_{n} \psi_{\varepsilon}^{\prime}(x, T) \geq \gamma>0; \tag{5.20}$$

notice that (5.18) ensures that $D_{t} f_{\varepsilon}>0$; (5.20) guarantees that we can solve for $f_{\varepsilon}\left(x^{\prime}, T+t\right)$ from $F=0$; and (5.19) implies that $\Omega_{\varepsilon T}$ is connected and is of class $C^{1+\alpha}$. Inequality (5.18) holds if M is suitable large, independently of ε. Set

$$2 \omega_{\delta}=1-f_{0}\left(T_{0}-\delta\right)>0.$$

By continuity, (5.19) holds for $\omega=\omega_{\delta}$ if T is small (depending only on the initial data, since $D_{t} f_{\varepsilon}$ is bounded by a constant depending on the initial data). Next, since by (5.11)

$$D_{n} \psi_{\varepsilon, n}=\left(1+\frac{p_{0} t^{2}}{2}\right)+\varepsilon D_{n} \psi_{\varepsilon, n}^{(1)},$$

(5.20) holds for $\gamma = 1/2$ if ε is small. We can now apply the proof of Theorem 2.1 with slight modifications (since $\psi_\varepsilon(x,T) \neq x$ in general) to extend the solution to $0 \le t \le T + \Delta T$ where ΔT is a positive number which depends only on ω, γ, and the $C^{1+\alpha}$-norms of the data at $t = T$. If $T + \Delta T < T_0 - \delta$, then

$$1 - f_0(T + \Delta T) \geq 1 - f_0(T_0 - \delta) = 2\omega_\delta$$

since f_0 is monotone increasing. By Lemma 5.2, $f_\varepsilon^{(1)}$ and $\psi_\varepsilon^{(1)}$ are bounded, uniformly with respect to ε. Hence for ε small, (5.18)–(5.20) hold with $\omega = \omega_\delta, \gamma = 1/2$, at $t = T + \Delta T$, where ΔT is independent of ε. By applying Theorem 2.1 once again, we can extend the solution up to $0 \le t \le T + 2\Delta T$ if ε is small enough. Repeating the above procedure a finite number of times, we obtain a solution for $0 \le t \le T_0 - \delta$. Note that at each step we may need to decrease the size of ε. However, the length of the time interval is fixed, until we reach $T_0 - \delta$. □

6. Asymptotic expansion. In this section, we consider system (2.1)–(2.9) with data (5.1) and seek a more precise asymptotic expansion than (5.11); namely,

$$f_\varepsilon(x', t) = f_0(t) + \varepsilon f^{(1)}(x', t) + \varepsilon^2 f_\varepsilon^{(2)}, \tag{6.1}$$

$$\psi_\varepsilon(x, t) = \psi_0(x, t) + \varepsilon\psi^{(1)}(x, t) + \varepsilon^2\psi_\varepsilon^{(2)}, \tag{6.2}$$

$$\varphi_\varepsilon(x, t) = \varphi_0(x, t) + \varepsilon\varphi^{(1)}(x, t) + \varepsilon^2\varphi_\varepsilon^{(2)}. \tag{6.3}$$

For simplicity we shall work in R^2, instead of R^n, and denote the point in R^2 by (x, y). By elementary calculation,

$$J\left(\psi_\varepsilon^{-1}(x,t)\right) = J\left(\psi_0^{-1}(x,t)\right) + \varepsilon H(x,y,t) + O\left(\varepsilon^2\right) \tag{6.4}$$

and

$$\nabla\varphi_\varepsilon(\psi_\varepsilon(x,y,t), t) = \nabla\varphi_0(\psi_0(x,y,t), t)$$
$$+\varepsilon\begin{pmatrix} D_x\varphi^{(1)}(\psi_0(x,t), t) \\ -p_0 Q(t)\,\psi_2^{(1)}(x,y,t) + D_y\varphi^{(1)}(\psi_0(x,t), t)\end{pmatrix} + O\left(\varepsilon^2\right),$$

where

$$H(x,y,t) = -Q(t)\,D_x\psi_1^{(1)}\left(\psi_0^{-1}(x,y,t), t\right) - Q(t)^2 D_y\psi_2^{(1)}\left(\psi_0^{-1}(x,y,t), t\right) \tag{6.5}$$

and

$$Q(t) = \left(1 + \frac{p_0 t^2}{2}\right)^{-1}.$$

Substituting these expressions in the system (2.1)–(2.7) and comparing the coefficients of the ε^0 and ε^1 terms, we find that

$$\frac{d^2\psi_1^{(1)}(x,y,t)}{dt^2} = -D_x\varphi^{(1)}(\psi_0, t), \tag{6.6}$$

$$\frac{d^2\psi_2^{(1)}(x,y,t)}{dt^2} = p_0 Q(t)\,\psi_2^{(1)}(x,y,t) - D_y\varphi^{(1)}(\psi_0, t), \tag{6.7}$$

$$\psi^{(1)}(x,y,0) = \psi_t^{(1)}(x,y,0) = 0, \tag{6.8}$$

$$\Delta\varphi^{(1)}(x,y,t) = \begin{cases} -p_0 H(x,y,t) & \text{for } 0 < y < \psi_{0,2}(1,t), \\ 0 & \text{for } \psi_{0,2}(1,t) < y < 1, \end{cases} \tag{6.9}$$

$$\varphi^{(1)}(x,y,t) = -\left(A(t) + M\right)h(x) \quad \text{on } \Gamma_\varepsilon, \tag{6.10}$$

$$\varphi^{(1)}(x,y,t) = 0 \quad \text{on } \Gamma_1, \tag{6.11}$$

where $A(t)$ is defined in (4.9). Furthermore, differentiating the relation

$$\psi_{\varepsilon,2}(x, f_\varepsilon(x,t), t) = \varepsilon h(x)\, \psi_{\varepsilon,1}(x, f_\varepsilon(x,t), t)$$

with respect to ε and taking $\varepsilon = 0$, we get

$$D_y \psi_{0,2}(f_0(t), t)\, f^{(1)}(x,t) + \psi^{(1)}(x, f_0(t), t) = h(x).$$

Since, by (4.25), $D_y \psi_{0,2}(y,t) = Q(t)^{-1}$, we conclude that

$$f^{(1)}(x,t) = \Big(h(x) - \psi^{(1)}(x, f_0(t), t)\Big)\, Q(t). \tag{6.12}$$

We have thus formally obtained a linearized system (6.5)–(6.12). We shall prove that this system has a unique solution and that the asymptotic formulas (6.1)–(6.3) hold with $f_\varepsilon^{(2)}, \psi_\varepsilon^{(2)}, \varphi_\varepsilon^{(2)}$ bounded. Note that $f^{(1)}$ does not appear in the system (6.5)–(6.11), and, once the system is solved, we can express $f^{(1)}$ in terms of $\psi^{(1)}$ by the formula (6.12).

THEOREM 6.1. *Let* (ψ_0, f_0) *be the solution* (4.25)–(4.28) *for* $0 \le t \le T_0$ *such that* $f_0(T_0) = 1$. *Then there exists a unique solution* $\left(\psi^{(1)}, f^{(1)}\right)$ *of* (6.5)–(6.11) *for* $0 \le t < T_0$. *Furthermore,* $\psi^{(1)}$ *and* $D_t\psi^{(1)}$ *can be continuously extended to* $\Omega_0 \times [0, T_0)$, *and* $\psi^{(1)}(\cdot,\cdot,t)$, $D_t\psi^{(1)}(\cdot,\cdot,t)$ *belong to* $C^{1+\alpha}(\Omega_0)$.

Proof. We first fix a $T_1 < T_0$ and derive a priori estimates. Clearly, $\psi_{0,2}(1,t) = Q(t)^{-1}(1 - f_0(t)) \ge \delta > 0$ if $0 \le t \le T_1$. Hence the region

$$\Omega_t^{\psi_0} = \{(x,y) : \ 0 < y < \psi_{0,2}(1,t)\}$$

contains an infinite horizontal strip and then, by Lemma 5.1,

$$\left\|\varphi^{(1)}\right\|_{C^{2+\alpha}\left(\Omega_t^{\psi_0}\right)} \le C\left(\|H\|_{C^\alpha\left(\Omega_t^{\psi_0}\right)} + 1\right),$$

where C depends on T_1 (or δ), $\psi_{0,2}(1,t)$, $A(t)$, and M. From (6.5),

$$\|H\|_{C^\alpha\left(\Omega_t^{\psi_0}\right)} \le C\left\|\nabla\psi^{(1)}\right\|_{C^\alpha(\Omega_t)}.$$

It follows that

$$\left\|\varphi^{(1)}(\psi_0, t)\right\|_{C^{2+\alpha}(\Omega_t)} \le C\left(\left\|\nabla\psi^{(1)}\right\|_{C^\alpha(\Omega_t)} + 1\right).$$

Differentiating (6.6) and (6.7) in the spatial variables and using the above estimate, we obtain, by Gronwall's inequality,

$$\left\|\nabla\psi^{(1)}\right\|_{C^\alpha(\Omega_t)} + \left\|\nabla\psi_t^{(1)}\right\|_{C^\alpha(\Omega_t)} \le C, \tag{6.13}$$

where C depends on T_1, $\psi_{0,2}(1,t)$, $A(t)$, and M.

We now proceed to prove existence for a small time interval $0 \le t \le T_2$. For any $\psi^{(1)}(\cdot,\cdot,t) \in C^{1+\alpha}(\Omega_t)$, $0 \le t \le T_2 \le T_1$ such that $\psi^{(1)}$ and $D_t\psi^{(1)}$ are continuous in (x,y,t), we define H by (6.5) and $\varphi^{(1)}$ by (6.9)–(6.11). Substituting $\psi^{(1)}$ and $\varphi^{(1)}$ into the right-hand sides of (6.6) and (6.7), we solve the left-hand sides by integration, using the initial condition (6.8); we denote the solution by $\hat\psi^{(1)} = (\hat\psi_1^{(1)}, \hat\psi_2^{(1)})$. By Lemma 5.1 (and the derivation of (6.13)), it is easily seen that the mapping $\psi^{(1)} \mapsto \hat\psi^{(1)}$ is contraction if T_2 is small. Therefore, there exists a unique fixed point, which is a solution of the linearized problem (6.5)–(6.11) for $0 \le t \le T_2$.

Using (6.13), we can further extend the solution to $0 \le t \le T_2 + \beta$ with β depending only on the constant C on the right-hand side of (6.13). By a step-by-step argument, the solution can be extended up to $t = T_1$. Uniqueness can be proved in the same way as in Theorem 2.1. Since T_1 can be chosen arbitrarily close to T_0, the theorem follows. $\square$

Remark 6.1. By applying Lemma 5.1 to φ_x, we deduce that if $q \in C^{1+\alpha}$, $b \in C^{3+\alpha}$, then $\varphi \in C^{3+\alpha}$

In Theorem 5.3 we have established existence of a uniqueness solution $(\psi_\varepsilon, f_\varepsilon, \varphi_\varepsilon)$ for (2.1)–(2.9) for $0 \le t \le T_\varepsilon$, where $T_\varepsilon \le T_0$, $T_\varepsilon \to T_0$ as $\varepsilon \to 0$. We next wish to justify that the formal expansions (6.1)–(6.3). It will be convenient to correspond Ψ_ε, Ψ_0, and $\Psi^{(1)}$ to ψ_ε, ψ_0, and $\psi^{(1)}$ as in (2.13).

THEOREM 6.2. *Let* $(\psi_\varepsilon, f_\varepsilon, \varphi_\varepsilon)$ *be a solution of system* (2.1)–(2.9) *with data* (5.1) *in* $0 \le t \le T_\varepsilon$. *Then there exists a constant* C *such that for* $0 \le t \le T_\varepsilon$,

$$\left\| f_\varepsilon - \left(f_0 + \varepsilon f^{(1)} \right) \right\|_{C^{1+\alpha}} \le C\varepsilon^2, \tag{6.14}$$

$$\left\| \Psi_\varepsilon - \left(\Psi_0 + \varepsilon \Psi^{(1)} \right) \right\|_{C^{1+\alpha}} \le C\varepsilon^2. \tag{6.15}$$

Proof. For simplicity, we assume that both ψ_ε and $\psi^{(1)}$ are defined in $\Omega_{\varepsilon t}^{\psi_\varepsilon}$. Then (6.15) is equivalent to

$$\left\| \psi_\varepsilon - \left(\psi_0 + \varepsilon \psi^{(1)} \right) \right\|_{C^{1+\alpha}\left(\Omega_{\varepsilon t}^{\psi_\varepsilon}\right)} \le C\varepsilon^2. \tag{6.16}$$

In $\Omega_{\varepsilon t}$, $\psi_\varepsilon^{(2)}$ satisfies

$$\varepsilon^2 \frac{d^2 \psi_{\varepsilon,1}^{(2)}}{dt^2} = -D_x \varphi_\varepsilon (\psi_\varepsilon) + D_x \varphi_0 (\psi_0) + \varepsilon D_x \varphi^{(1)} (\psi_0, t), \tag{6.17}$$

$$\begin{aligned} \varepsilon^2 \frac{d^2 \psi_{\varepsilon,2}^{(2)}}{dt^2} = & -D_y \varphi_\varepsilon (\psi_\varepsilon) + D_y \varphi_0 (\psi_0) - p_0 Q(t) \psi_2^{(1)} (x, y, t) \\ & + D_y \varphi^{(1)} (\psi_0, t). \end{aligned} \tag{6.18}$$

Substituting (6.3) into (6.17) and (6.18), we obtain

$$\varepsilon^2 \frac{d^2 \psi_{\varepsilon,1}^{(2)}}{dt^2} = -\varepsilon^2 D_x \varphi_\varepsilon^{(2)} (\psi_\varepsilon) + I_1, \tag{6.19}$$

$$\varepsilon^2 \frac{d^2 \psi_{\varepsilon,2}^{(2)}}{dt^2} = -\varepsilon^2 D_y \varphi_\varepsilon^{(2)} (\psi_\varepsilon) + I_2, \tag{6.20}$$

where

$$\begin{aligned} I_1 &= D_x \varphi_0 (\psi_0) - D_x \varphi_0 (\psi_\varepsilon) + \varepsilon D_x \varphi^{(1)} (\psi_0) - \varepsilon D_x \varphi^{(1)} (\psi_\varepsilon), \\ I_2 &= D_y \varphi_0 (\psi_0) - D_y \varphi_0 (\psi_\varepsilon) - \varepsilon p_0 Q \psi_2^{(1)} + \varepsilon D_y \varphi^{(1)} (\psi_0) - \varepsilon D_y \varphi^{(1)} (\psi_\varepsilon). \end{aligned}$$

Since $\varphi_0 (x, y, t) = \varphi_0 (y, t)$,

$$I_1 = \varepsilon D_x \varphi^{(1)} (\psi_0) - \varepsilon D_x \varphi^{(1)} (\psi_\varepsilon).$$

By (6.2),

$$\|\nabla I_1(\cdot,t)\|_{L^\infty(\Omega_{\varepsilon t})} \le C\varepsilon^2\left(\left\|\nabla\psi_\varepsilon^{(2)}(\cdot,t)\right\|_{L^\infty(\Omega_{\varepsilon t})}+1\right),$$

$$\|\nabla I_1(\cdot,t)\|_{C^\alpha(\Omega_{\varepsilon t})} \le C\varepsilon^2\left(\left\|\psi_\varepsilon^{(2)}(\cdot,t)\right\|_{C^{1+\alpha}(\Omega_{\varepsilon t})}^{1+\alpha}+1\right),$$

where C depends also on $C^{1+\mu}$-norm of ψ_ε. Analogously, since $D_y^2\varphi_0(\psi_0) = -p_0Q$ and consequently

$$D_y\varphi_0(\psi_0) - D_y\varphi_0(\psi_\varepsilon) - \varepsilon p_0 Q\psi_2^{(1)} = -\varepsilon^2 p_0 Q\psi_\varepsilon^{(2)},$$

we obtain (using the regularity of φ_0 in Remark 6.1)

$$\|\nabla I_2(\cdot,t)\|_{L^\infty(\Omega_{\varepsilon t})} \le C\varepsilon^2\left(\left\|\nabla\psi_\varepsilon^{(2)}(\cdot,t)\right\|_{L^\infty(\Omega_{\varepsilon t})}+1\right),$$

$$\|\nabla I_2(\cdot,t)\|_{C^\alpha(\Omega_{\varepsilon t})} \le C\varepsilon^2\left(\left\|\psi_\varepsilon^{(2)}(\cdot,t)\right\|_{C^{1+\alpha}(\Omega_{\varepsilon t})}+1\right).$$

Using these estimates in (6.19) and (6.20), we find that

$$\begin{aligned}&\left\|\psi_\varepsilon^{(2)}(\cdot,t)\right\|_{C^{1+\alpha}(\Omega_{\varepsilon t})}\\ (6.21)\quad &\le C + C\int_0^t\int_0^s\left(\left\|\varphi_\varepsilon^{(2)}(\psi_\varepsilon,\tau)\right\|_{C^{2+\alpha}(\Omega_{\varepsilon s})}+\left\|\psi_\varepsilon^{(2)}(\cdot,\tau)\right\|_{C^{1+\alpha}(\Omega_{\varepsilon s})}\right)d\tau ds.\end{aligned}$$

We next estimate $\varphi_\varepsilon^{(2)}$. By (6.2),

$$\left\|J\left(\psi_\varepsilon^{-1}\right) - J\left(\psi_0^{-1}\right) - \varepsilon H\right\|_{C^\alpha} \le C\varepsilon^2\left(\left\|\psi_\varepsilon^{(2)}(\cdot,t)\right\|_{C^{1+\alpha}(\Omega_{\varepsilon s})}+1\right), \tag{6.22}$$

where C depends on $\|\psi_\varepsilon(\cdot,t)\|_{C^{1+\alpha}(\Omega_{\varepsilon t})}$. Substituting (6.3) into (2.1) and using (6.9), one sees that $\varphi_\varepsilon^{(2)}$ solves

$$\varepsilon^2\Delta\varphi_\varepsilon^{(2)} = J\left(\psi_\varepsilon^{-1}\right)\mathcal{X}_{\Omega_{\varepsilon t}^{\psi_\varepsilon}} - J\left(\psi_0^{-1}\right)\mathcal{X}_{\Omega_{0t}^{\psi_0}} - \varepsilon H\mathcal{X}_{\Omega_{0t}^{\psi_0}}.$$

Using (6.22) and Lemma 5.1, we then obtain

$$\left\|\varphi_\varepsilon^{(2)}(\psi_\varepsilon)\right\|_{C^{2+\alpha}(\Omega_{\varepsilon s})} \le C\left(\left\|\psi_\varepsilon^{(2)}(\cdot,t)\right\|_{C^{1+\alpha}(\Omega_{\varepsilon s})}+1\right). \tag{6.23}$$

Substituting (6.23) into (6.21) and applying Gronwall's inequality, we find that

$$\left\|\varphi_\varepsilon^{(2)}(\psi_\varepsilon)\right\|_{C^{2+\alpha}(\Omega_{\varepsilon s})} + \left\|\psi_\varepsilon^{(2)}(\cdot,t)\right\|_{C^{1+\alpha}(\Omega_{\varepsilon s})} \le C, \tag{6.24}$$

which implies, in particular, the assertion (6.16). Expanding the relation

$$\psi_{\varepsilon,2}(x,f_\varepsilon) = \varepsilon h(\psi_{\varepsilon,1}(x,f_\varepsilon))$$

in ε (by (6.1) and (6.2)) and using the equations

$$\begin{aligned}&\psi_{0,2}(x,f_0) = \varepsilon h(\psi_{0,1}(x,f_0)),\\ &(D_y\psi_{0,2})f_1 + \psi_2^{(1)}(x,f_0) = h(\psi_{0,1}(x,f_0)),\end{aligned}$$

we can express $f_\varepsilon^{(2)}$ in terms of the derivatives of $\psi_\varepsilon^{(2)}$ and, by the estimate for $\psi_\varepsilon^{(2)}$ in (6.24),

$$\left\|f_\varepsilon^{(2)}(\cdot,t)\right\|_{C^{1+\alpha}} \le C,$$

which is the assertion (6.14). $\square$

7. Numerical results. In this section, we take $n = 2$ and compute the thickness W_ε of the layer of the particles that have accumulated along the workpiece Γ_ε for small ε. We are interested in the relation between the local profile of W_ε and the geometry of Γ_ε.

The density along the boundary Γ_ε is $P(x, \varepsilon h(x), t)$. By (2.3) and (6.4), the normal thickness $W_\varepsilon(x)$ at $(x, \varepsilon h(x)) \in \Gamma_\varepsilon$ can be written as

$$W_\varepsilon(x) = \left(1 + \varepsilon^2 h(x)^2\right)^{-\frac{1}{2}} \int_0^T P(x, \varepsilon h(x), t)\, dt = W_0 + \varepsilon W_1(x) + O\left(\varepsilon^2\right), \tag{7.1}$$

where

$$W_0 = \int_0^T p_0 Q(t)\, dt = \text{const}, \quad W_1(x) = \int_0^T p_0 H(x, 0, t)\, dt. \tag{7.2}$$

We want to compute $W_1(x)$.

Differentiating (6.6) and (6.7) in x and y, respectively, and introducing the functions

$$u(x, y, t) = D_x \psi_1^{(1)}(x, y, t), \quad v(x, y, t) = D_y \psi_2^{(1)}(x, y, t),$$

we obtain the system

$$\frac{d^2 u(x, y, t)}{dt^2} = -\varphi_{xx}^{(1)}(x, \psi_{0,2}(y, t), t), \tag{7.3}$$

$$\frac{d^2 v(x, y, t)}{dt^2} = p_0 Q(t) v - Q(t)^{-1} \varphi_{yy}^{(1)}(x, \psi_{0,2}(y, t), t), \tag{7.4}$$

with the initial data

$$u(x, y, 0) = v(x, y, 0) = u_t(x, y, 0) = v_t(x, y, 0) = 0, \tag{7.5}$$

where $\varphi^{(1)}$ satisfies (6.9). By (6.5)

$$H(x, y, t) = -Q(t)\, u\left(\psi_0^{-1}(x, y, t), t\right) - Q(t)^2\, v\left(\psi_0^{-1}(x, y, t), t\right). \tag{7.6}$$

To simplify the computation of $W_1(x)$ we introduce the function

$$G(x, y, t) = -Q(t)^{-2} H(\psi_0(x, y, t), t). \tag{7.7}$$

Substituting H from (7.6) into (7.7), differentiating (7.7) in t, and then using (6.9), (7.3), (7.4), we get

$$\frac{d^2 G(x, y, t)}{dt^2} = 2 p_0 t u_t(x, y, t) \tag{7.8}$$

and, by integration in t,

$$\begin{aligned} G_t &= p_0 \int_0^t u_t(x, y, t)\, dt \\ &= p_0 \left(t^2 u_t(x, y, t) + \int_0^t s^2 D_x^2 \varphi^{(1)}(\psi_0, s)\, ds \right). \end{aligned}$$

Integrating once more and using (7.3), (7.4), we obtain

$$\begin{aligned} G(x, y, t) &= \frac{p_0}{3} \int_0^t u_t(x, y, t)\, dt + p_0 \int_0^t \int_0^\tau s^2 D_x^2 \varphi^{(1)}(\psi_0, s)\, ds d\tau \\ &= \frac{p_0 t^3}{3} u_t - \frac{p_0}{3} \int_0^t s^3 u_{tt}(x, y, s)\, ds + p_0 \int_0^t \int_0^\tau s^2 D_x^2 \varphi^{(1)}(\psi_0, s)\, ds d\tau \\ &= -\frac{p_0 t^3}{3} \int_0^t \left(-\frac{p_0 t^3}{3} - \frac{2 p_0 s^3}{3} + p_0 t s^2 \right) D_x^2 \varphi^{(1)}(\psi_0, s)\, ds, \end{aligned} \tag{7.9}$$

for $f_0(t) < y < 1$, where we have used the identity

$$\int_0^t \int_0^\tau s^2 k(s)\, ds d\tau = t \int_0^t s^2 k(s)\, ds - \int_0^t s^3 k(s)\, ds.$$

Since $G(x, y, t)$ is defined only for $f_0(t) < y < 1$, it is more convenient to work with another function $\hat{G}$, defined by

$$\hat{G}(x, y, t) = G(x, (1 - f_0(t))\, y + f(t), t),\ 0 < y < 1.$$

Equation (7.9) is equivalent to

$$\hat{G}(x, y, t) = \int_0^t D_x^2 \varphi^{(1)} \left(x, \frac{(1 - f_0(t))\, y + f_0(t) - f_0(s)}{Q(s)}, s \right) \times \left(-\frac{p_0 t^3}{3} - \frac{2p_0 s^3}{3} + p_0 t s^2 \right) ds. \tag{7.10}$$

Equation (7.10) is supplemented by (6.9)–6.11), where

$$H(x, y, t) = -Q(t)^2\, \hat{G}\left(x, \frac{Q(t)\, y}{1 - f_0(t)}, t \right). \tag{7.11}$$

We can now express W_1 in the form

$$W_1(x) = -p_0 \int_0^T Q(s)^2\, \hat{G}(x, 0, s)\, ds.$$

The assumption $P_0(x) = p_0$ means that initially we have a uniform distribution of particles everywhere in the domain. In the problem of spray painting, initially there are no paint particles in the domain and, for a certain period of time, say $0 \le t \le t_0$, a cloud of particles with uniform distribution is injected from Γ_∞ at a uniform rate. In order to represent this situation more closely, we should replace the assumption $P_0(x) \equiv p_0$ by the assumption

$$P_0(x, y) = \begin{cases} 0 & \text{if } 0 \le y < a, \\ p_0 & \text{if } a \le y \le 1 \end{cases} \tag{7.12}$$

for some $0 < a < 1$. Note that although we do not include here continuous injection of particles across Γ_∞ for $t > 0$, the assumption (7.12) is an approximation to the spray painting model with t_0 proportional to a. The system (7.10), (7.11), (6.9)–(6.11) then take the form

$$\hat{G}(x, y, t) = P_0^2((1 - f_0(t))\, y + f_0(t)) \int_0^t \left(-\frac{p_0 t^3}{3} - \frac{2p_0 s^3}{3} + p_0 t s^2 \right) \times D_x^2 \varphi^{(1)}(x, \psi_{02}((1 - f_0(t))\, y + f_0(t), s), s)\, ds, \tag{7.13}$$

$$\Delta\varphi^{(1)}(x, y, t) = \begin{cases} Q\left(\psi_{02}^{-1}, t\right) \hat{G}\left(x, \dfrac{\psi_{02}^{-1}(y, t) - f_0}{1 - f_0(t)}, t \right), & \text{if } 0 < y < \psi_{02}(1, t), \\ 0 & \text{if } \psi_{02}(1, t) < y < 1, \end{cases} \tag{7.14}$$

$$\varphi^{(1)} = 0 \text{ on } \Gamma_1,\ \varphi^{(1)} = -(A(t) + M) h(x) \text{ on } \Gamma_0, \tag{7.15}$$

where

$$Q(y,t) = J(\psi_0(x,y,t))^{-1}, \tag{7.16}$$

$A(t)$, $f_0(t)$, φ_0, and ψ_0 are the solutions of (4.5), (4.8)–(4.11) corresponding to (7.12). The ε order term of W_ε is then

$$W(x) + W_{10}, \tag{7.17}$$

where

$$W_{10} = \int_0^T \left[Q_y(f_0(s), s) Q(f_0(s), s) P_0(f_0) + Q(f_0(s), s)^2 P_{0y}(f_0) \right] ds$$

is a constant and

$$W(x) = -\int_0^T Q(f_0(s), s)^2 \hat{G}(x, 0, s)\, ds. \tag{7.18}$$

Since W_{10} is constant, we shall only be interested in computing $W(x)$. The system (7.12)–(7.16) has a unique solution (the proof is similar to the system (7.10), (7.11), (6.9)–6.11)), although we do not establish here the global existence and the asymptotic estimates (as in sections 5 and 6) for the initial data (7.12).

Numerical computation. We now present some numerical results for $W(x)$, the nonconstant term of the painting thickness in (7.17). We choose $M = 50$, $p_0 = 10$, $a = 0.7$, and then solve ODE (4.11) to get a numerical solution $f_0(t)$ for $0 \leq t \leq T_0 = 0.202$. The function ψ_{02} is calculated numerically from (4.10). Since we are particularly interested in understanding how the thickness changes near the corners of the workpiece, we take three typical workpieces $\Gamma_0 = \{\, y = g(x) = \varepsilon h_i(x)\}$:

$$h_1(x) = \begin{cases} 60 \exp\left(-\dfrac{1}{0.5^2 - x^2} \right) & \text{for } |x| < 0.5, \\ 0 & \text{otherwise}, \end{cases} \tag{7.19}$$

$$h_2(x) = \begin{cases} 10 \exp\left(\dfrac{1}{0.5^4} - \dfrac{1}{0.5^4 - (x+0.5)^4} \right) & \text{for } -1 < x < -0.5, \\ 10 \exp\left(\dfrac{1}{0.5^4} - \dfrac{1}{0.5^4 - (x-0.5)^4} \right) & \text{for } 0.5 < x < 1, \\ 1 & \text{for } -0.5 < x < 0.5, \\ 0 & \text{otherwise}, \end{cases} \tag{7.20}$$

$$h_3(x) = \begin{cases} h_2(x - 0.5) & \text{for } x < 0, \\ 1 & \text{otherwise}. \end{cases} \tag{7.21}$$

The system (7.13), (7.14) is then solved by using the finite element method for the Poisson equation (7.14) in the truncated rectangular domain $(0,1)\times(-n,n)$ (for $n = 5$) and by the Euler forward method for the integral equation (7.13). The computational results show that the scheme is stable, and that the results would be the same (near the support of h_i) if n is increased. Figures 7.1–7.3 show the numerical value for $W(x)$ for three different workpieces with the uniform space and time meshes $dx = 10^{-2}$ and $dt = T_0/100$, respectively; the small figures within Figures 7.1–7.3 are the rescaled profiles of the corresponding $W(x)$.

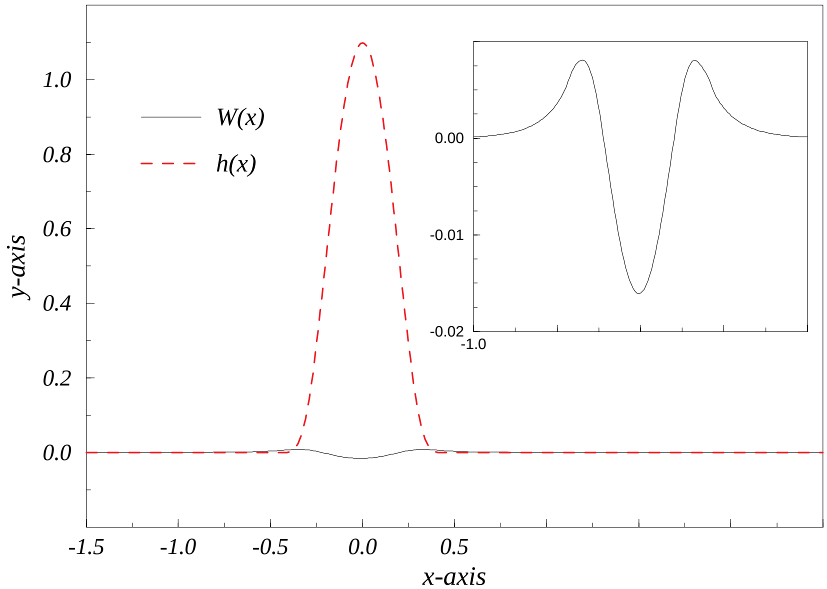

Fig. 7.1. $\Gamma_0 = \{y = \varepsilon h(x)\}, h(x) = h_1(x)$ *from* (7.19).

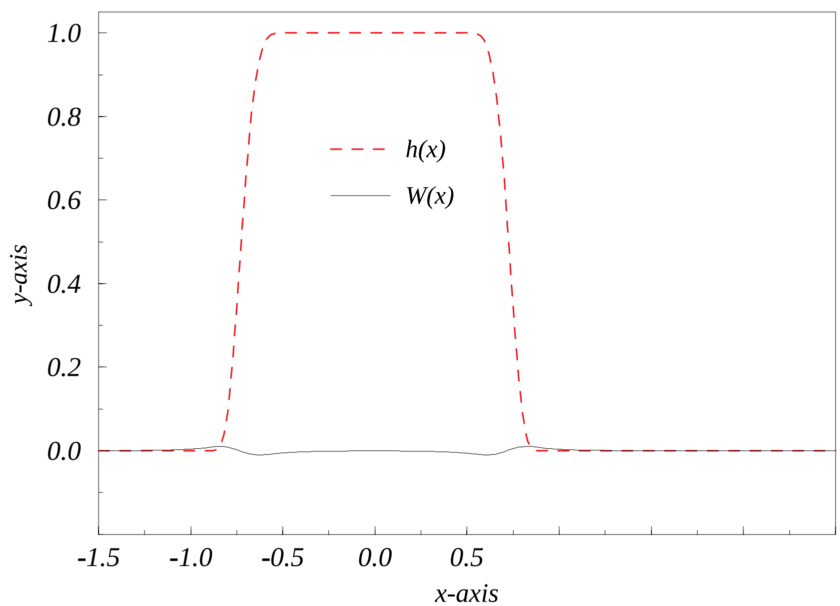

Fig. 7.2. $\Gamma_0 = \{y = \varepsilon h(x)\}, h(x) = h_2(x)$ *from* (7.20).

Conclusion. The paint coming from $y = 1$ is initially uniform. The hill-shaped geometry in Figure 7.1 is such that the electric field$-\nabla\varphi$ due to the high potential M on $y = 1$ and 0 on the workpiece steers the particles away from the peak point at $x = 0$. Consequently, the accumulation of the paint particles at the maximum point of the workpiece is smaller than that at the nearby slopes. The same "steering away" from the convex corners to the nearby slopes occurs for the workpieces in Figures 7.2

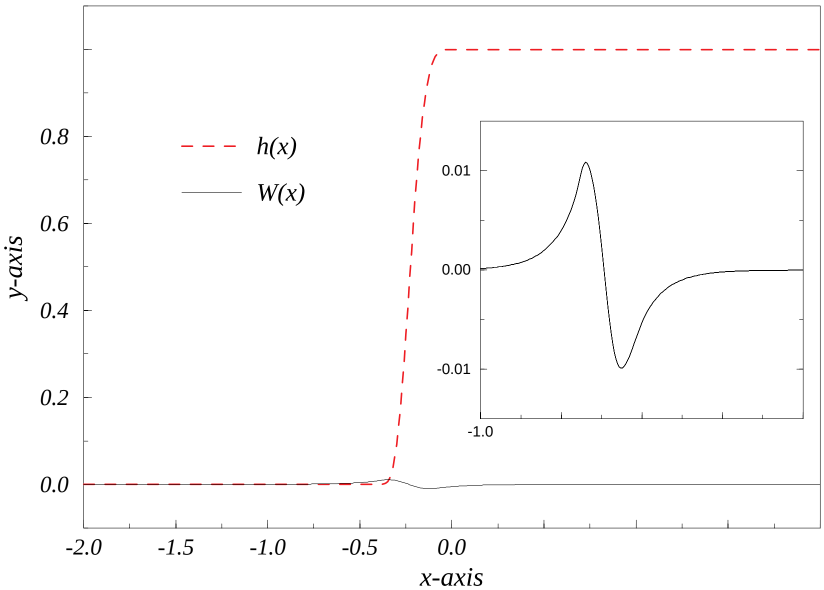

FIG. 7.3. $\Gamma_0 = \{y = \varepsilon h(x)\}, h(x) = h_3(x)$ *from* (7.21).

and 7.3. We also observe that the particles tend to accumulate around the concave corner points.

This result is in agreement with the numerical results obtained by Ellwood and Braslaw [1]; their method is based on a discrete model of a finite number of elastic particles.

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