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Axiomatic Approach for Quantification of Image Resolution

Ge Wang

Yi Li
Wright State University - Main Campus, yi.li@wright.edu

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[1]Ge Wang and [2]Yi Li
[1]Department of Radiology, [2]Department of Mathematics
University of Iowa, Iowa City, Iowa 52242, USA

Abstract: Image resolution is the primary parameter for performance characterization of any imaging system. In this communication, we present an axiomatic approach for quantification of image resolution, and demonstrate that a good image resolution measure should be proportional to the standard deviation of the point spread function of an imaging system.

Index Terms: Axiomatic approach, imaging system, point spread function, image resolution, image noise.

1 Introduction

Imaging science and technology play an increasingly significant role. There are numerous types of imaging systems, such as cameras, electronic and optic microscopes, medical and industrial tomographic scanners, and radars. Image resolution is the primary parameter for performance characterization of any imaging system. Various definitions have been introduced for quantification of image resolution. Generally speaking, image resolution, or resolving power, is the ability of an imaging system to separate two localized signals. Well-known resolution measures include the Rayleigh criterion, the bandwidth of modulation transfer function (MTF), the cut-off-frequency of MTF, the full-width-at-half-maximum (FWHM), full-width-at-tenth-maximum, and standard deviation of the point spread function (PSF) of the system. *Depending on which definition of image resolution is used, a better image resolution may correspond to either a larger value (for example, the wider bandwidth, the better resolution [1]) or a smaller value (the smaller FWHM, the better resolution [2]) of resolution measure.*

Although the output of an imaging system may be either one-dimensional (1D) or multi-dimensional, 1D resolution analysis is important not only by itself but also as a basis for multi-dimensional extension. In this communication, we focus on the 1D nonnegative case [3], postulate a set of axioms that a good definition of image resolution should satisfy, and demonstrate that such an image resolution measure must be proportional to the standard deviation of PSF.

2 Axioms

It is assumed that a 1D imaging system can be modeled as

$$o(t) = p(t) \otimes i(t), \tag{1}$$

where $p(t) \geq 0$ denotes the PSF with a finite standard deviation, $i(t)$ and $o(t)$ the input and output, respectively. Let $R[p(t)]$ denote a resolution measure. *We request that the smaller the value of $R[p(t)]$, the finer detail the system can resolve.*

Our axioms are as follows:

1. $R[p(t)] > 0$, if $p(t) \neq c\delta(t)$ (*nonzero*)
2. $R[p(t)]$ is a continuous function of p in the distribution sense (*continuity*),
3. $R[p(t-c)] = R[p(t)]$ (*translation*),
4. (a) $R[p(ct)] = \frac{R[p(t)]}{c}$, (b) $R[cp(t)] = R[p(t)]$ (*scaling*),
5. $R[p_1(t) \otimes p_2(t)] = F\left(R[p_1(t)], R[p_2(t)]\right)$ (*combination*),

   where c is any positive constant, and F a function.

A few comments on the heuristics behind these axioms are in order. The nonzero axiom excludes the trivial solution that $R\left[p(t)\right] \equiv 0$. The continuity axiom guarantees that image resolution should be insensitive to perturbation unavoidable in practice. The translation axiom requires that image resolution should not depend on where and when an image is formed. The two scaling axioms mean that image resolution is inversely proportional to the scale by which the PSF is compressed (Axiom 4a), and invariant under contrast magnification (Axiom 4b). Because of Axiom 4b, from now on we assume $\int_{-\infty}^{\infty} p((t)dt = 1$ without loss of generality. The combination axiom is rather powerful, relating the resolution measure of a composite serial system to the resolution measures of its sub-systems.

3 Quantification

According to the zero axiom, we denote that

$$\alpha \equiv R[N(t;0,1)] > 0, whereN(t;0,1)isthestandardGaussianPSF.$$

$$R[N(t;\mu,\sigma^2)] = \alpha\sigma, \tag{2}$$

where $N(t;\mu,\sigma^2)$ is the Gaussian PSF with mean μ and variance σ^2.

Proof: By the translation and scaling axioms,

$$R[N(t;\mu,\sigma^2)] = R[N(\frac{t-\mu}{\sigma};0,1)] = \sigma R[N(t;0,1)] = \alpha\sigma. \blacksquare \tag{3}$$

$$F(x,y) = \sqrt{x^2+y^2}. \tag{4}$$

Proof: Let $p_1(t) = N(t;0,(x\alpha)^2)$ and $p_2(t) = N(t;0,(y\alpha)^2)$, then

$$p_1(t) \otimes p_2(t) = N(t;0,(\sqrt{x^2+y^2}\alpha)^2). \tag{5}$$

By Lemma 1 and the combination axiom,

$$\begin{aligned} \sqrt{x^2+y^2} &= R[N(t;0,(\sqrt{x^2+y^2}\alpha)^2)] = R[p_1(t)\otimes p_2(t)] \\ &= F\left(R[p_1(t)], R[p_2(t)]\right) = F(R[N(t;0,(x\alpha)^2)], R[N(t;0,(y\alpha)^2)]) \\ &= F(x,y). \blacksquare \end{aligned}$$

Let $p(t)$ be any PSF. Then

$$R[p(t)] = \alpha\sigma, \tag{6}$$

where σ is the standard deviation of $p(t)$,

$$\sigma = \sqrt{\int_{-\infty}^{\infty} (t-\mu)^2 p(t) dt}, \tag{7}$$

and

$$\mu = \int_{-\infty}^{\infty} t p(t) dt\cdot \tag{8}$$

Proof: Let $q(t) = N(t;\mu,\sigma^2)$ where σ and μ are defined by (7, 8). If $p(t) = q(t)$, then the proof can be found in Lemma 1. For $p(t) \neq q(t)$, suppose

$$R[p(t)] = \alpha\sigma + e[p(t)], \tag{9}$$

where $e[p(t)]$ is a nonzero constant. Let

$$g_n(t) = \underbrace{\left\{\sqrt{n}p(\sqrt{n}t)\right\} \otimes \left\{\sqrt{n}p(\sqrt{n}t)\right\} \otimes \cdots \otimes \left\{\sqrt{n}p(\sqrt{n}t)\right\}}_{n}, \tag{10}$$

then by the Lėvy-Lindeberg central limiting theorem, the scaling and continuity axioms,

$$\lim_{N\to\infty} R[g_n(t)] = R[q(t)] = \alpha\sigma. \tag{11}$$

On the other hand, by the scaling and combination axioms,

$$R[g_n(t)] = \sqrt{\frac{n\left(\alpha\sigma + e[p(t)]\right)^2}{n}} = \alpha\sigma + e[p(t)]. \tag{12}$$

Therefore,

$$e[p(t)] = 0.\blacksquare \tag{13}$$

$$R[\delta(t)] = 0. \tag{14}$$

Proof: It follows directly from Lemma 1, the scaling and continuity axioms, as well as the fact

$$\delta(t) = \lim_{\sigma \to 0^+} N(0, \sigma^2).\blacksquare$$

This corollary quantifies the ideal imaging system, with which infinitely small details can be captured.

4 Discussion

Similar to the axiomatic quantification of image resolution, a good image noise definition can be axiomatically shown to be proportional to the standard deviation of an image of a homogeneous region. For this purpose, we also need five axioms, which are essentially the same as those stated above. In addition to the zero, continuity, translation and scaling axioms, which can be naturally established for quantification of image noise, the combination axiom should be formulated to quantify the noise of an image generated by adding a number of noisy component images. If the noise distributions of the component images are independent, the noise distribution of the composite image can be computed via convolution. Therefore, a similar combination axiom can be obtained. Given this outline, the details for the axiomatic quantification of image noise can be developed easily, which are not included here for brevity.

The axioms we have postulated are heuristically sounding and have led to the simple closed form solution - the standard deviation. *In the case of a complex-valued PSF $p(t)$, we recommend that the standard deviation of its norm $|p(t)|$ be used to measure the resolution (assuming existence of the standard deviation), similar to what was suggested for dispersion measurement [4].* Note that despite the axiom-based conclusions they may not always be the best choice in practice, because other factors may be significant in a specific application. Our findings indicate that it is preferable to measure image resolution and image noise with the standard deviation for performance evaluation and protocol optimization of imaging systems. The axiomatic approach should be also valuable in other aspects of imaging science and technology.

5 References

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