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Cyclic Operators, Commutators, and Absolutely Continuous Measures

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CYCLIC OPERATORS, COMMUTATORS, AND ABSOLUTELY CONTINUOUS MEASURES

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ABSTRACT. Commutator equations are used to study the relationship between the tridiagonal matrix structure of an unbounded cyclic selfadjoint operator and its spectrum. Sufficient conditions are given for absolute continuity. Results are related to the study of systems of orthogonal polyomials for which the measure of orthogonality is supported on an unbounded subset of the real line.

1. Introduction. Let C be a selfadjoint operator with unit cyclic vector Φ, defined on a separable Hilbert space $\mathscr{H}$. A basis $\{\Phi_n\}$ can be obtained for $\mathscr{H}$ by orthonormalizing $\{\Phi, C\Phi, C^2\Phi, \dots\}$. It follows that the matrix with (i, j) entry defined by $\langle C\Phi_i, \Phi_j\rangle$ has tridiagonal form:

$$(1.1) \qquad \begin{bmatrix} b_1 & a_1 & 0 & 0 & \cdots \\ a_1 & b_2 & a_2 & 0 & \cdots \\ 0 & a_2 & b_3 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{bmatrix}, \qquad a_n > 0,\ b_n \text{ real}.$$

This matrix, in turn, can be used to define an operator on the subset Ω of l^2 consisting of sequences for which matrix multiplication yields a sequence in l^2. If the given operator C is bounded, then $\Omega = l^2$ and the matrix operator is unitarily equivalent to C. In fact, the set of bounded cyclic selfadjoint operators on $\mathscr{H}$ can be identified with the set of tridiagonal matrix operators satisfying (1.1) with $\{a_n\}$ and $\{b_n\}$ bounded. In this case the sequences $\{a_n\}$ and $\{b_n\}$ provide information about the spectral measure. Applications can be given to the study of systems of orthogonal polynomials for which the measure of orthogonality is supported on a bounded set (see **[1–4]**).

It is the purpose of this paper to consider matrices of the form (1.1) which represent unbounded cyclic selfadjoint operators defined on a dense subset of a separable Hilbert space $\mathscr{H}$. The tridiagonal structure of these matrices will be used to obtain information about the spectral measure of the corresponding selfadjoint operator. Applications will be given to the study of systems of orthogonal polynomials for which the interval of orthogonality is infinite.

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It will be shown below that (1.1) defines a selfadjoint operator C if $\sum 1/a_n = \infty$. In this case the spectral resolution $C = \int \lambda \, dE_\lambda$ gives rise to a spectral measure $\mu(\beta) = \|E(\beta)\phi_1\|^2$, defined for the Borel subsets of the real line. The polynomials defined by

$$(1.2) \qquad \begin{gathered} P_1(\lambda) = 1, \qquad P_2(\lambda) = \frac{\lambda - b_1}{a_1}, \\ P_n(\lambda) = \frac{(\lambda - b_{n-1})P_{n-1}(\lambda) - a_{n-2}P_{n-2}(\lambda)}{a_{n-1}} \end{gathered}$$

are orthonormal with respect to the usual inner product on $L^2(\mu)$ (see [**6**, Chapter VII]). These polynomials will be used to show that if $b_n = 0$, $d_n = a_n - a_{n-1} \geqslant 0$, and $\sum_{n=1}^{\infty}|d_{n+1} - d_n| < \infty$ for all n, then C has no eigenvalues. Under additional restrictions on the sequence $\{d_n - d_{n-1}\}$ it will be shown that C is absolutely continuous, or equivalently, that μ is absolutely continuous.

From a theoretical point of view the techniques to be used are related to the work of C. R. Putnam on the use of commutator equations in the study of spectral measures (see [**5**], for example). The use of commutator equations is obviously complicated by the fact that the operators to be studied are unbounded and hence only densely defined.

2. Main results. The first lemma provides a sufficient condition for selfadjointness.

LEMMA 1. *The matrix operator defined by* (1.1) *on the subset* Ω *of* l^2 *consisting of those sequence for which matrix multiplication yields a sequence in* l^2, *is selfadjoint if* $\sum 1/a_n = \infty$.

PROOF. To show that the matrix operator C is symmetric it is necessary to show that $\langle Cx, y\rangle = \langle x, Cy\rangle$ for any two sequences x, y in Ω. Toward this end let $x = \{x_i\}_{i=1}^{\infty}$ and $y = \{y_i\}_{i=1}^{\infty}$. It follows that

$$\langle Cx, y\rangle = \lim_{N\to\infty}\left[(b_1x_1 + a_1x_2)\bar{y}_1 + \sum_{i=2}^{N}(a_{i-1}x_{i-1} + b_ix_i + a_ix_{i+1})\bar{y}_i\right]$$

and

$$\langle x, Cy\rangle = \lim_{N\to\infty}\left[(b_1\bar{y}_1 + a_1\bar{y}_2)x_1 + \sum_{i=2}^{N}(a_{i-1}\bar{y}_{i-1} + b_i\bar{y}_i + a_i\bar{y}_{i+1})x_i\right].$$

If α_N denotes the Nth partial sum of $\langle Cx, y\rangle$ and β_N denotes the Nth partial sum of $\langle x, Cy\rangle$, then $|\alpha_N - \beta_N| = |a_Nx_{N+1}\bar{y}_N - a_nx_N\bar{y}_{N+1}|$. Assume $\lim_{N\to\infty}|\alpha_N - \beta_N| = 2d$, with $d > 0$. Then for N sufficiently large

$$d \leqslant |\alpha_N - \beta_N| \leqslant |a_Nx_{N+1}\bar{y}_N| + |a_Nx_N\bar{y}_{N+1}|.$$

It follows that $|x_{N+1}\bar{y}_N| + |x_N\bar{y}_{N+1}| \geqslant d/a_N$ for N sufficiently large which contradicts the fact that $\sum_{N=1}^{\infty}[|x_{N+1}\bar{y}_N| + |x_N\bar{y}_{N+1}|]$ converges. Therefore $d = 0$ and C is symmetric. To show that C is selfadjoint, assume that $y = \{y_i\}_{i=1}^{\infty}$ is in the domain of C^* and that $C^*y = \omega = \{\omega_i\}$. Then $\langle C\Phi_i, y\rangle = \langle \Phi_i, C^*y\rangle$ implies that $\bar{\omega}_i = a_{i-1}\bar{y}_{i-1} + b_i\bar{y}_i + a_i\bar{y}_{i+1}$ and hence that $C^*y = Cy$. Therefore C is selfadjoint. $\square$

The objective now is to analyze the spectral measure. Throughout the remainder of the paper the following will be assumed:

(2.1)
(i) C is the cyclic selfadjoint operator defined on a dense subset of l^2 by the matrix (1.1) with $\Sigma 1/a_n = \infty$.
(ii) The diagonal entries in (1.1) vanish.
(iii) The subdiagonal sequence $\{a_n\}$ monotonically increases to infinity.

Also, the following notation will be used:

(2.2)
(i) $J = (T - T^*)/2i$ where $T\Phi_n = 2a_n\Phi_{n+1}$.
(ii) J_N is the bounded operator obtained from J by letting $a_n = a_N$ for $n \geqslant N$.
(iii) $CJ_N - J_NC = -2iK_N$ on Ω_N for $N = 1, 2, 3, \ldots$.
(iv) $d_n = a_n - a_{n-1}$ $(a_0 = 0)$.

Note that for each N, Ω_N is a dense linear subset of l^2 which contains the basis vectors $\{\Phi_n\}$. Also K_N, defined by (2.2)(iii), is bounded if $\{d_n\}$ is a bounded sequence.

LEMMA 2. *Assume* (2.1) *and* (2.2). *If* $\{d_n\}$ *is bounded and* $Cx = \lambda x$, *then* $\langle K_N x, x\rangle = 0$ *for* $N = 1, 2, \ldots$.

PROOF. If $Cx = \lambda x$ with $x \neq 0$, then x is a nonzero multiple of $(P_1(\lambda), P_2(\lambda), P_3(\lambda), \ldots)$ with $\Sigma|P_n(\lambda)|^2 < \infty$. Let $x_j = (P_1(\lambda), \ldots, P_j(\lambda), 0, \ldots)$. If $C_\lambda = C - \lambda I$ then $\langle C_\lambda J_N x_j, x\rangle - \langle J_N C_\lambda x_j, x\rangle = -2i\langle K_N x_j, x\rangle$. Since $C_\lambda x = 0$, $\langle J_N C_\lambda x_j, x\rangle = 2i\langle K_N x_j, x\rangle$. Thus since K_N is a bounded operator, $\lim_{j\to\infty}|\langle C_\lambda x_j, J_N x\rangle| = 2|\langle K_N x, x\rangle|$ exists. Assume $\lim|\langle C_\lambda x_j, J_N x\rangle| = 2p$, $p > 0$. If $J_N x = (t_1, t_2, \ldots)$ then

$$\begin{aligned}|\langle C_\lambda x_j, J_N x\rangle| &= |t_j a_{j-1} P_{j-1}(\lambda) - \lambda t_j P_j(\lambda) + a_j P_j(\lambda) t_{j+1}| \\ &= |-t_j a_j P_{j+1} + a_j t_{j+1} P_j(\lambda)|.\end{aligned}$$

If $d_n = a_n - a_{n-1} \leqslant M$ then for j sufficiently large

$$p < |\langle C_\lambda x_j, J_N x\rangle| \leqslant 2Mj \max\{|t_j P_{j+1}(\lambda)|, |t_{j+1} P_j(\lambda)|\}.$$

But this contradicts the fact that $\Sigma[|t_j P_{j+1}(\lambda)| + |t_{j+1} P_j(\lambda)|] < \infty$. It follows that $\lim_{j\to\infty}|\langle C_\lambda x_j, J_N x\rangle| = 2|\langle K_N x, x\rangle| = 0$. □

THEOREM 1. *Assume* (2.1). *If* $d_n = a_n - a_{n-1}$ *and* $\Sigma|d_{n+1} - d_n| < \infty$, *then* C *has no eigenvalues.*

PROOF. Suppose λ is an eigenvalue with corresponding eigenvector $x = (P_1(\lambda), P_2(\lambda), P_3(\lambda), \ldots)$. Choose N_0 sufficiently large such that for $n \geqslant N_0$, $d_n < \frac{1}{2}(a_n - |\lambda/2|)$ and $\Sigma_n^\infty |d_i - d_{i+1}| < \frac{1}{2}(a_n - |\lambda|/2)$. Let N be defined by $P_N^2(\lambda) = \max_{n \geqslant N_0} P_n^2(\lambda)$. Then $CJ_N - J_NC = -2iK_N$ on a dense linear subset Ω_N of $\mathscr{H}$. Note that $K_N = [k_{ij}]$ where $k_{ii} = a_i^2 - a_{i-1}^2$ for $i = 1, \ldots, N$, $k_{ii} = a_N(a_i - a_{i-1})$

for $i > N$, and $k_{i,i+2} = k_{i+2,i} = \frac{1}{2}a_N(a_{i+1} - a_i)$ for $i \geqslant N$. All other matrix entries of K_N are zeros. It is shown in [2] that

$$\sum_{n=1}^{N} (a_n^2 - a_{n-1}^2)P_n^2(\lambda) = \left[a_{N-1} P_{N-1}(\lambda) - \frac{\lambda}{2}P_N(\lambda)\right]^2 + \left(a_N^2 - \frac{\lambda^2}{4}\right)P_N^2(\lambda).$$

Thus

$$\begin{aligned}\langle K_N x, x\rangle &= \sum_1^N (a_i^2 - a_{i-1}^2)P_i^2(\lambda) + \sum_{N+1}^{\infty} a_N d_i P_i^2(\lambda) + \sum_{N+1}^{\infty} a_N d_i P_{i+1}P_{i-1} \\ &\geqslant \left[a_{N-1}P_{N-1}(\lambda) - \frac{\lambda}{2}P_N(\lambda)\right]^2 + \left(a_N^2 - \frac{\lambda^2}{4}\right)P_N^2(\lambda) + \sum_{N+1}^{\infty} a_N d_i P_i^2(\lambda) \\ &\quad - a_N \sum_{N+1}^{\infty} \frac{1}{2} d_i\left[P_{i+1}^2 + P_{i-1}^2\right] \\ &\geqslant \left[a_{N-1}P_{N-1}(\lambda) - \frac{\lambda}{2}P_N(\lambda)\right]^2 + \left(a_N^2 - \frac{\lambda^2}{4}\right)P_N^2(\lambda) + \frac{1}{2}a_N d_{N+1}P_{N+1}^2(\lambda) \\ &\quad - \frac{1}{2}a_N d_N P_N^2(\lambda) - a_N P_N^2(\lambda)\sum_{N+1}^{\infty} |d_i - d_{i-1}| \\ &> \left[a_{N-1}P_{N-1}(\lambda) - \frac{\lambda}{2}P_N(\lambda)\right]^2 + \frac{1}{4}a_N\left(a_N - \frac{|\lambda|}{2}\right)P_N^2(\lambda).\end{aligned}$$

If $P_N(\lambda) = P_{N-1}(\lambda) = 0$, then $P_{N-2}(\lambda) = \cdots = P_1(\lambda) = 0$ which cannot happen since $P_1 \equiv 1$. Therefore $\langle K_N x, x\rangle \neq 0$ and by the previous lemma, λ is not an eigenvalue. □

Sufficient conditions will now be given for absolute continuity. Another lemma is needed. All notation remains as before.

LEMMA 3. *Assume* (2.1) *and* (2.2). *If the sequence* $\{d_n\}$ *is bounded, then for any bounded interval* Δ *and positive integer* N,

$$\left|\left\langle K_N E(\Delta)\Phi_1, E(\Delta)\Phi_1\right\rangle\right| \leqslant \tfrac{1}{2}\|J_N\|\,|\Delta|\,\|E(\Delta)\Phi_1\|^2.$$

PROOF. Let λ be the midpoint of the bounded interval Δ and let $C_\lambda = C - \lambda I$. If $x_n = \sum_{i=1}^n \alpha_i\Phi_i$ where $\alpha_i = \langle E(\Delta)\Phi_1, \Phi_i\rangle$, then $C_\lambda J_N x_n - J_N C_\lambda x_n = -2iK_N x_n$ and $\langle J_N x_n, C_\lambda E(\Delta)\Phi_1\rangle - \langle C_\lambda x_n, J_N E(\Delta)\Phi_1\rangle = -2i\langle K_N x_n, E(\Delta)\Phi_1\rangle$. Since J_N and K_N are bounded, $\lim_{n\to\infty}\langle C_\lambda x_n, J_N E(\Delta)\Phi_1\rangle$ exists and it will be shown that this limit is $\langle C_\lambda E(\Delta)\Phi_1, J_N E(\Delta)\Phi_1\rangle$. If so then

$$\begin{aligned}&\left\langle J_N E(\Delta)\Phi_1, C_\lambda E(\Delta)\Phi_1\right\rangle - \left\langle C_\lambda E(\Delta)\Phi_1, J_N E(\Delta)\Phi_1\right\rangle \\ &\qquad = -2i\left\langle K_N E(\Delta)\Phi_1, E(\Delta)\Phi_1\right\rangle\end{aligned}$$

and since $\|C_\lambda E(\Delta)\Phi_1\| \leqslant \frac{1}{2}|\Delta|\|E(\Delta)\Phi_1\|$, the lemma readily follows.

To evaluate the required limit, let $y_n = \sum_{i=1}^n\langle C_\lambda E(\Delta)\Phi_1, \Phi_i\rangle\Phi_i$. Since $E(\Delta)\Phi_1$ is in the domain of C_λ, the sequence $\{y_n\}$ converges to $C_\lambda E(\Delta)\Phi_1$, and so $\{J_N y_n\}$ also converges. Let $\omega_n = C_\lambda x_n - y_{n-1} = (a_{n-1}\alpha_{n-1} - \lambda\alpha_n)\Phi_n + a_n\alpha_n\Phi_{n+1}$. Then $\lim_{n\to\infty}\langle J_N\omega_n, E(\Delta)\Phi_1\rangle$ must exist. Since $E(\Delta)\phi_1 = \sum_{i=1}^{\infty}\alpha_i\Phi_i$,

$$\left|\left\langle \omega_n, J_N E(\Delta)\Phi_1\right\rangle\right| = \left|(a_{n-1}\alpha_{n-1} - \lambda\alpha_n)q_n + a_n\alpha_n q_{n+1}\right|,$$

where $q_n = \langle J_N E(\Delta)\Phi_1, \Phi_n \rangle$. Assume $\lim|\langle \omega_n, J_N E(\Delta)\Phi_1 \rangle| = 2p$ where $p > 0$. Since $\{d_n\}$ is bounded there exists M such that $a_n \leqslant nM$. It follows that for n sufficiently large

$$p < \left|\langle \omega_n, J_N E(\Delta)\Phi_1 \rangle\right| < nM\left[|\alpha_{n-1}q_n| + |\alpha_n q_n| + |\alpha_n q_{n+1}|\right].$$

But this contradicts the fact that $\Sigma[|\alpha_{n-1}q_n| + |\alpha_n q_n| + |\alpha_n q_{n+1}|] < \infty$. Therefore $\lim|\langle \omega_n, J_N E(\Delta)\Phi_1 \rangle| = 0$. Hence

$$\lim_{n\to\infty} \langle C_\lambda x_n, J_N E(\Delta)\Phi_1 \rangle = \langle C_\lambda E(\Delta)\Phi_1, J_N E(\Delta)\Phi_1 \rangle$$

as was to be shown. □

Theorem 1 claims that if $\Sigma|d_{n+1} - d_n| < \infty$ then C has no eigenvalues. The next theorem provides a sufficient condition for absolute continuity. Note that the conditions of the theorem are satisfied if $\{d_n\}$ is monotone increasing, bounded above, and $d_{n+1} - d_n \leqslant d_n - d_{n-1}$. In fact, all examples of this theorem are of this type.

THEOREM 2. *Assume* (2.1). *Let* $d_n = a_n - a_{n-1}$. *If* $\{d_n\}$ *is bounded and* $d_{n+1} + d_{n-1} \leqslant 2d_n$ *for* $n \geqslant N \geqslant 2$ *then* C *is absolutely continuous.*

PROOF. The equation $CJ_N - J_N C = -2iK_N$ holds on a dense subset Ω_N where J_N is defined in (2.2) and $K_N = [k_{ij}]$ with $k_{ii} = a_i^2 - a_{i-1}^2$ for $i = 1, \ldots, N$, $k_{ii} = a_N(a_i - a_{i-1})$ for $i > N$, and $k_{i,i+2} = \frac{1}{2}a_N(a_{i+1} - a_i)$ for $i \geqslant N$. All other matrix entries of K_N are zeros. Note also that for any interval Δ

$$E(\Delta)\Phi_1 = \sum_{i=1}^{\infty} \langle E(\Delta)\Phi_1, \Phi_i \rangle \Phi_i, \quad \text{where } \langle E(\Delta)\Phi_1, \Phi_i \rangle = \int_\Delta P_i \, d\mu.$$

If $x_n = \Sigma_{i=1}^n \langle E(\Delta)\Phi_1, \Phi_i \rangle \Phi_i$ then for $n > N$

$$\begin{aligned}
\langle K_N x_n, x_n \rangle &= \sum_{i=1}^{N} (a_i^2 - a_{i-1}^2)\left|\int_\Delta P_i \, d\mu\right|^2 + \sum_{i=N+1}^{n} a_N(a_i - a_{i-1})\left|\int_\Delta P_i \, d\mu\right|^2 \\
&\quad + \sum_{i=N+1}^{n-1} a_N(a_i - a_{i-1})\int_\Delta P_{i+1}\int_\Delta P_{i-1} \\
&\geqslant \sum_{i=1}^{N} (a_i^2 - a_{i-1}^2)\left|\int_\Delta P_i \, d\mu\right|^2 + \sum_{i=N+1}^{n} a_N(a_i - a_{i-1})\left|\int_\Delta P_i \, d\mu\right|^2 \\
&\quad - \sum_{i=N+1}^{n-1} \frac{1}{2}a_N(a_i - a_{i-1})\left[\left|\int P_{i+1}\right|^2 + \left|\int P_{i-1}\right|^2\right] \\
&\geqslant a_1^2\left|\int_\Delta P_1 \, d\mu\right|^2.
\end{aligned}$$

This, together with the result of Lemma 3, implies that

$$a_1^2\|E(\Delta)\Phi_1\|^4 \leqslant \tfrac{1}{2}\|J_N\| \, |\Delta| \, \|E(\Delta)\Phi_1\|^2.$$

Let β be a Borel subset of the real line of Lebesgue measure zero. Then for any $\varepsilon > 0$ there exists a pairwise disjoint sequence of intervals $\{\Delta_j\}$ such that $\beta \subset \bigcup \Delta_j$ and $\Sigma|\Delta_j| < \varepsilon$. Since

$$\mu(\beta) \leqslant \sum \mu(\Delta_j) \leqslant \frac{1}{2a_1^2} \|J_N\| \sum |\Delta_j|$$

it readily follows that $\mu(\beta) = 0$. □

Another sufficient condition for absolute continuity is provided by the following theorem. It will be shown in the next section that many examples can be constructed with the sequence $\{d_n\}$ monotonically decreasing to a nonnegative limit.

THEOREM 3. *Assume* (2.1). *Let* $d_n = a_n - a_{n-1}$. *If* $\{d_n\}$ *is bounded and* $d_{n+1}^2 \leqslant d_{n+2} d_n$ *for* $n \geqslant N \geqslant 2$ *then C is absolutely continuous.*

PROOF. The proof is similar to that of Theorem 2. With J_N defined by (2.2) the operator K_N is obtained from the equation $CJ_N - J_N C = -2iK_N$ which holds on a dense subset of $\mathscr{H}$. For any interval Δ let $x_n = \Sigma_{i=1}^n \langle E(\Delta)\Phi_1, \Phi_i\rangle \Phi_i$. Then for $n > N$,

$$\begin{aligned} \langle K_N x_n, x_n\rangle &= \sum_{i=1}^{N} \left(a_i^2 - a_{i-1}^2\right)\left|\int_\Delta P_i\, d\mu\right|^2 + \sum_{i=N+1}^{n} a_N(a_i - a_{i-1})\left|\int_\Delta P_i\, d\mu\right|^2 \\ &\quad + \sum_{i=N+1}^{n-1} a_N(a_i - a_{i-1}) \int_\Delta P_{i+1}\, d\mu \int_\Delta P_{i-1}\, d\mu \\ &\geqslant \sum_{i=1}^{N} \left(a_i^2 - a_{i-1}^2\right)\left|\int_\Delta P_i\, d\mu\right|^2 + \sum_{i=N+1}^{n} a_N(a_i - a_{i-1})\left|\int_\Delta P_i\, d\mu\right|^2 \\ &\quad - \left|\sum_{i=N+1}^{n-1} a_N d_i \int_\Delta P_{i+1}\, d\mu \int_\Delta P_{i-1}\, d\mu\right|. \end{aligned}$$

Since

$$\begin{aligned} &\left|\sum_{i=N+1}^{n-1} a_N d_i \int_\Delta P_{i+1}\, d\mu \int_\Delta P_{i-1}\, d\mu\right| \\ &\qquad = a_N \sum_{i=N+1}^{n-1} \sqrt{d_{i-1}} \frac{d_i}{\sqrt{d_{i-1}}} \left|\int_\Delta P_{i+1}\, d\mu\right| \left|\int_\Delta P_{i-1}\, d\mu\right| \\ &\qquad \leqslant \sum_{i=N+1}^{n-1} \frac{a_N}{2}\left[d_{i-1}\left|\int_\Delta P_{i-1}\right|^2 + \frac{d_i^2}{d_{i-1}}\left|\int_\Delta P_{i+1}\, d\mu\right|^2\right] \end{aligned}$$

it follows that $\langle K_N x_n, x_n\rangle \geqslant a_1^2 |\int P_1\, d\mu|^2$. The proof ends with the same argument used in the proof of Theorem 2. □

3. Examples. The purpose of this final section is to illustrate the above results. Obviously all three theorems hold if the difference sequence $\{d_n\}$ is a constant sequence. As noted above, examples for Theorem 2 can be constructed by choosing

$\{d_n\}$ to be monotone increasing, bounded above with $d_{n+1} - d_n \leqslant d_n - d_{n-1}$. Choose, for example, $d_n = \sum_{i=1}^{n} 1/2^i$. For this choice of d_n the hypotheses of Theorem 2 are satisfied but those of Theorem 3 are not.

Theorem 3 requires that the sequence $\{d_n\}$ satisfy the condition $d_{n+1}^2 \leqslant d_n d_{n+2}$ for $n \geqslant N$. This condition implies that if $d_n \leqslant d_{n+1}$ then $d_{n+1} \leqslant d_{n+2}$. It follows that $\{d_n\}_{n=N}^{\infty}$ is either monotone decreasing or eventually monotone increasing. If $\{d_n\}$ is monotone increasing for $n \geqslant M$ then $d_{M+k} < d_{M+1}(d_{M+1}/d_M)^{k-1}$ for $k = 1, 2, \ldots,$ and so $\{d_n\}$ either diverges or becomes eventually constant. Since Theorem 3 also requires that $\{d_n\}$ be bounded it must be true that $\{d_n\}_{n=N}^{\infty}$ is montone decreasing.

Examples for Theorem 3 can be constructed by the following scheme. Choose $\frac{1}{2} \leqslant d_2 < d_1$. If $\imath_2 = d_2(d_2/d_1)$ then $\imath_2 < d_2$ and it is possible to choose d_3 with $\max\{\imath_2, \frac{1}{3}\} \leqslant d_3 < d_2$. If $d_1, \ldots, d_n$ have been chosen, let $\imath_n = d_n^2/d_{n-1}$ and choose $\max\{\imath_n, 1/(n+1)\} \leqslant d_{n+1} < d_n$. The end result is a monotone decreasing sequence $\{d_n\}$ with $d_{n+1}^2 \leqslant d_n d_{n+2}$ and $\Sigma d_n = \infty$. One specific example is $d_n = 1/n$.

Another, perhaps more significant, example comes from the normalized Hermite polynomials which are orthonormal on $(-\infty, \infty)$ with respect to $d\mu = w(x)dx$ where $w(x) = e^{-x^2}$. These polynomials satisfy a recursion formula of the form (1.2) with $b_n = 0$ and $a_n = \sqrt{n/2}$. If C is defined by (1.1) and J by (2.2) then $CJ - JC = -iI$ where I denotes the identity opeator (see [5, pp. 63–64] for an interesting interpretation of the operators C and J). In this case $d_n = (1/\sqrt{2})(\sqrt{n} - \sqrt{n-1})$ and it can be shown that $d_{n+1}^2 \leqslant d_n d_{n+2}$ so that the conditions of Theorem 3 are satisfied. Since $\sqrt{n} - \sqrt{n-1} = 1/(\sqrt{n} + \sqrt{n-1})$ the required inequality is equivalent to $(\sqrt{n+1} + \sqrt{n})^2 \geqslant (\sqrt{n} + \sqrt{n-1})(\sqrt{n+2} + \sqrt{n+1})$ which is easily verified.

In a related example, suppose $a_n^2 - a_{n-1}^2 = M$ for all n, so that $CJ - JC = -2iK$ where K is a multiple of the identity. If $d_n = a_n - a_{n-1}$ then $d_{n+1}^2 \leqslant d_n d_{n+2}$ is equivalent to $(a_{n+1} + a_n)^2 \geqslant (a_n + a_{n-1})(a_{n+2} + a_{n+1})$ which follows from the observations that $a_{n-1}a_{n+1} = (a_n - d_n)(a_n + d_{n+1}) \leqslant a_n^2 - d_{n+1}^2 \leqslant a_n^2$ and that $a_{n-1}a_{n+2} = (a_n - d_n)(a_{n+1} + d_{n+2}) \leqslant a_n a_{n+1}$.

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