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QUASITRIANGULAR MATRICES

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Abstract. It is shown that there exist quasitriangular operators which
cannot be represented as quasitriangular matrices.

Introduction. A quasitriangular matrix is an infinite matrix \( A = [a_{ij}] \) in
which all entries below the subdiagonal are zero and the subdiagonal entries
cluster at zero (i.e., \( a_{ij} = 0 \) for \( i > j + 1 \) and \( \lim \inf |a_{i,j+1}| = 0 \)). A quasitri-
angular operator is an operator which can be expressed as the sum of a
triangular matrix (\( a_{ij} = 0 \) for \( i > j \)) and a compact one. The relationship
between quasitriangular matrices and quasitriangular operators, and the
significance of studying that relationship in conjunction with the invariant
subspace problem, are discussed by Halmos in [2]. It is shown in [2] that
every bounded quasitriangular matrix defines a quasitriangular operator.
Halmos then asks whether every cyclic quasitriangular operator has a quasi-
triangular matrix. It will be shown below that the answer is no. A few
preliminary ideas are needed.

Let \( A = \int_{\eta} dE_{\eta} \) be a bounded selfadjoint operator defined on a separable
Hilbert space \( \mathcal{H} \). By Weyl's theorem \( A \) is the sum of a diagonal operator and
a compact one. Hence \( A \) is quasitriangular. Denote by \( \mathcal{K}_a(A) \) the set of
elements \( x \) in \( \mathcal{H} \) for which \( \| E_{\eta} x \|^2 \) is an absolutely continuous function of \( \eta \).
The subspace \( \mathcal{K}_a(A) \) reduces \( A \) [1, p. 104] and the restriction of \( A \) to \( \mathcal{K}_a(A) \)
is called the absolutely continuous part of \( A \). A result due to Kato [3] and
Rosenblum [4] asserts that the absolutely continuous part of the operator \( A \)
remains stable under a trace class perturbation. In particular, if \( C \) is
selfadjoint and of trace class, and if \( B = A + C \), then the absolutely continu-
ous parts of \( A \) and \( B \) are unitarily equivalent.

Main result. The main result to be established is as follows.

Proposition. A selfadjoint operator with a nontrivial absolutely continuous
part cannot be represented as a quasitriangular matrix.

Proof. Let \( A \) be a selfadjoint operator with a nontrivial absolutely continu-
ous part. Suppose that with respect to some orthonormal basis, \( A \) can be
represented as a quasitriangular matrix. Clearly this matrix takes the form
with some subsequence of \( \{a_n\} \) converging to zero. Furthermore, the subsequence \( \{a_{n_k}\} \) can be chosen so that \( \sum |a_{n_k}| < \infty \).

Let \( B \) be the matrix obtained from \( A \) by replacing each \( a_{n_k} \) by zero. Then \( B \) has finite dimensional invariant subspaces. In fact, \( B \) has a pure point spectrum.

If \( C = A - B \) then \( C \) is the real part of a weighted shift with weight sequence \( \{c_n\} \) satisfying \( \sum |c_n| < \infty \). Hence \( C \) is of trace class. Since \( A \) has an absolutely continuous part it follows, from the Kato-Rosenblum theorem, that \( A - C = B \) has an absolutely continuous part. But this contradicts the fact that \( B \) has a pure point spectrum.

**Corollary.** The real part of the unilateral shift does not have a quasitriangular matrix.

**References**


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