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SPECTRAL MEASURES, ORTHOGONAL POLYNOMIALS, AND ABSOLUTE CONTINUITY*

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Abstract. This paper studies the spectral measure of an unbounded tridiagonal matrix operator for which the matrix entries satisfy a certain growth condition, and presents a sufficient condition for the existence of an absolutely continuous part. The results are related to a class of orthogonal polynomials with exponential weights.

Key words. absolute continuity, commutators, orthogonal polynomials

AMS(MOS) subject classification. primary 47B15

1. Introduction. The purpose of this paper is to continue the study of unbounded tridiagonal matrix operators, measures and systems of orthogonal polynomials begun in [2]. A brief review of some known results in the bounded case will introduce the unbounded problem to be considered.

A bounded cyclic self-adjoint operator C, with cyclic vector ϕ, defined on a separable Hilbert space $\mathcal{H}$, can be represented as a tridiagonal matrix with respect to the basis obtained by orthonormalizing $\{C^n \phi\}_{n=0}^{\infty}$. The positive subdiagonal sequence $\{a_n\}$ and diagonal sequence $\{b_n\}$ in this matrix can be used to obtain information about the Borel measure $\mu(\beta) = \|E(\beta)\phi_1\|^2$, obtained from the spectral resolution $C = \int \lambda \, dE_\lambda$. It is shown in [1] and [4], for example, that if $\lim a_n = a$, $a \neq 0$, $\lim b_n = 0$, $\sum |a_n - a_{n-1}| < \infty$ and $\sum |b_n - b_{n-1}| < \infty$ then μ restricted to $(-2a, 2a)$ is absolutely continuous with respect to Lebesgue measure. This result was motivated by and is applicable to the study of orthogonal polynomials. For the spectral measure μ is also the measure of orthogonality for the sequence of polynomials $\{P_n\}$ recursively defined as follows:

$$P_1(\lambda) = 1, \qquad P_2(\lambda) = \frac{\lambda - b_1}{a_1},$$

(1.1)

$$P_n(\lambda) = \frac{(\lambda - b_n)P_{n-1}(\lambda) - a_{n-2}P_{n-2}(\lambda)}{a_{n-1}}.$$

In fact, the polynomials $\{P_n\}$ form an orthonormal basis for $L^2(\mu)$ and C is unitarily equivalent to the multiplication operator on $L^2(\mu)$ defined by $Mf(\lambda) = \lambda f(\lambda)$. Such systems of polynomials have been studied extensively in the literature. One item of interest, among many, has been the relationship between the recurrence coefficients and the nature of the measure of orthogonality.

Recently there has been considerable interest in the study of systems of the form (1.1) with $a_n > 0$, b_n real, for which the support of the measure of orthogonality is an unbounded set. In this case the sequence $\{a_n\}$ is unbounded. As discussed in [2], the corresponding tridiagonal matrix $C = [c_{ij}]$ with $c_{ii} = b_i$ and $c_{i,i+1} = c_{i+1,i} = a_i$ defines an unbounded operator on l^2 with domain consisting of those elements in l^2 for which matrix multiplication yields a vector in l^2. If $\sum (1/a_n) = \infty$ then C is self-adjoint and hence has a spectral decomposition $C = \int \lambda \, dE\lambda$. If $\{\phi_n\}$ is the standard basis for l^2, then ϕ_1 is a cyclic vector for C and $\mu(\beta) = \|E(\beta)\phi_1\|^2$ is the measure of orthogonality for the polynomials in (1.1). The purpose of this paper is to present a sufficient condition

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in terms of the sequences $\{a_n\}$ and $\{b_n\}$ for the existence of a nontrivial absolutely continuous part for the measure μ. The condition to be presented seems to be the natural generalization to the unbounded case of the condition given above for the bounded case. Whereas in the bounded case it is sufficient that limits exist and differences are absolutely summable, in the unbounded case it is sufficient that limits exist and differences of differences are absolutely summable. This will be made more precise below. The main result will be illustrated with a class of orthogonal polynomials introduced by G. Freud.

2. Main results. Henceforth C will denote an infinite matrix of the form

$$(2.1) \qquad C = \begin{bmatrix} 0 & a_1 & 0 & 0 & \cdots \\ a_1 & 0 & a_2 & 0 & \cdots \\ 0 & a_2 & 0 & a_3 & \cdots \\ 0 & 0 & a_3 & 0 & \cdots \\ \cdot & \cdot & \cdot & \cdot & \end{bmatrix}$$

with $a_n > 0$ and $\lim a_n = \infty$. It will further be assumed that $\sum (1/a_n) = \infty$ (so that C is self-adjoint), that $\sum_{n=2}^{\infty} [a_n^2 - a_{n-1}^2]^- < \infty$ and that $\sum |d_n - d_{n-1}| < \infty$, where $d_n = |a_n - a_{n-1}|$. The domain of C will consist of those vectors in l^2 for which matrix multiplication yields a vector in l^2. As shown below, such operators have no eigenvalues. Therefore the spectrum coincides with the essential spectrum and remains fixed if a finite number of terms in the sequence $\{a_n\}$ are changed. This is needed for the main result.

To establish the results on eigenvalues and the existence of an absolutely continuous part the following notation is needed. If $\{\phi_n\}$ is the standard basis for l^2 then $C\phi_n = \frac{1}{2}(T + T^*)\phi_n$ where $T\phi_n = 2a_n\phi_{n+1}$. Let $J\phi_n = (1/2i)(T - T^*)\phi_n$ and obtain the bounded operator J_N from J by substituting a_N for a_n when $n \geq N$. It follows that $CJ_N - J_NC = -2iK_N$ where $K_N = [k_{ij}]$ is a bounded operator with $k_{ii} = a_i^2 - a_{i-1}^2$ for $i = 1, \cdots, N$, $k_{ii} = a_N(a_i - a_{i-1})$ for $i > N$, $k_{i,i+2} = k_{i+2,i} = \frac{1}{2}a_N(a_{i+1} - a_i)$ for $i \geq N$ and all other entries equal to zero. This commutator equation, which holds only on a dense subset of H, is fundamental to the arguments to be presented.

The following result essentially appears in [2]. The proof is summarized to indicate the modifications needed for the general setting of this paper. Note that $d_n = |a_n - a_{n-1}|$. Recall from [1] that an induction argument can be used to show that the polynomials defined in (1.1) satisfy the equation

$$a_1^2 P_1^2(\lambda) + \sum_{n=2}^{N} (a_n^2 - a_{n-1}^2) P_n^2(\lambda) = \left[a_{N-1} P_{N-1}(\lambda) - \frac{\lambda}{2} P_N(\lambda) \right]^2 + \left(a_N^2 - \frac{\lambda^2}{4} \right) P_N^2(\lambda).$$

THEOREM 1. *If $\sum [a_n - a_{n-1}]^- < \infty$ and $\sum |d_n - d_{n-1}| < \infty$ then C has no eigenvalues.*

Proof. Assume λ is an eigenvalue. One corresponding eigenvector must be $x = \{P_n(\lambda)\}$. Choose N_0 such that for $n \geq N_0$, $\sum_n^{\infty} [a_i - a_{i-1}]^- < \frac{1}{8}(a_n - |\lambda|/2)$, $d_n < \frac{1}{4}(a_n - |\lambda|/2)$ and $\sum_n^{\infty} |d_i - d_{i-1}| < \frac{1}{4}(a_n - |\lambda|/2)$. Let N be defined by $P_N^2(\lambda) = \max_{n \geq N_0} P_n^2(\lambda)$. It then follows that

$$\langle K_N x, x \rangle \geq \left[a_{N-1} P_{N-1}(\lambda) - \frac{\lambda}{2} P_N(\lambda) \right]^2 + \left(a_N^2 - \frac{\lambda^2}{4} \right) P_N^2(\lambda) + a_N \sum_{N+1}^{\infty} d_i P_i^2(\lambda)$$

$$- 2a_N \sum_{N+1}^{\infty} [a_i - a_{i-1}]^- P_i^2(\lambda) - \frac{1}{2} a_N \sum_{N+1}^{\infty} d_i [P_{i+1}^2(\lambda) + P_{i-1}^2(\lambda)]$$

$$\geq \left[a_{N-1} P_{N-1}(\lambda) - \frac{\lambda}{2} P_N(\lambda) \right]^2 + \frac{1}{4} \left(a_N^2 - \frac{\lambda^2}{4} \right) P_N^2(\lambda).$$

But this contradicts the fact, established in [2], that if $\{d_n\}$ is bounded and $Cx = \lambda x$ then $\langle K_N x, x \rangle = 0$. □

If C given by (2.1) is self-adjoint with spectral resolution $C = \int \lambda \, dE_\lambda$ then the polynomials defined in (1.1) are orthonormal with respect to the measure $\mu(\beta) = \|E(\beta)\phi_1\|^2$. The following technical lemma about these polynomials is needed for the result on absolute continuity. Note that the lemma has content when a_1^2 is large relative to the sum $\sum_{n=2}^\infty [a_n^2 - a_{n-1}^2]^-$. Obviously, for example, the lemma provides information when $\{a_n\}$ is monotone increasing.

LEMMA 1. *Suppose there exists a subinterval* Δ *of* $[-2a_1, 2a_1]$ *and a* $\delta > 0$ *such that* $\lambda \in \Delta$ *implies that* $4a_1^2 - \lambda^2 \geqq \delta$ *and* $\sum_{n=2}^\infty [a_n^2 - a_{n-1}^2]^- < \delta/8$. *Then for* $n > 1$, $\int_\Delta P_n^2 \, d\mu \leqq 8a_1^2 \mu(\Delta)/\delta$.

Proof. Fix $n > 1$. Choose $N < n$ such that $\int_\Delta P_N^2 \, d\mu = \max_{1 < i \leqq n} \int_\Delta P_i^2 \, d\mu$. Then

$$a_1^2 \int_\Delta P_1 \, d\mu = a_1^2 \int_\Delta P_1 \, d\mu + \sum_{i=2}^N (a_i^2 - a_{i+1}^2) \int_\Delta P_i^2 \, d\mu - \sum_{i=2}^N (a_i^2 - a_{i-1}^2) \int_\Delta P_i^2 \, d\mu$$

$$\geqq \int_\Delta \left(a_N^2 - \frac{\lambda^2}{4} \right) P_N^2 \, d\mu - \sum_{i=2}^N (a_i^2 - a_{i-1}^2)^+ \int_\Delta P_i^2 \, d\mu.$$

Since $a_N^2 - a_1^2 = \sum_{i=1}^N [a_i^2 - a_{i-1}^2]^+ - \sum_{i=2}^N [a_i^2 - a_{i-1}^2]^-$ it follows that

$$a_1^2 \int_\Delta P_1 \, d\mu \geqq \int_\Delta \left(a_1^2 - \frac{\lambda^2}{4} \right) P_N^2 \, d\mu - \frac{\delta}{8} \int_\Delta P_N^2 \, d\mu.$$

Hence $(8a_1^2/\delta)\mu(\Delta) \geqq \int_\Delta P_N^2 \, d\mu \geqq \int_\Delta P_n^2 \, d\mu$ as was to be shown. □

This lemma will now be used to present the main result of this paper. The notation sp (C) will be used for the spectrum of C.

THEOREM 2. *If* $\sum_{n=2}^\infty [a_n^2 - a_{n-1}^2]^- < \infty$ *and* $\sum_{n=2}^\infty |d_n - d_{n-1}| < \infty$ *then* μ *has an absolutely continuous part with support* sp (C).

Proof. Fix $R \geqq 1$. By the Kato–Rosenblum Theorem [3], [7] it is enough to show that the spectral measure of the trace class perturbation of C obtained by changing a finite number of weights in $\{a_n\}$ is absolutely continuous on $[-R, R]$. Observe that since there are no eigenvalues, all such perturbations of C will have the same spectrum. Now choose N such that $2a_N - R \geqq 8 \sum_{n=N}^\infty |d_n - d_{n-1}|$, $2a_N - R \geqq 32 \sum_{n=N}^\infty [a_n^2 - a_{n-1}^2]^-$ and such that for $n \geqq N$, $a_n \geqq R$ and $a_n \geqq \frac{1}{2}a_N$. Note that if N is so chosen, then for $n \geqq N$, $\Delta \subset [-R, R]$ and $\lambda \in \Delta$, it follows that $4a_n^2 - \lambda^2 \geqq 4a_n^2 - R^2 \geqq 16a_n \sum_N^\infty |d_n - d_{n-1}|$. Similarly, $4a_n^2 - \lambda^2 \geqq 64a_n \sum_N^\infty [a_n^2 - a_{n-1}^2]^-$. Assume, for now, that $a_1 = a_2 = \cdots = a_N$ (i.e., consider a trace class perturbation of the given operator). If Δ is a subinterval of $[-R, R]$, then $E(\Delta)\phi_1 = \sum_{i=1}^\infty \langle E(\Delta)\phi_1, \phi_i \rangle \phi_i$ where $\langle E(\Delta)\phi_1, \phi_i \rangle = \int_\Delta P_i \, d\mu$. The commutator equation $CJ_N - J_N C = -2iK_N$ is used in [2] to show that $|\langle K_N E(\Delta)\phi_1, E(\Delta)\phi_1 \rangle| \leqq \frac{1}{2} \|J_N\| |\Delta| \|E(\Delta)\phi_1\|^2$. On the other hand,

$$\langle K_N E(\Delta)\phi_1, E(\Delta)\phi_1 \rangle = \sum_1^N (a_i^2 - a_{i-1}^2) \left| \int_\Delta P_i \, d\mu \right|^2 + \sum_{N+1}^\infty a_N(a_i - a_{i-1}) \left| \int_\Delta P_i \, d\mu \right|^2$$

$$+ \sum_{N+1}^\infty a_N(a_i - a_{i-1}) \int_\Delta P_{i+1} \, d\mu \int_\Delta P_i \, d\mu$$

$$\geqq a_1^2 \left| \int P_1 \, d\mu \right|^2 + \sum_{N+1}^\infty a_N(a_i - a_{i-1}) \left| \int_\Delta P_i \, d\mu \right|^2$$

$$- \frac{1}{2} \sum_{N+1}^\infty a_N d_i \left[\left| \int_\Delta P_{i+1} \, d\mu \right|^2 + \left| \int_\Delta P_{i-1} \, d\mu \right|^2 \right]$$

$$\geqq a_1^2 \left| \int_\Delta P_1 \, d\mu \right|^2 + a_N \left[(a_{N+1} - a_N) - \frac{1}{2} d_{N+1} \right] \left| \int_\Delta P_{N+1} \, d\mu \right|^2$$

$$+ \sum_{N+2}^\infty a_N [(a_i - a_{i-1}) - d_i] \left| \int_\Delta P_i \, d\mu \right|^2$$

$$- \sum_N^\infty a_N |d_i - d_{i-1}| \left| \int_\Delta P_i \, d\mu \right|^2$$

$$\geqq a_1^2 \left| \int_\Delta P_1 \, d\mu \right|^2 - 2 \sum_{N+1}^\infty a_N [a_i - a_{i-1}]^- \left| \int_\Delta P_i \, d\mu \right|^2$$

$$- \sum_N^\infty a_N |d_i - d_{i-1}| \left| \int_\Delta P_i \, d\mu \right|^2$$

$$\geqq a_1^2 \left| \int_\Delta P_1 \, d\mu \right|^2 - 2 \sum_{N+1}^\infty (a_i^2 - a_{i-1}^2)^- \left| \int_\Delta P_i d\mu \right|^2$$

$$- \sum_N^\infty a_N |d_i - d_{i-1}| \left| \int_\Delta P_i \, d\mu \right|^2.$$

Now apply Lemma 1 with $\delta = 4a_N^2 - R^2$. Recall that $\Delta \subset [-R, R]$. It follows that

$$\langle K_N E(\Delta)\phi_1, E(\Delta)\phi_1 \rangle \geqq \frac{a_N^2}{8} |\mu(\Delta)|^2.$$

Combining this result with the inequality obtained from the commutator equation implies that $|\mu(\Delta)| \leqq 4a_N^2 \|J_N\| |\Delta|$. Let β be a Borel subset of $[-R, R]$ of Lebesgue measure zero. Then for any $\varepsilon > 0$ there exists a pairwise disjoint sequence of intervals $\{\Delta_j\}$ such that $\beta \subset \cup \Delta_j$ and $\sum |\Delta_j| < \varepsilon$. Since $\mu(\beta) \leqq \sum \mu(\Delta_j) \leqq 4a_N^2 \|J_N\| \sum |\Delta_j|$ it follows that $\mu(\beta) = 0$. Hence it has been shown that the spectral measure of a trace class perturbation of C is absolutely continuous with respect to Lebesgue measure on $[-R, R]$. The theorem follows from an application of the Kato-Rosenblum Theorem.

3. Examples. In this section examples will be presented to illustrate the above results. The asymptotic expansions needed for these examples are developed in [5]. See also the references cited therein.

The simplest example of practical interest is obtained by letting $a_n = \sqrt{n/2}$. It is easily checked that the conditions of the above theorems are satisfied. The corresponding polynomials $\{P_n\}$ are the Hermite polynomials and it is well known that $\mu(\beta) = \int_\beta e^{-x^2} \, dx$.

For a related class of examples choose α to be an even positive integer and let $\{P_n(\lambda)\}$ be the sequence of polynomials obtained by orthonormalizing the sequence $\{\lambda^n\}_{n=0}^\infty$ with respect to the measure $\mu(\beta) = \int_\beta e^{-x^\alpha/\alpha} \, dx$. These polynomials satisfy a recursion formula of the form (1.1) with $b_n = 0$ and, as shown in [5],

$$a_n = n^{1/\alpha} \left[c_0 + \frac{c_2}{n^2} + O\left(\frac{1}{n^4}\right) \right].$$

Since

$$a_n - a_{n-1} = c_0 [n^{1/\alpha} - (n-1)^{1/\alpha}] + c_2 [n^{-2+(1/\alpha)} - (n-1)^{-2+(1/\alpha)}] + n^{1/\alpha} t_n - (n-1)^{1/\alpha} t_{n-1}$$

with $|t_n| \leqq M/n^4$ for all n, the Mean Value Theorem applied to $f(x) = x^{1/\alpha}$ shows that for large n, $a_n \geqq a_{n-1}$. That is, the term $c_0[n^{1/\alpha} - (n-1)^{1/\alpha}]$ is positive and, for large n, dominates the remaining terms of the sum. For large n, write $d_n = a_n - a_{n-1} = x_n + y_n$ with $x_n = c_0[n^{1/\alpha} - (n-1)^{1/\alpha}]$. Note that $\{x_n\}$ is decreasing and $\sum |y_n| < \infty$. It follows that $\sum |d_n - d_{n-1}| \leqq \sum |x_n - x_{n-1}| + \sum |y_n - y_{n-1}| < \infty$. Therefore the hypotheses of the above theorems are satisfied.

4. Final comments. The main theorem of this paper presents a sufficient condition for the existence of an absolutely continuous part for the measure μ. It does not, in contrast to the results presented in [2], claim that μ is absolutely continuous. This is, however, true for the examples cited above. It would be interesting to know if the hypotheses of the main theorem are sufficient to conclude that μ is indeed absolutely continuous.

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