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CONTROLLABILITY AND STABILIZABILITY OF COUPLED STRINGS WITH CONTROL APPLIED AT THE COUPLED POINTS*

L. F. HO†

Abstract. Controllability and stabilizability of a system of coupled strings with control applied at the coupled points is studied. By investigating the properties of certain exponential series, it is shown that the system is approximate controllable if and only if related systems of uncoupled strings do not share a common eigenvalue. A sufficient condition for exact controllability is also obtained in terms of the Riesz basis properties of those exponential series.

Key words. coupled strings, controllability, stabilizability, nonharmonic Fourier series, Riesz basis

AMS subject classification. 93B05

1. Introduction. Let $L > 0$ and $0 = x_0 < x_1 < \cdots < x_N = L$. We consider the system of coupled strings

$$(1.1) \quad \frac{\partial^2 y}{\partial t^2} = \frac{1}{\rho_i} \frac{\partial}{\partial x} \left(\tau_i \frac{\partial y}{\partial x} \right), \quad x \in (x_{i-1}, x_i),$$

with conditions at the boundary and at the coupled points

$$(1.2a) \quad y(0, t) = y(L, t) = 0,$$

$$(1.2b) \quad y(x_i^+, t) = y(x_i^-, t), \quad i = 1, \dots, N-1,$$

and control applied at the coupled points

$$(1.3) \quad \tau_{i+1}(x_i) \frac{\partial y(x_i^+, t)}{\partial x} - \tau_i(x_i) \frac{\partial y(x_i^-, t)}{\partial x} = u_i(t), \quad i = 1, \dots, N-1,$$

where $u_i \in L^2[0, T]$, $i = 1, \dots, N-1$, and the functions $\rho_i, \tau_i, i = 1, \dots, N$ are continuously differentiable and positive. We will consider the state space $X = H_0^1[0, L] \times L^2[0, L]$ and the space of controls $U = (L^2[0, T])^{N-1}$. We say that the system is *exactly controllable in time* $T > 0$ if given any initial state (f_1, f_2) and terminal state (g_1, g_2) , both in X , there exist controls $(u_1, \dots, u_{N-1})$ in U such that the corresponding solution of (1.1) with initial conditions

$$(1.4) \quad y(x, 0) = f_1(x); \quad \frac{\partial}{\partial t} y(x, 0) = f_2(x)$$

satisfies

$$y(x, T) = g_1(x); \quad \frac{\partial}{\partial t} y(x, T) = g_2(x).$$

We say that the system is *approximately controllable in time* $T, T > 0$, if given any initial state (f_1, f_2) and terminal state (g_1, g_2) both in X and any $\epsilon > 0$, there exist controls $(u_1, \dots, u_{N-1})$ in U such that the solution of (1.1) with initial condition (1.4) satisfies

$$\left\| \left(y(\cdot, T), \frac{\partial y(\cdot, T)}{\partial t} \right) - (g_1, g_2) \right\|_X < \epsilon.$$

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We also consider the related problem of stabilizability with feedback given by

$$u_i(x, t) = k_i \frac{\partial y(x_i, t)}{\partial t}, \quad i = 1, \dots, N-1,$$

where $k_i, i = 1, \dots, N-1$, are positive constants. Controllability of the system will then be established by the “controllability via stabilizability” method of Russell [15]. The stabilizability problem, for the case of constant wave speeds, has been considered by Chen, Coleman, and West [1], Liu [11], and Liu, Huang, and Chen [12]. There is some similarity between their results and the result we obtain in this paper. We will discuss that in the examples given in §5.

In §2, we will establish some results concerning exponential series that we will need for the proof of the main results. In §3, we will state and prove our main results on stabilizability and approximate controllability. Roughly speaking, it says that the whole system is approximately controllable if and only if the uncoupled strings do not share a common eigenvalue. In §4, we consider stabilizability with a uniform exponential decay rate and the related problem of exact controllability. A sufficient condition is given in terms of the Riesz basis property for certain sets of exponential functions. Some examples are discussed in §5.

2. Nonharmonic Fourier series. In this section, we study sequences of exponential functions of the form $\{e^{i\lambda_n t}\}_{n=-\infty}^{\infty}$ where $\lambda_n, -\infty < n < \infty$, are real. Properties of completeness and independence of series of the form $\sum_n a_n e^{i\lambda_n t}$, known as nonharmonic Fourier series, have been investigated by, e.g., Levinson [8] and Riesz and Nagy [13]. Russell has also studied the relationship between the properties of such series and results in control theory [14], [15]. We first recall a definition.

DEFINITION 2.1. Let X be a Hilbert space and let $\{\phi_n\}_{n=-\infty}^{\infty}$ be a sequence in X . We say that $\{\phi_n\}_{n=-\infty}^{\infty}$ is a *Riesz basis* of X if

(RB1) $\overline{\text{span}}\{\phi_n : -\infty < n < \infty\} = X$;

(RB2) there exist constants $d, D > 0$ such that

$$(2.1) \quad d \sum_n |a_n|^2 \leq \left\| \sum_n a_n \phi_n \right\|^2 \leq D \sum_n |a_n|^2$$

for any finite sequence $\{a_n\}_{n=-\infty}^{\infty}$.

If a sequence $\{\phi_n\}_{n=-\infty}^{\infty}$ satisfies the first inequality in (RB2), we say that it is *uniformly independent*. We also have the following weaker sense of independence. (See Levinson and McCalla [9].)

DEFINITION 2.2. Let $\{\phi_n\}_{n=-\infty}^{\infty}$ be a sequence in a Hilbert space X . Suppose for all n there exists a positive constant d_n such that

$$(2.2) \quad \left\| \sum_k a_k \phi_k \right\| \geq d_n |a_n|$$

for all finite sequences $\{a_n\}_{n=-\infty}^{\infty}$; then we say that $\{\phi_n\}_{n=-\infty}^{\infty}$ is *strongly independent* in X .

The following fact is well known.

A sequence $\{\phi_n\}_{n=-\infty}^{\infty}$ is strongly independent if and only if we can find a sequence $\{\psi_n\}_{n=-\infty}^{\infty}$ such that

$$[\psi_m, \phi_n] = \delta_{m,n}.$$

The sequence $\{\psi_n\}_{n=-\infty}^{\infty}$ is said to be *biorthogonal* to $\{\phi_n\}_{n=-\infty}^{\infty}$.

If $f : \mathbb{R} \rightarrow \mathbb{C}$ has compact support, we denote

$$\hat{f}(\lambda) = \int_{-\infty}^{\infty} f(t)e^{i\lambda t} dt.$$

Obviously, $\hat{f}$ is an entire function of λ .

Let $a < b$ and f be any function in $L^2[a, b]$; we may consider f as defined on the whole real line by extending it by zero outside $[a, b]$. In the rest of this paper, we will identify any such function f with its extension in this way.

LEMMA 2.3. *Let $f \in L^2[0, T]$ and let μ be a zero of $\hat{f}$ of order greater than or equal to m , then*

$$\frac{\hat{f}(\lambda)}{(\lambda - \mu)^m} = \hat{g}(\lambda)$$

for some function g in $L^2[0, T]$.

Proof. We prove by induction on m . For $m = 1$, we make the following straightforward calculation:

$$\begin{aligned} \frac{\hat{f}(\lambda)}{\lambda - \mu} &= \frac{\hat{f}(\lambda) - \hat{f}(\mu)}{\lambda - \mu} = \int_0^T f(t) \frac{e^{i\lambda t} - e^{i\mu t}}{\lambda - \mu} dt \\ &= \int_0^T f(t) i \int_0^t e^{i\lambda s + i\mu(t-s)} ds dt \\ &= \int_0^T i \int_s^T f(t) e^{i\mu(t-s)} dt e^{i\lambda s} ds \\ &= \int_0^T g(t) e^{i\lambda t} dt = \hat{g}(\lambda), \end{aligned}$$

where $g(t) = i \int_t^T f(s) e^{i\mu(s-t)} ds$.

Suppose the result holds for $m \leq k$. Let μ be a zero of order greater than or equal to $k + 1$ of $\hat{f}$. By induction hypothesis,

$$\frac{\hat{f}(\lambda)}{(\lambda - \mu)^k} = \hat{h}(\lambda)$$

for some h in $L^2[0, T]$. Then μ is a zero of order greater than or equal to 1 of $\hat{h}$. By induction hypothesis once again, we have

$$\frac{\hat{f}(\lambda)}{(\lambda - \mu)^{k+1}} = \frac{\hat{h}(\lambda)}{\lambda - \mu} = \hat{g}(\lambda)$$

for some g in $L^2[0, T]$. This completes the proof of the lemma.

LEMMA 2.4. *If $\{e^{i\lambda_n t} : -\infty < n < \infty\}$ is strongly independent in $L^2[0, T]$, then for any real number ξ and any $T_1 > T$, the set*

$$\{e^{i\lambda_n t} : -\infty < n < \infty\} \cup \{e^{i\xi t}\}$$

is strongly independent in $L^2[0, T_1]$.

Proof. If $\xi = \lambda_n$ for some n , the result is trivial. So let us assume that $\xi \neq \lambda_n$ for all n . Let $\{p_n\}_{n=-\infty}^{\infty}$ be functions in $L^2[0, T]$ biorthogonal to $\{e^{i\lambda_n t} : -\infty < n < \infty\}$. Let $\delta = T_1 - T$ and let $\{g_n : -\infty < n < \infty\}$ and h be functions in $L^2[0, \delta]$ such that

$$\int_0^\delta g_n(t) e^{i\lambda_n t} dt = 1,$$

$$\int_0^\delta g_n(t) e^{i\xi t} dt = 0, \quad -\infty < n < \infty,$$

and

$$\int_0^\delta h(t) e^{i\lambda_0 t} dt = 0.$$

Let $q_n = p_n^* g_n$. (Here $*$ denotes convolution). Then q_n is supported on $[0, T_1]$ and $\hat{q}_n(\lambda) = \hat{p}_n(\lambda) \hat{g}_n(\lambda)$. So we have

$$\int_0^{T_1} q_n e^{i\lambda_m t} dt = \left(\int_0^T p_n e^{i\lambda_m t} dt \right) \left(\int_0^\delta g_n e^{i\lambda_m t} dt \right) = \delta_{m,n},$$

$$\int_0^{T_1} q_n e^{i\xi t} dt = \left(\int_0^T p_n e^{i\xi t} dt \right) \left(\int_0^\delta g_n e^{i\xi t} dt \right) = 0.$$

Let $r_0 = p_0^* h$. Then again r_0 is supported on $[0, T]$ and $\hat{r}_0(\lambda) = \hat{p}_0(\lambda) \hat{h}(\lambda)$. Let ξ be a zero of order m of $\hat{r}_0$ (m may be zero). By Lemma 2.3, there exists r in $L^2[0, T_1]$ such that $\hat{r}(\lambda) = \hat{r}_0(\lambda)/(\lambda - \xi)^m$. Since $\hat{r}_0(\lambda_n) = 0$ for all n , we have

$$\int_0^{T_1} r(t) e^{i\lambda_n t} dt = \hat{r}(\lambda_n) = 0 \quad \text{for all } n.$$

Because $\hat{r}_0^{(m)}(\xi) \neq 0$,

$$\int_0^{T_1} r(t) e^{i\xi t} dt = \hat{r}(\xi) = \hat{r}_0^{(m)}(\xi)/m! \neq 0.$$

It follows that $\{q_n : -\infty < n < \infty\} \cup \{m!r/\hat{r}_0^{(m)}(\xi)\}$ is biorthogonal to $\{e^{i\lambda_n t} : -\infty < n < \infty\} \cup \{e^{i\xi t}\}$ in $L^2[0, T_1]$. Hence the latter is strongly independent in $L^2[0, T_1]$.

Remark. Clearly, Lemma 2.4 remains valid if the set $\{e^{i\xi t}\}$ is replaced by any finite set $\{e^{i\xi_n t} : 1 \leq n \leq m\}$.

THEOREM 2.5. Suppose that $\{e^{i\lambda_n t}\}_{n=-\infty}^{\infty}$ and $\{e^{i\mu_n t}\}_{n=-\infty}^{\infty}$ are strongly independent in $L^2[0, T_1]$ and $L^2[0, T_2]$, respectively. Then for any $T > T_1 + T_2$, the set

$$S = \{e^{i\lambda_n t} : -\infty < n < \infty\} \cup \{e^{i\mu_n t} : -\infty < n < \infty\}$$

is strongly independent in $L^2[0, T]$.

Proof. It suffices to show the existence of a sequence biorthogonal to S . Let the sequences $\{p_n\}_{n=-\infty}^{\infty}$ and $\{q_n\}_{n=-\infty}^{\infty}$ be biorthogonal to $\{e^{i\lambda_n t}\}_{n=-\infty}^{\infty}$ and $\{e^{i\mu_n t}\}_{n=-\infty}^{\infty}$, respectively. Let $e^{i\eta \cdot} \in S$. We need to find a square integrable function r vanishing outside $[0, T]$ such that

$$(2.3) \quad \int_0^T r(t) e^{i\xi t} dt = \hat{r}(\xi) = 0 \quad \text{if } e^{i\xi \cdot} \in S \quad \text{and} \quad \xi \neq \eta$$

and

$$(2.4) \quad \int_0^T r(t)e^{i\eta t} dt = \hat{r}(\eta) = 1.$$

We consider three cases.

Case 1. $\eta = \lambda_n = \mu_m$ for some m and n .

Let $r = p_n^* q_m$. (Here $*$ again denotes convolution.) Then r vanishes outside $[0, T]$ and $\hat{r}(\lambda) = \hat{p}_n(\lambda)\hat{q}_m(\lambda)$ for any complex number λ . It is easy to see that (2.3) and (2.4) hold.

Case 2. $\eta = \lambda_n$ but does not equal any μ_m . Since $T - T_1 > T_2$, by Lemma 2.4, the set $\{e^{i\mu_m t} : -\infty < m < \infty\} \cup \{e^{i\eta t}\}$ is strongly independent in $L^2[0, T - T_1]$. In particular, we can find a function f in $L^2[0, T - T_1]$ such that $\hat{f}(\mu_m) = 0$ for all m and $\hat{f}(\eta) = 1$. Let $r = p_n^* f$. Then r is in $L^2[0, T]$ and (2.3) and (2.4) hold.

Case 3. $\eta = \mu_m$ for some m but not equal to any λ_n . We can find r in a way similar to that in Case 2.

This completes the proof of the theorem.

It follows immediately from the remark after Lemma 2.4 that Theorem 2.5 can be strengthened slightly as follows.

THEOREM 2.6. *Suppose $\{e^{i\lambda_n t}\}_{n=-\infty}^{\infty}$ and $\{e^{i\mu_n t}\}_{n=-\infty}^{\infty}$ are strongly independent in $L^2[0, T_1]$ and $L^2[0, T_2]$, respectively, and $\{\xi_n : 1 \leq n \leq m\}$ is any finite set of real numbers. Then for any $T > T_1 + T_2$, the set*

$$S = \{e^{i\lambda_n t} : -\infty < n < \infty\} \cup \{e^{i\mu_n t} : -\infty < n < \infty\} \cup \{e^{i\xi_n t} : 1 \leq n \leq m\}$$

is strongly independent in $L^2[0, T]$.

3. Approximate controllability. We first consider the stability of the following closed-loop system:

$$(3.1) \quad \frac{\partial^2 y}{\partial t^2} = \frac{1}{\rho_i} \frac{\partial}{\partial x} \left(\tau_i \frac{\partial y}{\partial x} \right), \quad x \in (x_{i-1}, x_i);$$

$$(3.2a) \quad y(0, t) = y(L, t) = 0,$$

$$(3.2b) \quad y(x_i^+, t) = y(x_i^-, t),$$

$$(3.3) \quad \tau_{i+1}(x_i) \frac{\partial y(x_i^+, t)}{\partial x} - \tau_i(x_i) \frac{\partial y(x_i^-, t)}{\partial x} = k_i \frac{\partial y(x_i, t)}{\partial t}, \quad i = 1, \dots, N-1,$$

$$(3.4) \quad y(x, 0) = f_1(x), \quad \frac{\partial y(x, 0)}{\partial t} = f_2(x),$$

where $k_i, i = 1, \dots, N-1$, are positive constants.

We will show that the stability of the above system is related to the properties of a system of N uncoupled strings.

$$(3.5) \quad \frac{\partial^2 w}{\partial t^2} = \frac{1}{\rho_i} \frac{\partial}{\partial x} \left(\tau_i \frac{\partial w}{\partial x} \right), \quad x \in (x_{i-1}, x_i);$$

$$(3.6) \quad w(x_{i-1}, t) = w(x_i, t) = 0,$$

$i = 1, \dots, N$. We will refer to the i th string in this system as (S_i) , $i = 1, \dots, N$. Recall that the functions ρ_i, τ_i , $i = 1, \dots, N$, are assumed to be continuously differentiable and positive. We first give a result of strong observability for such a system. When the wave speed is constant or when the domain and the wave speeds are both analytic, this result is known, even in higher dimensions. (See Bardos, Lebeau, and Rauch [1], Ho [4], [5], and Kormonik [7]).

THEOREM 3.1. *The systems (S_i) , $i = 1, \dots, N - 1$ are strongly observable in some time $T_i > 0$ with observation*

$$(3.7) \quad \theta_i(t) = \frac{\partial}{\partial x} w(x_{i-1}, t)$$

or

$$(3.8) \quad \theta_i(t) = \frac{\partial}{\partial x} w(x_i, t).$$

In other words, for $i = 1, \dots, N$ the inequality

$$\int_0^T \theta_i(t)^2 dt \geq K_i \int_0^L \tau_i \left(\frac{\partial}{\partial x} w(x, 0) \right)^2 + \rho_i \left(\frac{\partial}{\partial t} w(x, 0) \right)^2$$

holds for some positive constants K_i .

Proof. We use a multiplier method similar to that used in Ho [6]. For simplicity of notation, let us drop the subscript i and assume that the string extends from $x = 0$ to $x = L$. We first consider the observation

$$(3.9) \quad \theta(t) = \frac{\partial}{\partial x} w(L, t).$$

Let h be the function satisfying the linear differential equation

$$(3.10) \quad h'(x) = 1 + \max \left\{ -\frac{\rho'}{\rho}, \frac{\tau'}{\tau} \right\} h(x)$$

with initial condition

$$(3.11) \quad h(0) = 0.$$

It is clear that $h(x) > 0$ for $x > 0$. Multiplying (3.1) by $\rho h(\partial w / \partial x)$ and integrating, we have

$$\int_0^T \int_0^L h \frac{\partial w}{\partial x} \left(\rho \frac{\partial^2 w}{\partial t^2} - \frac{\partial}{\partial x} \left(\tau \frac{\partial w}{\partial x} \right) \right) dx dt = 0.$$

Integrating by parts and making use of the boundary conditions for w and h , we have

$$\begin{aligned} \frac{1}{2} \int_0^T \int_0^L \frac{d}{dx} (h\rho) \left(\frac{\partial w}{\partial t} \right)^2 + \frac{d}{dx} \left(\frac{h}{\tau} \right) \left(\tau \frac{\partial w}{\partial x} \right)^2 dx dt + \int_0^L h\rho \frac{\partial w}{\partial x} \frac{\partial w}{\partial t} dx \Big|_{t=0}^{t=T} \\ - \frac{1}{2} h(L)\tau(L) \int_0^T \theta(t)^2 dt = 0. \end{aligned}$$

Then by the differential equation (3.9) and conservation of energy, we have

$$(3.12) \quad \frac{1}{2}h(L)\tau(L) \int_0^T \theta(t)^2 dt \leq TE_0 - KE_0,$$

where

$$E_0 = \frac{1}{2} \int_0^L \tau \left(\frac{\partial}{\partial x} w(x, 0) \right)^2 + \rho \left(\frac{\partial}{\partial t} w(x, 0) \right)^2 dx$$

and K is a constant that only depends on h and ρ . It follows from (3.12) that the system (3.5), (3.6) with observation (3.9) is strongly observable in any T such that $T > K$. The proof for strong observability for the observation $\theta(t) = (\partial/\partial x)w(0, t)$ is similar.

We let $\{\lambda_{n,i}\}_{n=1}^\infty$ be the sequence of eigenvalues of the system (S_i) and $\{\phi_{n,i}\}_{n=1}^\infty$ be the corresponding set of eigenfunctions satisfying

$$(3.13) \quad \frac{1}{\rho_i} \frac{\partial}{\partial x} \left(\tau_i \frac{\partial}{\partial x} \phi_{n,i} \right) + \lambda_{n,i}^2 \phi_{n,i} = 0, \quad \phi_{n,i}(x_{i-1}) = \phi_{n,i}(x_i) = 0,$$

and we assume that $\phi_{n,i}$ is normalized so that

$$(3.14) \quad \int_{x_{i-1}}^{x_i} \tau_i \left(\frac{\partial}{\partial x} \phi_{n,i} \right)^2 dx = \int_{x_{i-1}}^{x_i} \lambda_{n,i}^2 \rho_i \phi_{n,i}^2 dx = 1.$$

It is well known that strong observability is related to the Riesz basis property (more precisely, the property (RB2) in §1) of the exponential functions formed from the eigenvalues of the system (see Russell [14], [15]). The following proposition follows easily from known results but we supply its proof for completeness.

PROPOSITION 3.2. *If the system (S_i) is strongly observable in time T_i with observation $\theta_i(t)$ given by (3.8) or (3.9), then the sequence of exponential functions $\{e^{\pm i\lambda_{n,i}t}\}_{n=1}^\infty$ satisfies condition (RB2) in the space $L^2[0, T_i]$.*

Proof. Suppose the system is strongly observable in time T_i with observation $\theta_i(t)$ given by (3.9), we then have the inequality

$$(3.15) \quad \int_0^{T_i} |\theta_i(t)|^2 dt \geq K_1 E_0,$$

where

$$\begin{aligned} E_0 &= \text{total energy of } w \text{ at } t = 0 \\ &= \frac{1}{2} \int_{x_{i-1}}^{x_i} \rho_i \left(\frac{\partial w(x, 0)}{\partial t} \right)^2 + \tau_i \left(\frac{\partial w(x, 0)}{\partial x} \right)^2 dx, \end{aligned}$$

w is any solution of (3.5), (3.6), and K_1 is a constant independent of w . Also by regularity results (Lasiecka, Lions, and Triggiani [10]), there exists a constant $K_2 > 0$ such that

$$(3.16) \quad \int_0^{T_i} |\theta_i(t)|^2 dt \leq K_2 E_0.$$

Because every function $e^{i\lambda_{n,i}t} \phi_{n,i}(x)$, $1 \leq n < \infty$, is a solution of (3.5), (3.6) and by the normalization condition (3.13), the total energy $E_0 = 1$, it follows that

$$(3.17) \quad K_1 \leq \int_0^{T_i} |e^{i\lambda_{n,i}t} \phi'_{n,i}(x_{i-1})|^2 dt \leq K_2.$$

Hence

$$(3.18) \quad K_1/T_i \leq |\phi'_{n,i}(x_{i-1})|^2 \leq K_2/T_i.$$

Let $\{a_n\}_{n=-\infty, n \neq 0}^\infty$ by a finite sequence and let $b_{\pm n} = a_{\pm n}/\phi'_{n,i}(x_{i-1})$, $n = 1, 2, \dots$. The function

$$(3.19) \quad w(x, t) = \sum_{n=1}^{\infty} (b_n e^{i\lambda_n, i t} + b_{-n} e^{-i\lambda_n, i t}) \phi_{n,i}(x)$$

is a solution of (3.5), (3.6) with energy

$$(3.20) \quad E_0 = \sum_{n=1}^{\infty} (|b_n|^2 + |b_{-n}|^2)$$

and observation

$$(3.21) \quad \theta_i(t) = \sum_{n=1}^{\infty} a_n e^{i\lambda_n, i t} + a_{-n} e^{-i\lambda_n, i t}.$$

By (3.18) and (3.20), we have

$$(3.22) \quad (T_i/K_2) \sum_{n=1}^{\infty} |a_n|^2 + |a_{-n}|^2 \leq E_0 \leq (T_i/K_1) \sum_{n=1}^{\infty} |a_n|^2 + |a_{-n}|^2.$$

Hence by (3.15) and (3.16), we have

$$\begin{aligned} \frac{T_i K_1}{K_2} \sum_{n=1}^{\infty} |a_n|^2 + |a_{-n}|^2 &\leq \int_0^{T_i} \left| \sum_{n=1}^{\infty} a_n e^{i\lambda_n, i t} + a_{-n} e^{-i\lambda_n, i t} \right|^2 dt \\ &\leq \frac{T_i K_2}{K_1} \sum_{n=1}^{\infty} |a_n|^2 + |a_{-n}|^2. \end{aligned}$$

This completes the proof of the proposition when θ_i is given by (3.9). The proof for θ_i given by (3.8) is similar.

Let us now consider the system (3.1)–(3.3). Let $T > 0$. Given any initial state $(f_1, f_2) \in H_0^1[0, L] \times L^2[0, L]$, we can find the solution of (3.1)–(3.3) with initial conditions (3.4). Denote

$$\begin{aligned} \mathcal{E}y(t) &= \text{the total energy of } y \text{ at time } t \\ (3.23) \quad &= \frac{1}{2} \sum_{k=1}^{N-1} \int_{x_i}^{x_{i+1}} \rho_i(x) \left(\frac{\partial y(x, t)}{\partial t} \right)^2 = \tau_i(x) \left(\frac{\partial y(x, t)}{\partial x} \right)^2 dx. \end{aligned}$$

We have

$$\begin{aligned} \frac{d\mathcal{E}y(t)}{dt} &= \sum_{k=1}^{N-1} \frac{\partial y(x_i, t)}{\partial t} \left(\tau_i(x_i) \frac{\partial y(x_i^-, t)}{\partial x} - \tau_{i+1}(x_i) \frac{\partial y(x_i^+, t)}{\partial x} \right) \\ (3.24) \quad &= - \sum_{k=1}^{N-1} k_i \left(\frac{\partial y(x_i, t)}{\partial t} \right)^2 \end{aligned}$$

It follows that for any $t_1, t_2 > 0$, we have

$$(3.25) \quad \mathcal{E}y(t_1) - \mathcal{E}y(t_2) = - \sum_{k=1}^{N-1} \int_{t_1}^{t_2} k_i \left(\frac{\partial y(x_i, t)}{\partial t} \right)^2 dt.$$

Hence in particular,

$$\begin{aligned} \sum_{k=1}^{N-1} \int_0^T k_i \left(\frac{\partial y(x_i, t)}{\partial t} \right)^2 dt &= \mathcal{E}y(0) - \mathcal{E}y(T) \leq \mathcal{E}y(0) \\ &= \frac{1}{2} \sum_{k=1}^{N-1} \int_{x_i}^{x_{i+1}} \rho_i(x) f_2(x)^2 + \tau_i(x) f_1'(x)^2 dx \\ &\leq K \|f_1\|_{H_0^1[0, L]}^2 + \|f_2\|_{L^2[0, L]}^2, \end{aligned}$$

where K is a positive constant. So if we define

$$(3.26) \quad \mathcal{N}_T(f_1, f_2) = \left(\sum_{k=1}^{N-1} \int_0^T k_i \left(\frac{\partial y(x_i, t)}{\partial t} \right)^2 dt \right)^{1/2},$$

then $\mathcal{N}_T(\cdot, \cdot)$ is a seminorm on $H^1[0, L] \times L^2[0, L]$ and by (3.22), we have

$$(3.27) \quad \mathcal{E}y(t) - \mathcal{E}y(t+T) = \mathcal{N}_T \left(y(\cdot, t), \frac{\partial y(\cdot, t)}{\partial t} \right)^2$$

for all $t > 0$.

The following theorem says that a necessary and sufficient condition for $\mathcal{N}_T$ to be a norm for some $T > 0$ is that the following holds:

(C1) The systems $(S_i), i = 1, \dots, N$, do not have a common eigenvalue.

In what follows, $T_i, i = 1, \dots, N$, will denote times for which the conclusion of Theorem 3.1 holds.

THEOREM 3.3. *If condition (C1) does not hold, then we can find nonzero $(f_1, f_2) \in H_0^1[0, L] \times L^2[0, L]$ such that $\mathcal{N}_T(f_1, f_2) = 0$ for all $T > 0$. Conversely, if condition (C1) holds, then $\mathcal{N}_T(\cdot, \cdot)$ defines a norm on $H_0^1[0, L] \times L^2[0, L]$ for any $T > T_0 = \max\{T_i + T_{i+1} : 1 \leq i \leq N-1\}$.*

Proof. Suppose that all the $(S_i), i = 1, \dots, N$ have a common eigenvalue λ and with eigenfunctions $\phi_i, i = 1, \dots, N$. Let $f_1(x) = c_i \phi_i(x)$ if $x \in (x_{i-1}, x_i)$ and let $f_2 \equiv 0$. With appropriate choice of the constants $c_i, i = 1, \dots, N$, we can make

$$c_i \tau_i(x_i) \phi_i'(x_i) = c_{i+1} \tau_{i+1}(x_i) \phi_{i+1}'(x_i), \quad i = 1, \dots, N-1.$$

Then

$$y(x, t) = c_i \cos(\lambda t) \phi_i(x) \quad \text{if } x \in (x_{i-1}, x_i)$$

would be the solution of (3.1)–(3.3) with initial condition $y(x, 0) = f_1(x), \partial y(x, 0)/\partial t = f_2(x)$. Since $y(x_i, t) = 0$, for $i = 1, \dots, N-1$ and $t > 0$, it follows that $\mathcal{N}_T(f_1, f_2) = 0$ for all $T > 0$.

Next, suppose that the system (S_i) has eigenvalues

$$\wedge_i = \{\lambda_{n,i} : 1 \leq n < \infty\}$$

and normalized eigenfunctions (see (3.14))

$$\{\phi_{n,i}(x) : 1 \leq n < \infty\}$$

$i = 1, \dots, N$. Suppose $\mathcal{N}_T(f_1, f_2) = 0$. Let $y(x, t)$ be the solution of (3.1), (3.4). Then $\partial y(x_i, t)/\partial t = 0$ for $i = 1, \dots, N-1, 0 \leq t \leq T$. So we have $y(x_i, t) = \text{constant} = \alpha_i, i = 1, \dots, N-1$. Let

$$w(x, t) = y(x, t) - \alpha_{i-1} - \frac{(\alpha_i - \alpha_{i-1})(x - x_{i-1})}{x_i - x_{i-1}}.$$

(We take $\alpha_0 = \alpha_N = 0$.) Then w is a solution of $(S_i), i = 1, \dots, N$. So we can expand

$$(3.28) \quad w(x, t) = \sum_{n=1}^{\infty} (a_{n,i} e^{i\lambda_{n,i}t} + a_{-n,i} e^{i\lambda_{-n,i}t}) \phi_{n,i}(x),$$

where $\lambda_{-n,i} = -\lambda_{n,i}$. It follows that

$$\begin{aligned} & \sum_{n=1}^{\infty} (a_{n,i+1} e^{i\lambda_{n,i+1}t} + a_{-n,i+1} e^{i\lambda_{-n,i+1}t}) \phi'_{n,i+1}(x_i) \tau_{i+1}(x_1) \\ & - \sum_{n=1}^{\infty} (a_{n,i} e^{i\lambda_{n,i}t} + a_{-n,i} e^{i\lambda_{-n,i}t}) \phi'_{n,i}(x_i) \tau_i(x_i) \\ (3.29) \quad & = \tau_{i+1}(x_i) \frac{\partial}{\partial x} w(x_i^+, t) - \tau_i(x_i) \frac{\partial}{\partial x} w(x_i^-, t) \\ & = \tau_{i+1}(x_i) \frac{\partial}{\partial x} y(x_i^+, t) - \tau_i(x_i) \frac{\partial}{\partial x} y(x_i^-, t) - \tau_{i+1}(x_i) \frac{\alpha_{i+1} - \alpha_i}{x_{i+1} - x_i} \\ & + \tau_i(x_i) \frac{\alpha_i - \alpha_{i-1}}{x_i - x_{i-1}} \\ & = -\tau_{i+1}(x_i) \frac{\alpha_{i+1} - \alpha_i}{x_{i+1} - x_i} + \tau_i(x_i) \frac{\alpha_i - \alpha_{i-1}}{x_i - x_{i-1}}. \end{aligned}$$

By Theorem 2.6, the set $\{e^{\pm i\lambda_{n,i}t} : 0 < n < \infty\} \cup \{e^{\pm i\lambda_{n,i+1}t} : 0 < n < \infty\} \cup \{1\}$ is strongly independent in $L^2[0, T]$. So by (3.29), we have

$$\begin{aligned} a_{n,i} &= 0 & \text{if } \lambda_{n,i} \notin \Lambda_i \cap \Lambda_{i+1}, \\ a_{n,i+1} &= 0 & \text{if } \lambda_{n,i+1} \notin \Lambda_i \cap \Lambda_{i+1}, \end{aligned}$$

and

$$(3.30) \quad \tau_{i+1}(x_i) \frac{\alpha_{i+1} - \alpha_i}{x_{i+1} - x_i} = \tau_i(x_i) \frac{\alpha_i - \alpha_{i-1}}{x_i - x_{i-1}}.$$

Suppose w is not identically zero. Then there exist some n and i such that $a_{n,i} \neq 0$. It follows that $\lambda_{n,i} \in \Lambda_{i+1}$. Let $\lambda_{n,i} = \lambda_{m,i+1}$. Then we must have $a_{m,i+1} \neq 0$. Hence $\lambda_{n,i} = \lambda_{m,i+1} \in \Lambda_{i+2}$. Proceeding in this way, we can then show that $\lambda_{n,i} \in \Lambda_j$ for all $j > i$. In a similar way, we can prove that $\lambda_{n,i} \in \Lambda_j$ for all $j < i$. Hence $\lambda_{n,i}$ is a common eigenvalue of the systems $(S_i), i = 1, \dots, N$, a contradiction. So we must have $w \equiv 0$. Also, since (3.30) holds for $i = 1, \dots, N-1$, it follows that $\alpha_i - \alpha_{i-1}, i = 1, \dots, N$, all have the same sign. Because $\alpha_0 = \alpha_N = 0$. This forces $\alpha_i = 0$ for $i = 1, \dots, N-1$. Hence we have proved that $y \equiv 0$ and therefore that $(f_1, f_2) = 0$. Thus $\mathcal{N}_T$ is indeed a norm.

Remark. We see from equality (3.27) that if (C1) holds, then $\mathcal{N}_T$ is a norm weaker than the $H_0^1[0, L] \times L^2[0, L]$ norm.

Theorem 3.4 is a consequence of Theorem 3.3.

THEOREM 3.4. *If condition (C1) does not hold, then the system (3.1)–(3.3) is not asymptotically stable. Conversely, if condition (C1) holds, then the solution of the system (3.1)–(3.3) satisfies*

$$\lim_{t \rightarrow +\infty} \mathcal{N} \left(y(\cdot, t), \frac{\partial y(\cdot, t)}{\partial t} \right) = 0,$$

where $\mathcal{N}(\cdot, \cdot)$ is a norm on $H_0^1[0, L] \times L^2[0, L]$.

Proof. The proof of the first part follows easily from the corresponding part in Theorem 3.3. Conversely, if the systems (S_i) do not have a common eigenvalue, then by Theorem 3.3, we can find some $T > 0$ such that $\mathcal{N}_T$ is a norm on $H_0^1[0, L] \times L^2[0, L]$. From equality (3.24), we know that $\mathcal{E}y(t)$ is a nonnegative decreasing function of t . Hence $\alpha = \lim_{t \rightarrow +\infty} \mathcal{E}y(t)$ exists. Also, by (3.26), we have

$$\mathcal{N}_T \left(y(\cdot, t), \frac{\partial y(\cdot, t)}{\partial t} \right)^2 = \mathcal{E}y(t) - \mathcal{E}y(t + T).$$

Hence

$$\lim_{t \rightarrow +\infty} \mathcal{N}_T \left(y(\cdot, t), \frac{\partial y(\cdot, t)}{\partial t} \right)^2 = \alpha - \alpha = 0.$$

This completes the proof of the theorem.

We will use the following elementary result of functional analysis.

PROPOSITION 3.5. *If $\mathcal{T}$ is a bounded linear operator on a Hilbert space X satisfying*

$$\|\mathcal{F}x\| < \|x\| \quad \text{for all } x \in X, x \neq 0,$$

then $I - \mathcal{T}$ has dense range.

THEOREM 3.6. *If condition (C1) does not hold, then the system (1.1)–(1.3) is not approximately controllable in any time $T > 0$. Conversely, if condition (C1) holds, then the system (1.1)–(1.3) is approximately controllable in any time $T > T_0 = \max\{t_i + T_{i+1} : 1 \leq i \leq N - 1\}$.*

Proof. Suppose that the systems (S_i) , $i = 1, \dots, N$, have a common eigenvalue λ with eigenfunctions ϕ_i , $i = 1, \dots, N$. Let

$$z(x, t) = c_i \phi_i(x) \cos \lambda t \quad \text{if } x \in [x_{i-1}, x_i], i = 1, \dots, N.$$

Then z satisfies

$$\begin{aligned} \frac{\partial^2 z}{\partial t^2} &= \frac{1}{\rho_i} \frac{\partial}{\partial x} \left(\tau_i \frac{\partial z}{\partial x} \right), & x \in (x_{i-1}, x_i); \\ z(0, t) &= z(L, t) = 0, \\ z(x_i^+, t) &= z(x_i^-, t) = 0, & i = 1, \dots, N - 1. \end{aligned}$$

Also, we can choose the constants c_i , $i = 1, \dots, N - 1$, so that

$$\tau_{i+1}(x_i) \frac{\partial z(x_i^+, t)}{\partial x} = \tau_i(x_i) \frac{\partial z(x_i^-, t)}{\partial x}, \quad i = 1, \dots, N - 1.$$

(See the proof of Theorem 3.3.) Let y be a solution of (1.1)–(1.3). It is easy to see that

$$\frac{d}{dt} \sum_{i=1}^{x_i} \rho_i \left(z \frac{\partial y}{\partial t} - y \frac{\partial z}{\partial t} \right) dx = 0.$$

So if y satisfies the initial condition

$$y(x, 0) = \frac{\partial y(x, 0)}{\partial t} = 0,$$

then

$$\sum_{i=1}^N \int_{x_{i-1}}^{x_i} \rho_i(x) \left(z(x, T) \frac{\partial y(x, T)}{\partial t} - y(x, T) \frac{\partial z(x, T)}{\partial t} \right) dx = 0.$$

Hence we have found a nonzero pair of functions orthogonal, in $L^2[0, L] \times L^2[0, L]$, to the set of terminal states $(y(x, T), \partial y(x, T)/\partial x)$. So this set is not dense in $L^2[0, L] \times L^2[0, L]$. Therefore, it is not dense in $H^1[0, L] \times L^2[0, L]$ either and the system is not approximately controllable in time T .

Conversely, suppose that the systems $(S_i), i = 1, \dots, N-1$ do not have a common eigenvalue. Then by Theorem 3.3, the seminorm $\mathcal{N}_T$ is a norm if $T > T_0 = \max\{T_i + T_{i+1} : 1 \leq i \leq N-1\}$. Consider the norm $\|\cdot\|_E$ on $H_0^1[0, L] \times L^2[0, L]$ defined by

$$(3.31) \quad \|(f_1, f_2)\|_E = \left(\frac{1}{2} \sum_{k=1}^{N-1} \int_{x_k}^{x_{k+1}} \rho_k(x) f_1(x)^2 + \tau_k(x) f_2'(x)^2 dx \right)^{1/2}.$$

This norm is equivalent with the usual $H_0^1[0, L] \times L^2[0, L]$ norm. Now consider given initial state (f_1, f_2) in $H_0^1[0, L] \times L^2[0, L]$. For (h_1, h_2) in $H_0^1[0, L] \times L^2[0, L]$, let y_1 be the solution of (3.1)–(3.3) with the initial conditions

$$y_1(x, 0) = h_1(x) \quad \text{and} \quad \frac{\partial y_1(x, 0)}{\partial t} = -h_2(x)$$

and let y_2 be the solution of (3.1)–(3.3) with initial conditions

$$y_2(x, 0) = y_1(x, T) - f_1(x) \quad \text{and} \quad \frac{\partial y_2(x, 0)}{\partial t} = -\frac{\partial y_1(x, T)}{\partial t} - f_2(x).$$

Let $\mathcal{T}$ be the mapping that carries (h_1, h_2) to $(y_2(\cdot, T), \partial y_2(\cdot, T)/\partial t)$. It follows from equality (3.27) that because $\mathcal{N}_T$ is a norm, we have $\|\mathcal{T}(h_1, h_2)\|_E < \|(h_1, h_2)\|_E$ for $(h_1, h_2) \neq 0$. Then by Proposition 3.5, $I - \mathcal{T}$ has dense range in $X = H_0^1[0, L] \times L^2[0, L]$. Now let $y(x, t) = y_1(x, T - t) - y_2(x, t)$. Then y is a solution of (1.1)–(1.3) with initial conditions

$$y(x, 0) = f_1(x) \quad \text{and} \quad \frac{\partial y(x, 0)}{\partial t} = f_2(x)$$

and control functions

$$u_i(t) = k_i \left(\frac{\partial y_1(x, T - t)}{\partial t} - \frac{\partial y_2(x, t)}{\partial t} \right), \quad i = 1, \dots, N-1.$$

Because both y_1 and y_2 have finite energy for all time t , by (3.25) we have $u_i \in L^2[0, T], i = 1, \dots, N-1$. Also, the terminal state of y is

$$y(x, T) = h_1(x) - y_2(x, T), \quad \frac{\partial y(x, T)}{\partial t} = h_2(x) - \frac{\partial y_2(x, T)}{\partial t}.$$

Because $I - \mathcal{T}$ has dense range, the set of such terminal states is dense in $H_0^1[0, L] \times L^2[0, L]$. Hence the system (1.1)–(1.3) is approximately controllable.

4. Stabilizability with a uniform exponential decay rate and exact controllability.

In this section, we still consider the closed-loop system (3.1)–(3.3) and the related systems of uncoupled strings $(S_i), i = 1, \dots, N$, with sets of eigenvalues $\{\lambda_{n,i} : 1 \leq n < \infty\}$ and eigenfunctions $\{\phi_{n,i} : 1 \leq n < \infty\}, i = 1, \dots, N$. We will give a sufficient condition under which the system (3.1)–(1.3) will decay with a uniform exponential rate. Then by the “controllability via stabilizability method” of Russell [15], this will, in turn, give us a sufficient condition for the open-loop system (1.1)–(1.3) to be exactly controllable. We consider the following condition:

(C2) There exists $i, 1 \leq i \leq N - 1$, and $T^* > 0$ such that

$$\{\lambda_{n,i} : 1 \leq n < \infty\} \cap \{\lambda_{n,i+1} : 1 \leq n < \infty\} = \emptyset$$

and the set of exponential functions

$$\{e^{\pm i\lambda_{n,i}t} : 1 \leq n < \infty\} \cup \{e^{\pm i\lambda_{n,i+1}t} : 1 \leq n < \infty\}$$

satisfies condition (RB2) in $L^2[0, T^*]$.

Clearly (C2) is a stronger condition than (C1). Let $\mathcal{N}_T$ and $T_1, \dots, T_N$ be as defined in the previous section.

THEOREM 4.1. *If condition (C2) holds, then the seminorm $\mathcal{N}_T$ is a norm equivalent with the $H_0^1[0, L] \times L^2[0, L]$ norm if $T > T_0 = \max\{T^*, T_1, T_2, \dots, T_N\}$.*

To prove Theorem 4.1, we need the following lemma.

LEMMA 4.2. *Let $T > 0$. There exists a positive constant K such that for any v satisfying*

$$(4.1) \quad \rho \frac{\partial^2 v}{\partial t^2} = \frac{\partial}{\partial x} \left(\tau \frac{\partial v}{\partial x} \right), \quad t > 0, a < x < b;$$

$$(4.2) \quad v(x, 0) = \frac{\partial v(x, 0)}{\partial t} = 0, \quad a \leq x \leq b;$$

we have

$$(4.3) \quad \int_0^T \left(\frac{\partial v(a, t)}{\partial x} \right)^2 + \left(\frac{\partial v(b, t)}{\partial x} \right)^2 dt \leq K \int_0^T \left(\frac{\partial v(a, t)}{\partial t} \right)^2 + \left(\frac{\partial v(b, t)}{\partial t} \right)^2 dt.$$

Proof. By a multiplier method, (using, for example, the multiplier $(x - (a+b)/2)\partial w/\partial x$) we can easily prove the following inequality:

$$(4.4) \quad \int_0^s \left(\frac{\partial w(a, t)}{\partial x} \right)^2 + \left(\frac{\partial w(a, t)}{\partial t} \right)^2 + \left(\frac{\partial w(b, t)}{\partial x} \right)^2 + \left(\frac{\partial w(b, t)}{\partial t} \right)^2 dt \leq K_1 \left(\mathcal{E}w(0) + \mathcal{E}w(s) + \int_0^s \mathcal{E}w(t) dt \right)$$

for any function w satisfying

$$(4.5) \quad \rho \frac{\partial^2 w}{\partial t^2} = \frac{\partial}{\partial x} \left(\tau \frac{\partial w}{\partial x} \right), \quad 0 \leq t \leq s, \quad a \leq x \leq b.$$

Here K_1 is a constant independent of w and s . Suppose that in addition,

$$(4.6) \quad w(a, t) = k_1, \quad w(b, t) = k_2,$$

for some constants k_1 and k_2 . Then by conservation of energy and (4.4) we have

$$(4.7) \quad \int_0^s \left(\frac{\partial w(a, t)}{\partial x} \right)^2 + \left(\frac{\partial w(b, t)}{\partial x} \right)^2 dt \leq K_1(2+s)\mathcal{E}w(0) = K_1(2+s)\mathcal{E}w(s).$$

If v satisfies (4.1), (4.2), then

$$\begin{aligned} \frac{d}{dt} \int_a^b \rho \frac{\partial w}{w} \frac{\partial v}{\partial t} + \tau \frac{\partial w}{\partial x} \frac{\partial v}{\partial x} dx \\ = \tau(b) \frac{\partial w(b, t)}{\partial x} \frac{\partial v(b, t)}{\partial t} - \tau(a) \frac{\partial w(a, t)}{\partial x} \frac{\partial v(a, t)}{\partial t}. \end{aligned}$$

Hence

$$\begin{aligned} \int_a^b \rho(x) \frac{\partial w(x, s)}{\partial t} \frac{\partial v(x, s)}{\partial t} + \sigma(x) \frac{\partial w(x, s)}{\partial x} \frac{\partial v(x, s)}{\partial x} dx \\ = \tau(b) \int_0^s \frac{\partial w(b, t)}{\partial x} \frac{\partial v(b, t)}{\partial t} dt \\ - \tau(a) \int_0^s \frac{\partial w(a, t)}{\partial x} \frac{\partial v(a, t)}{\partial t} dt \\ \leq \tau(a) \left(\int_0^s \left(\frac{\partial w(a, t)}{\partial x} \right)^2 dt \right)^{1/2} \left(\int_0^s \left(\frac{\partial v(a, t)}{\partial t} \right)^2 dt \right)^{1/2} \\ + \tau(b) \left(\int_0^s \left(\frac{\partial w(b, t)}{\partial x} \right)^2 dt \right)^{1/2} \left(\int_0^s \left(\frac{\partial v(b, t)}{\partial t} \right)^2 dt \right)^{1/2} \\ \leq m(K_1(2+s)\mathcal{E}w(s))^{1/2} \left(\int_0^s \left(\frac{\partial v(a, t)}{\partial t} \right)^2 \right. \\ \left. + \left(\frac{\partial v(b, t)}{\partial t} \right)^2 dt \right)^{1/2}, \end{aligned}$$

where $m = \max\{\tau(a), \tau(b)\}$. Since this holds for any w satisfying (4.5) and (4.6), we conclude that

$$(4.8) \quad \mathcal{E}v(s) \leq K_2(2+s) \int_0^s \left(\frac{\partial v(a, t)}{\partial t} \right)^2 + \left(\frac{\partial v(b, t)}{\partial t} \right)^2 dt,$$

where K_2 is a constant independent of v and s . Noting that this holds for any arbitrary s , we have, by (4.4) again, that

$$\begin{aligned} \int_0^T \left(\frac{\partial v(a, t)}{\partial x} \right)^2 + \left(\frac{\partial v(b, t)}{\partial x} \right)^2 dt \\ \leq K_1 \left(\int_0^T \mathcal{E}v(t) dt + \mathcal{E}v(T) \right) \\ \leq K_1 K_2 \left(\int_0^T (2+t) \int_0^t \left(\frac{\partial v(a, s)}{\partial t} \right)^2 + \left(\frac{\partial v(b, s)}{\partial t} \right)^2 ds dt \right. \\ \left. + (2+T) \int_0^T \left(\frac{\partial v(a, t)}{\partial t} \right)^2 + \left(\frac{\partial v(b, t)}{\partial t} \right)^2 dt \right) \end{aligned}$$

$$\leq K_1 K_2 (1+T)(2+T) \int_0^T \left(\frac{\partial v(a,t)}{\partial t} \right)^2 + \left(\frac{\partial v(b,t)}{\partial t} \right)^2 dt.$$

This completes the proof of the lemma.

Proof of Theorem 4.1. Let y satisfy (3.1)–(3.3) with initial condition (3.4). Let v satisfy

$$\begin{aligned} \rho_i \frac{\partial^2 v}{\partial t^2} &= \frac{\partial}{\partial x} \left(\tau_i \frac{\partial v}{\partial x} \right), \quad t > 0, \quad x_{i-1} < x < x_i, \quad i = 1, \dots, N; \\ v(x, 0) &= \frac{\partial v(x, 0)}{\partial t} = 0, \quad \text{for } 0 < x < L; \\ v(x_i^-, t) &= v(x_i^+, t) = y(x, t), \quad i = 1, \dots, N-1; \\ v(0, t) &= v(L, t) = 0. \end{aligned}$$

Then $w = y - v$ satisfies

$$\begin{aligned} \rho_i \frac{\partial^2 w}{\partial t^2} &= \frac{\partial}{\partial x} \left(\tau_i \frac{\partial w}{\partial x} \right), \quad t > 0, \quad x_{i-1} < x < x_i, \quad i = 1, \dots, N; \\ w(x, 0) &= f_1(x), \quad \frac{\partial w(x, 0)}{\partial t} = f_2(x) \quad \text{for } 0 < x < L; \\ w(x_i^-, t) &= w(x_i^+, t) = 0, \quad i = 1, \dots, N-1; \\ w(0, t) &= w(L, t) = 0. \end{aligned}$$

Thus w is a solution of the uncoupled strings $(S_i), i = 1, \dots, N$. Let $1 \leq j \leq N-1$ be such that for $i = j$, condition (C2) is satisfied. We expand

$$(4.9) \quad w(x, t) = \sum_{n=1}^{\infty} (a_{n,i} e^{i\lambda_{n,i}t} + a_{-n,i} e^{i\lambda_{-n,i}t}) \phi_{n,i}(x).$$

Denote $\omega_i(t) = \tau_{i+1}(x_i)(\partial/\partial x)w(x_i^+, t) - \tau_i(x_i)(\partial/\partial x)w(x_i^-, t)$. Then

$$\begin{aligned} \omega_i(t) &= \sum_{n=1}^{\infty} (a_{n,i+1} e^{i\lambda_{n,i+1}t} + a_{-n,i+1} e^{i\lambda_{-n,i+1}t}) \phi'_{n,i+1}(x_i) \tau_{i+1}(x_i) \\ &\quad - \sum_{n=1}^{\infty} (a_{n,i} e^{i\lambda_{n,i}t} + a_{-n,i} e^{i\lambda_{-n,i}t}) \phi'_{n,i}(x_i) \tau_i(x_i). \end{aligned}$$

By the observability result Theorem 3.1 and the fact that the eigenfunctions are normalized, it follows that for all $i, 1 \leq i \leq N$, the sequences $\{\phi'_{n,i}(x_{i-1})\}_{n=1}^{\infty}$ and $\{\phi'_{n,i}(x_i)\}_{n=1}^{\infty}$ are bounded and bounded away from zero. So since the set of exponential functions

$$\{e^{\pm i\lambda_{n,j}t} : 1 \leq n < \infty\} \cup \{e^{\pm i\lambda_{n,j+1}t} : 1 \leq n < \infty\}$$

satisfies condition (RB2) in $L^2[0, T]$, there exist constant $d > 0$ such that

$$(4.10) \quad \int_0^{T^*} |\omega_j(t)|^2 dt \geq d \sum_{n=1}^{\infty} |a_{n,j+1}|^2 + |a_{-n,j+1}|^2 + |a_{n,j}|^2 + |a_{-n,j}|^2.$$

Hence, there exists constant $d_1 > 0$ such that

$$(4.11) \quad \int_0^{T^*} |\omega_j(t)|^2 dt \geq d_1 (E_{j+1} + E_j),$$

where E_i denotes the total (constant) energy of the system (S_i) , $i = 1, \dots, N$. By the observability result Theorem 3.1, it follows that for each i , $1 \leq i \leq N$, there exists constant $K_i > 0$ such that

$$E_i \leq K_i \int_0^{T_i} \left| \frac{\partial}{\partial x} w(x_{i-1}^+, t) \right|^2 dt$$

and

$$E_i \leq K_i \int_0^{T_i} \left| \frac{\partial}{\partial x} w(x_i^-, t) \right|^2 dt.$$

and, by regularity results [10], there exists constant $M_i > 0$ such that

$$\int_0^{T_i} \left| \frac{\partial}{\partial x} w(x_{i-1}^+, t) \right|^2 dt \leq M_i E_i$$

and

$$\int_0^{T_i} \left| \frac{\partial}{\partial x} w(x_i^-, t) \right|^2 dt \leq M_i E_i$$

So for $i, j+1 < i \leq N$,

$$\begin{aligned} E_i &\leq K_i \int_0^{T_i} \left| \frac{\partial}{\partial x} w(x_{i-1}^+, t) \right|^2 dt \\ (4.12) \quad &\leq 2K_i/\tau_i(x_{i-1})^2 \left(\int_0^{T_i} \left| \tau_{i-1}(x_{i-1}) \frac{\partial}{\partial x} w(x_{i-1}^-, t) \right|^2 + |\omega_{i-1}(t)|^2 dt \right) \\ &\leq 2K_i/\tau_i(x_{i-1})^2 \left(M_i \tau_{i-1}(x_{i-1})^2 E_{i-1} + \int_0^{T_i} |\omega_{i-1}(t)|^2 dt \right). \end{aligned}$$

Hence, we have

$$(4.13) \quad E_{j+2} + \dots + E_N \leq C_1 \left(\sum_{i=j+1}^{N-1} \int_0^{T_i} |\omega_i(t)|^2 dt + E_{j+1} \right)$$

for some positive constant C_1 . In a similar way, we can prove that

$$(4.14) \quad E_1 + \dots + E_{j-1} \leq C_2 \left(\sum_{i=1}^{j-1} \int_0^{T_i} |\omega_i(t)|^2 dt + E_j \right)$$

for some positive constant C_2 . Combining (4.11), (4.13), and (4.14), we have

$$(4.15) \quad \|(f_1, f_2)\|_E^2 = \mathcal{E}w(0) = E_1 + E_2 + \dots + E_N \leq C_3 \sum_{i=1}^N \int_0^{T_i} |\omega_i(t)|^2 dt$$

for some positive constant C_3 . But by the definition of ω_i and the conditions on y at the coupled points, we have

$$\omega_i(t) = k_i \frac{\partial}{\partial t} y(x_i, t) - \left(\tau_{i+1}(x_i) \frac{\partial}{\partial x} v(x_i^+, t) - \tau_i(x_i) \frac{\partial}{\partial x} v(x_i^-, t) \right).$$

Hence

$$(4.16) \quad \sum_{i=1}^N \int_0^{T_i} |\omega_i(t)|^2 dt \leq C_4 \sum_{i=1}^N \int_0^{T_i} \left| \frac{\partial y(x_i, t)}{\partial t} \right|^2 + \left| \frac{\partial v(x_{i-1}^+, t)}{\partial x} \right|^2 + \left| \frac{\partial v(x_i^-, t)}{\partial x} \right|^2 dt$$

for some constant $C_4 > 0$. By Lemma 4.2 and the fact that $\partial v(x_i, t)/\partial t = \partial y(x_i, t)/\partial t$, we have

$$(4.17) \quad \sum_{i=1}^N \int_0^{T_i} |\omega_i(t)|^2 dt \leq C_5 \sum_{i=1}^N \int_0^{T_i} \left| \frac{\partial y(x_i, t)}{\partial t} \right|^2 dt \leq C_6 \mathcal{N}_T(f_1, c_2),$$

where C_5 and C_6 are positive constants. Combining (4.15) and (4.17), we see that the norm $\mathcal{N}_T$ is stronger than the $H_0^1[0, L] \times L^2[0, L]$ norm. Since we already know that the norm $\mathcal{N}_T$ is weaker than the $H_0^1[0, L] \times L^2[0, L]$ norm (see the remark after Theorem 3.3) this completes the proof of Theorem 4.1.

From Theorem 4.1, we immediately have the following theorem.

THEOREM 4.3. *If condition (2) holds, then the system (3.1)–(3.3) is asymptotically stable with a uniform exponential rate of decay. In other words, we can find positive constants M and k such that*

$$\mathcal{E}y(t) \leq M e^{-kt} \mathcal{E}y(0)$$

for all y satisfying (3.1)–(3.3).

Proof. Suppose (C2) holds. Let T_0 be as in the statement of Theorem 4.1. Then for $T > T_0$, $\mathcal{N}_T$ is equivalent with the $H_0^1[0, L] \times L^2[0, L]$ norm. Hence we can find a constant $c > 0$ such that

$$(4.18) \quad \mathcal{N}_T(f_1, f_2) \geq c \|(f_1, f_2)\|_E.$$

Then by (3.27), we have, for any $t > 0$,

$$(4.19) \quad \begin{aligned} \mathcal{E}y(t+T) - \mathcal{E}y(t) &= -\mathcal{N}_T \left(y(\cdot, t), \frac{\partial y(\cdot, t)}{\partial t} \right)^2 \\ &\leq -c^2 \left\| \left(y(\cdot, t), \frac{\partial y(\cdot, t)}{\partial t} \right) \right\|_E^2 = -c^2 \mathcal{E}y(t). \end{aligned}$$

Because $\mathcal{E}y(t)$ decreases with t , it follows from the semigroup property that

$$\mathcal{E}y(t) \leq M e^{-kt} \mathcal{E}y(0),$$

where $M = (1 - c^2)^{-1}$ and $k = -\ln M/T$. (Note that from (4.19), we must have $c^2 < 1$.)

Using the “controllability via stabilizability method” of Russell, we can then prove the following theorem.

THEOREM 4.4. *If condition (C2) holds, then the system is exactly controllable in any time $T > T_0 \max\{T^*, T_1, T_2, \dots, T_n\}$.*

5. Examples. We consider a special situation where $N = 2$ and ρ_i, τ_i are constants, $i = 1, 2$. Denoting $c_i = (\tau_i/\rho_i)^{1/2}$ and $L_i = x_i - x_{i-1}$, $i = 1, 2$, the system (1.1)–(1.3) becomes

$$(5.1) \quad \frac{\partial^2 y}{\partial t^2} = c_i^2 \frac{\partial^2 y}{\partial x^2}, \quad x \in (x_{i-1}, x_i), \quad i = 1, 2;$$

$$(5.2) \quad y(0, t) = y(L, t) = 0,$$

$$(5.3) \quad y(x_1^+, t) = y(x_1^-, t),$$

$$(5.4) \quad \tau_2 \frac{\partial y(x_1^+, t)}{\partial x} - \tau_1 \frac{\partial y(x_1^-, t)}{\partial x} = u_1(t).$$

The eigenvalues of the uncoupled strings are

$$\lambda_{n,i} = n\pi c_i / L_i, \quad n = 1, 2, 3, \dots, \quad \text{and} \quad i = 1, 2.$$

It follows that $\lambda_{n,1} = \lambda_{m,2}$ for some m, n if and only if $c_1 L_2 / c_2 L_1$ is rational. Hence from Theorem 3.6, we conclude that *the system (5.1)–(5.4) is approximately controllable (in some time $T > 0$) if and only if $c_1 L_2 / c_2 L_1$ is irrational*. The time T can be any number greater than

$$(5.5) \quad T_0 = 2 \left(\frac{L_1}{c_1} + \frac{L_2}{c_2} \right).$$

However, we can show that even when $c_1 L_2 / c_2 L_1$ is irrational, *the system is not exactly controllable*.

To prove that, let us first consider a function defined by the formula

$$f_\mu(x) = \begin{cases} \sin \frac{\mu L_2}{c_2} \sin \frac{\mu x}{c_1} & \text{if } 0 < x < x_1, \\ \sin \frac{\mu L_1}{c_1} \sin \frac{\mu(L-x)}{c_2} & \text{if } x_1 < x < L. \end{cases}$$

Here μ is a real parameter. It is easy to verify that

$$\tilde{y}(x, t) = \sin \mu(T-t) f_\mu(x)$$

is a solution of (5.1)–(5.4) with

$$\tilde{u}_1(t) = -\mu \frac{F(\mu) \sin \mu(T-t)}{c_1 c_2},$$

where

$$F(\mu) = c_1 \tau_1 \sin \frac{\mu L_1}{c_2} \cos \frac{\mu L_2}{c_2} + c_2 \tau_2 \sin \frac{\mu L_2}{c_2} \cos \frac{\mu L_1}{c_1}.$$

If y is any solution of (5.1)–(5.4), we have

$$(5.6) \quad \frac{d}{dt} \sum_{i=1}^2 \int_{x_{i-1}}^{x_i} \rho_i \frac{\partial y}{\partial t} \frac{\partial \tilde{y}}{\partial t} + \tau_i \frac{\partial y}{\partial x} \frac{\partial \tilde{y}}{\partial x} dx = -\frac{\partial y(x_1, t)}{\partial t} \tilde{u}_1(t) - \frac{\partial \tilde{y}(x_1, t)}{\partial t} u_1(t).$$

Note that if $F(\mu) = 0$, then $\tilde{u}_1(t) = 0$. We now construct a sequence $\{\mu_n\}_{n=0}^\infty$ such that $F(\mu_n) = 0$ for all n .

From a result in number theory, we know that if c_1L_2/c_2L_1 is an irrational number, there exist two sequences of integers $\{p_n\}_{n=0}^\infty$ and $\{q_n\}_{n=0}^\infty$ tending to infinity as n tends to infinity, such that

$$(5.7) \qquad \left| \frac{c_1L_2}{c_2L_1} - \frac{p_n}{q_n} \right| < 1/q_n^2$$

for all n . (See, e.g., Hardy and Wright [3, Thm. 171, p. 140].) Let $\lambda_n = p_n c_2 \pi / L_2$. Then

$$(5.8) \qquad \sin \frac{\lambda_n L_2}{c_2} = 0.$$

also, by (5.7), we have

$$\left| \frac{\lambda_n L_1}{c_1} - q_n \pi \right| < \epsilon_n,$$

where

$$\epsilon_n = \frac{c_2 L_1 \pi}{c_1 L_2 q_n}.$$

So

$$(5.9) \qquad \left| \sin \frac{\lambda_n L_1}{c_1} \right| < \epsilon_n \quad \text{for all } n.$$

It follows from (5.8) and (5.9) that we have

$$(5.10) \qquad |F(\lambda_n)| < c_1 \tau_1 \epsilon_n \quad \text{for all } n$$

and

$$(5.11) \qquad \lim_{n \rightarrow \infty} |F'(\lambda_n)| = \tau_1 L_1 + \tau_2 L_2.$$

Because F'' is a bounded function, we can then find a constant $\delta > 0$, independent of n , such that for all n sufficiently large,

$$(5.12) \qquad |F'(\lambda)| > (\tau_1 L_1 + \tau_2 L_2)/2 \quad \text{whenever } |\lambda - \lambda_n| < \delta.$$

From (5.10), (5.12) and the fact that $\lim_{n \rightarrow \infty} \epsilon_n = 0$, we conclude that for all n sufficiently large, there exists μ_n such that

$$(5.13) \qquad |\mu_n - \lambda_n| < \frac{2c_1 \tau_1 \epsilon_n}{\tau_1 L_1 + \tau_2 L_2}$$

and

$$(5.14) \qquad F(\mu_n) = 0.$$

Let

$$g_n(x) = f_{\mu_n}(x).$$

Let y be a solution of (5.1)–(5.4) with zero initial conditions. Then setting $\mu = \mu_n$ (which makes $\tilde{u}_1 = 0$) and integrating (5.6) from $t = 0$ to $t = T$ and canceling μ_n , we have

$$(5.15) \quad - \sum_{i=1}^2 \int_{x_{i-1}}^{x_i} \rho_i \frac{\partial y(x, T)}{\partial t} g_n(x) dx = g_n(x_1) \int_0^T \cos \mu_n(T-t) u_1(t) dt.$$

If the system (5.1)–(5.4) is exactly controllable, there exists a bounded linear mapping $\mathcal{C}$ from $H_0^1[0, L] \times L^2[0, L]$ into $L^2[0, T]$ that maps the terminal state $(y(\cdot, T), \partial y(\cdot, T)/\partial t)$ to a control function u_1 that steers the zero initial state to this terminal state. Hence (5.15) implies that

$$\begin{aligned} \sum_{i=1}^2 \int_{x_{i-1}}^{x_i} \rho_i \frac{\partial y(x, T)}{\partial t} g_n(x) dx &\leq K |g_n(x_1)| \left(\mathcal{E} y(T) \int_0^T |\cos \mu_n(T-t)|^2 dt \right)^{1/2} \\ &= K |g_n(x_1)| \left(\mathcal{E} y(T) \left(\frac{T}{2} - \frac{\sin 2\mu_n(T-t)}{4\mu_n} \right) \right)^{1/2} \\ &\leq K |g_n(x_1)| \left(\mathcal{E} y(T) \left(\frac{T}{2} + \frac{1}{4\mu_n} \right) \right)^{1/2}. \end{aligned}$$

Here K is a positive constant. Since the terminal state can be arbitrary, this implies that there exists a constant K_1 such that

$$\sum_{i=1}^2 \int_{x_{i-1}}^{x_i} |g_n(x)|^2 dx \leq K_1 |g_n(x_1)|^2.$$

In other words

$$\begin{aligned} \left(\sin \frac{\mu_n L_2}{c_2} \right) \left(\frac{L_1}{2} - \frac{\sin(2\mu_n L_1/c_1)}{4\mu_n/c_1} \right) + \left(\sin \frac{\mu_n L_1}{c_1} \right)^2 \left(\frac{L_2}{2} - \frac{\sin(2\mu_n L_2/c_2)}{4\mu_n/c_2} \right) \\ \leq K_1 \left(\sin \frac{\mu_n L_1}{c_1} \right)^2 \left(\sin \frac{\mu_n L_2}{c_2} \right)^2. \end{aligned}$$

(5.16)

For n sufficiently large, both terms on the left-hand side of the above inequality are nonnegative. Furthermore, because $c_1 L_2 / c_2 L_1$ is irrational, $\sin(\mu_n L_1 / c_1)$ and $\sin(\mu_n L_2 / c_2)$ cannot vanish simultaneously. Suppose there exist infinitely many n such that $\sin(\mu_n L_2 / c_2) \neq 0$. For such n , n sufficiently large, we must have

$$(5.17) \quad \frac{L_1}{2} - \frac{\sin(2\mu_n L_1 / c_1)}{4\mu_n / c_1} \leq K_1 \left(\sin \frac{\mu_n L_1}{c_1} \right)^2.$$

But the inequalities (5.9) and (5.13) together imply that

$$\lim_{n \rightarrow \infty} \sin \frac{\mu_n L_1}{c_1} = 0.$$

So letting n tend to infinity in (5.17), we have $L_1/2 \leq 0$, a contradiction. In a similar way, we can show that if there exist infinitely many n such that $\sin(\mu_n L_2 / c_2) \neq 0$, then $L_2/2 \leq 0$, again a contradiction. So the system (5.1)–(5.4) cannot be exactly controllable.

The situation would be different if we change the boundary condition at one of the endpoints (say $x = L$) to

$$(5.18) \quad \frac{\partial}{\partial x} z(L, t) = 0$$

but keeping the boundary condition at the other end the same as before. With obvious modifications in their proofs, we see that Theorems 3.6 and 4.4 remain valid when the boundary condition at $x = L$ is replaced by the condition (5.18). Now, the eigenvalues of the corresponding uncoupled strings are

$$\lambda_{n,1} = n\pi c_1/K_1 \quad \text{and} \quad \lambda_{n,2} = (n - \tfrac{1}{2})\pi c_2/L_2, \quad n = 1, 2, \dots$$

Obviously, if $c_1 L_2/c_2 L_1$ is irrational, $\lambda_{n,1} \neq \lambda_{m,2}$ for any n and m . So the system (5.1)–(5.4) is approximately controllable in time T , with $T > T_0$, T_0 again given by (5.5). But note that if

$$(5.19) \quad \frac{c_1 L_2}{c_2 L_1} = \text{an integer/an odd integer} = \frac{r}{2s-1},$$

where r and s are positive integers, then

$$|\lambda_{n,1} - \lambda_{m,2}| = \left| \frac{\pi c_2}{L_2 2(2s-1)} (2nr - (2m-1)(2s-1)) \right| \geq \frac{\pi c_2}{L_2 2(2s-1)} > 0$$

for any n and m . Therefore, the system (5.1)–(5.4) is approximately controllable in time $T > T_0$, T_0 given by (5.5). Furthermore, all the $\lambda_{n,1}$ and $\lambda_{n,2}$, $n = 1, 2, \dots$, are multiples of a fixed number

$$\alpha = (\pi c_2)/(2L_2(2s-1)).$$

So the set of exponential functions

$$S = \{e^{\pm i\lambda_{n,1}t} : 1 \leq n < \infty\} \cup \{e^{\pm i\lambda_{n,2}t} : 1 \leq n < \infty\}$$

is a subset of the exponential functions

$$\{e^{in\alpha t} : -\infty < n < \infty\},$$

which is orthogonal in $L^2[0, 2\pi/\alpha]$. Hence so is S . So S satisfies (RB2). It follows that if assumption (5.19) is satisfied, then the system (5.1)–(5.4) is exactly controllable in any time T such that

$$T > T^* = \max \left\{ 2 \left(\frac{L_1}{c_1} + \frac{L_2}{c_2} \right), \frac{2\pi}{\alpha} \right\} = \max \left\{ 2 \left(\frac{L_1}{c_1} + \frac{L_2}{c_2} \right), \frac{4L_2(2s-1)}{c_2} \right\}.$$

Because $L_2(2s-1)/c_2 = L_1 r/c_1$, it follows that

$$\frac{L_2(2s-1)}{c_2} = \frac{1}{2} \left(\frac{L_2(2s-1)}{c_2} + \frac{L_1 r}{c_1} \right) \geq \frac{1}{2} \left(\frac{L_1}{c_1} + \frac{L_2}{c_2} \right).$$

Hence $T^* = 4(L_2(2s-1)/c_2)$. If either one of s or r is greater than 1, then T^* is greater than T_0 , the infimum time for approximate controllability. Of course, we have not shown that the time T^* given in Theorem 4.4 is optimal. So we do not know whether we actually

need a greater time for exact controllability than for just approximate controllability, when r or s is greater than 1.

The results in the above examples are consistent with those in [2] and [11]. For example, it was shown in [11] that for the case of “symmetric” boundary conditions

$$(5.20) \quad z(0, t) = z(2, t) = 0,$$

with feedback

$$(5.21) \quad z_t(1^+, t) - z_t(1^-, t) = -K_1 \tau_1 u_x(1^-, t), \quad K_1 > 0$$

the system is asymptotically stable if and only if the ratio of the wave speeds is irrational. (In [11], Liu took $L_1 = L_2 = 1, L = 2$.) However, the system never decays with a uniform exponential rate.

For the case of “unsymmetric” boundary conditions

$$(5.22) \quad z(0, t) - \frac{\partial}{\partial x} z(2, t) = 0$$

the system is asymptotically stable, with feedback (5.21), if and only if

$$c_1/c_2 = \text{an odd integer/an even integer.}$$

However, it decays with a uniform exponential rate if

$$c_1/c_2 = \text{an integer/an odd integer.}$$

This last condition is, of course, the same as (5.19).

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