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## DOMAIN FUNCTIONALS AND EXIT TIMES FOR BROWNIAN MOTION

CHAOCHENG HUANG AND DAVID MILLER

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ABSTRACT. Two domain functionals describing the averaged expectation of exit times and averaged variance of exit times of Brownian motion from a domain, respectively, are studied. We establish the variational formulas for maximizing the functionals over  $C^k$  domains with a volume constraint, and characterize the stationary points and maximizers.

### 1. INTRODUCTION

Let  $W_t$  be a standard  $n$ -dimensional Brownian motion satisfying  $W_0 = x \in R^n$ . For any bounded domain  $\Omega$ , denote by  $\tau_\Omega(\omega)$  the (first) exit time of the Brownian motion from  $\Omega$ , i.e.,  $\tau_\Omega(\omega) = \inf \{t \geq 0 : W_t(\omega) \in \partial\Omega\}$ , where  $\partial\Omega$  is the boundary of  $\Omega$ . Consider the following domain functionals:

$$\mathfrak{F}_E(\Omega) = \frac{1}{|\Omega|} \int_{\Omega} E_x(\tau_\Omega) dx,$$
$$\mathfrak{F}_V(\Omega) = \frac{1}{|\Omega|} \int_{\Omega} Var_x(\tau_\Omega) dx,$$

where  $E_x(\tau_\Omega)$  and  $Var_x(\tau_\Omega)$  are the expectation and variance of  $\tau_\Omega$ , respectively. The values  $\mathfrak{F}_E(\Omega)$  and  $\mathfrak{F}_V(\Omega)$  may be understood as the averaged expected exit time and averaged variance of exit time from  $\Omega$  for the Brownian motion initiated in  $\Omega$ .

We are interested in characterizing stationary points for the following optimization problems ( $P_E$ ) and ( $P_V$ ):

$$(P_E) \quad \text{find } \Omega \text{ that maximizes } \mathfrak{F}_E(\Omega) \text{ over } \mathfrak{D}^k,$$
$$(P_V) \quad \text{find } \Omega \text{ that maximizes } \mathfrak{F}_V(\Omega) \text{ over } \mathfrak{D}^k$$

where  $\mathfrak{D}^k$  is the collection of bounded  $C^k$  ( $k \geq 2$ ) domains  $\Omega \subset R^n$  satisfying the volume constraint

$$(1.1) \quad \text{volume of } \Omega = A_0, \quad A_0 \text{ is a positive constant.}$$

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A domain  $\Omega$  in  $\mathfrak{D}^k$  is called a stationary point of any domain functional  $\mathfrak{F}$  over  $\mathfrak{D}^k$  if for any  $\{\Omega_\varepsilon\}_{\varepsilon|\leq\varepsilon_0} \subset \mathfrak{D}^k$  with  $\Omega_0 = \Omega$ , for some  $\varepsilon_0 > 0$ , we have  $d\mathfrak{F}(\Omega_\varepsilon)/d\varepsilon = 0$  at  $\varepsilon = 0$  if it exists. We show the following results.

**Theorem 1.1.** *Let  $\Omega \in \mathfrak{D}^k$  be a stationary point of functional  $\mathfrak{F}_E$  over  $\mathfrak{D}^k$ . Then there exists a solution to the overdetermined boundary value problem*

$$\begin{aligned} (1.2) \quad & \Delta u(x) = -2 \text{ in } \Omega, \\ (1.3) \quad & u(x) = 0 \text{ on } \partial\Omega, \\ (1.4) \quad & \frac{\partial u(x)}{\partial v} \text{ is constant on } \partial\Omega, \end{aligned}$$

where  $v$  is the exterior unit normal vector. In particular,  $\Omega \in \mathfrak{D}^k$  is a maximizer of  $\mathfrak{F}_E$  over  $\mathfrak{D}^k$  if and only if  $\Omega$  is a ball.

**Theorem 1.2.** *Let  $\Omega \in \mathfrak{D}^k$  be a stationary point of functional  $\mathfrak{F}_V$  over  $\mathfrak{D}^k$ . Then there exists a solution to the overdetermined boundary value problem*

$$\begin{aligned} (1.5) \quad & \Delta u(x) = -2 \text{ in } \Omega, \\ (1.6) \quad & \Delta w(x) = -2u(x) \text{ in } \Omega, \\ (1.7) \quad & u(x) = w(x) = 0 \text{ on } \partial\Omega, \\ (1.8) \quad & \frac{\partial u(x)}{\partial v} \frac{\partial w(x)}{\partial v} \text{ is constant on } \partial\Omega, \end{aligned}$$

where  $v$  is the exterior unit normal vector. In particular,  $\Omega \in \mathfrak{D}^k$  is a maximizer of  $\mathfrak{F}_V$  over  $\mathfrak{D}^k$  if and only if  $\Omega$  is a ball.

In the case  $k = \infty$ , the optimization problems  $(P_E)$  and  $(P_V)$  have been studied in [4, 5], where the same results with a different definition of stationary points (called critical points there) were established. See also [1, 2, 6, 8] for more general domain functionals. However, the notion of “critical point” adopted in these papers is with respect to infinitesimal volume preservation. More precisely, a domain  $\Omega$  is called a critical point in [4, 5, 6] if  $d\mathfrak{F}_E(\Omega_\varepsilon)/d\varepsilon = 0$  at  $\varepsilon = 0$  for any normal domain variations  $\Omega_\varepsilon$  ( $\Omega_0 = \Omega$ ) satisfying  $d|\Omega_\varepsilon|/d\varepsilon = 0$  at  $\varepsilon = 0$ . This same definition was also employed in [1, 2] for interior domain variations. In general, this domain variation  $\Omega_\varepsilon$  may not satisfy constraint (1.1), i.e.,  $\Omega_\varepsilon \notin \mathfrak{D}^k$ . Strictly speaking, a maximizer of  $\mathfrak{F}_E$  or  $\mathfrak{F}_V$  over  $\mathfrak{D}^k$  may not be a critical point in this sense.

The novelty of this note is the introduction of a family of volume-preserving domain variations  $\Omega_\varepsilon$ , i.e.,  $\Omega_\varepsilon \in \mathfrak{D}^k$  for all  $\varepsilon$ . This guarantees that a maximizer of  $\mathfrak{F}_E$  or  $\mathfrak{F}_V$  over  $\mathfrak{D}^k$  is necessarily a stationary point. In addition, we weaken the regularity requirement on  $\Omega$  which cannot be achieved by using the method of normal domain variations.

## 2. VOLUME-PRESERVING DOMAIN VARIATIONS

For simplicity, throughout the paper, we assume that in (1.1),  $A_0 = 1$ . Let  $\Omega$  be a bounded  $C^k$  domain. For any  $V \in C^k(\Omega', R^n)$ , where  $\Omega'$  is a neighborhood of  $\Omega$ , we denote by  $\Phi(x, \varepsilon)$  the solution of the initial value problem, for fixed  $x \in \Omega$ ,

$$(2.1) \quad \frac{d\Phi(x, \varepsilon)}{d\varepsilon} = V(\Phi(x, \varepsilon)), \quad \Phi(x, 0) = x.$$

A standard ODE theorem shows that  $\Phi(x, \varepsilon)$  is a  $C^k$  diffeomorphism for  $|\varepsilon| < \varepsilon_0$  for some  $\varepsilon_0 > 0$ . We define a variation  $\Omega_{\varepsilon, V}$  of the domain  $\Omega$  by the vector field  $V$ , by

$$\Omega_{\varepsilon, V} = \Phi(\Omega, \varepsilon) = \{\Phi(x, \varepsilon) : x \in \Omega\}.$$

When no confusion arises, we shall omit the subscript  $V : \Omega_\varepsilon = \Omega_{\varepsilon, V}$ . Obviously,  $\Omega_0 = \Omega$ , and  $\Omega_\varepsilon \subset \Omega'$  (for small  $\varepsilon$ ) is a  $C^k$  domain. Also,  $\Omega_\varepsilon \rightarrow \Omega$  as  $\varepsilon \rightarrow 0$ , in the sense of the  $C^k$  topology.

**Lemma 2.1.** *Suppose  $\Omega \in \mathfrak{D}^k$ . Let  $\Omega' \supset \bar{\Omega}$  be a neighborhood of  $\bar{\Omega}$ , and let  $V \in C^k(\Omega', \mathbb{R}^n)$  be a divergence free vector field, i.e., it satisfies  $\nabla \cdot V(x) = 0$  for  $x \in \Omega'$ . Let  $\Omega_\varepsilon = \Phi(\Omega, \varepsilon)$  be the variation of  $\Omega$  by the vector field  $V$ . Then  $\Omega_\varepsilon \in \mathfrak{D}^k$ .*

*Proof.* By the ODE theory and the fact that  $\nabla \cdot V(x) = 0$ , its Jacobian  $J(\Phi(x, \varepsilon))$  satisfies

$$\frac{dJ(\Phi(x, \varepsilon))}{d\varepsilon} = (\nabla \cdot V)(\Phi(x, \varepsilon)) J(\Phi(x, \varepsilon)) = 0.$$

Hence

$$(2.2) \quad J(\Phi(x, \varepsilon)) = J(\Phi(x, 0)) = 1.$$

This immediately leads to  $|\Omega_\varepsilon| = |\Omega| = 1$ . □

Let  $V$  be such a divergence free  $C^k$  vector field and let  $\Omega_\varepsilon$  be the variation of  $\Omega$  by  $V$ . Consider the Dirichlet problem

$$\Delta u_\varepsilon(x) = -2 \text{ in } \Omega_\varepsilon, \quad u_\varepsilon(x) = 0 \text{ on } \partial\Omega_\varepsilon.$$

Set  $u = u_0$ . We define the first variation  $\delta_V(u)$  of  $u$  along the vector field  $V$  by

$$(2.3) \quad \delta_V(u)(x_0) = \left. \frac{d}{d\varepsilon} u_\varepsilon(\Phi(x_0, \varepsilon)) \right|_{\varepsilon=0},$$

for  $x_0 \in \bar{\Omega}$ .

**Lemma 2.2.** *Let  $u$  be the solution of (1.2) and (1.3) and  $G_0(x, y)$  be the Green's function for the Laplacian  $\Delta$  for  $\Omega$ . Then for any divergence free  $C^k$  vector field  $V$ , defined in a neighborhood of  $\bar{\Omega}$ , and  $x_0 \in \bar{\Omega}$ , we have*

$$(2.4) \quad \begin{aligned} \delta_V(u)(x_0) = & - \int_{\partial\Omega} (V(\sigma) \cdot \nu(\sigma)) \frac{\partial u(\sigma)}{\partial \nu} \frac{\partial G_0(x_0, \sigma)}{\partial \nu} d\sigma \\ & + V(x_0) \cdot \nabla u(x_0), \text{ for } x_0 \text{ in } \bar{\Omega}. \end{aligned}$$

*Proof.* Let  $G_\varepsilon(x, y)$  be the Green's function for the Laplacian  $\Delta$  for  $\Omega_\varepsilon$ . By Green's formula and (2.2), we have

$$\begin{aligned} u_\varepsilon(\Phi(x_0, \varepsilon)) &= -2 \int_{\Omega_\varepsilon} G_\varepsilon(\Phi(x_0, \varepsilon), x) dx \\ &= -2 \int_{\Omega} G_\varepsilon(\Phi(x_0, \varepsilon), \Phi(x, \varepsilon)) dx. \end{aligned}$$

By [3, 2.4.28] (an early argument of this type was given by Hadamard) and direct computations, we have the formula

$$\begin{aligned} & \left. \frac{\partial}{\partial \varepsilon} G_\varepsilon (\Phi (x_0, \varepsilon), \Phi (x, \varepsilon)) \right|_{\varepsilon=0} \\ &= - \int_{\Omega} \nabla_y G_0 (x_0, y) \cdot \left( \nabla V (y) + (\nabla V (y))^T \right) \nabla_y G_0 (x, y) dy. \end{aligned}$$

Using the fact that

$$\nabla u (y) = -2 \int_{\Omega} \nabla_y G_0 (x, y) dx,$$

it follows from (2.3) that

$$\begin{aligned} \delta_V (u) (x_0) &= -2 \int_{\Omega} \left. \frac{\partial}{\partial \varepsilon} G_\varepsilon (\Phi (x_0, \varepsilon), \Phi (x, \varepsilon)) \right|_{\varepsilon=0} dx \\ &= - \int_{\Omega} (\nabla_y G_0 (x_0, y)) \cdot \left( \nabla V (y) + (\nabla V (y))^T \right) \nabla u (y) dy. \end{aligned}$$

Since  $\nabla^2 u$  and  $\nabla^2 G_0$  are symmetric matrices and since  $V$  is divergence free, a direct computation leads to

$$\begin{aligned} & \nabla_y G_0 (x_0, y) \cdot \left( \nabla V (y) + (\nabla V (y))^T \right) \nabla u (y) \\ &= \nabla G_0 \cdot \nabla (V \cdot \nabla u) + \nabla u \cdot \nabla (V \cdot \nabla G_0) - \nabla \cdot ((\nabla u \cdot \nabla G_0) V), \end{aligned}$$

for  $y \neq x_0$ . Since  $u$  solves (1.2) and (1.3),  $\Delta G_0 (x_0, y)$  is the Dirac measure at  $y = x_0$ ,  $\nabla \cdot V (x) = 0$ , and  $G_0 (x_0, \sigma) = 0$  for  $\sigma \in \partial \Omega$ , the divergence theorem yields

$$\begin{aligned} \delta_V (u) (x_0) &= - \int_{\partial \Omega} (V (\sigma) \cdot \nabla u) \frac{\partial G_0 (x_0, \sigma)}{\partial v} d\sigma + V (x_0) \cdot \nabla u (x_0) \\ &\quad - \int_{\partial \Omega} (V (\sigma) \cdot \nabla G_0 (x_0, \sigma)) \frac{\partial u (\sigma)}{\partial v} d\sigma - 2 \int_{\Omega} V (y) \cdot \nabla G_0 (x_0, y) dy \\ &\quad + \int_{\partial \Omega} (\nabla u (\sigma) \cdot \nabla G_0 (x_0, \sigma)) V (\sigma) \cdot v (\sigma) d\sigma \\ &= - \int_{\partial \Omega} (V (\sigma) \cdot v (\sigma)) \frac{\partial u (\sigma)}{\partial v} \frac{\partial G_0 (x_0, \sigma)}{\partial v} d\sigma + V (x_0) \cdot \nabla u (x_0). \end{aligned}$$

In deriving the last equality, we have used the fact that for  $\sigma \in \partial \Omega$ ,

$$\nabla u (\sigma) = \frac{\partial u (\sigma)}{\partial v} v (\sigma), \quad \nabla G_0 (x_0, \sigma) = \frac{\partial G_0 (x_0, \sigma)}{\partial v} v (\sigma),$$

and the fact that  $V$  is divergence free. □

We point out that the mappings defining interior variations used in [2, 3] are actually the first order approximation to  $\Phi (x, \varepsilon)$ .

### 3. VARIATIONAL FORMULATIONS FOR FUNCTIONALS

In this section we develop variational formulations for problems  $(P_E)$  and  $(P_V)$ , utilizing the results established in the previous section. Define the first variation of any domain functional  $\mathfrak{F}$  along a divergence free vector field  $V$  by

$$\delta_V (\mathfrak{F}) (\Omega) = \left. \frac{d\mathfrak{F} (\Omega_\varepsilon)}{d\varepsilon} \right|_{\varepsilon=0} = \lim_{\varepsilon \rightarrow 0} \frac{\mathfrak{F} (\Omega_\varepsilon) - \mathfrak{F} (\Omega)}{\varepsilon},$$

where  $\Omega_\varepsilon$  is the variation of  $\Omega$  along  $V$ .

**Lemma 3.1.** *Let  $\Omega \in \mathfrak{D}^k$  and  $V$  be a divergence free  $C^k$  vector field defined in a neighborhood of  $\bar{\Omega}$ . Then*

$$(3.1) \quad \delta_V (\mathfrak{F}_E) (\Omega) = \frac{1}{2} \int_{\partial\Omega} (V(\sigma) \cdot v) \left( \frac{\partial u(\sigma)}{\partial v} \right)^2 d\sigma$$

where  $u$  is the solution of (1.2) and (1.3), and

$$(3.2) \quad \delta_V (\mathfrak{F}_V) (\Omega) = \int_{\partial\Omega} (V(\sigma) \cdot v) \frac{\partial u(\sigma)}{\partial v} \frac{\partial w(\sigma)}{\partial v} d\sigma$$

where  $u$  and  $w$  satisfy (1.5)-(1.7).

The formulas (3.1) and (3.2) were previously derived in [6] for normal variations of smooth domains, and in [2] for interior variations of Lipschitz domains. By the same techniques used in [6], together with Lemma 2.2, one can show that (3.1) and (3.2) remain true for our domain variations by any divergence free vector field  $V$ . We omit the details of the proof to Lemma 3.1.

We are now in a position to prove Theorem 1.1 and Theorem 1.2.

*Proof of Theorem 1.1.* Suppose that the functional  $\mathfrak{F}_E$  has a stationary point  $\Omega \in \mathfrak{D}^k$ . Then for any divergence free  $C^k$  vector field  $V$ ,  $\delta_V (\mathfrak{F}_E) (\Omega) = 0$ . Let  $u$  solve (1.2), (1.3). By Lemma 3.1, it follows that

$$(3.3) \quad \int_{\partial\Omega} (V(\sigma) \cdot v(\sigma)) \left( \frac{\partial u(\sigma)}{\partial v} \right)^2 d\sigma = 0.$$

To show (1.4), it suffices to show that for any smooth function  $\varphi$  defined in a neighborhood of  $\partial\Omega$  and satisfying

$$(3.4) \quad \int_{\partial\Omega} \varphi(\sigma) d\sigma = 0,$$

the following equality holds:

$$(3.5) \quad \int_{\partial\Omega} \varphi(\sigma) \left( \frac{\partial u(\sigma)}{\partial v} \right)^2 d\sigma = 0.$$

To verify (3.5), we select a sequence of decreasing smooth domains  $\{\Omega^{(m)}\}_{m=0}^\infty$  with  $\Omega^{(0)} = \Omega$  such that  $\Omega^{(m)} \supset \bar{\Omega}^{(m+1)}$ ,  $\Omega^{(m)} \supset \bar{\Omega}$  for  $m \geq 1$ , and as  $m \rightarrow \infty$ ,  $\Omega^{(m)} \rightarrow \Omega$  in the  $C^k$  topology. Let  $\varphi$  be a fixed smooth function satisfying (3.4). We extend  $\varphi$  smoothly to  $R^n$ . Define a sequence of constants  $\theta_m$  by

$$\theta_m = -\frac{1}{\int_{\partial\Omega^{(m)}} d\sigma} \int_{\partial\Omega^{(m)}} \varphi(\sigma) d\sigma.$$

It is easy to see that  $\theta_m \rightarrow 0$  (by (3.4)) as  $m \rightarrow \infty$  and that

$$(3.6) \quad \int_{\partial\Omega^{(m)}} (\varphi(\sigma) + \theta_m) d\sigma = 0.$$

Consider the Neumann boundary value problem

$$\begin{aligned} \Delta g_m &= 0 \text{ in } \Omega^{(m)}, \\ \frac{\partial g_m}{\partial v} &= \varphi + \theta_m \text{ on } \partial\Omega^{(m)}. \end{aligned}$$

Note that (3.6) is the solvability condition for the Neumann problem. Hence, the above boundary value problem admits a smooth solution  $g_m$  (which is unique up to a constant). Furthermore, the  $H^1(\Omega)$  norm of the function  $g_m$  is bounded uniformly in  $m$ . Set  $V_m = \nabla g_m$ . Obviously  $V_m$  is a divergence free vector field. Let  $\eta(x)$  be a  $C^k$  extension of  $\partial u(\sigma)/\partial v$ . Then, by (3.3) and the divergence theorem (in  $\Omega_m \setminus \Omega$ ), we have

$$\begin{aligned} 0 &= \int_{\partial\Omega} \frac{\partial g_m(\sigma)}{\partial v} \left( \frac{\partial u(\sigma)}{\partial v} \right)^2 d\sigma \\ &= \int_{\Omega_m \setminus \Omega} \nabla \cdot (\eta^2(x) \nabla g_m(x)) dx + \int_{\partial\Omega_m} \frac{\partial g_m(\sigma)}{\partial v} \eta^2(\sigma) d\sigma \\ &= \int_{\Omega_m \setminus \Omega} \nabla \eta^2(x) \cdot \nabla g_m(x) dx + \int_{\partial\Omega_m} (\varphi(\sigma) + \theta_m) \eta^2(\sigma) d\sigma \\ &\rightarrow \int_{\partial\Omega} \varphi(\sigma) \left( \frac{\partial u(\sigma)}{\partial v} \right)^2 d\sigma \quad (\text{as } m \rightarrow \infty). \end{aligned}$$

Our previous assertion (3.5) follows.

The rest of the assertions follow from the fact that any solution of (1.2)-(1.4) is radially symmetric [6] (see [7] for the original results) and that any ball is a maximizer [4].  $\square$

*Proof of Theorem 1.2.* Suppose that the functional  $\mathfrak{F}_V$  has a stationary point  $\Omega \in \mathfrak{D}^k$ . Then for any divergence free  $C^k$  vector field  $V$ ,  $\delta_V(\mathfrak{F}_V)(\Omega) = 0$ . Let  $u$  and  $w$  be the solutions of (1.5)-(1.7). By Lemma 3.1, it follows that

$$\int_{\partial\Omega} (V(\sigma) \cdot v(\sigma)) \frac{\partial u(\sigma)}{\partial v} \frac{\partial w(\sigma)}{\partial v} d\sigma = 0.$$

Using exactly the same argument as in the proof of Theorem 1.1, we may show that for any smooth function  $\varphi$  defined in a neighborhood of  $\bar{\Omega}$  and satisfying  $\int_{\partial\Omega} \varphi(\sigma) d\sigma = 0$ , we have

$$\int_{\partial\Omega} \varphi(\sigma) \frac{\partial u(\sigma)}{\partial v} \frac{\partial w(\sigma)}{\partial v} d\sigma = 0.$$

This immediately leads to (1.8). The rest of the assertions follow from [6].  $\square$

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