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Investigating Problems with Mathematical Proof Skills

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Abstract

This research paper is focused on identifying and exploring problems that high school students have with the mathematical proof process. The study described in this report was designed to answer the question, do students who tend to struggle with mathematical proofs also struggle with understanding what is known and unknown in a proof, making assumptions about figures, and understanding mathematical converses? To answer these questions, students at three Dayton area high schools completed tasks relating to proofs and converse statements as well as a questionnaire. Contrary to the initial hypothesis, students who were perceived to have greater proof abilities did not always excel with various aspects of proofs. Similarly, students who were perceived to have lesser proof abilities did not always struggle with aspects of proofs. It was found that students tend to struggle with two major aspects of proof: Making assumptions when part of an if-then statement is not satisfied, and understanding that a statement and its converse are not equivalent. In general, this study found that the topic of if-then statements, their converses, and the differences between the two needs more attention by students of all levels of ability.

Introduction

While there has been a lot of investigation regarding the topic of mathematical proofs and how students struggle with proof, this study examined some typical and basic proof skills and how these related to other struggles students might have with proofs. Some of the basic proof skills that this research examined are understanding what is known and unknown, understanding statements and their converses, and understanding what is given in figures. An important part of becoming a teacher is realizing and working to help with student struggles. This research was done to look at the common struggle with proofs that many students share to help better understand it. The hope of this study is to bring new understanding to some of the struggles that students have with proofs in an attempt to find ways to adjust future teaching skills to fix the problems.

Literature Review

There are different aspects that are part of reasoning and proof. One aspect is understanding, which is what this research focuses on. In the aspect of understanding, students should, for example, understand theorems, postulates, definitions, converses, etc., understand and interpret diagrams and their markings, memorize definitions, properties, theorems, converses, etc., and understand what is known and unknown when looking at a problem. Another aspect is the aspect of logic/order. This aspect involves the ability of the student to put together the order of steps in a proof and figure out the structure of the proof, as well as the ability to allow the proof to flow in a way that makes sense. Another aspect is inductive reasoning, where the students use examples to “prove”, or convince themselves, that something is true. In this aspect, students generally see examples as enough to convince someone that something is true, and is usually where many students are stuck and are not able to generalize how or why it is true for all cases. Another aspect is application/generalization. This aspect involves the students being able to apply theorems, postulates, etc. to multiple situations, as well as to manipulate variables to make arguments applicable to multiple situations. Deductive reasoning is the final aspect we will discuss. This aspect tends to pull together all of the previously stated aspects. In this aspect, students should have full proofs that are logical and applicable to multiple scenarios with correct reasoning.

As stated by Healy and Hoyles (2000), mathematical proofs involve students understanding many different and complex aspects of mathematics and logic. This includes things like “identifying assumptions, isolating given properties and structures, and organizing logical arguments”. In their study of high-attaining 14- and 15-year old students, Healy and Hoyles (2000) concluded that students tend to think empirically and use empirical reasoning when asked to prove something, that is to say that the students tended to use more inductive reasoning in place of mathematical proofs. However, the students in their study also noted that they understood that their proofs were not general, complete, or enough to receive the highest grades by their teachers.

In another study about proofs, Hoyles and Kuchemann (2002) analyzed high-attaining students’ written responses that were a part of a nationwide survey on proof conceptions in English schools. They found that students tended to struggle with the fundamental aspect of proofs such as the if-then statement. In the study, an overwhelming number of students, 69% for Year 8 and 62% Year 9, indicated that they were indeed equivalent (this does not include the 16% of Year 8 and 19% of Year 9 students who changed their answer from “yes, they are equivalent” to “no, they are not equivalent”) (Hoyles & Kuchemann, 2002). These findings showed a confusion of whether or not an if-then statement and its converse are equivalent. These findings also reveal that there was very little clarity given between Year 8 and Year 9 on this issue.

Proof is present in the Common Core and Ohio Learning Standards (Ohio Department of Education, 2017). In these standards, mathematical practices describe how students are to “reason abstractly”, “look for and make use of structure”, “attend to precision”, and “construct viable arguments and critique the reasoning of others” (Ohio Department of Education, 2017). These practices go hand-in-hand with the skills required to prove a theorem or other fact in geometry. However, in the research done by Dreyfus (1999), it was concluded that proof may be difficult for students due to the lack of exposure to the different elements that are involved in the process, such as the lack of understanding geometric content, problem-solving skills, and reasoning skills. This is consistent with a study by Chinnappan, Ekanayake, and Brown (2011) which studied the knowledge on geometric proof constructions of Sri Lankan students. They also found that students need to understand the geometric content well, but that they also need to practice reasoning and problem-solving skills to be able to successfully create proofs (Chinnappan, Ekanayake, & Brown, 2011). Antonini and Mariotti (2008) also agree that students should develop some foundational skills, such as producing conjectures, and conclude that to effectively teach mathematical proofs, there must be a corresponding teaching of mathematical logic (Kanellos, 2014).

The research of student understanding are also consistent with The Van Hiele Levels of Geometric Thinking and Usiskin’s Dimensions of Mathematical Understanding. The Van Hiele levels describe the sequence of levels that students progress through. In the outline, it is stated that the levels must be moved through in order with a mastery in one level before moving to the next. The five levels are described more in Table 1 (Mayberry, 1983 & Vojkuvkova, 2012).

Table 1:

Level 0: Visual	Figures are recognized as a whole, without specific characteristics associated with it. Abilities are limited to being able to recognized similar shapes and recalling names.
Level 1: Analysis	Properties are being put with shapes and are being used to recognize shapes. There is still no connection between the properties and different shapes.
Level 2: Abstraction	Relationships between properties are becoming evident. Logical implication is beginning to be understood, but not deduction.
Level 3: Deduction	Students can construct proofs and understand how to use logical reasoning along with geometric definitions, axioms, and postulates as reasoning for steps in proofs.
Level 4: Rigor	Formal deduction is understood. Understands the role of proofs and can manipulate symbols according to formal logical laws.

These levels break down the process of forming proofs from the most basic geometrical understanding to the most complex. This breakdown of levels can help understand that the students in Healy and Hoyles's (2000) study have not mastered formal deduction (level 3) and rigor (level 4), leaving them with an incomplete understanding of proofs.

Pairing with Van Hiele's idea that there are different levels of understanding that a student has is Usiskin's dimensions of mathematical understanding. Unlike Van Hiele, Usiskin does not claim that the student will move through the dimensions in a particular way, but rather that the student can obtain one or more of the dimensions independently from other dimensions. The dimensions are outlined by Usiskin (2012) in Table 2.

Table 2:

Dimension	Description
Skill-Algorithm	From the rote application of an algorithm through the selection and comparison of algorithms to the invention of new algorithms (calculators and computers included)
Property-Proof	From the rote justification of a property through the derivation of properties to the proofs of new properties.
Use-Application	From the rote application of mathematics in the real world through the use of mathematical models to the invention of new models.
Representation-Metaphor	From the rote representations of mathematical ideas through the analysis of such representations to the invention of new representations.
History-Culture	From rote facts through the analysis and comparison of mathematics in cultures to the discovery of new connections or historical themes.

In this article, Usiskin examines each dimension more carefully in both algebra and geometry situations, where they conclude that a true level of understanding comes from mastery of more than one dimension (Usiskin, 2012). Usiskin (2012) also explains that the different dimensions are generally learned in situations that are isolated from the other dimensions.

Method

Participants

The participants in this study consisted of fourteen high school students from three different high schools in the Dayton, Ohio area. The students all have either taken a geometry course already or are currently in a geometry course. Each student was put into one of three categories by their teacher - above-average, average, or below-average - in terms of their performance with regards to mathematical proofs. At each different high school, there was at least one student in each of the categories.

Materials

The only materials used by participants in this study were a pencil, and the documents provided by the researcher: The worksheet of tasks (Attachment A), the questionnaire (Attachment B), and the consent form (Attachment C) that had to be completed and returned before the students could participate in this study.

Procedure

First, the researcher's faculty advisor contacted a list of teachers in the Dayton, Ohio area explaining the project and inquiring if any of them were interested in helping with the study. Four teachers responded that they were interested, but one had to withdraw at the last minute, leaving the researcher with three schools and teachers to participate in the study.

The researcher followed up with each of the teachers that responded to set up a time to conduct the tasks and questionnaire with students. The teachers were also asked to choose at least three students for the study: One that they deemed as above-average with regards to proofs, one that was deemed average, and one that was deemed below-average. Consent forms were also sent ahead of time so that they could be distributed to, and completed by, the chosen students.

The researcher then went into the schools on the mutually agreed upon dates. The researcher reviewed all consent forms for all of the students participating in the study, prior to implementing the tasks and questionnaire with the students. The group of students and researcher were taken to a place that would not disturb other students. The task worksheets were distributed first to the group of students. Once these were completed and collected, students were given the questionnaire. Finally, when they completed the questionnaire, they were released back to their assigned class. This process was repeated at all three schools in the study.

In the scoring process, each student's responses were given a point score out of three. For the scoring, the student was given zero points if they gave no answer, or if they incorrectly responded to both parts of a task. They were given one point if they got the decision correct but had the wrong reasoning, or if they did not have a decision written down but got part of the correct reasoning. They were given two points if they had the wrong decision or no decision but correct reasoning; or they had the correct decision and only partially correct reasoning. They were given three points if they got the answer completely correct.

Findings

Table 3:

Group	**Number of Students**	**Average Score**	**Experience with Proofs**	**Difficulty with Tasks (Scale of 1-4)**
Above- Average	5	16.8	Very little: 0 Some: 1 Average: 0 A lot: 4	Level 1: 1 Level 2: 4 Level 3: 0 Level 4: 0
Average	4	11.5	Very little: 0 Some: 0 Average: 1 A lot: 3	Level 1: 0 Level 2: 3 Level 3: 1 Level 4: 0
Below- Average	5	10.8	Very little: 0 Some: 2 Average: 3 A lot: 0	Level 1: 1 Level 2: 3 Level 3: 1 Level 4: 0

Table 3 is a summary of some of the data from the study. The three groups that are listed in the left-hand column are the groups that students were assigned into by the teachers. The number of students in each group is shown in the next column. The third column shows the average score students in each category received on the proof tasks. The fourth column shows how much exposure the students reported that they have had with proofs. On the questionnaire the students had four options for this question; Very Little (1-2 days), Some (3-4 days), Average (5-6 days), and A lot (more than 1 week). Finally, Table 3 shows a column of how much difficulty the students reported to have had with the tasks. The scale they were given in the questionnaire is a scale from one to four, where one is no difficulty and four is extreme difficulty.

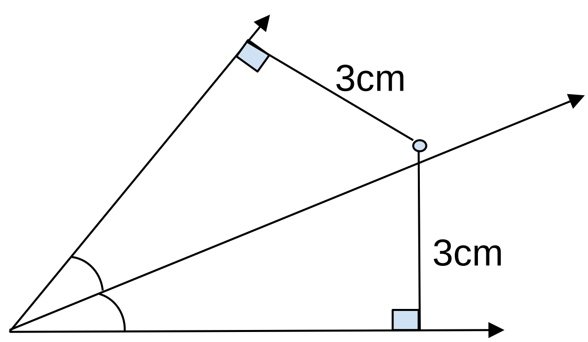
If we look at the average scores for each group, there were some interesting findings. While the average scores went up between the groups, from Below-Average to Above-Average, there was a very small difference between the Average and Below-Average groups. Their average scores only differed by 0.7 points.

For the Above-Average group, the majority of students claimed that they had a lot of exposure to proofs. Similarly for the Average group, again the majority said that they had a lot of exposure to proofs. However, for the Below-Average group, none of them claimed to have more than an average amount of exposure to proofs. When the perceived difficulty levels were compared between all of the groups, the majority of the students indicated a difficulty level of 2, which was surprising that this was so common between the groups. This shows that despite the perceived ability of the students by the teachers, most of them still saw the tasks as equally challenging.

Discussion

Initially, it was hypothesized that the average scores would have very defined differences between the groups. However, this was not the case. The less defined differences could have been due to the fact that students across all groups tended to have similar struggles. When analyzing the student answers for the worksheet of tasks, a few anticipated problems were noticed, as well as some that were not anticipated. One problem was students using the original statement and the converse interchangeably. One of the hypotheses of this study was that those who tend to struggle with proofs also tend to struggle with mathematical skills involved in proofs, such as understanding a statement and its converse. When looking into the data, it was revealed that students in each group, but in mostly the Average and Below-Average groups, often used the original statement and its converse interchangeably in their reasoning. This provided clear evidence that students are confused about whether or not statements and their converses are equivalent, which coincides with Hoyles and Kuchemann's (2002) similar findings. Many of the students were assuming that they knew more than just the given statement. Specifically, the students continually used "if not A, then not B" or "if not B, then not A" for the statement or converse given in the form of "if A, then B". This confusion about what is truly known and unknown seemed to be very common, especially with the angle bisector problems in the tasks. For example, the students were given that they only know that "If a point is equidistant from the two sides of an angle (perpendicularly), then it is on the angle bisector." They were given Figure 1.

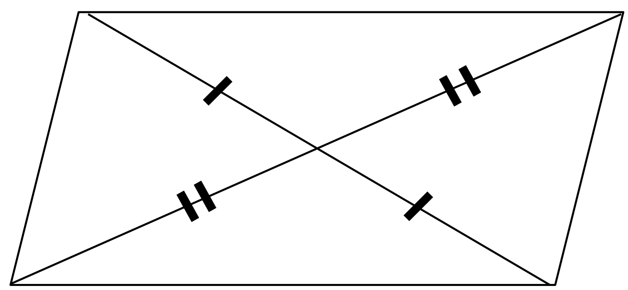
Figure 1:



In this problem, the student was asked again if there was anything wrong with this figure based on what is known. A typical response was that in this figure, "the lines should not be the same length because in order to be the same length, the point has to be on the bisector." Here, the student is showing the process of thinking, "if not B, then not A". This student, as well as others, also used the reasoning of "if not A, then not B" when knowing the same statement with a diagram that showed a point being on the angle bisector but that was not equidistant (perpendicularly) from both sides of the angle.

One problem that was unexpected was with the special quadrilateral task. One of the questions read to decide if anything was wrong with the following figure if we only know that "if a quadrilateral has diagonals that bisect each other, then it is a parallelogram":

Figure 2:



For this problem, many of the students claimed that nothing was wrong with this figure because the figure showed that the diagonals bisect each other, so with that fact and what we know, it must be a parallelogram. However, the correct answer was that something was wrong because we do not know that it is a parallelogram because of the lack of parallel markings on the figure. Despite this, half of the students gave this same, technically incorrect, answer. This answer was first identified as the students assuming something about the figure that was not given in that it was a parallelogram because it looked like one. When it was dissected further, however, it came to light that the students were following and understanding the process of an if-then statement in proofs. Though it was odd that the students only used this reasoning for this one particular question, it was important to note that it seems that students generally understand how if-then statements can be used to move from one property to another in a proof. However, it also may have been an issue showing order or direction of an if-then statement in one drawing.

This finding brought to attention a possible improvement to the study where a progression of the markings in the image could be shown. By giving multiple figures showing the markings being added we could better see how the students interpret the if-then statements because they would see what comes first and what comes afterwards. Seeing the progression may give greater detail into what the students are truly struggling with.

Conclusion

Teachers' perceived ability of students with respect to proof was not directly related to their struggles with various aspects of mathematical proofs. Students in general tend to struggle with understanding what the differences are between a statement and its converse and whether or not they are not able to be used as equivalent statements. They also tend to struggle with assuming that if one part of a conditional statement is not satisfied, then the other part of the conditional statement is then assumed to be not satisfied as well.

Due to these findings, it can be concluded that the topic of conditionals and their converses are a topic that needs to be addressed more in depth in classrooms. These struggles were much more prevalent in the students who tended to be considered Average or Below-Average with regards to their proof ability, however, this study shows that it needs more attention across all performance levels of students.

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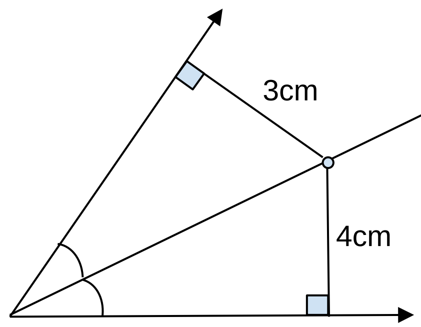
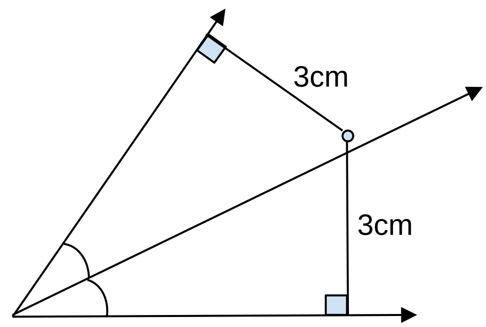
Attachment A:

Assume that you only know that *if a point is on the angle bisector, then it is equidistant from the two sides of the angle (perpendicularly)*. Determine if there is anything wrong with each figure. Explain why or why not.

1



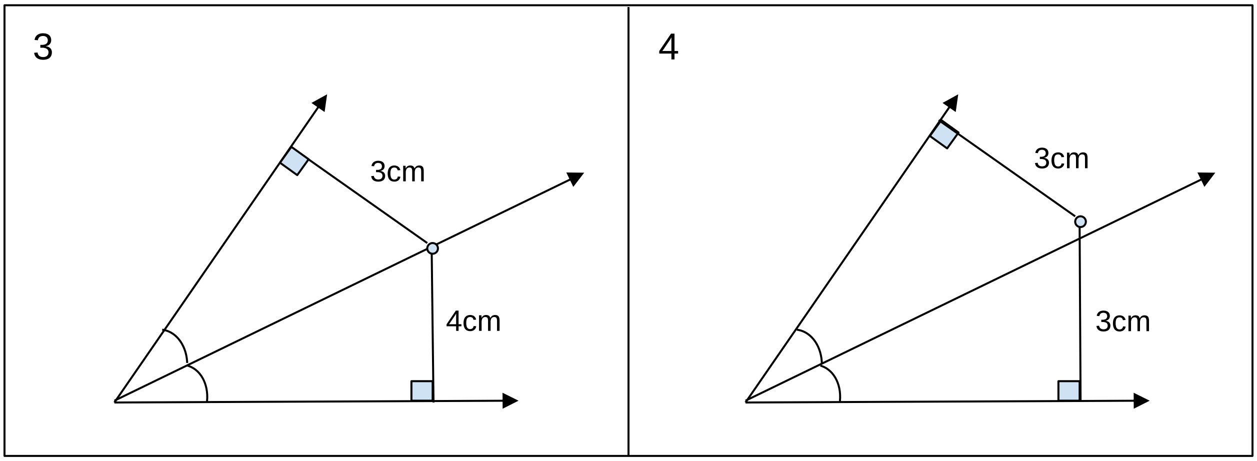
2



1. ____

2. ____

Now, assume you only know that the converse is true: *If a point is equidistant from the two sides of an angle (perpendicularly), then it is on the angle bisector*. Again, determine if anything is wrong with each figure and explain why or why not.

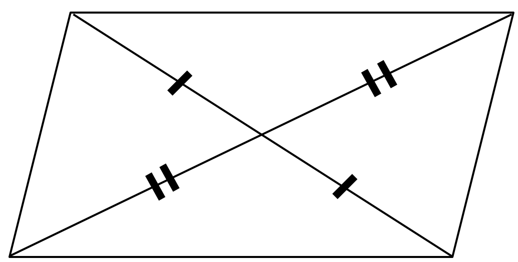


3. ____

4. ____

Assume that you only know that *if a quadrilateral is a parallelogram, then it has diagonals that bisect each other*. Determine if there is anything wrong with each figure. Explain why or why not.

5



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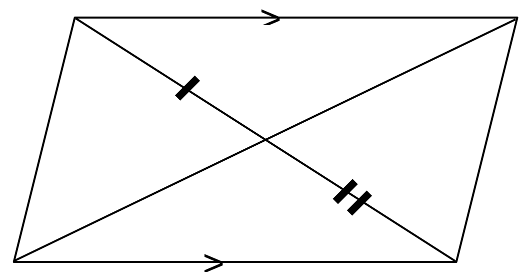
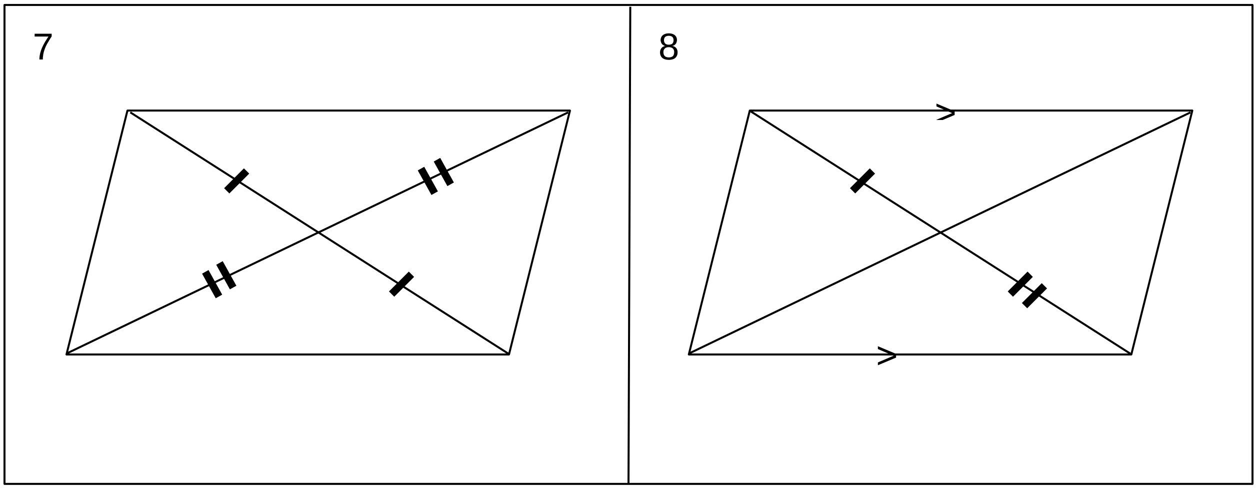


5.

6.

Now, assume you only know that the converse is true: *If a quadrilateral has diagonals that bisect each other, then it is a parallelogram*. Again, determine if anything is wrong with each figure and explain why or why not.



7. ____

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8. ____

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Attachment B:

Questionnaire

1. Did you notice any certain parts of the tasks that you struggled with? If so, please explain.

2. On the scale below with 1 being no difficulty and 4 being extreme difficulty, circle how much difficulty you had with the set of tasks.

   1 2 3 4

3. How much exposure have you had to proofs? (Circle one.)

   Very little (1-2 days) Some (3-4 days) Average (5-6 days) A lot (more than 1 week)

4. In your opinion, do you think that you tend to struggle with understanding theorems/postulates/definitions and their converses when trying to prove something in mathematics?

5. What is the converse of the statement "If two angles have a sum of 90°, then those angles are complementary."?

Attachment C:

Consent Form

My name is Samantha Collier and I am a senior Mathematics Education major at Wright State University. I am doing a research project for my honors program that looks at students' struggles with some skills involved in mathematical proofs. For this project, I need students who are willing to participate in my study. Participation would include the student completing a worksheet of tasks and a questionnaire relating to the tasks. All of the data would be collected and would be analyzed and written up in a paper.

Keeping your child/dependent's identity confidential would be a priority. My faculty advisor and I would be the only two people who knew the identity of the students and in the paper the student's name will not be reported to ensure that each participant would remain anonymous. No names will ever be released from this study.

I, ___________________________, the parent/guardian of _________________________, am signing this form to (check one):

___ give consent for my child/dependent to participate in this study.

___ not give consent for my child/dependent to participate in this study.

Parent/Guardian Signature: _________________________________ Date: ______________