On the Product of Two Generalized Derivations

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ABSTRACT. Two elements \( A \) and \( B \) in a ring \( \mathfrak{R} \) determine a generalized derivation \( \delta_{A,B} \) on \( \mathfrak{R} \) by setting \( \delta_{A,B}(X) = AX - XA \) for any \( X \) in \( \mathfrak{R} \). We characterize when the product \( \delta_{C,D} \delta_{A,B} \) is a generalized derivation in the cases when the ring \( \mathfrak{R} \) is the algebra of all bounded operators on a Banach space \( \mathcal{E} \), and when \( \mathfrak{R} \) is a \( C^* \)-algebra \( \mathfrak{A} \). We use these characterizations to compute the commutant of the range of \( \delta_{A,B} \).

1. INTRODUCTION

Let \( \mathfrak{B} \) be a ring or an algebra. A derivation on \( \mathfrak{B} \) is an additive (linear) map \( \delta : \mathfrak{B} \to \mathfrak{B} \) satisfying \( \delta(XY) = \delta(X)Y + X\delta(Y) \) for all \( X \) and \( Y \) in \( \mathfrak{B} \). Consequences of assuming

\[(*) \quad \text{the product } \delta_1 \delta_2 \text{ of two derivations } \delta_1 \text{ and } \delta_2 \text{ again is a derivation}\]

have been investigated in a number of papers. Posner [Po] proved that if \( \mathfrak{B} \) is a prime ring of characteristic not 2, then (*) implies \( \delta_1 = 0 \) or \( \delta_2 = 0 \). This result has been reproved in other papers, e.g., [FS] and [Wi], under the stronger assumption that \( \mathfrak{B} = \mathcal{L}(\mathcal{H}) \), the algebra of all bounded linear operators on the Hilbert space \( \mathcal{H} \). If \( \mathfrak{B} = \mathfrak{A} \) is a \( C^* \)-algebra, then (*) implies \( \delta_1 \delta_2 = 0 \) [Mal] (a result of this type is proved in [Ped] for unbounded densely defined derivations), and (*) together with \( \delta_1 = \delta_2 = \delta \) implies \( \delta = 0 \). In [Wi] the range of a derivation on \( \mathcal{L}(\mathcal{H}) \) is investigated, a major tool in [Wi] is the result mentioned above. It is known (e.g., [Ch], [Ka], [Sa1], [Se]) that a derivation of \( \mathcal{L}(\mathcal{E}) \) is inner, i.e., there exists an element \( A \) in \( \mathcal{L}(\mathcal{E}) \), the algebra of all bounded linear operators on the Banach space \( \mathcal{E} \), so that \( \delta(X) = \delta_A(X) = AX -XA \) for all \( X \) in \( \mathcal{L}(\mathcal{E}) \). It is also known (a result of Sakai) [Pe], [Sa2] that, if \( \delta \) is a derivation of a \( C^* \)-algebra \( \mathfrak{A} \) then, \( \delta = \delta_A \) for some \( A \) in the enveloping von Neumann algebra of \( \mathfrak{A} \).

For two operators \( A \) and \( B \) in \( \mathcal{L}(\mathcal{E}) \) (or in the multiplier algebra \( M(\mathfrak{A}) \) of \( \mathfrak{A} \)), we say that the operator \( \delta_{A,B}(X) = AX - XB \) is a generalized derivation. It is shown in [Br] that, if the product of two derivations of an prime ring of characteristic not 2 is a generalized derivation, then one of the derivations must be zero. We characterize when the product \( \delta_{C,D} \delta_{A,B} \) of a derivation and a generalized derivation
on $\mathcal{L}(\mathcal{E})$ is a derivation. In contrast to the product of two derivations, the product $\delta_C \delta_{A,B}$ can be a non-zero derivation. As a consequence of the characterization we obtain information about the commutant of the range of a generalized derivation. We also apply the results for $\mathfrak{B} = \mathcal{L}(\mathcal{E})$ to obtain a characterization for the case when $\mathfrak{B} = \mathfrak{A}$ is a $C^*$-algebra. The later characterization is strong enough to yield the results, mentioned above, characterizing when the products $\delta \delta_1$ and $\delta_1 \delta_2$ are derivations on a $C^*$-algebra as easy corollaries. In the final section we consider the product of two generalized derivations.

We refer the reader to [Pe] for background information about $C^*$-algebras, and to [Pe] and [Sa2] for the theory of derivations in operator algebras. Generalized derivations have been studies extensively, see e.g., [CF], [FS], and [Ma2].

2. THE COMMUTANT OF THE RANGE

In this part of the paper we consider bounded linear operators on a Banach space $\mathcal{E}$, we show that there exist $C$ in $\mathcal{L}(\mathcal{E}) \setminus CI$ such that $\delta_C \delta_{A,B}$ is a derivation if and only if $A + B \in CI$. We use this to compute the commutant, in $\mathcal{L}(\mathcal{E})$, of the range of $\delta_{A,B}$. The first result is an immediate consequence of Theorem 5 below.

**Theorem 1.** Let $\mathcal{E}$ be a Banach space and let $A$ and $B$ be in $\mathcal{L}(\mathcal{E})$. There exist $C$ in $\mathcal{L}(\mathcal{E}) \setminus CI$ so that $\delta_C \delta_{A,B}$ is a derivation if and only if $A + B \in CI$. More precisely, when $A + B \in CI$, the possible choices for $C$ are given by

1. if $A - B \in CI$, then $C$ can be any element of $\mathcal{L}(\mathcal{E}) \setminus CI$
2. if $A - B \notin CI$, then $C = a(A - B) + bI$, where $a, b \in \mathbb{C}$ and $a \neq 0$.

**Remark.** Theorem 1 remains true for standard algebras. A standard algebra is a subalgebra of $\mathcal{L}(\mathcal{E})$ containing all the finite rank operators.

**Corollary 2.** If $A$ and $B$ are in $\mathcal{L}(\mathcal{E})$, then the commutant in $\mathcal{L}(\mathcal{E})$ of the range of $\delta_{A,B}$ is:

1. CI, if $A + B \notin CI$ or if $A + B \in CI$ and either $A - B \in CI \setminus \{0\}$ or $(A - B)^2 \notin CI$;
2. $\mathcal{L}(\mathcal{E})$, if $A + B \in CI$ and $A - B = 0$;
3. $\{ a(A - B) + bI : a, b \in \mathbb{C} \}$, if $A + B \in CI$, $A - B \notin CI$ and $(A - B)^2 \in CI$.

**Proof.** First observe that an element $C$ of $\mathcal{L}(\mathcal{E})$ is in the commutant $(\text{Ran} \delta_{A,B})'$ if and only if $\delta_C \delta_{A,B} = 0$. Hence, if $C$ is in the commutant of the range of $\delta_{A,B}$ then $\delta_C \delta_{A,B}$ is a derivation. The proof of the corollary is divided into three cases.

1. $A + B \notin CI$. In this case Theorem 1 implies any $C$ in the commutant of the range of $\delta_{A,B}$ is in CI. Clearly, CI is a subset of the commutant.
2. $A + B \in CI$ and $A - B \in CI$. In this case $A = ai$ and $B = bi$, hence $\delta_{A,B}(X) = (a - b)X$ for all $X$ in $\mathcal{L}(\mathcal{E})$, the remaining details are left for the reader.
3. $A + B \in CI$ and $A - B \notin CI$. If follows from Theorem 1 that

$$(\text{Ran} \delta_{A,B})' \subseteq \{ a(A - B) + bI : a, b \in \mathbb{C} \}.$$
It follows from (4), (5) and (7) below that \( A = aC + bI \) and \( B = -aC + cI \) for some scalars \( a, b, \) and \( c \), hence we have

\[
(Ran \delta_{A,B})' \ni C \text{ iff } \\
\delta_C \delta_{A,B} = \delta_{aC^2 + (b-c)C} = 0 \text{ iff } \\
aC^2 + (b - c)C \in CI \text{ iff } \\
(2aC)^2 + 2(b-c)2aC \in CI \text{ iff } \\
(A - B - (b-c)I)^2 + 2(b-c)(A - B - (b-c)I) \in CI \text{ iff } \\
(A - B)^2 \in CI.
\]

Again a few easy details are left for the reader. \( \square \)

3. APPLICATIONS TO C*-ALGEBRAS

Let \( \mathfrak{A} \) be a C*-algebra, and let \( A \) and \( B \) be in the multiplier algebra \( M(\mathfrak{A}) \) of \( \mathfrak{A} \). Let \( \pi \) be the reduced atomic representation of \( \mathfrak{A} \), that is, \( \pi = \sum_{t \in \hat{\mathfrak{A}}} \pi_t \) on the Hilbert space \( \mathcal{H} = \sum_{t \in \hat{\mathfrak{A}}} \mathcal{H}_t \). It is well known that \( \pi \) is faithful, and that the double commutant, \( \pi(\mathfrak{A})'' \), of \( \pi(\mathfrak{A}) \) satisfies

\[
\pi(\mathfrak{A})'' = \prod_{t \in \hat{\mathfrak{A}}} \mathcal{L}(\mathcal{H}_t).
\]

See, for example, [Pe] for more details about the reduced atomic representation. Let \( C \) in the enveloping von Neumann algebra of \( \mathfrak{A} \) be so that \( \delta_C \delta_{A,B} \) is a derivation of \( \mathfrak{A} \), and pick \( D \) in the enveloping von Neumann algebra, so that \( \delta_C \delta_{A,B} = \delta_D \). Then for \( X \) in \( \mathfrak{A} \) we have

\[
(AX - XB) - (AX - XB)C = DX - XD.
\]

For \( t \in \hat{\mathfrak{A}} \) we can apply \( \pi_t \) to (1) and get

\[
\pi_t(C)(\pi_t(A)\pi_t(X) - \pi_t(X)\pi_t(B)) \\
- (\pi_t(A)\pi_t(X) - \pi_t(X)\pi_t(B))\pi_t(C) \\
= \pi_t(D)\pi_t(X) - \pi_t(X)\pi_t(D).
\]

If \( \pi_t(C) \) is not a scalar multiple of the identity on \( \mathcal{H}_t \), then we can apply Theorem 1 to the operators \( \delta_{\pi_t(C)} \) and \( \delta_{\pi_t(A),\pi_t(B)} \) to conclude that \( \pi_t(A + B) = \pi_t(A) + \pi_t(B) \) is a scalar times the identity on \( \mathcal{H}_t \). It follows from the faithfulness of the reduced atomic representation that \( A + B \) is a scalar times the identity in \( M(\mathfrak{A}) \). It is clear that if \( \pi_t(C) \) is a scalar multiple of the identity on \( \mathcal{H}_t \), then \( \pi_t(A) \) and \( \pi_t(B) \) can be arbitrary operators on \( \mathcal{H}_t \). We have shown:

**Corollary 3.** If \( A, B \) and \( C \) are operators in the multiplier algebra \( M(\mathfrak{A}) \) of some C*-algebra \( \mathfrak{A} \), then \( \delta_C \delta_{A,B} \) is a derivation on \( \mathfrak{A} \) if and only if, for each \( t \in \hat{\mathfrak{A}} \), either \( \pi_t(A + B) \) or \( \pi_t(C) \) is a scalar multiple of the identity on \( \mathcal{H}_t \).

Immediate consequences of Corollary 3 are the known results that if \( \delta \) is a derivation on a C*-algebra so that \( \delta \delta \) also is a derivation, then \( \delta = 0 \); and if \( \delta_1, \delta_2, \) and \( \delta_1 \delta_2 \) all are derivations on a C*-algebra, then \( \delta_1 \delta_2 = 0 \).
Corollary 4. Let $\mathfrak{A}$ be a $C^*$-algebra and let $A$ and $B$ be in the multiplier algebra $M(\mathfrak{A})$ of $\mathfrak{A}$. Then $(\text{Ran } \delta_{A,B})' \neq CI$ if and only if there is an irreducible representation $\pi$ of $\mathfrak{A}$ so that either $\pi(A) = \pi(B) \in CI$ or $\pi(A + B) \in CI$, $\pi(A - B) \notin CI$ and $\pi(A - B)^2 \in CI$.

Proof. The proof is based on Corollary 2 instead of Theorem 1, otherwise it is similar to the proof of Corollary 3. $\square$

When defining $\delta_{A,B}$ on a $C^*$-algebra $\mathfrak{A}$ is it not really necessary that $A$ and $B$ are in the multiplier algebra $M(\mathfrak{A})$. We only need that $A$ and $B$ are in the enveloping von Neumann algebra and that $\delta_{A,B}$ maps $\mathfrak{A}$ into itself.

4. THE PRODUCT ON $\mathcal{L}(\mathcal{E})$

In this section we investigate when the product of two generalized derivations again is a generalized derivation. In particular we prove the following generalization of Theorem 1.

Theorem 5. Let $A$, $B$, $C$, and $D$ be bounded operators on a Banach space $\mathcal{E}$.

1. If $A \notin CI$ and $B \notin CI$, then $\delta_{C,D}\delta_{A,B}$ is a generalized derivation if and only if $C = aA + cI$ and $D = -aB + dI$ for some scalars $a$, $c$, and $d$;
2. If $A \in CI$ and $B \notin CI$, then $\delta_{C,D}\delta_{A,B}$ is a generalized derivation if and only if $C \in CI$;
3. If $A \notin CI$ and $B \in CI$, then $\delta_{C,D}\delta_{A,B}$ is a generalized derivation if and only if $D \in CI$;
4. If $A \in CI$ and $B \in CI$, then $\delta_{C,D}\delta_{A,B}$ is a generalized derivation.

Proof. We will only prove part 1, the proofs of parts 2 and 3 are similar, but simpler, and part 4 is trivial. Suppose $A$, $B \notin CI$ and $\delta_{C,D}\delta_{A,B} = \delta_{E,F}$. Let $G = CA - E$ and $H = BD - F$; then

$$CXBy + AXDy = GXy - XHy$$

for all $X \in \mathcal{L}(\mathcal{E})$. Consider the element $X = x \otimes f$ of $\mathcal{L}(\mathcal{E})$ determined by $Xy = f(y)x$. Then (2) becomes

$$f(By)Cx + f(Dy)Ax = f(y)Gx - f(Hy)x$$

for all $x$ and $y$ in $\mathcal{E}$ and all $f \in \mathcal{E}'$. Since $B \notin CI$ we can find $y$ in $\mathcal{E}$ so that $y$ and $By$ are linearly independent, hence we can pick $f$ in $\mathcal{E}'$ so that $f(y) = 0$ and $f(By) = -1$. Using these choices in (3) we see that $Cx = f(Dy)Ax + f(Hy)x$ for all $x$ in $\mathcal{E}$. It follows that

$$C = aA + cI$$

for some scalars $a$ and $c$. Considering the adjoint of (2) it follows similarly that

$$D = bB + dI$$

for some scalars $b$ and $d$. It remains to show that $a + b = 0$. Let $K = G - dA$ and $L = H + cB$. Substituting (4) and (5) into (2) we get

$$(a + b)AXBy = KXY - XLy$$

for all $y$ in $\mathcal{E}$ and all $X \in \mathcal{L}(\mathcal{E})$. If $y$ and $f$ are so that $f(y) = 1$ and $f(By) = 0$, then letting $X = x \otimes f$ in (6) we get $K \in CI$. Similarly, considering adjoints, we
get $L \in CI$. Hence, if $X = x \otimes f$ and $f(y) = 0$, and $f(By) = 1$ then it follows from (6) that $(a + b)A \in CI$. Since, $A \not\in CI$ we conclude that

$$a + b = 0.$$  

(7)

The converse is trivial. \hfill \Box

**Remark.** It is easy to obtain a variety of special cases of Theorem 5 by asking when $\delta_{C,D}\delta_{A,B} = \delta_{E,F}$ and one or more of the following hold: $A = B$, $C = D$, or $E = F$. For example

1. if $\delta_{C,C}\delta_{A,A} = \delta_{E,F}$, then $C \in CI$ or $A \in CI$ and $E = F \in CI$ (this generalization of Posner's theorem was also observed in [Br]);
2. if $\delta_{C,C}\delta_{A,B} = \delta_{E,F}$, then $A + B \in CI$ and $E = F$ (this is a generalization of Theorem 1 above);
3. if $\delta_{C,D}\delta_{A,A} = \delta_{E,F}$, then $C + D \in CI$ and $E = F$.

As in section 3 one can extend the results in this section to obtain a characterization of when the product of two generalized derivations of a $C^*$-algebra is a generalized derivation.

**REFERENCES**


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