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PERTURBATION OF PERIODIC BOUNDARY CONDITIONS*

LAWRENCE TURYN†

Abstract. We consider perturbations of the problem $(*) -x'' + bx = \lambda ax$, $x(0) - x(1) = 0 = x'(0) - x'(1)$ both by changes of the boundary conditions and by addition of nonlinear terms. We assume that at $\lambda = \lambda_0$ there are two linearly independent solutions of the unperturbed problem $(*)$ and that $a(\cdot)$ is bounded away from zero. When only the boundary conditions are perturbed either the Hill's discriminant or the method of Lyapunov-Schmidt reduces the problem to $0 = \det((\lambda - \lambda_0)A - \varepsilon H) + \text{higher order terms}$, where A and H are real 2×2 constant matrices. We analyse the existence of curves $(\lambda(\varepsilon), \varepsilon)$ of eigenvalues for this problem of linear perturbation and give as an example a heat problem with $H = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

The method of Lyapunov-Schmidt is used to analyse the full nonlinear problem. In a sequel to this paper we will analyse the bifurcation problem from a "generic" point of view and we will present some numerical examples.

1. Introduction. We consider boundary value problems which are perturbations of a linear boundary value problem with periodic boundary conditions. The first such problem we consider, in §2, is

$$\begin{aligned} x'' + (\lambda a - b)x &= 0, \\ x(0) - x(1) &= \varepsilon \text{ (terms linear in } x(1), x'(1)), \\ x'(0) - x'(1) &= \varepsilon \text{ (terms linear in } x(1), x'(1)). \end{aligned}$$

For this linear boundary value problem with a parameter ε we establish a condition for the local splitting of a double eigenvalue λ_0 for $\varepsilon = 0$ into two curves $\lambda = \lambda^*(\varepsilon)$ for $\varepsilon \neq 0$. This condition can be established either using the Hill's discriminant or the method of Lyapunov-Schmidt, and the boundary conditions can be permitted to be nonself-adjoint for $\varepsilon \neq 0$. For problems where the boundary conditions remain self-adjoint for $\varepsilon \neq 0$ some classical perturbation results of Rellich can be applied to the above situation, even though the differential operator has domain varying with ε .

In §3 we consider nonlinear perturbations, i.e.

$$x'' + (\lambda a - b)x = (\text{nonlinear function of } x),$$

with the boundary conditions also having terms nonlinear in x . When the linearisation has for $\varepsilon = 0$ a double eigenvalue λ_0 the method of Lyapunov-Schmidt is used to reduce the problem to a system of two equations in four unknowns, $\mathbf{u} \in \mathbb{R}^2$, $\lambda \in \mathbb{R}$, $\varepsilon \in \mathbb{R}$.

2. Linear perturbation of the periodic boundary value problem. In this section we consider the boundary value problem

$$(2.1) \quad \tau x = \lambda ax,$$

$$(2.2) \quad Mx = \varepsilon N(\varepsilon)x$$

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where $\tau x = -x'' + b(t)x$, $' = d/dt$, $0 \leq t \leq 1$, $b(\cdot)$ and $a(\cdot)$ are continuous on $[0, 1]$, $a(t) \geq a_0 > 0$ for $0 \leq t \leq 1$, $Mx = (x(0) - x(1), x'(0) - x'(1))^T$, T denoting transpose, and $N(\epsilon)x = (H + \epsilon \bar{H} + O(\epsilon^2))(x(1), x'(1))^T$, where $H, \bar{H}$ are 2×2 real matrices, $H = (h_{ij})_{i,j=1,2}$. We will assume that $H \neq 0$, ϵ is a real parameter, and that $N(\epsilon)$ is real-valued and two times continuously differentiable.

When $\epsilon = 0$ (2.2) is called the periodic boundary conditions. Since (2.1)–(2.2) is linear in x we call this problem a linear perturbation of the periodic boundary value problem, although the periodic boundary conditions are perturbed by terms nonlinear in ϵ .

Denote by $X(t, \lambda)$ the principal fundamental matrix of solutions of (2.1), i.e. that matrix satisfying

$$X(0, \lambda) = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and the differential equation $X' = A(t, \lambda)X$, where

$$A(t, \lambda) = \begin{pmatrix} 0 & 1 \\ b(t) - \lambda a(t) & 0 \end{pmatrix}.$$

Since $\text{tr} A(t, \lambda) = 0$ for all t, λ it follows that $\det X(t, \lambda) = 1$ for all t, λ , in particular that $\det X(1, \lambda) = 1$ for all λ . Also, $X(1, \cdot)$ is analytic; see, for example, Hale [8, p. 82].

Define

$$\Delta(\lambda, \epsilon) = \det(I - (I + \epsilon N(\epsilon))X(1, \lambda)).$$

This function is analytic in λ and three times continuously differentiable in ϵ . Given λ, ϵ the linear boundary value problem (2.1)–(2.2) has (a) no nontrivial solutions when $\Delta(\lambda, \epsilon) \neq 0$, (b) at least one linearly independent solution when $\Delta(\lambda, \epsilon) = 0$, (c) exactly two linearly independent solutions when $I - (I + \epsilon N(\epsilon))X(1, \lambda) \equiv 0$.

For $\epsilon = 0$ the hypothesis $a(t) \geq a_0 > 0$ assures that (2.1)–(2.2) has an infinity of eigenvalues, i.e. values of λ for which there is at least one linearly independent solution. For this fact, see Birkhoff [1], Coddington and Levinson [6], or Magnus and Winkler [11]. Let us assume henceforth in this section that we are given an eigenvalue λ_0 for (2.1)–(2.2) at $\epsilon = 0$. We will examine the question of the existence of a curve or curves $\lambda^*(\epsilon)$ of eigenvalues for (2.1)–(2.2) passing through the point $(\lambda_0, 0)$.

Let $D_\lambda = \partial/\partial\lambda$, $D_\epsilon = \partial/\partial\epsilon$. From Birkhoff [1] or Magnus and Winkler [11] it is known that (2.1)–(2.2) at $(\lambda_0, 0)$ has (a) exactly one linearly independent solution when $\Delta(\lambda_0, 0) = 0 \neq D_\lambda \Delta(\lambda_0, 0)$, (b) exactly two linearly independent solutions when $\Delta(\lambda_0, 0) = 0 = D_\lambda \Delta(\lambda_0, 0)$. So, if at $(\lambda_0, 0)$ there is exactly one linearly independent solution then the implicit function theorem implies that there is a unique curve $(\lambda^*(\epsilon), \epsilon)$ in $\mathbb{R}^2$ passing through $(\lambda_0, 0)$ with $\Delta(\lambda^*(\epsilon), \epsilon) = 0$.

So let us consider case (b), i.e., $\Delta(\lambda_0, 0) = 0 = D_\lambda \Delta(\lambda_0, 0)$. By suitably modifying the calculations of Magnus and Winkler [11, p. 18] it follows that $\rho \stackrel{\text{def}}{=} D_{\lambda\lambda} \Delta(\lambda_0, 0) = 2(\sigma_{11}\sigma_{22} - \sigma_{12}\sigma_{21})$ where

$$\sigma_{ij} \stackrel{\text{def}}{=} \int_0^1 a(t) x_i(t, \lambda_0) x_j(t, \lambda_0) dt, \quad X(t, \lambda) = \begin{pmatrix} x_1(t, \lambda) & x_1(t, \lambda) \\ x_1'(t, \lambda) & x_2'(t, \lambda) \end{pmatrix}.$$

Since $\sigma_{12} = \sigma_{21}$ and $a(t) \geq a_0 > 0$, the Schwarz inequality and the linear independence of $x_1(\cdot, \lambda_0)$, $x_2(\cdot, \lambda_0)$ imply that $\rho > 0$. We will need some further notation:

$$\Xi \stackrel{\text{def}}{=} D_\lambda X(1, \lambda_0) = (\xi_{ij})_{i,j=1,2} \quad \text{and} \quad \Sigma = (\sigma_{ij})_{i,j=1,2}.$$

Again, by suitably modifying the calculations of Magnus and Winkler [11, p. 18] one can conclude that $\Xi = J\Sigma$ where $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Let $\sigma = \det \Sigma$. We see then that $\rho = 2\sigma = 2 \det \Sigma = 2 \det \Xi$.

Let $\nu = \lambda - \lambda_0$. Since $X(1, \lambda) = I + \nu \Xi + O(\nu^2)$, one can calculate that

$$\begin{aligned} \Delta(\lambda, \varepsilon) &= \det(I - (I + \varepsilon H)(I + \nu \Xi)) + O((|\varepsilon| + |\nu|)^3), \\ &= \nu^2 \cdot \det \Xi + \varepsilon \nu \cdot \gamma + \varepsilon^2 \cdot \det H + O((|\varepsilon| + |\nu|)^3) \end{aligned}$$

where

$$\begin{aligned} \gamma &\stackrel{\text{def}}{=} h_{11}\xi_{22} + h_{22}\xi_{11} - h_{12}\xi_{21} - h_{21}\xi_{12} \\ &= -h_{11}\sigma_{12} + h_{22}\sigma_{12} + h_{12}\sigma_{11} - h_{21}\sigma_{22}, \end{aligned}$$

since $\Xi = J\Sigma$. The validity of this asymptotic expansion follows from $\Delta(\lambda, \varepsilon)$ being analytic in λ and three times continuously differentiable in ε . Let $\delta = \det H$. The question of existence of a curve or curves passing through $(\nu, \varepsilon) = (0, 0)$, i.e. $(\lambda, \varepsilon) = (\lambda_0, 0)$ is thus equivalent to the question of the existence of solutions to the equation

$$(2.3) \quad 0 = \Delta = \sigma \nu^2 + \gamma \varepsilon \nu + \delta \varepsilon^2 + O((|\varepsilon| + |\nu|)^3).$$

This can be re-written directly as

$$(2.3') \quad 0 = \Delta = \det(\nu \Sigma - \varepsilon JH) + O((|\varepsilon| + |\nu|)^3).$$

The fact that the terms of second degree of Δ are equal to $\det(\nu \Sigma - \varepsilon JH)$ will also be derived, quite independently, in §3. There the method of Lyapunov-Schmidt will be used to find the bifurcation equations when the boundary value problem (2.1)–(2.2) is also subjected to nonlinear perturbation. When the nonlinear perturbation is taken to be identically zero, the bifurcation equations reduce to

$$(2.4) \quad (\nu \Sigma - \varepsilon JH)u = O((|\varepsilon| + |\nu|)^2 |u|)$$

where $u \in \mathbb{R}^2$, $|u| = |u_1| + |u_2|$. System (2.4) has solutions $u \neq 0$ if and only if

$$0 = \det(\nu \Sigma - \varepsilon JH) + O((|\varepsilon| + |\nu|)^3).$$

Thus we see that the terms of second degree of Δ can be found by the method of Lyapunov-Schmidt just as well as by calculation of the Hill's discriminant.

THEOREM 2.1. *Assume that at $(\lambda_0, 0)$ there are two linearly independent solutions of (2.1)–(2.2). If $\gamma^2 - 4\delta\sigma > 0$ then there are two distinct continuously differentiable curves of eigenvalues $\lambda = \lambda_0 + \nu^*(\varepsilon)$ for all $|\varepsilon|$ sufficiently small, with $\nu^*(0) = 0$.*

Proof. This follows from (2.3) and $\sigma > 0$. $\square$

In the above work no assumption was made about the self-adjointness of the boundary conditions for $\varepsilon \neq 0$. In fact we will give self-adjointness a privileged position only when applying the classical perturbation results of Rellich in Theorem 2.7 and in the following paragraphs.

From Coddington and Levinson [6, p. 297] it is known that (2.2) is self-adjoint if and only if

$$1 = \det(I + \varepsilon N(\varepsilon)) = 1 + \varepsilon \operatorname{tr} H + \varepsilon^2 (\det H + \operatorname{tr} \bar{H}) + O(|\varepsilon|^3).$$

So, self-adjointness of the boundary conditions requires at least that $\text{tr} H = 0$. When $\text{tr} H = 0$, the 2×2 real matrix JH appearing in (2.4) is Hermitian. So, we see that self-adjointness of the boundary conditions implies self-adjointness of the matrix in (2.4) corresponding to the term of lowest order in ε .

Remark 2.2. When $\text{tr} H = 0$, $\gamma^2 - 4\delta\sigma \geq 0$.

Proof. Σ is always positive definite and self-adjoint because $a(t) \geq a_0 > 0$ by assumption. When $\text{tr} H = 0$, JH is self-adjoint. It follows that the eigenvalues β_1 of the generalised eigenvalue problem

$$(2.5) \quad JH\mathbf{u} = \beta_1 \Sigma \mathbf{u} \in \mathbb{C}^2$$

are real. Equivalent to β_1 being an eigenvalue is $0 = \det(\beta_1 \Sigma - JH)$; reality of the eigenvalues β_1 implies $\gamma^2 - 4\delta\sigma \geq 0$. $\square$

So we see that when $\text{tr} H = 0$ the double eigenvalue λ_0 will usually split into two smooth curves of real eigenvalues for $\varepsilon \neq 0$; the exceptional case would be when $\gamma^2 - 4\delta\sigma = 0$.

Example 2.3. When $a \equiv 1$, $b = 0$, $\text{tr} H = 0$, and $\lambda_0 = 4\pi^2 n^2$ for some positive integer n , we calculate $\sigma_{12} = 0$ and $\gamma^2 - 4\delta\sigma = \frac{1}{4}((h_{12} + \lambda_0^{-1} h_{21})^2 + \lambda_0^{-1} h_{11}^2) \geq 0$ since $H \neq 0$ for nontriviality. It follows that Theorem 2.1 is applicable, except in the exceptional case $h_{11} = h_{22} = 0$, $h_{12} + \lambda_0^{-1} h_{21} = 0$.

As a specific sub-example, take $a \equiv 1$, $b \equiv 0$, $\lambda_0 = 4\pi^2 n^2$ for some $n \geq 1$, and $H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Theorem 2.1 is applicable; in fact, one can calculate explicitly $\Delta(\lambda, \varepsilon) = \varepsilon^2 - 2(1 - \cos \lambda^{1/2})$ for $\lambda > 0$. Explicitly, the curves are $\lambda = (\lambda_0^{1/2} \pm 2 \arcsin(\varepsilon/2))^2$, which clearly cease to exist for $|\varepsilon| > 2$. Note that the corresponding boundary conditions (2.2) are not self-adjoint for $\varepsilon \neq 0$, since $\det(I + \varepsilon H) = 1 - \varepsilon^2$.

Application 2.4. Consider a ring of metal obtained by joining the endpoints $\xi = 0$, $\xi = 1$. If the joining is not perfect then there will be some "contact resistance". Appropriate boundary conditions for the temperature $v(\xi)$ are then (see Özişik [12, p. 283] or Carslaw and Jaeger [3, p. 23])

$$u(0) - u(1) = \varepsilon u'(1),$$

$$u'(0) - u'(1) = 0,$$

where $\varepsilon = k/h = (\text{Biot number})^{-1}$ can be taken to be small and positive if the heat transfer coefficient h is large. These boundary conditions have as a consequence a temperature drop across the join, this phenomenon being well known in practise. See Holman [10, pp. 45–48] for more details on the causes of contact resistance. These boundary conditions are self-adjoint for all ε , with $H = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Since $\gamma = \sigma_{11} > 0$, Theorem 2.1 guarantees the existence of two curves $\lambda = \lambda_0 + \nu_{\pm}^*(\varepsilon)$. In fact, for the example $a \equiv 1$, $b \equiv 0$ one can see that $\nu_{\pm}^*(\varepsilon) = 0$ for all ε , since the second row of H is trivial, and further one can calculate that $D_{\varepsilon} \nu^*(0) = -2\lambda_0$.

It is useful to consider further the generalised eigenvalue problem (2.5). A *simple eigenvalue* β_1^0 for a pencil $(L_2; L_1)$ of two $n \times n$ matrices is a value of β_1 for which $\dim \mathcal{N}(A) = 1 = \text{codim } \mathcal{R}(A)$ and $L_1 \mathbf{z} \notin \mathcal{R}(A)$ where $A = L_2 + \beta_1^0 L_1$ and $\mathbf{0} \neq \mathbf{z} \in \mathcal{N}(A)$. The generalisation of the concept of simple eigenvalue to Banach space operators $(B; A_1, \dots, A_N)$ originated in Hale [9]. Bibliographic references and extensive material on the use of simple eigenvalues in the analysis of linear and nonlinear problems can be found in Chow and Hale [4].

Remark 2.5. If $n \times n$ matrices L_2, L_1 are Hermitian and L_1 is definite then β_1^0 is simple for $(L_2; L_1)$ whenever $\dim \mathcal{N}(L_2 + \beta_1^0 L_1) = 1$.

Proof. Let $A = L_2 + \beta_1^0 L_1$ and $0 \neq z \in \mathcal{U}(A)$. The hypotheses imply that $\beta_1^0 = -z^* L_2 z / z^* L_1 z$ is real, so that A is also Hermitian. The Fredholm alternative implies that $\mathcal{R}(A) = \{a \in \mathbb{C}^n : z^* a = 0\}$; since L_1 is definite, $z^* L_1 z \neq 0$, so that $L_1 z \notin \mathcal{R}(A)$.

□

With this background we can return to (2.5). If the discriminant $\gamma^2 - 4\delta\sigma > 0$, then whenever $0 = \det(-JH + \beta_1^0 \Sigma)$ necessarily $\dim \mathcal{U}(-JH + \beta_1^0 \Sigma) = 1$. This and Remark 2.5 prove

Remark 2.6. If $\text{tr } H = 0$ and $\gamma^2 - 4\delta\sigma > 0$ then there are exactly two eigenvalues of $(-JH; \Sigma)$ and both are simple.

The simplicity of the eigenvalues of $(-JH; \Sigma)$ is required for the application of the results of Chow and Hale [4, Chap. 7] for nonlinear bifurcation problems. In a sequel to this present paper we will consider such problems.

Remark. Given enough differentiability in ϵ for the original problem (2.1)–(2.2), Newton's polygon helps one to calculate solution(s) of $\Delta = 0$. Consider the case $\gamma = 0 = \delta$: If $D_\lambda D_\epsilon^k \Delta(\lambda_0, 0) \neq 0$ or $D_\epsilon^k \Delta(\lambda_0, 0) \neq 0$ for some $k \geq 3$ then Newton's polygon guarantees the existence of a curve of the approximate form $\lambda \sim \lambda_0 + c\epsilon^p$ for some $p \geq \frac{3}{2}$. Of course, this assumes a sufficient amount of differentiability (in ϵ) of the boundary conditions.

Yet another approach, besides those utilising the Hill's discriminant or the method of Lyapunov–Schmidt, is to set problem (2.1)–(2.2) in a Hilbert space. This approach for the self-adjoint case will use the now-classic method of Rellich [13].

Let H be the Hilbert space of Lebesgue measurable functions $x: [0, 1] \rightarrow \mathbb{C}$ constructed by completion of $C[0, 1]$ with respect to the weighted inner product $(x, y) = \int_0^1 ax\bar{y}$. Define operators $T(\epsilon): \mathcal{U}(\epsilon) \subset H \rightarrow H$ by $T(\epsilon)x = (a^{-1}\tau - \lambda_0)$ on the domains $\mathcal{U}(\epsilon) = \{x \in H: T(\epsilon)x \in H, x \text{ satisfies boundary conditions (2.2)}\}$. We will assume the self-adjointness of the boundary conditions, i.e. $1 = \det(I + \epsilon H + \epsilon^2 \bar{H} + \dots)$, from which it follows that the operators $T(\epsilon)$ on $\mathcal{U}(\epsilon)$ are symmetric.

Let $X(t, \lambda)$ denote the fundamental matrix for the differential equation (2.1) rewritten as a system. One can show that if the matrix $S_\lambda \stackrel{\text{def}}{=} X(1, \lambda) - I$ is invertible then the equation $(T(\epsilon) + (\lambda - \lambda_0)I)x = p \in H$ has the unique solution

$$x_\epsilon(t) = \int_0^t [-x_1(t)x_2(s) + x_2(t)x_1(s)] p(s) \\ + (x_1(t), x_2(t)) [S_\lambda - E_{\epsilon, \lambda}]^{-1} E_{\epsilon, \lambda} \begin{pmatrix} -\int_0^1 x_2 p \\ \int_0^1 x_1 p \end{pmatrix},$$

where $E_{\epsilon, \lambda} = \epsilon N(\epsilon) X(1, \lambda)$, for all $|\epsilon|$ sufficiently small. From study of Hill's equation one knows that in fact S_λ is invertible at all but discrete and isolated values of $\lambda \in \mathbb{R}$. In particular, $S_{\lambda_0 \pm i}$ is invertible, hence the operators $T(\epsilon)$ are self-adjoint.

Therefore, from Rellich [13, pp. 71–72] we can conclude that $T(\epsilon)$ on $\mathcal{U}(\epsilon)$ is a so-called regular family of self-adjoint operators. The next result follows from Rellich [13, p. 74].

THEOREM 2.7. Assume that at $(\lambda_0, 0)$ there are two linearly independent solutions of (2.1)–(2.2). If the boundary conditions (2.2) are self-adjoint for all ϵ then there are two (counting multiplicity) real continuous curves $\lambda = \lambda_0 + \nu^*(\epsilon)$ of eigenvalues for (2.1)–(2.2) with $\nu^*(0) = 0$. The case where there is a curve of double eigenvalues is not precluded.

We remark that one could allow the parameter ϵ into the linear differential equation (2.1) without substantially altering any of the discussion in §2. The same

cannot be said for allowing λ into the linear boundary conditions (2.2). Perturbation of problems with λ in separated boundary conditions has been considered in [17], and ε in the differential equation appeared in [16].

3. Nonlinear perturbation and Lyapunov-Schmidt. In this section we consider the boundary value problem

$$(3.1) \quad \tau x = \lambda a x + f_0(\varepsilon, \lambda; t, x, x'),$$

$$(3.2) \quad Mx = \varepsilon N(\varepsilon)x + f(\varepsilon, \lambda; x),$$

where ε, a, M , and $N(\varepsilon)$ are as in §2 and $f(\lambda, x) \in \mathbb{R}^2$ represents nonlinear contributions to the boundary conditions. Assume that both f_0 and f are $O(|\alpha|^2 \|x\| + \|x\|^n)$ for some integer $n \geq 2$ as $|\alpha|, \|x\| \rightarrow 0$, where $\nu = \lambda - \lambda_0$, $\alpha = (\varepsilon, \nu) \in \mathbb{R}^2$, $|\alpha| = |\varepsilon| + |\nu|$, $\|x\| = |x|_\infty + |x'|_\infty + |x''|_\infty$, $|x|_\infty = \sup_{0 \leq t \leq 1} |x(t)|$, the estimate on f_0 holding uniformly for $t \in [0, 1]$. Here the symbol $O(s)$ for $s \in \mathbb{R}^+$ denotes any quantity $F(s)$ which satisfies $F(s)/s \rightarrow \text{constant}$ as $s \rightarrow 0+$. The term f may include things like $\int_0^1 w(t)x^2(t)dt$, but we do assume that $f_0(\cdot, \cdot; t, \cdot, \cdot), f(\cdot, \cdot, \cdot): (-\eta, \eta) \times (\lambda_0 - \eta, \lambda_0 + \eta) \times C^2[0, 1] \times C^1[0, 1] \rightarrow \mathbb{R}^2$ is $(n+1)$ times continuously differentiable for some $\eta > 0$, this being true uniformly in $t \in [0, 1]$ for f_0 . Let us assume that for $\lambda = \lambda_0$, $\varepsilon = 0 \equiv f_0 \equiv f$ there are two linearly independent solutions of (3.1)–(3.2), as was assumed in the latter part of §2. As before, let $X(t, \lambda)$ denote the principal fundamental matrix of solutions.

To pose (3.1)–(3.2) as a bifurcation problem it will help to write that problem as a nonlinear equation in Banach spaces $\bar{Y} = C^2[0, 1]$, $Z = C[0, 1] \times \mathbb{R}^2$ with norms $\|x\|$, as above, and $\|(v; c, d)\| = |v|_\infty + |c| + |d|$, respectively. Then (3.1)–(3.2) is equivalent to the abstract problem

$$(3.3) \quad (B - \nu A - \varepsilon C)x = G(x) + O(|\alpha|^2 \|x\| + |\alpha| \|x\|^n + \|x\|^{n+1})$$

where $x \in \bar{Y}$, $Bx = (\tau x - \lambda_0 a(\cdot)x; Mx)$, $Ax = (a(\cdot)x, 0)$, $Cx = (0; H(x(1), x'(1))^T)$, and $G(x) = (D_\lambda f_0(0, \lambda_0; \cdot, x, x'), D_\lambda f(0, \lambda_0; x))$. Note that $G(x) = O(\|x\|^n)$ as $\|x\| \rightarrow 0$.

Recall that $x_1 = x_1(\cdot, \lambda_0)$, $x_2 = x_2(\cdot, \lambda_0)$ are linearly independent solutions of the linearisation of (3.3). One can show that $\mathcal{R}(B) = \{(v; c, d): l_j(v; c, d) = 0 \text{ for } j = 1, 2\}$ where $l_1(v; c, d) = -d + \int_0^1 v x_1$, $l_2(v; c, d) = c + \int_0^1 v x_2$ are linear functionals on Z . Setting $z_1 = (x_1; c_1, 0)$, $z_2 = (x_2; 0, c_2)$, $\alpha_j = \int_0^1 x_j^2$, one can define a projection $Q: Z \rightarrow \mathcal{R}(B)$ by

$$Qz = z - \alpha_1^{-1} l_1(z) z_1 - \alpha_2^{-1} l_2(z) z_2.$$

The constants c_1, c_2 must be chosen in such a way as to assure that $l_j(Qz) = 0$ for $j = 1, 2$ for all $z \in Z$, and this is equivalent to requiring $l_1(z_2) = 0 = l_2(z_1)$, a sort of Gram-Schmidt manipulation. One sees then that $c_2 = \int_0^1 x_1 x_2 = -c_1$ satisfy this requirement. Further, let us define a projection $P: \bar{Y} \rightarrow Y_0 = (\text{linear span of } x_1, x_2)$ by $Px = \alpha_1^{-1} (\int_0^1 x x_1) x_1 + \alpha_2^{-1} (\int_0^1 x x_2) x_2$.

The method of Lyapunov-Schmidt consists of replacing (3.3) by the pair of equations

$$(3.4) \quad Q(B - \nu A - \varepsilon C)(Px + (I - P)x) = QG(x) + O(|\alpha|^2 \|x\| + |\alpha| \|x\|^n + \|x\|^{n+1}),$$

$$(3.5)$$

$$(I - Q)(B - \nu A - \varepsilon C)(Px + (I - P)x) = (I - Q)G(x) + O(|\alpha|^2 \|x\| + |\alpha| \|x\|^n + \|x\|^{n+1}).$$

Rewrite $Px = u_1x_1 + u_2x_2$ for real numbers u_1, u_2 . Since $Bx_i = 0$ for $i=1, 2$ and $QB(I-P): \bar{Y} \rightarrow \mathcal{R}(B)$ is a linear operator with bounded inverse, equation (3.4) can be solved by $(I-P)x = w^*(\mathbf{u}, \alpha) \in \bar{Y}$ for all sufficiently small $|\mathbf{u}|, |\alpha|$, where $|\mathbf{u}| = |u_1| + |u_2|$. Furthermore $w^* = O(|\alpha||\mathbf{u}| + |\mathbf{u}|^n)$ as $|\mathbf{u}|, |\alpha| \rightarrow 0$, where n is the same integer n as in the estimate that both f_0 and $\mathbf{f}$ are $O(|\alpha|^2\|x\| + \|x\|^n)$ as $|\alpha|, \|x\| \rightarrow 0$.

Substitute $x = u_1x_1 + u_2x_2 + w^*$ into (3.5) to arrive at the bifurcation equation

$$(3.6) \quad (I-Q)(B - \nu A - \varepsilon C)(u_1x_1 + u_2x_2 + w^*(\mathbf{u}, \alpha)) \\ - (I-Q)G(u_1x_1 + u_2x_2 + w^*) = O(|\alpha|^2|\mathbf{u}| + |\alpha||\mathbf{u}|^{n+1}).$$

Now, $(I-Q)B \equiv 0$ by design of the projection Q . Using the linear independence of z_1, z_2 one can separate (3.6) into a system of two equations, after first multiplying through by -1 :

$$(3.7) \quad (\nu \Sigma - \varepsilon JH)\mathbf{u} + \mathbf{p}(\mathbf{u}) = \mathbf{R}(\alpha, \mathbf{u})$$

where $\Sigma = (\sigma_{ij})_{i,j=1,2}$, $\sigma_{ij} = \int_0^1 ax_i x_j$, $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $\mathbf{p}(\cdot)$ is homogeneous of degree n , $\mathbf{R}(\alpha, \mathbf{u}) = O(|\alpha|^2|\mathbf{u}| + |\alpha||\mathbf{u}|^n + |\mathbf{u}|^{n+1})$, n as above, and H is as in §2 and the definition of the operator C .

As one can see, all of the information in §2 concerning the linear perturbation of the periodic boundary value problem is found also in (3.7). So, Lyapunov-Schmidt for an equation in Banach spaces correctly abstracts the linear problem. The author has shown [14] that the method of Fulton [7], Walter [15], Browne and Sleeman [2] et al., abstracting perturbations of the separated boundary conditions into the Hilbert space $L_2[0, 1] \times \mathbb{C}^2$, fails to preserve the self-adjoint features of perturbation of the boundary value problem. Specifically, the analogue of the operator B for the $L_2[0, 1] \times \mathbb{C}^2$ setting has either (i) $\text{codim } \mathcal{R}(B) \geq 2$ for all $\lambda_0 \in \mathbb{C}$, not just eigenvalues, or (ii) B not self-adjoint. Cases (i), (ii) correspond to different definitions of $\mathcal{D}(B)$; one may recall that in Fulton et al. the dependence of the boundary conditions on a parameter is arranged by an efficacious choice of $\mathcal{D}(B)$. This is a definite distinction between the periodic and separated boundary conditions.

Note added in proof. See also Robert Magnus, *Topological equivalence in bifurcation theory*, in Lecture Notes in Mathematics 799, Springer-Verlag, 1980.

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