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# The expected wet period of finite dam with exponential inputs

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**Abstract**

We use martingale methods to obtain an explicit formula for the expected wet period of the finite dam of capacity $V$, where the amounts of inputs are i.i.d exponential random variables and the output rate is one, when the reservoir is not empty. As a consequence, we obtain an explicit formula for the expected hitting time of either 0 or $V$ and a new expression for the distribution of the number of overflows during the wet period, both without the use of complex analysis.


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*Keywords:* Queue; Finite queue; Busy period; Martingales; Dam; Finite dam; Capacity

## 1. Introduction

In this paper, we consider a finite dam of capacity $V > 0$, where the input process is formed by a compound Poisson process with amounts of inputs which are i.i.d. exponential random variables and the output rate is one, when the reservoir is not empty. We use martingale methods to obtain explicit formulas for the expected wet period and other interesting quantities, including the expected hitting time of either 0 or $V$ and the distribution of the number of overflows during the wet period.

Notice that the results of this paper are also results for the M/M/1 queue whose virtual waiting time is uniformly bounded by a positive constant $V$ (cf. Cohen, 1969). The wet period is called the busy period in queueing theory.

The use of martingale methods for the queue is seen in Rosenkrantz (1983), in which a formula for the Laplace transform of the length of the busy period for the M/G/1 queue ($V = \infty$) is derived. The result on the length of the wet period of the infinite dam

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prior to Rosenkrantz (1983) can be found, for example, in Cohen (1969) or Takacs (1967), but the derivation is a technical argument involving complex analysis.

Let $A_t$ denote the arrival process, a Poisson process with intensity parameter $v > 0$, let $S_1, S_2, \ldots$ be independent inputs distributed exponential with mean $\mu$, and let

$$X_t = S_0 + \sum_{i=1}^{A_t} S_i - t, \tag{1}$$

where $S_0$ has distribution exponential with mean $\mu$ on $[0, V]$ and $V$ with probability $e^{-V/\mu}$.

Then the amount of water in the reservoir at time $t$ is given by

$$Z_t = X_t - \max\left(0, \sup_{0<s\leqslant t} X_s - V\right),$$

if we ignore the initial delay.

Let $\tau$ denote the length of the wet period of the dam

$$\tau = \inf\{t > 0\colon Z_t = 0\}.$$

We use martingale methods to compute the expected length of the wet period, $E(\tau)$. Let $E(\cdot \mid x)$ denote the given expectation $S_0 = x$ and $P(\cdot \mid x)$ likewise denote the given probability $S_0 = x$. Let $T = \inf\{t > 0\colon X_t \notin (0, V]\}$. We prove

**Theorem 1.** *Let* $\mu v \neq 1$. $E(\tau \mid x) = \frac{x p_0^V - \mu p_V^x}{(1-\mu v) p_0^V}$, *where*

$$p_V^a = P(X_T > V \mid a) = \frac{e^{\theta a} - 1}{(e^{\theta V}/\mu v) - 1},$$

$$p_0^a = P(X_T = 0 \mid a) = 1 - p_V^a,$$

*where* $\theta = (1 - \mu v)/\mu$. *Moreover*,

$$E(\tau \mid x) \to \frac{x(2V + 2\mu - x)}{2\mu} \quad \text{as } v \to \frac{1}{\mu}.$$

**Remark.** The unconditional expected length of the wet period can be computed through the double expectation formula $E(\tau) = E(E(\tau \mid S_0))$.

Let $N$ denote the number of overflows during the wet period. As a consequence of the martingale methods which will be used to prove Theorem 1, we obtain the exact distribution of $N$, without the use of complex analysis (see Cohen, 1969, p. 518, for the complex analysis approach).

**Corollary 1.** *Let* $\mu v \neq 1$. *Let* $N$ *denote the number of overflows during the wet period. Then*

$$P(N = 0 \mid x) = p_0^x \quad \text{and} \quad P(N = k \mid x) = p_V^x (p_V^V)^{k-1} p_0^V, \quad \text{for } k = 1, 2, \ldots .$$

*Moreover*, $E(N \mid x) = p_V^x p_0^V / (1 - p_V^V)^2$.

## 2. Proofs

In order to prove Theorem 1 and Corollary 1, we will need the following results.

**Lemma 1.** *Let $X_t$ be as in* (1). *Then the following stochastic processes are martingales*:

$$W_t = \exp\{\theta X_t\}, \quad \text{where } \theta = \frac{1-\mu v}{\mu}, \tag{2}$$

$$U_t = X_t - (\mu v - 1)t. \tag{3}$$

**Proof.** In order to prove that $W_t$ is a martingale, first recall that the moment generating function $M(\alpha)$ of $S_1$ is $1/(1-\mu\alpha)$. A similar argument to that of Rosenkrantz (1983) shows that $Y_t = \exp\{\alpha X_t - v(M(\alpha)-1)t + \alpha t\}$ is a martingale. That $W_t$ is a martingale follows by noting that $W_t$ is the martingale $Y_t$ with $\alpha = \theta$.

To prove that $U_t$ is a martingale, denoted by $\{\mathscr{F}_t\}$ the filtration $\sigma\{U_s: s \leqslant t\}$. Let $s \leqslant t$ and observe

$$\begin{aligned} E(X_t - (\mu v - 1)t \mid \mathscr{F}_s) &= E((X_t - X_s) + X_s - (\mu v - 1)t \mid \mathscr{F}_s) \\ &= (\mu v - 1)(t-s) + X_s - (\mu v - 1)t \\ &= U_s, \end{aligned}$$

using properties of the compound Poisson process.

**Corollary 2.** *Let $\mu v \neq 1$. Let $X_t$ be as in* (1) *and $\theta$ as in* (2). *Let $T = \inf\{t > 0: X_t \notin (0, V]\}$. Then*

$$P(X_T > V \mid x) = \frac{e^{\theta x} - 1}{(e^{\theta V}/\mu v) - 1},$$

$$P(X_T = 0 \mid x) = \frac{e^{\theta V} - \mu v e^{\theta x}}{e^{\theta V} - \mu v},$$

*and*

$$P(X_T > V \mid x) \to \frac{x}{V+\mu} \quad \text{as } v \to \frac{1}{\mu}.$$

**Proof of Corollary 2.** Let $T_n = \min\{T, n\}$.

We begin by proving that $E(T \mid x) < \infty$ (and hence that $T < \infty$ a.s.). Since $U_t$ is a martingale and $T_n$ is a bounded stopping time, we can apply the Optional Stopping Theorem to obtain

$$x = E(U_0 \mid x) = E(U_{T_n} \mid x) = E(X_{T_n} \mid x) - (\mu v - 1)E(T_n \mid x). \tag{4}$$

Moreover,

$$|E(X_{T_n} \mid x) - x| \leqslant \max\{x, V + \mu - x\} \leqslant V + \mu. \tag{5}$$

(See the appendix for a proof that $|E(X_{T_n} \mid x)| \leqslant V + \mu$.) From (4) and (5), we see that

$$E(T_n \mid x) = \frac{E(X_{T_n} \mid x) - x}{\mu v - 1} \leqslant \frac{V+\mu}{|\mu v - 1|}.$$

By Monotone Convergence Theorem and the fact that $T_n$ increases to $T$, we can conclude $E(T\,|\,x)=\lim_n E(T_n\,|\,x)<\infty$.

Since $W_t$ is a martingale and $T_n$ is a bounded stopping time, we can apply the Optional Stopping Theorem to get that

$$e^{\theta x}=E(W_0\,|\,x)=E(W_{T_n}\,|\,x).$$

Moreover,

$$W_{T_{nx}}=e^{\theta X_{T_n}}\leqslant e^{\theta X_T}\quad \text{a.s.}$$

and

$$E(e^{\theta X_T}\,|\,x)\leqslant e^{\theta V}E(e^{\theta S_1}\,|\,x)+1=e^{\theta V}(1-\mu\theta)^{-1}+1,$$

for all $n$. The first inequality in the above line follows from $X_T =^d \xi+V$, as seen in the appendix, where $\xi$ is an independent exponential random variable with mean $\mu$. Since $W_{T_n}\to W_T$ a.s., we apply the Dominated Convergence Theorem to conclude $\lim_n E(W_{T_n}\,|\,x)=E(W_T\,|\,x)$. Hence $e^{\theta x}=E(W_T\,|\,x)$. Using the memoryless property of the exponential distribution we see that

$$E(W_T\,|\,x)=P(X_T=0\,|\,x)+P(X_T>V\,|\,x)\int_V^\infty e^{\theta y}\frac{1}{\mu}e^{-(y-V)/\mu}\,dy.$$

Solving for $P(X_T>V\,|\,x)$ finishes the proof.

**Corollary 3.** *Let $\mu v\neq 1$. Let $T=\inf\{t>0\colon X_t\notin(0,V]\}$. Then*

$$E(T\,|\,x)=\frac{1}{\mu v-1}\left\{(\mu+V)\frac{e^{\theta x}-1}{(e^{\theta V}/\mu v)-1}-x\right\}, \tag{6}$$

*where $\theta=(1-\mu v)/\mu$.*

*Also,*

$$E(T\,|\,x)\to\frac{x(V^2+(2\mu-x)(V+\mu))}{2\mu(\mu+V)}, \tag{7}$$

*as $v\to 1/\mu$.*

**Proof of Corollary 3.** We first apply the Dominated Converegence Theorem to $U_{T_n}$. Now $|U_{T_n}|\leqslant X_T+|\mu v-1|T$ for all $n$ a.s. and

$$E(X_T+|\mu v-1|T|x)\leqslant V+\mu+|\mu v-1|E(T|x)<\infty.$$

Since $U_{T_n}\to U_T$ a.s., we can use the Dominated Convergence Theorem to obtain $\lim_n E(U_{T_n}\,|\,x)=E(U_T\,|\,x)$. Since $U_t$ is a martingale and $T_n$ is a bounded stopping time, we can apply the Optional Stopping Theorem and get $x=E(U_0\,|\,x)=E(U_{T_n}\,|\,x)$, and so $x=E(U_T\,|\,x)$. To finish the proof, compute

$$E(U_T\,|\,x)=P(X_T>V\,|\,x)\int_V^\infty y\frac{1}{\mu}e^{-(y-V)/\mu}\,dy-(\mu v-1)E(T\,|\,x).$$

Integrating and then solving results in

$$E(T\,|\,x)=\frac{1}{\mu v-1}\{(\mu+V)P(X_T>V\,|\,x)-x\}.$$

Combining this with Corollary 2 gives us (6). We prove (7) by taking the limit and using L'Hospital's Rule as needed.

**Proof of Theorem 1.** To prove Theorem 1, we observe that $\tau = T1_{\{X_T=0\}} + (T + \tau \cdot \Theta(V))1_{\{X_T>V\}}$, where $\Theta(V)$ denotes the shift operator for which $(X \cdot \Theta(V))_0 = V$. Using the strong Markov propery, $E(\tau \mid x) = E(T \mid x) + E(\tau \mid V)P(X_T > V \mid x)$. Recall that

$$p_V^a = P(X_T > V \mid a) = \frac{e^{\theta a} - 1}{(e^{\theta V}/\mu v) - 1} \quad \text{and} \quad p_0^a = P(X_T = 0 \mid a) = 1 - p_V^a.$$

Rewrite the above as $E(\tau \mid x) = E(T \mid x) + E(\tau \mid V)p_V^x$ and, similarly, $E(\tau \mid V) = E(T \mid V) + E(\tau \mid V)p_V^V$. Using algebra and applying Corollaries 2 and 3, the proof is completed. □

**Proof of Corollary 1.** It is easy to see that $P(N = 0 \mid x) = P(X_T = 0 \mid x) = p_0^x$. Next, using the strong Markov property and the notation described above,

$$P(N = 1 \mid x) = P(X_T > V, (X \cdot \Theta(V))_T = 0) = p_V^x p_0^V$$

and

$$\begin{aligned} P(N = 2 \mid x) &= P(X_T > V, (X \cdot \Theta(V))_T > V, ((X \cdot \Theta(V) \cdot \Theta(V))_T = 0) \\ &= p_V^x p_V^V p_0^V. \end{aligned}$$

The same technique applies for higher $k$, and $E(N \mid x)$ follows from a straightforward computation. □

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## Appendix

In this section, we prove that $E(X_{T_n} \mid x) \leqslant \mu + V$. Let $\xi$ be an exponential random variable with mean $\mu$. Additionally, we prove that $X_T =^d \xi + V$ on $\{X_T > V\}$.

Let $T_k$= time of the $k$th input (or arrival). Then for $y > V$,

$$P(X_T > y) = P\left(x + S_{A_T} + \sum_{i=1}^{A_{T-}} S_i - T > y\right),$$

$$= \sum_{k=1}^{\infty} P\left(x + S_k + \sum_{i=1}^{k-1} S_i - T_k > y,\ A_T = k\right),$$

$$= \sum_{k=1}^{\infty} P\left(x + S_k + \sum_{i=1}^{k-1} S_i - T_k > y,\ x + S_k + \sum_{i=1}^{k-1} S_i - T_k > V,\right.$$

$$\left. x + \sum_{i=1}^{k-1} S_i - T_k < V\right),$$

$$
\begin{aligned}
&=\sum_{k=1}^{\infty}\int\int_{\Gamma} P(S_k > y+t-s-x,\\
&\qquad\qquad S_k > V+t-s-x)f_k(t)g_{k-1}(s)\,\mathrm{d}t\,\mathrm{d}s,\\
&=\sum_{k=1}^{\infty}\int\int_{\Gamma} P(S_k > y+t-s-x \mid S_k > V+t-s-x)\\
&\qquad\qquad \times P(S_k > V+t-s-x)f_k(t)g_{k-1}(s)\,\mathrm{d}t\,\mathrm{d}s,\\
&=\sum_{k=1}^{\infty}\int\int_{\Gamma} P(\xi > y-V)P(S_k > V+t-s-x)f_k(t)g_{k-1}(s)\,\mathrm{d}t\,\mathrm{d}s
\end{aligned}
$$

using memoryless property

$$
\begin{aligned}
&=\sum_{k=1}^{\infty} P(\xi > y-V)P\left(x+S_k+\sum_{i=1}^{k-1} S_i - T_k > V,\right.\\
&\qquad\qquad\qquad \left. x+\sum_{i=1}^{k-1} S_i - T_k < V\right),\\
&=\sum_{k=1}^{\infty} P(\xi > y-V)P(A_T=k),\\
&=P(\xi > y-V),
\end{aligned}
$$

where $f_k$ is the density function for $T_k$ and $g_{k-1}$ is the density function for $\sum_{i=1}^{k-1} S_i$, and where $\Gamma = \{s>0, t>0\colon x+s-t<V\}$. We also used the independence of $S_k, \sum_{i=1}^{k-1} S_i$, and $T_k$ for each $k$. This proves that $X_T =^d \xi + V$ on $\{X_T > V\}$.

Then we can say $E(X_{T_n} \mid x) \leqslant V+\mu$ by noting that $X_{T_n} \leqslant V$ on $\{X_T = 0\}$, $X_{T_n} \leqslant X_T$ on $\{X_T > V\}$, and

$$\int_V^{\infty} P(X_T > y)\,\mathrm{d}y = \int_V^{\infty} P(\xi > y-V)\,\mathrm{d}y = \mu + V.$$

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