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THE FULL GROUP OF A COUNTABLE MEASURABLE EQUIVALENCE RELATION

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ABSTRACT. We study the group of all “ R -automorphisms” of a countable equivalence relation R on a standard Borel space, special Borel automorphisms whose graphs lie in R . We show that such a group always contains periodic maps of each order sufficient to generate R . A construction based on these periodic maps leads to totally nonperiodic R -automorphisms all of whose powers have disjoint graphs. The presence of a large number of periodic maps allows us to present a version of the Rohlin Lemma for R -automorphisms. Finally we show that this group always contains copies of free groups on any countable number of generators.

1. INTRODUCTION

Measurable equivalence relations play a fundamental role in operator algebras and more generally in understanding group actions. Much of the motivation for studying them comes from their use in representing von Neumann algebras with Cartan subalgebras, the so-called “Feldman-Moore representation” [3, 6]. Two fundamental questions concerning this representation are the following: (1) Does every separable von Neumann algebra have a Cartan subalgebra? (2) Does every countable measurable equivalence relation arise from the action of a freely acting countable discrete group? Both appear to be very difficult questions; this paper studies the automorphism group of a countable measurable equivalence relation in the hope that better understanding of this group will lead to progress on these questions.

A *measurable equivalence relation* on a Borel space X is an equivalence relation R on X where R is a Borel subset of $X \times X$. If $(x, y) \in R$ we write xRy . We write $R(x)$ for the equivalence class containing x and $R(Y)$ for the saturation of any $Y \subset X$. If $R(x)$ is countable for each x we say that R is *countable*. If each $R(x)$ is finite we say R is *finite*, and if $|R(x)| = n$ for each x , we say R has *cardinality* n . In this paper we will deal with countable relations on a standard Borel space X .

Two canonical projections are defined from R to X by $\pi_l(x, y) = x$ and $\pi_r(x, y) = y$ and an inversion map on R by $\theta(x, y) = (y, x)$. These, of

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course, are Borel maps, but π_l and π_r also map Borel sets to Borel sets since they are countable-to-one [5, §39, Corollary 5, p. 498].

A *partial Borel isomorphism* is a one-to-one Borel map φ between Borel subsets of X such that φ^{-1} is also a Borel map. If also $\Gamma(\varphi) \subset R$, φ is called a *partial R -isomorphism*. The domain and range of φ are denoted by $d(\varphi)$ and $r(\varphi)$, respectively. If φ is a partial R -isomorphism and $d(\varphi) = r(\varphi) = X$, then φ is called an *R -automorphism*. The group of all R -automorphisms is called the *full group* of R , denoted $G(R)$.

The following easy lemma is given for the record.

Lemma 1.1. *Let φ be a one-to-one map defined on a subset of countably separated Borel space X . Then the following are equivalent:*

- (i) φ is a Borel map.
- (ii) φ^{-1} is a Borel map.
- (iii) $\Gamma(\varphi)$ is a Borel subset of $X \times X$.

Proof. Suppose $\Gamma(\varphi)$ is Borel. If $B \subset X$ is Borel, then

$$\varphi(B) = \pi_r(B \times X \cap \Gamma(\varphi))$$

is also Borel. Since $\Gamma(\varphi^{-1}) = \theta(\Gamma(\varphi))$ is also Borel, $\varphi^{-1}(B)$ is Borel as well. Hence (iii) implies (i) and (ii). By [1, Proposition 3.3.1], (i) and (ii) each imply (iii). \square

Now assume X has a finite Borel measure μ . We will assume throughout this paper that μ is *quasi-invariant* in the sense that for $Y \subset X$, $\mu(Y) = 0$ implies $\mu(R(Y)) = 0$. In the study of measurable equivalence relations it is generally only the measure class of μ that is important. We define a “left counting measure” ν on R by $\nu(B) = \int_X |\pi_l^{-1}(x) \cap B| d\mu(x)$. Null sets in (X, μ) and (R, ν) are ignored in the usual way, for example, members of $G(R)$ are considered to be defined up to sets of measure zero.

If $\varphi, \psi \in G(R)$ then let $d(\varphi, \psi) = \mu(\{x \in X | \varphi(x) \neq \psi(x)\})$. $d(\varphi, \psi)$ is easily checked to be a metric on $G(R)$. This metric induces the *uniform topology* on $G(R)$.

Lemma 1.2. *$(G(R), d)$ is a complete metric space. Furthermore replacing μ by an equivalent finite measure will result in an equivalent metric.*

Proof. Suppose $\{\varphi_n\}$ is a Cauchy sequence in $G(R)$. Then we can choose a subsequence $\{\varphi_{n_k}\}$ such that $d(\varphi_{n_k}, \varphi_{n_{k+1}}) < 2^{-k}$. Let $C = \{x \in X | \varphi_{n_k}(x) \text{ is eventually constant}\}$. Then for each m ,

$$\mu(X \setminus C) \leq \sum_{k=m}^{\infty} d(\varphi_{n_k}, \varphi_{n_{k+1}}) < \sum_{k=m}^{\infty} 2^{-k} = 2^{-m+1}.$$

Therefore $\mu(X \setminus C) = 0$ and $\varphi_{n_k}(x)$ is eventually constant except on a null set. Define $\varphi(x)$ to be the eventually constant value of $\varphi_{n_k}(x)$ for all $x \notin C$. Given $\varepsilon > 0$, choose M so that $d(\varphi_n, \varphi_m) < \varepsilon/2$ whenever $n, m \geq M$, and also $2^{-M} < \varepsilon/2$. Then if $m \geq M$, choose k so that $n_k \geq m$. Then

$$d(\varphi, \varphi_m) \leq d(\varphi, \varphi_{n_k}) + d(\varphi_{n_k}, \varphi_m) < 2^{-n_k} + \varepsilon/2 < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Hence $\{\varphi_n\}$ converges to φ in the uniform metric. That φ is an R -automorphism is easily deduced from the fact that $\Gamma(\varphi)$ differs from $\Gamma(\varphi_n)$ on an arbitrarily small set.

The second claim is trivial given that equivalent finite measures control each other's sizes as in [4, Theorem 3.5]. \square

In the rest of this paper we assume that μ is a measure on the standard Borel space X . Since X is Borel isomorphic to $[0, 1]$, we may regard μ as a measure on $[0, 1]$; if μ is nonatomic, it may be taken to be Lebesgue measure on $[0, 1]$. We also assume that the countable measurable equivalence relation R has infinite equivalence classes and that μ is quasi-invariant for R . In the absence of these two assumptions there are easy counterexamples to most of the results of this paper. The assumption of infinite equivalence classes is used in the discussion following Proposition 2.1. The assumption of quasi-invariance is needed to define the measure ν on R and is used in the discussion following Proposition 2.2 to define the measures ν^n on R_n .

2. EXISTENCE OF PERIODIC ELEMENTS

We say that a partial R -isomorphism φ is n -periodic if $d(\varphi) = r(\varphi)$ and $\varphi^n(x) = x$ for all $x \in d(\varphi)$. In addition, φ is said to be strictly n -periodic if $\Gamma(\varphi^k) \cap \Delta = \emptyset$ for $k = 0, 1, \dots, n - 1$. An R -automorphism φ is said to be free if $\Gamma(\varphi) \cap \Delta = \emptyset$, and totally free (or totally nonperiodic) if $\Gamma(\varphi^n) \cap \Delta = \emptyset$ for all n . Here $\Delta = \{(x, x) : x \in R\}$, the diagonal of R .

Define R_n to be the set of all $(n + 1)$ -tuples of elements of X whose entries are all equivalent; R_n is a subset of $X_{n+1} = X \times \dots \times X$ ($n + 1$ factors). For each $i, j = 0, 1, \dots, n$ let $\pi_i : R_n \rightarrow X$ be the i th canonical projection map, and let $\pi_{ij} : R_n \rightarrow R$ be the projection onto the i th and j th coordinates.

Define a symmetry map θ_n on R_n by

$$\theta_n(x_0, x_1, \dots, x_n) = (x_1, x_2, \dots, x_n, x_0).$$

If φ is a partial R -isomorphism, define $\Gamma_n(\varphi)$ to be the set of $(n + 1)$ -tuples (x_0, x_1, \dots, x_n) in R_n with $\varphi(x_i) = x_{i+1}$, $i = 0, \dots, n - 1$. ($\Gamma_n(\varphi)$ may well be empty, as it will be if $d(\varphi) \cap r(\varphi) = \emptyset$.)

Our first goal is to show the existence of sufficiently many n -periodic partial R -isomorphisms to generate the full group of R . The case $n = 2$ was done by Feldman and Moore in [3, Theorem 1]. The argument here is a generalization of their approach, but the details are interesting.

Proposition 2.1. *Let Γ be any θ_n -invariant subset of R_n on which each π_i is one-to-one. Then there is a partial R -isomorphism φ of period $n + 1$ such that $\Gamma = \Gamma_n(\varphi)$. Conversely if φ is a partial R -isomorphism of period $n + 1$, then $\Gamma_n(\varphi)$ is a θ_n -invariant set on which each π_i is one-to-one.*

Proof. Given Γ as described, define the map φ on X by $\varphi(x) = y$ if $(x, y, x_2, \dots, x_n) \in \Gamma$ for some $x_2, \dots, x_n \in X$. To see that φ is well defined, let (x, y, \dots) and (x, z, \dots) both be in Γ . Then $y = z$ because π_1 is one-to-one on Γ . φ is one-to-one by a similar argument: if $\varphi(x) = \varphi(y) = z$ then (x, z, \dots) and (y, z, \dots) are both in Γ , and since π_0 is one-to-one on Γ , we have $x = y$. Since by definition $\Gamma(\varphi) = \pi_{01}(\Gamma)$ (where π_{01} is the canonical projection onto the first two coordinates), $\Gamma(\varphi)$ is a Borel set and, therefore, φ is a Borel map by Lemma 1.1. It also follows from the definition of φ that $\Gamma(\varphi) \subset R$, so φ is a partial R -isomorphism.

If $(x_0, x_1, \dots, x_n) \in \Gamma$, then so is $\theta_n^k(x_0, x_1, \dots, x_n) = (x_k, x_{k+1}, \dots, x_{k-1})$, so $x_{k+1} = \varphi(x_k)$ for $k = 0, \dots, n - 1$. Hence $(x_0, x_1, \dots, x_n) \in$

$\Gamma_n(\varphi)$. On the other hand, if $(x_0, x_1, x_2, \dots, x_n) \in \Gamma_n(\varphi)$ then $\varphi(x_0) = x_1$. Therefore $(x_0, x_1, y_2, \dots, y_n) \in \Gamma$ for some $y_2, \dots, y_n \in X$. But we have just shown $\Gamma \subset \Gamma_n(\varphi)$, so also $(x_0, x_1, y_2, \dots, y_n) \in \Gamma_n(\varphi)$. Therefore $y_k = \varphi^k(x_0) = x_k, k = 2, \dots, n$. Hence

$$(x_0, x_1, x_2, \dots, x_n) = (x_0, x_1, y_2, \dots, y_n) \in \Gamma,$$

and so $\Gamma = \Gamma_n(\varphi)$.

If $x_0 \in d(\varphi), (x_0, x_1, \dots, x_n) \in \Gamma$ for some $x_1, \dots, x_n \in X$, so $\varphi^{n+1}(x_0) = \varphi(\varphi^n(x_0)) = \varphi(x_n) = x_0$, the last equality following from the fact that $\theta_n^n(x_0, x_1, \dots, x_n) = (x_n, x_0, \dots, x_{n-1}) \in \Gamma$. The domain of φ is equal to $\pi_0(\Gamma)$ and its range is equal to $\pi_1(\Gamma)$, but $\pi_1(\Gamma) = \pi_0 \circ \theta_n(\Gamma) = \pi_0(\Gamma)$. Therefore φ is $(n + 1)$ -periodic. \square

Let $X_{n+1}^{(0)}$ be the set of $(n + 1)$ -tuples in which all entries are distinct, and let $R_n^{(0)} = R_n \cap X_{n+1}^{(0)}$. Because R has infinite equivalence classes, $R_n^{(0)}$ is nonempty. As in [3, Theorem 1], it follows from Kuratowski [5, §39, VI, Corollary 5] the fact that π_0 is countable-to-one, and the assumption that R is standard, that R_n and hence $R_n^{(0)}$ may be partitioned into sets D_i such that π_0 is one-to-one on each D_i . By taking all sets of the form $D_{i_0} \cap \theta_n(D_{i_1}) \cap \theta_n^2(D_{i_2}) \cap \dots \cap \theta_n^n(D_{i_n})$, we can construct sets $\{E_j\}$ partitioning $R_n^{(0)}$ such that each π_i is one-to-one on each E_j . By considering X Borel isomorphic to $[0, 1]$, we can write $X_{n+1}^{(0)}$ as a countable union of product rectangles. In each product the factor sets are then mutually disjoint. By taking the intersections of these product rectangles with each E_j we create sets $\{F_k\}$ such that

- (i) π_i is one-to-one on F_k for each i and k ;
- (ii) for each $k, \pi_i(F_k)$ are mutually disjoint;
- (iii) the union of the F_k is $R_n^{(0)}$.

For any $F \subset R_n$ define $\Theta(F) = \bigcup_{k=0}^n \theta_n^k(F)$. Note that $\Theta(F)$ is invariant under θ_n . We claim that if F satisfies (i) and (ii), then each π_i is still one-to-one on $\Theta(F)$. To see this we use the identity $\pi_k = \pi_0 \circ \theta_n^k$. The sets $\pi_0 \circ \theta_n^k(F) = \pi_k(F)$ are disjoint by (ii). Therefore, the sets $\theta_n^k(F)$ must themselves be disjoint. It follows that π_0 is one-to-one on $\Theta(F)$ if and only if π_0 is one-to-one on each $\theta_n^k(F)$ separately. Since $\pi_k = \pi_0 \circ \theta_n^k$, this is guaranteed by (i). It then follows that the other π_i are one-to-one on $\Theta(F)$ as well.

For any of the sets in the collection $\{F_k\}, \Theta(F_k)$ satisfies the conditions of Proposition 2.1. We therefore have

Proposition 2.2. *Let $S \subseteq R_n^{(0)}$. Then there is a partition of S into sets $\{S_k\}$ such that each S_k is contained in $\Gamma_n(\varphi_k)$ where each φ_k is a strictly $(n + 1)$ -periodic partial R -isomorphism.*

Proof. We need only take $S_k = S \cap F_k$. Since $S_k \subseteq F_k, S_k$ also satisfies (i) and (ii). By (iii), the union of the S_k is S . By the preceding discussion, $\Theta(S_k)$ satisfies the conditions of Proposition 2.1 for each k and hence $\Theta(S_k) = \Gamma_n(\varphi_k)$ for some $(n + 1)$ -periodic partial R -isomorphism φ_k . Since $\Gamma_n(\varphi_k) \subseteq R_n^{(0)}, \varphi_k$ must be strictly periodic. \square

The development in this section has thus far been independent of the measure μ . Given a quasi-invariant μ , we can construct a measure ν^n on R_n as

in [3, Proposition 2.3] by $\nu^n(C) = \int_X |\pi_0^{-1}(t) \cap C| d\mu(t)$. It is then clear that $\nu^n(R_n^{(0)}) = \infty$: since the equivalence class containing t is infinite, the cardinality of $\pi_0^{-1}(t) \cap R_n^{(0)}$ will always be infinite. In Proposition 2.2 we may then discard the S_k for which $\nu^n(S_k) = 0$. The partition would then be a partition modulo null sets.

Consider now the partial order on n -periodic maps given by graph inclusion modulo null sets. With this order, a chain of n -periodic maps is necessarily countable. Therefore, given an increasing chain of n -periodic maps, we can take the union of the graphs to obtain a Borel set, which is also the graph of an n -periodic map. The n -periodic map thus created is an upper bound for the chain, so by Zorn's Lemma there exist maximal n -periodic maps on R . The same argument can be used to show the existence of maximal strictly n -periodic maps.

Proposition 2.3. *Let φ be a (strictly) periodic partial R -isomorphism. Then φ extends to a (strictly) periodic R -automorphism.*

Proof. Suppose that φ is a maximal (strictly) $(n+1)$ -periodic partial R -isomorphism, but the domain $d(\varphi)$ of φ is a proper subset of X . Since $r(\varphi) = d(\varphi)$, $\Gamma(\varphi) \subset d(\varphi) \times d(\varphi)$. Let $Y = X \setminus d(\varphi)$. Then $R_Y = R \cap (Y \times Y)$ is a countable standard equivalence relation on Y . By Proposition 2.2 there exists a strictly $(n+1)$ -periodic partial R -isomorphism g on R_Y . Define ψ by $\Gamma(\psi) = \Gamma(\varphi) \cup \Gamma(g)$. ψ is then a (strictly) $(n+1)$ -periodic partial R -isomorphism and a nontrivial extension of φ , contradicting the assumption of maximality. Therefore $d(\varphi) = X$, and φ is an R -automorphism. Since every $(n+1)$ -periodic partial R -isomorphism extends to a maximal one, the proof is complete. \square

Theorem 2.4. *Let $T \subseteq R \setminus \Delta$. Then for any $n \geq 1$ there is a partition of T into nonnull sets $\{T_k\}$ such that each T_k is contained in some $\Gamma(\varphi_k)$, where φ_k is a strictly $(n+1)$ -periodic R -automorphism.*

Proof. Apply Proposition 2.2 to $\pi_{01}^{-1}(T) \cap R_n^{(0)}$ and project the resulting sets into R via π_{01} . This will result in sets $T_k \subseteq \Gamma(\varphi_k) = \pi_{01}(\Gamma_n(\varphi_k))$ that cover T . Taking set differences will then yield a partition. By Proposition 2.3 each φ_k can be extended to a strictly periodic R -automorphism. \square

Theorem 2.5. *Let R be a countable standard equivalence relation with infinite equivalence classes. For each $n \geq 2$, $G(R)$ is generated by strictly n -periodic R -automorphisms.*

Suppose φ is a Borel automorphism of X such that $\Gamma(\varphi) \cap \Delta = \emptyset$ (i.e., φ is not the identity on any nonnull set). Given a nonnull set $A \subset X$, there exists a nonnull set $B \subset A$ such that $\varphi(B) \setminus B$ is nonnull. (This requires the assumption that X is a standard Borel space.) Let $C = B \setminus \varphi^{-1}(B)$. Then $C \subset A$ and $\varphi(C) = \varphi(B) \setminus B$ is disjoint from C . We can easily extend this to the following claim:

(*) If $\Gamma(\varphi^k) \cap \Gamma(\varphi^m) = \emptyset$ and A is a nonnull set in X , there exists a nonnull $C \subset A$ such that $\varphi^k(C) \cap \varphi^m(C) = \emptyset$.

To see this we assume $k < m$, note that $\varphi^k(C) \cap \varphi^m(C) = \emptyset$ if and only if $C \cap \varphi^{m-k}(C) = \emptyset$, and apply the above argument to φ^{m-k} .

Proposition 2.6. *Let φ be an R -automorphism of period n . Then the following are equivalent:*

- (i) φ is strictly n -periodic.
- (ii) $\{\Gamma(\varphi^k) : k = 0, \dots, n-1\}$ are mutually disjoint.
- (iii) There is a set A with $\mu(A) \leq 1/n$ such that $\{\varphi^k(A) : k = 0, \dots, n-1\}$ is a partition of X .
- (iv) $F = \bigcup_{k=0}^{n-1} \Gamma(\varphi^k)$ is a finite subequivalence relation of R with $|F(x)| = n$ for all $x \in X$.

Proof. (i) \Leftrightarrow (ii) If $\varphi^k(x) = x$ for x in some nonnull set then $\Gamma(\varphi^k)$ and $\Gamma(\varphi^0) = \Delta$ are not disjoint. If $\Gamma(\varphi^k) \cap \Gamma(\varphi^m) \neq \emptyset$ with $k > m$ then $\varphi^k(x) = \varphi^m(x)$ and hence $\varphi^{k-m}(x) = x$ for x in some nonnull set, so φ is not strictly periodic.

(iii) \Rightarrow (ii) Assume there is such a set A , and suppose that $\Gamma(\varphi^k) \cap \Gamma(\varphi^m) \neq \emptyset$. Then $B = \{x | \varphi^k(x) = \varphi^m(x)\}$ is a nonnull set, so $B \cap \varphi^j(A)$ is nonnull for some j . But then $\varphi^k(B \cap \varphi^j(A)) = \varphi^m(B \cap \varphi^j(A))$. The first is contained in $\varphi^{k+j}(A)$ and the second in $\varphi^{m+j}(A)$, so it follows that $k + j \equiv m + j \pmod{n}$, hence $k \equiv m \pmod{n}$.

(ii) \Rightarrow (iii) First we show the existence of a set B such that $\{\varphi^k(B) : k = 0, \dots, n-1\}$ are disjoint sets. For this we need only use (*) repeatedly to select smaller and smaller nonnull sets guaranteeing that each intersection $\varphi^k(B) \cap \varphi^m(B)$ is empty.

Since the union of an increasing chain of sets $\{B_\alpha\}$ each with $\{\varphi^k(B_\alpha) : k = 0, \dots, n-1\}$ disjoint is another of this type, there is a set A that is maximal with respect to this property.

If $\bigcup_{k=0}^{n-1} \varphi^k(A) \neq X$ then $C = X \setminus \bigcup_{k=0}^{n-1} \varphi^k(A)$ is a set invariant under φ , so we may apply the above argument to φ acting on C and find a set $A_1 \subset C$ with $\{\varphi^k(A_1) : k = 0, \dots, n-1\}$ disjoint. Then $\{\varphi^k(A \cup A_1) : k = 0, \dots, n-1\}$ are disjoint, contradicting the maximality of A , and hence $\bigcup_{k=0}^{n-1} \varphi^k(A) = X$. Note that $\mu(\varphi^k(A)) \leq 1/n$ must be true for some k , and if necessary we can then choose $\varphi^k(A)$ to replace A .

(ii) \Leftrightarrow (iv) F is reflexive because $(x, x) = (x, \varphi^0 x) \in F$ for each x . F is transitive because if $(x, y), (y, z) \in F$, then $y = \varphi^m x$ and $z = \varphi^k y$, so $z = \varphi^{m+k} x$. Choose j with $0 \leq j < n$ so that $j \equiv m+k \pmod{n}$; then $z = \varphi^j x$ and hence $(x, z) \in F$. F is symmetric since if $(x, y) \in F$ with $y = \varphi^m x$ ($0 \leq m < n$), then $x = \varphi^{-m} y = \varphi^{n-m} y$ and hence $(y, x) \in F$. Therefore F is an equivalence relation with equivalence classes of the form $\{\varphi^k x : k = 0, \dots, n-1\}$. These all have cardinality n if and only if φ is strictly periodic. \square

Corollary 2.7. *Let $T \subseteq R$. Then there is a partition of T into sets $\{T_k\}$ such that each T_k is contained in a finite subequivalence relation of cardinality n .*

The relation R is *hyperfinite* if it is the nested union of finite subequivalence relations. Hyperfinite relations are associated with hyperfinite von Neumann algebras, and the finite subrelations are associated with finite-dimensional subalgebras. This corollary shows that a countable standard relation R is always the countable union of finite subequivalence relations even when it is not a nested union. This result gives the possibility of bringing finite-dimensional

techniques into play even in nonhyperfinite situations. For an example of this technique, see [7].

3. CONSTRUCTIONS WITH PERIODIC MAPS

We now give a construction of a large class of totally free R -automorphisms.

Let φ be a strictly n -periodic R -automorphism. By Proposition 2.6 there is a set A with $\mu(A) \leq 1/n$ such that $\{\varphi^k(A) : k = 0, \dots, n - 1\}$ is a partition of X . Let $R_A = R \cap (A \times A)$, and let ψ_A be a strictly m -periodic R_A -automorphism. By Proposition 2.6 again we can choose A_0 such that $\{\psi_A^j(A_0) : j = 0, \dots, m - 1\}$ is a partition of A . Define φ' on X by

$$\varphi'(x) = \begin{cases} \varphi \circ \psi_A(x), & x \in A, \\ \varphi(x), & x \notin A. \end{cases}$$

Proposition 3.1. φ' is a strictly mn -periodic R -automorphism with $d(\varphi, \varphi') \leq 1/n$.

Proof. Since φ' differs from φ only on A , $d(\varphi, \varphi') \leq 1/n$ is trivial. φ' is easily checked to be one-to-one and onto, and its graph lies in R since this is true of φ and $\varphi \circ \psi_A$ separately; hence φ' is an R -automorphism. To check the periodicity, let $x \in A$. Then we calculate

$$(\varphi')^r(x) = \begin{cases} \varphi^r \circ \psi_A(x), & 1 \leq r \leq n, \\ \varphi^{r-n} \circ \psi_A^2(x), & n + 1 \leq r \leq 2n, \dots, \\ \varphi^{r-jn} \circ \psi_A^j(x), & (j - 1)n + 1 \leq r \leq jn, 1 \leq j \leq m, \dots, \end{cases}$$

and hence $(\varphi')^{mn}(x) = \varphi^n \circ \psi_A^m(x) = x$. A similar argument applies for $x \notin A$. To see that φ' is strictly periodic, note that $(\varphi')^r(A_0)$ are disjoint for $0 \leq r < mn$ and apply Proposition 2.6. \square

To construct totally free R -automorphisms, we iterate this construction, beginning for simplicity with a 2-periodic R -automorphism φ_1 and doubling the periodicity at each step to obtain a sequence of strictly periodic R -automorphisms $\{\varphi_i\}$, each having period 2^i and differing from the previous on a set of measure at most 2^{-i} . At each stage the powers will have disjoint graphs by Proposition 2.6.

Proposition 3.2. The sequence $\{\varphi_i\}$ converges in the uniform topology to an R -automorphism ψ such that $\{\Gamma(\psi^n) : -\infty < n < +\infty\}$ are disjoint.

Proof. Since each φ_i differs from φ_{i-1} on a set of measure at most 2^{-i} , $\{\varphi_i\}$ is easily seen to be a Cauchy sequence in the metric d , and so it converges in the uniform topology to some R -automorphism ψ . Suppose that $\nu(\Gamma(\psi^m) \cap \Gamma(\psi^n)) \neq 0$ for integers m, n with $m < n$. Then there is a nonnull set B such that $\psi^m(x) = \psi^n(x)$ whenever $x \in B$. Choose a positive integer i so that $2^{-i} < \mu(B)$ and $n - m < 2^i$. Then ψ differs from φ_i on a set of measure at most $\sum_{k=i+1}^\infty 2^{-k} = 2^{-i}$, so ψ agrees with φ_i on a nonnull subset of B . We also have $\psi^i(x) = \psi^{n-m+i}(x)$ whenever $x \in B$. Hence $\varphi_i^i = \varphi_i^{n-m+i}(x)$ on a nonnull subset of B . Since $n - m < 2^i$, this contradicts the fact that φ_i is strictly 2^i -periodic; therefore, $\nu(\Gamma(\psi^m) \cap \Gamma(\psi^n)) = 0$ for distinct integers m, n . \square

The disjointness of $\{\Gamma(\psi^n) : -\infty < n < +\infty\}$ is equivalent to $\Gamma(\psi^n) \cap \Delta = \emptyset$ for all n , therefore

Corollary 3.3. $G(R)$ contains totally free R -automorphisms.

The above construction clearly yields a vast quantity of totally free R -automorphisms, yet they do not seem to form a very “large” set. There does not seem to be any hope that they are generic in any sense; they are not even dense in the uniform topology because of R -automorphisms that are the identity on a nonnull subset. The Rohlin Lemma (Lemma 4.5 to follow) tells us that the strictly periodic maps are dense at each totally free map, so it may be that the periodic maps should be thought of as a “larger” set than the totally free maps.

A group acting on a measure space acts *freely* if no group element other than the identity acts as the identity on any nonnull set. This corresponds to being strictly periodic for elements of finite order or totally free for elements of infinite order. Thus we have seen that $G(R)$ contains many freely acting cyclic subgroups of both finite and infinite order, a very modest contribution to the second problem mentioned in the introduction.

A modified version of the above construction yields integer roots for periodic maps. The details are similar to Proposition 3.1 and are omitted.

Proposition 3.4. Let φ be a strictly n -periodic R -automorphism, and let A , ψ_A , and A_0 be defined as above. Define

$$\hat{\varphi}(x) = \begin{cases} \varphi^k \circ \psi_A \circ \varphi^{-k}(x), & x \in \varphi^k(A \setminus A_0), \\ \varphi(x), & x \in \varphi^k(A_0), \end{cases} \quad 0 \leq k \leq n - 1.$$

Then $\hat{\varphi}$ is a strictly mn -periodic R -automorphism, and $\hat{\varphi}^m = \varphi$.

4. APPLICATIONS

In this section we assume that μ is a nonatomic standard Borel measure.

Lemma 4.1. Let φ be a partial R -isomorphism. Then given $\varepsilon > 0$, there is a $\delta > 0$ such that whenever B is a Borel set in X with $\mu(B) < \delta$, $\mu(\varphi(B)) < \varepsilon$.

Proof. By [3, Proposition 2.2], the measure $\mu \cdot \varphi$ is absolutely continuous with respect to μ . The result then follows from standard measure theory, e. g., [4, Theorem 3.5]. \square

Lemma 4.2. Multiplication is jointly continuous in the uniform topology on $G(R)$.

Proof. Let $\varphi_0, \psi_0 \in G(R)$ be fixed and $\varepsilon > 0$ be given. Choose δ so that $\mu(\psi_0^{-1}(B)) < \varepsilon/2$ whenever $\mu(B) < \delta$, and also $\delta < \varepsilon/2$. Suppose $\varphi, \psi \in G(R)$ with $d(\varphi, \varphi_0) < \delta$ and $d(\psi, \psi_0) < \delta$. If $\varphi \circ \psi(x) \neq \varphi_0 \circ \psi_0(x)$ then either $\psi(x) \neq \psi_0(x)$ or $\varphi(y) \neq \varphi_0(y)$ where $y = \psi_0(x)$. Therefore

$$\begin{aligned} d(\varphi \circ \psi, \varphi_0 \circ \psi_0) &\leq \mu(\{x : \psi(x) \neq \psi_0(x)\}) + \mu(\psi_0^{-1}(\{y | \varphi(y) \neq \varphi_0(y)\})) \\ &< \delta + \varepsilon/2 < \varepsilon. \quad \square \end{aligned}$$

Corollary 4.3. Let $r(\alpha_1, \dots, \alpha_n)$ be a finite word in n variables on $G(R)$. Then given $\varepsilon > 0$, there exists $\delta > 0$ so that $d(r(\varphi_1, \dots, \varphi_n), r(\alpha_1, \dots, \alpha_n)) < \varepsilon$ whenever $d(\varphi_k, \alpha_k) < \delta$ for $1 \leq k \leq n$.

We present a version of the well-known Rohlin Lemma, adapted for R -automorphisms. We begin with a slight modification of a lemma from [2]; although the result is stated there for Lebesgue measure on $[0, 1]$, it is equally valid for any nonatomic measure on $[0, 1]$.

Lemma 4.4. *Given a totally free Borel automorphism φ of X , an integer n , and $\delta > 0$, there exists a Borel set E such that the sets $\varphi^k(E)$, $0 \leq k \leq n - 1$, are pairwise disjoint, $\mu(\varphi^{n-1}(E)) \leq 1/n$, and $\mu(X \setminus \bigcup_{k=0}^{n-1} \varphi^k(E)) < \delta$.*

Proof. Except for the condition that $\mu(\varphi^{n-1}(E)) \leq 1/n$, this is precisely Lemma 4 of [2]. If it is not immediately true that $\mu(\varphi^{n-1}(E)) \leq 1/n$, then nevertheless at least one of the sets $\varphi^k(E)$, $0 \leq k \leq n - 1$, must satisfy $\mu(\varphi^k(E)) \leq 1/n$.

By Lemma 4.1 we choose δ_0 so that $\mu(B) < \delta_0$ implies $\mu(\varphi^m(B)) < \delta$, $-n \leq m \leq 0$. Then apply Lemma 4 of [2] with δ_0 . Suppose $\mu(\varphi^{j-1}(E)) \leq 1/n$ for some j ; then consider the sets $\varphi^{j-n+k}(E)$, $0 \leq k \leq n - 1$. These sets are also mutually disjoint, and since $\mu(X \setminus \bigcup_{k=0}^{n-1} \varphi^k(E)) < \delta_0$,

$$\mu\left(X \setminus \bigcup_{k=0}^{n-1} \varphi^{j-n+k}(E)\right) = \mu\left(\varphi^{j-n}\left(X \setminus \bigcup_{k=0}^{n-1} \varphi^k(E)\right)\right) < \delta.$$

The lemma is now satisfied if we replace E by $\varphi^{j-n}(E)$. \square

Lemma 4.5 (Rohlin Lemma for R -automorphisms). *Given a totally free R -automorphism φ of X and $\varepsilon > 0$, there exists a strictly periodic R -automorphism π such that $d(\pi, \varphi) < \varepsilon$. The period of π may be chosen to be any sufficiently large integer.*

Proof. Choose $\delta = \varepsilon/2$ and choose an integer n so that $1/n < \varepsilon/2$. Apply Lemma 4.4 to obtain a set E . We define a map π_0 by letting $\pi_0(x) = \varphi(x)$ on $\bigcup_{k=0}^{n-2} \varphi^k(E)$ and $\pi_0(x) = \varphi^{-n+1}(x)$ on $\varphi^{n-1}(E)$. Then π_0 is an n -periodic partial R -isomorphism with domain (and range) $\bigcup_{k=0}^{n-1} \varphi^k(E)$. In particular, π_0 is 1-1 because $\pi_0(\varphi^{n-1}(E)) = E$ is disjoint from $\pi_0(\bigcup_{k=0}^{n-2} \varphi^k(E)) = \bigcup_{k=1}^{n-1} \varphi^k(E)$, and π_0 is a partial R -isomorphism because its graph consists of pieces of the graphs of φ and φ^{-n+1} , which are contained in R . By Proposition 2.3 π_0 can be extended to a strictly n -periodic R -automorphism π defined on all of X . π differs from φ on the set $F = \varphi^{n-1}(E) \cup D$, where $D = X \setminus \bigcup_{k=0}^{n-1} \varphi^k(E)$. Then $\mu(F) \leq \mu(\varphi^{n-1}(E)) + \mu(D) < \varepsilon/2 + \delta < \varepsilon$. \square

Given an arbitrary R -automorphism φ , X may be decomposed into φ -invariant sets on which φ is periodic of various periods and a set on which φ is totally free. Because $\mu(X)$ is finite, we may ignore sets on which φ is periodic of sufficiently large period, as these may be chosen to have arbitrarily small combined measure. Using Proposition 3.1, the period of π can then be chosen to be a large multiple of the remaining periods occurring in φ , so we have

Corollary 4.6. *Given an R -automorphism φ of X and $\varepsilon > 0$, there exists a periodic R -automorphism π such that $d(\pi, \varphi) < \varepsilon$. The period of π may be chosen to be an arbitrarily large integer and to be a multiple of any given integer.*

Proposition 4.7. *Let R be a nonatomic countable standard equivalence relation. Suppose $\alpha = \{\alpha_1, \dots, \alpha_n\} \in G(R)$ generates a group of R -automorphisms isomorphic to a free group on n generators. Then there exists a dense G_δ subset $B \subseteq G(R)$ such that for each $\beta \in B$ the subgroup of $G(R)$ generated by $\{\alpha_1, \dots, \alpha_n, \beta\}$ is isomorphic to the free group on $n + 1$ generators.*

Proof. For each finite word r on $n + 1$ letters, let $F_r(\alpha) = \{\varphi \in G(R) \mid r(\varphi, \alpha) = e\}$, where e is the identity element of $G(R)$ and $r(\varphi, \alpha) = r(\varphi, \alpha_1, \dots, \alpha_n)$.

We will show that the complement $F'_r(\alpha)$ is an open dense set in $G(R)$. Since there are countably many finite words on $n+1$ letters, the set of $\varphi \in G(R)$ such that $\{\varphi, \alpha_1, \dots, \alpha_n\}$ satisfy no relations at all (i.e., generate a free group) will be the intersection of the $F'_r(\alpha)$ over all words r . Since $G(R)$ is completely metrizable by Lemma 1.2, it will follow from the Baire category theorem that this intersection is nonempty and a dense G_δ .

We first note that $F_r(\alpha)$ is closed in $G(R)$ due to the joint continuity of multiplication (Lemma 4.2). Therefore we need only show that $F'_r(\alpha)$ is dense for each r . Given $\varepsilon > 0$ and $\alpha_0 \in G(R)$, we need to show that there exists an R -automorphism φ with $d(\varphi, \alpha_0) < \varepsilon$ such that $r(\varphi, \alpha) \neq e$.

Suppose $r(\alpha_0, \alpha_1, \dots, \alpha_n) = r(\alpha_0, \alpha)$ is a word on $n+1$ letters where powers of α_0 occur q times. Then $r(\alpha_0, \alpha_1, \dots, \alpha_n) = \beta_q \alpha_0^{k_q} \cdots \beta_2 \alpha_0^{k_2} \beta_1 \alpha_0^{k_1} \beta_0$, where each β_i is a product of powers of $\alpha_1, \dots, \alpha_n$ and $\beta_1, \dots, \beta_{q-1}$ are not the identity. The proof will be by induction on q . If $\beta_0 \neq e$ we may begin the induction with the trivial case $q = 0$, while if $\beta_0 = e$ we may begin the induction with the case $q = 1$, which can be proven using a variation of the following argument.

Assume then that there exists an R -automorphism φ_{q-1} with $d(\varphi_{q-1}, \alpha_0) < \varepsilon$ such that $\gamma = \beta_{q-1} \varphi_{q-1}^{k_{q-1}} \cdots \beta_2 \varphi_{q-1}^{k_2} \beta_1 \varphi_{q-1}^{k_1} \beta_0 \neq e$. If $\delta = \varepsilon - d(\varphi_{q-1}, \alpha_0)$, we will show that there exists an R -automorphism φ_q with $d(\varphi_q, \varphi_{q-1}) < \delta/2$ (and hence $d(\varphi_q, \alpha_0) < \varepsilon$) such that $r(\varphi_q, \alpha) \neq e$. By continuity and the density of periodic R -automorphisms (Corollaries 4.3 and 4.6), we may assume that φ_{q-1} is periodic.

There exist disjoint nonnull sets A_1 and A_2 such that γ maps A_1 onto A_2 ; also define $A_3 = \beta_q^{-1}(A_2)$. Let D be the union of the orbits of A_1, A_2 , and A_3 under φ_{q-1} . Since φ_{q-1} is periodic, D is a finite union of iterates of φ_{q-1} applied to these sets. It follows from repeated application of Lemma 4.1 that we may choose $\mu(A_1)$ sufficiently small to guarantee that $\mu(D) < \delta$. Let $R_D = R \cap (D \times D)$ be the restriction of R to D .

Choose an integer $m > k_q$, and consider the set $Q = A_2 \times D \times \cdots \times D \times A_3 \times D \times \cdots \times D \subset (R_D)_{m-1}$, where A_3 occurs in the k_q th position. Since A_2 and A_3 are nonnull, it follows that Q is nonnull in $(R_D)_{m-1}$ (i.e., $\nu^{m-1}(Q) > 0$), and applying Proposition 2.2 with $X = D$ yields a strictly m -periodic partial R_D -automorphism ψ_0 on D such that $\Gamma(\psi_0) \subset Q$. If $C_2 = A_2 \cap d(\psi_0)$ then $\psi_0^{k_q}(C_2) \subseteq A_3$. By Proposition 2.3 ψ_0 extends to a strictly m -periodic R_D -automorphism ψ on D , for which $\psi_0^{k_q}(C_2) \subseteq A_3$ still holds. Let $C_1 = \gamma^{-1}(C_2) \subseteq A_1$; then C_1 and A_2 are disjoint.

Define φ_q by $\varphi_q = \varphi_{q-1}$ on $X \setminus D$ and $\varphi_q = \psi$ on D . Then $r(\varphi_q, \alpha)(C_1) = \beta_q \circ \varphi_q^{k_q} \circ \gamma(C_1) = \beta_q \circ \varphi_q^{k_q}(C_2) = \beta_q \circ \psi^{k_q}(C_2) \subseteq \beta_q(A_3) = A_2$. As C_1 and A_2 are disjoint, this shows that $r(\varphi_q, \alpha) \neq e$.

The variation of this argument needed to prove the alternate initial case $q = 1$ is simply to let $\beta_q = e$ and $\gamma = \beta_0$. This concludes the proof that $F'_r(\alpha)$ is dense. \square

Theorem 4.8. *Let R be a nonatomic countable standard equivalence relation. Then $G(R)$ contains subgroups isomorphic to free groups on any countable number of generators. Furthermore, all the generators may be chosen to lie in an arbitrarily small neighborhood of any nonperiodic element of $G(R)$.*

Proof. The case of a single generator is satisfied by any nonperiodic member of $G(R)$. (If nothing else, we can use a totally free map as guaranteed by Corollary 3.3.) The previous proposition provides the induction step to prove this result for any finite number of generators. For a countably infinite number of generators, simply take a sequence $\{\alpha_1, \alpha_2, \alpha_3, \dots\}$ obtained by starting with a nonperiodic map α_1 and applying the previous proposition repeatedly to the sequence already obtained. The sequence of generators $\{\alpha_1, \alpha_2, \alpha_3, \dots\}$ will satisfy no finite relation and hence generates a free group on a countably infinite number of generators. The last claim follows from the fact that each generator may be chosen from a dense G_δ . \square

This result emphasizes the highly nonabelian structure of $G(R)$. In the case of a single generator we were able in Corollary 3.3 to prove much more, namely, that α could be chosen so that each word had a graph disjoint from the diagonal. One might hope that we could prove the same for any number of generators, i.e., that $\{\alpha_1, \alpha_2, \alpha_3, \dots\}$ could be chosen so that for each word $r(\alpha_1, \alpha_2, \alpha_3, \dots)$ in the generators the graph of $r(\alpha_1, \alpha_2, \alpha_3, \dots)$ is disjoint from the diagonal.

However, this would imply that for any two words r_1 and r_2 , the graphs of $r_1(\alpha_1, \alpha_2, \alpha_3, \dots)$ and $r_2(\alpha_1, \alpha_2, \alpha_3, \dots)$ would be disjoint, and from this it would follow that R is nonamenable, which is not necessarily true.

REFERENCES

1. W. Arveson, *An invitation to C^* -algebra*, Springer, New York, 1976.
2. R. Chacon and N. Friedman, *Approximation and invariant measures*, *Z. Wahrsch. Verw. Gebiete* **3** (1965), 286–295.
3. J. Feldman and C. C. Moore, *Ergodic equivalence relations, cohomology, and von Neumann algebras*. I, II, *Trans. Amer. Math. Soc.* **234** (1977), 289–359.
4. G. Folland, *Real analysis*, Wiley, New York, 1984.
5. K. Kuratowski, *Topology*, Academic Press, New York, 1966.
6. P. Muhly, K. Saito, and B. Solel, *Coordinates for triangular operator algebras*, *Ann. of Math.* (2) **127** (1988), 245–278.
7. R. Mercer, *Isometric isomorphisms of Cartan bimodule algebras*, *J. Funct. Anal.* **101**, 10–24.

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