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VARIATIONAL PRINCIPLES FOR AVERAGE EXIT TIME MOMENTS FOR DIFFUSIONS IN EUCLIDEAN SPACE

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ABSTRACT. Let D be a smoothly bounded domain in Euclidean space and let X_t be a diffusion in Euclidean space. For a class of diffusions, we develop variational principles which characterize the average of the moments of the exit time from D of a particle driven by X_t, where the average is taken over all starting points in D.

1. Introduction

In this note we study diffusions on $\mathbb{R}^d$ and properties of their corresponding exit times from smoothly bounded, connected, open domains in $\mathbb{R}^d$ with compact closure. We will denote by X_t a diffusion in $\mathbb{R}^d$ with corresponding generator L a uniformly elliptic operator of divergence form. We will write $Lf = \operatorname{div}(a\nabla f)$ where the coefficient matrix $a = a_{ij}(x)$ is smooth and symmetric.

Let $\tau = \tau(\omega) = \inf\{t \geq 0 : X_t(\omega) \notin D\}$ be the first exit time of X_t from $D(S)$.

We study the average kth moment of the exit time for a particle driven by X_t, starting in D:

$$\mathcal{E}_k = \mathcal{E}_k(D) = \int_D E_x(\tau^k)dx$$

where E_x denotes expectation under the measure P_x satisfying $P_x\{X_0 = x\} = 1$, for all $x \in \mathbb{R}^d$. Note that $\mathcal{E}_k$ is invariant under Euclidean motions.

We give a variational characterization of $\mathcal{E}_k$ for each positive integer value of k in the following theorem:

Theorem 1.1. *Let X_t be a diffusion on $\mathbb{R}^d$ with generator L a uniformly elliptic operator of divergence form, $Lf = \operatorname{div}(a\nabla f)$, where the coefficient matrix a is smooth and symmetric. Let D be a smoothly bounded open domain in $\mathbb{R}^d$ with compact closure, $\bar{D}$. Define $\mathcal{E}_k$ as above and let $\mathcal{F}_k$ be defined by*

$$\mathcal{F}_k = \left\{ f \in C^\infty(\bar{D}); \int_D f(x)dx \neq 0,\ f = Lf = \cdots = L^{k-1}f = 0 \text{ on } \partial D \right\}.$$

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Let $[\![\frac{k}{2}]\!]$ *be the greatest integer of* $\frac{k}{2}$. *Then, for* k *even,*

$$\mathcal{E}_k = k! \sup_{f\in\mathcal{F}_k} \frac{\left(\int_D f\right)^2}{\int_D |L^{\frac{k}{2}} f|^2} \tag{1.1}$$

and for k *odd,*

$$\mathcal{E}_k = k! \sup_{f\in\mathcal{F}_k} \frac{\left(\int_D f\right)^2}{\int_D \left|\nabla L^{[\![\frac{k}{2}]\!]} f\right|_L^2} \tag{1.2}$$

where $\langle \nabla f, \nabla g\rangle_L = \sum_{i,j} a_{ij} \frac{\partial f}{\partial x_j}\frac{\partial g}{\partial x_i}$ *is the inner product associated with* L.

The proof of Theorem 1.1 is an application of the generalized Dynkin formula [AK] (cf. also [P1]), followed by an explicit computation. That smooth minimizers for the variational principles cited in Theorem 1.1 exist is explicit in our computations.

Our study of the sequence $\{\mathcal{E}_k\}$ is largely motivated by the now classic work in spectral analysis concerning to what extent a smoothly bounded domain in Euclidean space is determined by its Dirichlet spectrum. More precisely, when the diffusion is standard Brownian motion with generator $L = \frac{1}{2}\Delta$, we are interested in studying to what extent the sequence $\{\mathcal{E}_k\}$ determines the geometry of the underlying domain. There are a number of preliminary results in this direction. For example, in [KMM] the authors prove that among domains of a fixed volume, each element of the sequence is maximized if and only if the underlying domain is a ball of the appropriate volume.

For the case $k = 1$, it is known that the functional $\mathcal{E}_1$ computes the *torsional rigidity* of a domain. The St. Venant torsion problem, a problem with a long and distinguished history, is to determine those domains of a given volume which maximize torsional rigidity. The problem was settled by Polya (cf. [P2]) who proved that among domains of a fixed volume, the torsional rigidity is maximized by a ball. This result can be recovered using (1.2) and properties of the quotient given in (1.2) under symmetric rearrangement (cf. also [KM1]).

2. Basic results and definitions

Let $(\Omega, \mathcal{B})$ be a measurable space and $\{P_x\}_{x\in\mathbb{R}^d}$ a family of probability measures on $(\Omega, \mathcal{B})$. Let $\{X_t\}_{t\geq 0}$ denote a d-dimensional diffusion with generator L, a uniformly elliptic operator in divergence form and for which $P_x\{X_0 = x\} = 1$, for $x \in \mathbb{R}^d$.

Let D be a smoothly bounded, connected, open domain with compact closure. As in the introduction, we define the first exit time for a particle driven by X_t from D by $\tau = \tau(\omega) = \inf\{t : X_t(\omega) \notin D\}$. For each $x \in \mathbb{R}^d$, we will denote the expected value of a random variable Y under the probability measure P_x by $E_x(Y)$.

There is a useful relationship between the solution of a certain Poisson problem on the domain D and the expected value of the kth power of the first exit time of a particle driven by X_t from D starting at $x \in D$. Suppose u_k solves the problem

$$L^k u_k + (-1)^{k-1} k! = 0 \text{ on } D,$$
$$u_k = Lu_k = \cdots L^{k-1} u_k = 0 \text{ on } \partial D.$$

Note that u_k can be defined inductively by

$$\begin{aligned} Lu_1 + 1 &= 0 \text{ on } D, \\ u_1 &= 0 \text{ on } \partial D \end{aligned} \tag{2.1}$$

and

$$\begin{aligned} Lu_k + ku_{k-1} &= 0 \text{ on } D, \\ u_k &= 0 \text{ on } \partial D. \end{aligned} \tag{2.2}$$

Using the generalized Dynkin formula [H] (cf. also [AK] and [P1]) we have

$$\begin{aligned} E_x[u_k(X_0)] - E_x[u_k(X_\tau)] = &\sum_{j=1}^{k-1} \frac{(-1)^j}{j!} E_x[\tau^j L^j u_k(X_\tau)] \\ &+ \frac{(-1)^k}{(k-1)!} E_x\left[\int_0^\tau s^{k-1} L^k u_k(X_s) ds\right]. \end{aligned}$$

Using the definition of u_k and τ, this gives $u_k(x) = E_x\left[\tau^k\right]$ and $\mathcal{E}_k$ can be expressed in terms of u_k by $\mathcal{E}_k(D) = \int_D u_k(x)dx$.

We will need a number of integral formulae involving the function u_1 and the geometry of the diffusion L. To ease notation in the sequel we define, for α and β tangent vectors at $x \in D$, a scalar product, $\langle \alpha, \beta \rangle_L$, by

$$\langle \alpha, \beta \rangle_L = \alpha^T a(x) \beta \tag{2.3}$$

where α^T denotes the transpose of α.

Let u_1 be as defined in (2.1) and let $f \in \mathcal{F}_k$. Let ν be the outward pointing unit normal vector to ∂D. By the Divergence Theorem,

$$\int_D fLu_1 - u_1 Lf = \int_{\partial D} f\langle \nabla u_1, \nu \rangle_L - u_1 \langle \nabla f, \nu \rangle_L = 0.$$

We conclude

$$\int_D f = -\int_D u_1 Lf. \tag{2.4}$$

If X is a vectorfield on D and $f \in \mathcal{F}_k$, then $\operatorname{div}(fX) = f \operatorname{div}(X) + \langle \nabla f, X \rangle$ where $\langle \alpha, \beta \rangle$ is the standard scalar product. By the Divergence Theorem,

$$\int_D \operatorname{div}(fX) = \int_{\partial D} f\langle X, \nu \rangle = 0$$

and we conclude that

$$\int_D f \operatorname{div}(X) = -\int_D \langle \nabla f, X \rangle.$$

In particular, if u_1 is as defined in (2.1) and $X = a_{ij}(x) \nabla u_1$, then

$$\int_D f = \int_D \langle \nabla u_1, \nabla f \rangle_L. \tag{2.5}$$

3. Variational characterizations

Throughout this section let D be as above and let $\mathcal{F}_k$ be given as in Theorem 1.1. We begin with a lemma which generalizes (2.4) and (2.5).

Lemma 3.1. *Let u_n be as defined by (2.2), let k be a positive integer, and let $f \in \mathcal{F}_k$. If $k = 2n$, then*

$$\int_D f = \frac{(-1)^n}{n!} \int_D u_n L^n f. \tag{3.1}$$

If $k = 2n+1$, then

$$\int_D f = \frac{(-1)^n}{(n+1)!} \int_D \langle \nabla u_{n+1}, \nabla L^n f \rangle_L \tag{3.2}$$

where the scalar product is as given in (2.3).

Proof. Suppose that $k = 2n$ and for $0 \le l \le n-1$, define

$$P_l = (L^l u_n)(L^{n-l} f) - (L^{l+1} u_n)(L^{n-(l+1)} f).$$

Then

$$\sum_{l=0}^{n-1} P_l = u_n L^n f - f L^n u_n. \tag{3.3}$$

Let ν be the outward pointing unit normal vector along ∂D. By the Divergence Theorem and the fact that $L^l u_n = 0$ on ∂D, for $l = 0, \dots, n-1$,

$$\int_D P_l = \int_{\partial D} (L^l u_n) \langle \nabla L^{n-l-1} f, \nu \rangle_L - (L^{n-(l+1)} f) \langle \nabla L^l u_n, \nu \rangle_L = 0. \tag{3.4}$$

Combining (3.3) and (3.4) and using that $L^n u_n = (-1)^n n!$, we have established (3.1).

Suppose $k = 2n+1$ and for $0 \le l \le n$, define

$$R_l = (L^l u_{n+1})(L^{n+1-l} f) - (L^{l+1} u_{n+1})(L^{n+1-(l+1)} f).$$

Then

$$\sum_{l=0}^{n} R_l = u_{n+1} L^{n+1} f - f L^{n+1} u_{n+1}.$$

As above, we use the Divergence Theorem to see that

$$\int_D R_l = \int_{\partial D} (L^l u_{n+1}) \langle \nabla L^{n-l} f, \nu \rangle_L - (L^{n-l} f) \langle \nabla L^l u_{n+1}, \nu \rangle_L = 0.$$

Since $L^{n+1} u_{n+1} = (-1)^{n+1}(n+1)!$, we conclude

$$\int_D f = \frac{(-1)^{n+1}}{(n+1)!} \int_D u_{n+1} L(L^n f).$$

If X is the vectorfield given by $X = a\nabla(L^n f)$, then following the argument used to establish (2.5),

$$\begin{aligned} \int_D u_{n+1} L(L^n f) &= \int_D u_{n+1} \operatorname{div}(X) \\ &= -\int_D \langle \nabla u_{n+1}, \nabla L^n f \rangle_L \end{aligned}$$

and we have established (3.2). □

We now prove Theorem 1.1. Suppose $k = 2n$ and, for $f \in \mathcal{F}_k$, consider the quotient

$$Q_k(f) = \frac{\left(\int_D f\right)^2}{\int_D |L^n f|^2}.$$

From (3.1)

$$Q_k(f) = \frac{\left(\frac{1}{n!}\right)^2 \left(\int_D u_n L^n f\right)^2}{\int_D |L^n f|^2}.$$

Let $\mathcal{G}_k = \{g \in \mathcal{F}_n : g = L^n f \text{ for some } f \in \mathcal{F}_k\}$. Let $\mathcal{H}_k$ be the completion of $\mathcal{G}_k$ in the Hilbert space, $\mathcal{L}^2$, of square integrable functions on D. If we denote the inner product of g and $h \in \mathcal{L}^2$ by $\langle g, h\rangle$ and by $\|g\|$ the $\mathcal{L}^2$ norm of g, then we can view Q_k as a map $Q_k : \mathcal{G}_k \subset \mathcal{H}_k \to \mathbb{R}$,

$$Q_k(g) = \left(\frac{1}{n!}\right)^2 \left(\frac{\langle u_n, g\rangle}{\|g\|}\right)^2.$$

Clearly, the domain of Q_k can be extended to nonzero elements of $\mathcal{H}_k$ and $Q_k(cg) = Q_k(g)$ for every nonzero scalar c. It follows that Q_k is maximized when $g \in \mathcal{H}_k$ is in the direction of $u_n \in \mathcal{H}_k$. If $g = cu_n$ we have $L^n(c' u_{2n}) = g$, and computing $Q_k(cu_n)$ we see that

$$\begin{aligned}\sup_{g\in\mathcal{H}_k} Q_k(g) &= Q_k(cu_n)\\ &= \frac{\left(\int_D u_{2n}\right)^2}{\int_D (L^n u_{2n})^2}\end{aligned}$$

where we have applied (3.1) of Lemma 3.1 to the numerator. Note that $(L^n u_{2n})^2 = \frac{(-1)^n}{n!}(2n)! u_n L u_{2n}$. Applying Lemma 3.1 to the denominator we obtain

$$\begin{aligned}\sup_{g\in\mathcal{H}_k} Q_k(g) &= \frac{\left(\int_D u_{2n}\right)^2}{(2n)! \int_D u_{2n}}\\ &= \frac{1}{k!}\mathcal{E}_k(D)\end{aligned}$$

which establishes (1.1) of Theorem 1.1.

The proof of (1.2) of Theorem 1.1 is similar. Suppose $k = 2n+1$ and, for $f \in \mathcal{F}_k$, consider the quotient

$$\tilde{Q}_k(f) = \frac{\left(\int_D f\right)^2}{\int_D |\nabla L^n f|_L^2}.$$

From (3.2) of Lemma 3.1,

$$\tilde{Q}_k(f) = \frac{\left(\frac{1}{(n+1)!}\right)^2 \left(\int_D \langle \nabla u_{n+1}, \nabla L^n f\rangle_L\right)^2}{\int_D |\nabla L^n f|_L^2}.$$

Let $C^\infty(\bar{D}, \mathbb{R}^d)$ be the space of smooth vectorfields on $\bar{D}$. Let

$$\tilde{\mathcal{G}}_k = \{X \in C^\infty(\bar{D}, \mathbb{R}^d) : X = \nabla g \text{ for some } g \in \mathcal{F}_{n+1} \text{ with } g = L^n f \text{ for some } f \in \mathcal{F}_k\}.$$

Let $\tilde{\mathcal{H}}_k$ be the completion of $\tilde{\mathcal{G}}_k$ in the space of vectorfields square integrable with respect to the inner product $\langle \alpha, \beta \rangle_L$. We can view $\tilde{Q}_k$ as a map $\tilde{Q}_k : \tilde{\mathcal{G}}_k \subset \tilde{\mathcal{H}}_k \to \mathbb{R}$,

$$\tilde{Q}_k(g) = \left(\frac{1}{(n+1)!} \right)^2 \left(\frac{\langle \nabla u_{n+1}, g \rangle_L}{\|g\|_L} \right)^2 .$$

It is clear that the domain of $\tilde{Q}_k$ extends to nonzero vectors in the space $\tilde{\mathcal{H}}_k$ and that for all nonzero scalars c, $\tilde{Q}_k(cg) = \tilde{Q}_k(g)$. It follows that $\tilde{Q}_k$ is maximized when $g = c\nabla u_{n+1}$ where c is some nonzero constant. Computing $\tilde{Q}_k(\nabla u_{n+1})$ we see that

$$\begin{aligned} \sup_{g \in \tilde{\mathcal{H}}_k} \tilde{Q}_k(g) &= \tilde{Q}_k(c\nabla u_{n+1}) \\ &= \frac{\left(\int_D u_{2n+1} \right)^2}{\int_D \|\nabla L^n u_{2n+1}\|_L^2} \end{aligned}$$

where we have used (3.2) on the numerator.

Note that $\|\nabla L^n u_{2n+1}\|_L^2 = \frac{(-1)^n}{(n+1)!}(2n+1)! \langle \nabla u_{n+1}, \nabla L^n u_{2n+1} \rangle_L$. Applying (3.2) of Lemma 3.1 to the denominator we obtain

$$\begin{aligned} \sup_{g \in \tilde{\mathcal{H}}_k} \tilde{Q}_k(g) &= \frac{\left(\int_D u_{2n+1} \right)^2}{(2n+1)! \int_D u_{2n+1}} \\ &= \frac{1}{k!} \mathcal{E}_k(D) \end{aligned}$$

which establishes (1.2) of Theorem 1.1.

References

[AK] Athreya, K. B. and Kurtz, T. G. (1973), *A generalization of Dynkin's identity*, Ann. of Prob. **1**, 570–579. MR **50**:1342

[H] Ha'sminskii, R. Z. (1960), *Probabilistic representation of the solution of some differential equations*, Proc. 6th All Union Conf. on Theor. Probability and Math. Statist. (Vilnius 1960) (See Math Rev. 3127 v.32 (1966) pg. 632). MR **32**:3127

[KM1] Kinateder, K. K. J. and McDonald, P. (1997), *Brownian functionals on hypersurfaces in Euclidean space*, Proc. Amer. Math. Soc. **125** **6**, 1815–1822. MR **97g**:60109

[KMM] Kinateder, K. K. J., McDonald, P., and Miller, D. F., *Boundary Value Problems and Exit Times for Diffusions in Euclidean Space*. Prob. Th. Rel. **111** (1998), pp. 469–487. CMP 99:01

[P1] Pinsky, M., *Mean exit times and hitting probabilities of Brownian motion in geodesic balls and tubular neighborhoods*, Stochastic Processes - Mathematics and Physics, Lecture Notes in Mathematics, vol. 1158, Springer Verlag, New York, pp. 216–223. MR **87g**:58129

[P2] Polya, G. (1948), *Torsional rigidity, principal frequency, electrostatic capacity and symmetrization*, Quart. Appl. Math. **6**, 267–277. MR **10**:206b

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