

Wright State University

CORE Scholar

---

Mathematics and Statistics Faculty  
Publications

Mathematics and Statistics

---

4-15-2011

## Multiple Solutions for an Elliptic Problem Related to Vortex Pairs

Yi Li

Wright State University - Main Campus, yi.li@csun.edu

Shuangjie Peng

Follow this and additional works at: <https://corescholar.libraries.wright.edu/math>



Part of the [Applied Mathematics Commons](#), [Applied Statistics Commons](#), and the [Mathematics Commons](#)

---

### Repository Citation

Li, Y., & Peng, S. (2011). Multiple Solutions for an Elliptic Problem Related to Vortex Pairs. *Journal of Differential Equations*, 250 (8), 3448-3472.

<https://corescholar.libraries.wright.edu/math/66>

This Article is brought to you for free and open access by the Mathematics and Statistics department at CORE Scholar. It has been accepted for inclusion in Mathematics and Statistics Faculty Publications by an authorized administrator of CORE Scholar. For more information, please contact [library-corescholar@wright.edu](mailto:library-corescholar@wright.edu).

# Multiple Solutions to an Elliptic Problem Related to Vortex Pairs

Yi Li

Department of Mathematics,  
the University of Iowa  
Iowa City, IA, 52242  
U.S.A

E-mail: yli@math.uiowa.edu

Shuangjie Peng

School of Mathematics and Statistics  
Central China Normal University  
Wuhan, 430079  
P. R. China

E-mail: sjpeng@mail.ccnu.edu.cn

## 1 Introduction

We consider the problem

$$(1.1) \quad \begin{cases} -\Delta u = \lambda(u - \varphi)_+^{p-1}, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases}$$

where  $\Omega$  is a bounded domain in  $\mathbb{R}^N$  ( $N \geq 2$ ),  $\lambda \in \mathbb{R}_+^N$ ,  $2 < p < 2N/(N-2)$  for  $N \geq 3$  and  $2 < p < \infty$  for  $N = 2$ ,  $\varphi$  is a positive harmonic function in  $\bar{\Omega}$ .

Problem (??) is related to steady vortex pairs: a steady vortex ring corresponding mathematically to a Stokes stream function  $\Psi$  defined on a bounded domain  $\bar{\Omega}$ , and an open set  $A \subset \Omega \subset \mathbb{R}^N$  called the cross-section of a steady vortex ring and unknown a priori. The function  $\Psi \in C^1(\bar{\Omega}) \cap C^2(\Omega \setminus A)$  and the open set  $A$  satisfy the following free-boundary problem in cylindrical coordinates (see, for example, [?])

$$(1.2) \quad -L\Psi = \begin{cases} \lambda f(\Psi), & x \in A, \\ 0, & x \in \Omega \setminus \bar{A}, \end{cases}$$

$$(1.3) \quad \Psi|_{\partial A} = 0, \quad \Psi|_{\partial\Omega} = -\frac{1}{2}B|x'|^{N-1} - \kappa < 0,$$

where  $L = 1/r^{(N-2)}(\partial/\partial r)(r^{N-2}\partial/\partial r) + \partial^2/\partial x_N^2$ ,  $x' = (x_1, \dots, x_{N-1})$ . The vorticity function  $f(t)$  is positive if  $t > 0$  and is zero if  $t \leq 0$ , while  $B > 0$  and  $\kappa$  are prescribed constants. In general, Problems (??) and (??) can be written as the following free boundary problem

$$(1.4) \quad -\Delta\Psi = \begin{cases} \lambda f(\Psi), & x \in A, \\ 0, & x \in \Omega \setminus \bar{A}, \end{cases}$$

$$(1.5) \quad \Psi|_{\partial A} = 0, \quad \Psi|_{\partial\Omega} = -\varphi_0 < 0,$$

where  $\varphi_0$  is a  $C^1$  function defined on  $\partial\Omega$ . Let  $\varphi$  be the solution of

$$(1.6) \quad \begin{cases} -\Delta\varphi = 0, & x \in \Omega, \\ \varphi = \varphi_0, & x \in \partial\Omega. \end{cases}$$

Then,  $\varphi > 0$  achieves its maximum and minimum on  $\Omega$ .

Let  $u = \Psi + \varphi$  and  $A = \{x \in \Omega : u > \varphi\}$ , then Problem (??) and (??) become

$$(1.7) \quad \begin{cases} -\Delta u = \lambda f(u - \varphi), & x \in \Omega, \\ \varphi = 0, & x \in \partial\Omega. \end{cases}$$

In this paper, we investigate Problem (??) to obtain its solution pairs  $(u_\lambda, A_\lambda)$  for  $\lambda$  sufficiently large, where  $A_\lambda = \{x \in \Omega : u_\lambda > \varphi\}$ .

There are many existence results for Problem (??) under various assumptions. In [?, ?, ?, ?, ?, ?], the solutions were obtained by using the mountain pass lemma for various nonlinearities  $f(x, u)$  and any  $\lambda > 0$ . In [?, ?, ?, ?, ?], to find the solutions, the constrained variation methods were used, but the vorticity function  $f$  is unknown a priori. Moreover, in [?, ?, ?], the solutions were obtained by regarding  $\lambda$  as eigenvalue, so  $\lambda$  is not arbitrary.

The asymptotic behavior of the solution pair  $(u_\lambda, A_\lambda)$  of Problem (??) was investigated in [?, ?, ?, ?, ?, ?, ?]. More precisely, it is verified that the cross-section  $A_\lambda$  of a steady vortex ring shrinks to a point, and a vortex ring degenerates into a singular vortex circle as  $\lambda \rightarrow +\infty$ . Moreover, the Stokes stream function  $\Psi_\lambda$  of a vortex ring converges to the Stokes stream function of the filament, which is the Green's function of the operator  $-\Delta$  in  $\Omega$ . In [?], Ambrosetti and Yang discussed the existence of solutions by using the mountain pass lemma and investigated the asymptotic behavior of the solution pair by estimating the upper bound of the critical value of the functional corresponding to Problem (??). Recently, Li, Yan and Yang[?] studied Problem (??) in the case  $N = 2$ . More precisely, they proved that the ‘‘vortex core’’  $A_\lambda$  shrinks to point  $x_0$  which is exactly the minimum point of  $\varphi_0$  on the boundary as  $\lambda \rightarrow +\infty$ . Furthermore, they verified that  $\frac{u_\lambda}{\lambda \int_\Omega f(u_\lambda - \varphi)}$  tends to the Green's function of  $\Omega$  both in  $W^{1,p}(\Omega)$  and  $C_{loc}^{1,\alpha}(\Omega \setminus x_0)$  as  $\lambda \rightarrow +\infty$ . To obtain their results, Li, Yan and Yang gave very delicate estimates to the upper and lower bounds of the least energy solution. In [?], the limiting behavior of the mountain pass solution to Problem (??) with  $N \geq 3$  and  $\varphi \equiv 1$  was given.

We want to point out that most of the solutions mentioned here are in some sense the least energy solutions and the ‘‘vortex core’’ shrinks to a single point. In this paper, we want to find some high energy solutions whose ‘‘vortex core’’ consists of multiple components which shrink to distinct points on  $\partial\Omega$  as  $\lambda \rightarrow +\infty$ .

In the paper, we assume that the harmonic function  $\varphi \in C^1(\overline{\Omega})$  satisfies that

$$(H) \quad \varphi \text{ has } k \text{ } (k \geq 1) \text{ strictly local minimum points } \bar{z}_1, \dots, \bar{z}_k \in \partial\Omega.$$

Our main result is

**Theorem 1.1.** *Let  $\varphi$  satisfy (H). Then exists  $\lambda_0 > 0$  such that, for any  $\lambda \in [\lambda_0, \infty)$ , Problem (??) has solution pair  $(u_\lambda, A_\lambda)$  satisfying that*

(i) *the “vortex core”  $A_\lambda$  has exactly  $k$  components  $A_{\lambda,j}$ ,  $j = 1, \dots, k$  which shrink to the points  $\bar{z}_1, \dots, \bar{z}_k$  respectively as  $\lambda \rightarrow +\infty$ . Moreover,  $A_{\lambda,j}$  is approximately a ball and*

$$\text{diam}(A_{\lambda,j}) \sim \begin{cases} 2 \left( \frac{|\phi'(1)|}{(N-2)\varphi(\bar{z}_j)} \right)^{\frac{p-2}{2}} \lambda^{-\frac{1}{2}}, & N \geq 3 \\ 2 \left( \frac{|\phi'(1)|}{\varphi(\bar{z}_j)} \right)^{\frac{p-2}{2}} \left( \lambda^{-\frac{1}{2}} |\ln \lambda|^{\frac{p-2}{2}} \right), & N = 2, \end{cases}$$

where  $\phi$  is the function defined by (??);

(ii)  $u_\lambda$  has exactly  $k$  local maximum points  $z_1 \in A_{\lambda,1}, \dots, z_k \in A_{\lambda,k}$  which satisfy

$$\begin{cases} |z_j - \bar{z}_j| = O(\lambda^{-\frac{N-2}{2(N-1)}}), \quad \text{dist}(z_j, \partial\Omega) = O(\lambda^{-\frac{N-2}{2(N-1)}}), & N \geq 3 \\ |z_j - \bar{z}_j| = O\left(\frac{\ln \ln \lambda}{\ln \lambda}\right), \quad \text{dist}(z_j, \partial\Omega) = O\left(\frac{1}{(\ln \lambda)^c}\right), & N = 2, \end{cases}$$

where  $j = 1, \dots, k$ ,  $c > 0$  is a constant.

Let us outline the proof of the main result of this paper. The solutions in [?] were obtained by finding a minimizer of the corresponding functional in a suitable function space. These method is hard to obtain solutions whose “vortex core” has several components. In the present paper, we will use a reduction argument to this kind of solutions.

To apply a reduction argument to prove Theorems ??, we need to construct an approximate solution for (??). In the case  $N \geq 3$ , we can scale the solution to the corresponding limit problem. But, in the case  $N = 2$ , the corresponding “limit” problem in  $R^2$  has no bounded solution. So, we will follow the method in [?] to construct an approximate solution. We should point out here, as we can see later, in the both two cases, since the local maximum points of the solution approach the boundary, the regular part of the green’s function tends to infinity, which causes a difficulty to us. We must take this influence into careful consideration and give a very exact estimate on the distance between the maximum points of the solutions and the boundary when we construct the approximate solutions. Compared with the known results, our result provides an exact description to the profile of the solution pair  $(u_\lambda, A_\lambda)$ .

This paper is organized as follows. In section 2, we construct the approximate solution for (??). We will carry out a reduction argument in section 3 and the main results will be proved in section 4 and section 5.

## 2 Approximate solutions and main result

It is convenient to change (??) to an equivalent problem. Let  $\varepsilon^2 = \frac{1}{\lambda}$ , that is, we consider the following problem

$$(2.1) \quad \begin{cases} -\varepsilon^2 \Delta u = (u - \varphi)_+^{p-1}, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases}$$

In this section, we will construct the approximate solution for (??).

Firstly, we consider the case  $N \geq 3$ . For  $c > 0$ , the following equation

$$(2.2) \quad \begin{cases} -\Delta u = (u - c)_+^{p-1}, & x \in \mathbb{R}^N, \\ u(0) = \max_{y \in \mathbb{R}^N} u(x), \\ u(x) \rightarrow 0, & \text{as } |x| \rightarrow +\infty, \end{cases}$$

has a unique solution  $W_c(x) = W_c(|x|)$  in  $D^{1,2}(\mathbb{R}^N)$ , which can be written as

$$W_c = \begin{cases} R_c^{-\frac{2}{p-2}} \phi\left(\frac{x}{R_c}\right) + c, & |x| \leq R_c = \left[\frac{\phi'(1)}{(2-N)c}\right]^{\frac{p-2}{2}}, \\ cR_c^{N-2}|x|^{2-N}, & |x| \geq R_c, \end{cases}$$

where  $\phi(x) = \phi(|x|)$  is the unique solution of

$$(2.3) \quad -\Delta \phi = \phi^{p-1}, \quad \phi > 0, \quad \phi \in H_0^1(B_1(0)).$$

From Flucher and Wei[?], we see that  $W_c$  is also non-degenerate. This is, the kernel of the operator  $Lv = -\Delta v - (p-1)(W_c - c)_+^{p-2}v$ ,  $v \in D^{1,2}(\mathbb{R}^N)$  is spanned by  $\left\{\frac{\partial W_c}{\partial x_1}, \dots, \frac{\partial W_c}{\partial x_N}\right\}$ . Note the operator  $L$  is not non-degenerate in the space of bounded functions.

For any  $z \in \Omega$ , define  $W_{\varepsilon,z,c}(x) = W_c\left(\frac{x-z}{\varepsilon}\right)$ . Because  $W_{\varepsilon,z,c}$  does not belong to  $H_0^1(\Omega)$ , we need to make a projection. Let  $PW_{\varepsilon,z,c}$  satisfies

$$(2.4) \quad \begin{cases} -\varepsilon^2 \Delta v = (W_{\varepsilon,z,c} - c)_+^{p-1}, & x \in \Omega, \\ v = 0, & x \in \partial\Omega. \end{cases}$$

Then

$$(2.5) \quad PW_{\varepsilon,z,c} = W_{\varepsilon,z,c} - \varepsilon^{N-2} c R_c^{N-2} h(x, z),$$

where  $h(x, z)$  is the regular part of the Green's function in  $\Omega$  and satisfies

$$\begin{cases} -\Delta h = 0, & \text{in } \Omega, \\ h = |x - z|^{2-N}, & \text{on } \partial\Omega. \end{cases}$$

Now, we consider the case  $N = 2$ . Since (2.1) has no solutions in this case, we use the approximate solution in [?]. Let  $R > 0$  be a large constant, such that for any  $x \in \Omega$ ,  $\Omega \subset B_R(x)$ . Consider the following problem

$$(2.6) \quad \begin{cases} -\varepsilon^2 \Delta u = (u - a)_+^{p-1}, & \text{in } B_R(0), \\ u = 0, & \text{on } \partial B_R(0), \end{cases}$$

where  $a > 0$  is a constant. Then, (??) has a unique solution  $U_{\varepsilon,a}$  with the form

$$(2.7) \quad U_{\varepsilon,a}(x) = \begin{cases} a + \varepsilon^{2/(p-2)} s_\varepsilon^{-2/(p-2)} \phi\left(\frac{|x|}{s_\varepsilon}\right), & |x| \leq s_\varepsilon \\ a \ln \frac{|x|}{R} / \ln \frac{s_\varepsilon}{R}, & s_\varepsilon \leq |x| \leq R, \end{cases}$$

where  $\phi$  solves (??), and  $s_\varepsilon \in (0, R)$  satisfies

$$(2.8) \quad \varepsilon^{2/(p-2)} s_\varepsilon^{-2/(p-2)} \phi'(1) = \frac{a}{\ln(s_\varepsilon/R)},$$

which implies

$$\frac{s_\varepsilon}{\varepsilon |\ln \varepsilon|^{(p-2)/2}} \rightarrow \left( \frac{|\phi'(1)|}{a} \right)^{(p-2)/2} > 0, \text{ as } \varepsilon \rightarrow 0.$$

For any  $z \in \Omega$ , define  $U_{\varepsilon,z,a}(x) = U_{\varepsilon,a}(x - z)$ . Let  $PU_{\varepsilon,z,a}$  be the solution of

$$\begin{cases} -\varepsilon^2 \Delta w = (U_{\varepsilon,z,a} - a)_+^{p-1}, & \text{in } \Omega, \\ w = 0, & \text{on } \partial\Omega. \end{cases}$$

Then

$$(2.9) \quad PU_{\varepsilon,z,a} = U_{\varepsilon,z,a} - \frac{a}{\ln \frac{R}{s_\varepsilon}} g(x, z),$$

where  $g(x, z)$  satisfies

$$\begin{cases} -\Delta g = 0, & \text{in } \Omega; \\ g = \ln \frac{R}{|x-z|}, & \text{on } \partial\Omega. \end{cases}$$

It is easy to see that

$$g(x, z) = \ln R + 2\pi h(x, z),$$

where  $h(x, z)$  is the regular part of the Green's function in  $\Omega$ .

We will construct solutions for (??) of the form

$$\sum_{j=1}^k PW_{\varepsilon, z_{\varepsilon,j}, c_{\varepsilon,j}} + \omega_\varepsilon, \quad (N \geq 3) \text{ or } \sum_{j=1}^k PU_{\varepsilon, z_{\varepsilon,j}, a_{\varepsilon,j}} + \omega_\varepsilon, \quad (N = 2),$$

where  $\omega_\varepsilon$  is a perturbation term. To obtain an appropriately approximate solution, we need to choose suitable  $c_{\varepsilon,j}$  and  $a_{\varepsilon,j}$ .

Denote  $Z = (z_1, \dots, z_k) \in R^{Nk}$ . In the sequel, we always assume that  $z_j \in \Omega$  ( $j = 1, \dots, k$ ) satisfies

$$(2.10) \quad \begin{aligned} &|z_j - \bar{z}_j| < \delta, \\ &d_j \triangleq d(z_j, \partial\Omega) \geq \gamma \varepsilon^{(N-2)/(N-1)} \quad (N \geq 3), \text{ or } d_j \triangleq d(z_j, \partial\Omega) \geq \frac{1}{|\ln \varepsilon|^\alpha}, \quad (N = 2), \end{aligned}$$

where  $\gamma, \delta > 0$  are small constants and  $\alpha > 0$  is a large number.

For  $i = 1, \dots, k$ , set

$$(2.11) \quad c_{\varepsilon,i} = \varphi(z_i) + \varepsilon^{N-2} \varphi(z_i) R_{\varphi(z_i)}^{N-2} h(z_i, z_i)$$

and let  $a_{\varepsilon,i}(Z)$  be the solution of the following problem

$$(2.12) \quad a_i = \varphi(z_i) + \frac{a_{\varepsilon,i}}{\ln \frac{R}{\varepsilon}} g(z_i, z_i) - \sum_{j \neq i} \frac{1}{\ln \frac{R}{\varepsilon}} a_{\varepsilon,j} G(z_i, z_j),$$

where  $G(x, z_j) = \ln \frac{|x-z_j|}{R} - g(y, z_j)$ . We can solve (??) and find

$$(2.13) \quad a_{\varepsilon,i}(Z) = \frac{\varphi(z_i) - \frac{1}{\ln \frac{R}{\varepsilon}} \sum_{j \neq i} a_{\varepsilon,j}(Z) G(z_i, z_j)}{1 - \frac{g(z_i, z_i)}{\ln \frac{R}{\varepsilon}}}.$$

Moreover,  $c_{\varepsilon,i}$  and  $a_{\varepsilon,i}(Z)$  are smooth in  $z_j, j = 1, \dots, k$ .

Define

$$(2.14) \quad P_{\varepsilon,i} = PW_{\varepsilon, z_i, c_{\varepsilon,i}} \quad \text{and} \quad V_{\varepsilon, Z, j} = PU_{\varepsilon, z_j, a_{\varepsilon,j}(Z)}.$$

Set  $s_{\varepsilon,i}$  to be the solution of

$$\varepsilon^{2/(p-2)} s_{\varepsilon,i}^{-2/(p-2)} \phi'(1) = \frac{a_{\varepsilon,i}}{\ln(s_{\varepsilon,i}/R)},$$

then, we see

$$\frac{1}{\ln(R/s_{\varepsilon,i})} = \frac{1}{\ln(R/\varepsilon)} + O\left(\frac{\ln |\ln \varepsilon|}{|\ln \varepsilon|^2}\right).$$

Now, using the facts (see [?] and [?]) that for  $j = 1, \dots, k$ ,

$$h(z_j, z_j) = \begin{cases} \frac{C}{d_j^{N-2}}, & N \geq 3, \\ C |\ln d_j|, & N = 2, \end{cases}$$

$$\frac{\partial h(z_j, z_j)}{\partial z_{j,l}} = \begin{cases} \frac{C}{d_j^{N-1}}, & l = 1, \dots, N, \quad N \geq 3, \\ \frac{C}{d_j}, & l = 1, 2, \quad N = 2, \end{cases}$$

we find that for any fixed constant  $L > 0$ ,

$$\begin{aligned} & P_{\varepsilon,i}(x) - \varphi(x) \\ &= W_{\varepsilon,z_i,c_{\varepsilon,i}} - \varphi(z_i) - \varepsilon^{N-2} \varphi(z_i) R_{\varphi(z_i)}^{N-2} h(z_i, z_i) \\ &\quad - (\varphi(x) - \varphi(z_i)) - \varepsilon^{N-2} \varphi(z_i) R_{\varphi(z_i)}^{N-2} \langle Dh(x, z_i), x - z_i \rangle + O(\varepsilon^{N/(N-1)}) \\ &= W_{\varepsilon,z_i,c_{\varepsilon,i}} - c_{\varepsilon,i} + O(\varepsilon), \quad \text{for } x \in B_{L\varepsilon}(z_i), \end{aligned}$$

$$\begin{aligned} V_{\varepsilon,Z,i}(x) - \varphi(x) &= U_{\varepsilon,z_i,a_{\varepsilon,i}}(Z)(x) - \varphi(x) - \frac{a_{\varepsilon,i}}{\ln \frac{R}{s_{\varepsilon,i}}} g(x, z_i) \\ &= U_{\varepsilon,z_i,a_{\varepsilon,i}}(Z)(x) - \varphi(z_i) - \frac{a_{\varepsilon,i}}{\ln \frac{R}{s_{\varepsilon,i}}} g(z_i, z_i) + O(s_{\varepsilon,i} |\ln \varepsilon|^{\alpha-1}) + O(s_{\varepsilon,i}) \\ &= U_{\varepsilon,z_i,a_{\varepsilon,i}}(Z)(x) - \varphi(z_i) - \frac{a_{\varepsilon,i}}{\ln \frac{R}{\varepsilon}} g(z_i, z_i) + O\left(\frac{\ln |\ln \varepsilon|}{|\ln \varepsilon|^2}\right) g(z_i, z_i) \\ &\quad + O(\varepsilon |\ln \varepsilon|^{\alpha-2+p/2}) \\ &= U_{\varepsilon,z_i,a_{\varepsilon,i}}(Z)(x) - \varphi(z_i) - \frac{a_{\varepsilon,i}}{\ln \frac{R}{\varepsilon}} g(z_i, z_i) + O\left(\frac{(\ln |\ln \varepsilon|)^2}{|\ln \varepsilon|^2}\right), \quad \text{for } x \in B_{Ls_{\varepsilon,i}}(z_i), \end{aligned}$$

and for  $j \neq i$ , by (??)

$$\begin{aligned} P_{\varepsilon,j}(x) &= W_{\varepsilon,z_j,c_{\varepsilon,j}} - \varepsilon^{N-2} \varphi(z_j) R_{\varphi(z_j)}^{N-2} h(x, z_j) = O(\varepsilon^{N-2}), \quad \text{for } x \in B_{L\varepsilon}(z_i), \\ V_{\varepsilon,Z,j}(x) &= U_{\varepsilon,z_j,a_{\varepsilon,j}}(Z)(x) - \frac{a_{\varepsilon,j}}{\ln \frac{R}{s_{\varepsilon}}} a_{\varepsilon,j} g(y, z_j) = \frac{a_{\varepsilon,j}}{\ln \frac{R}{s_{\varepsilon}}} G(y, z_j) \\ &= \frac{1}{\ln \frac{R}{s_{\varepsilon}}} a_{\varepsilon,j} G(z_i, z_j) + \frac{1}{\ln \frac{R}{s_{\varepsilon}}} a_{\varepsilon,j} \langle DG(z_i, z_j), y - z_i \rangle + O(\varepsilon^2 |\ln \varepsilon|^{p-1}) \\ &= \frac{1}{\ln \frac{R}{\varepsilon}} a_{\varepsilon,j} G(z_i, z_j) + O\left(\frac{\ln |\ln \varepsilon|}{|\ln \varepsilon|^2}\right), \quad \text{for } x \in B_{Ls_{\varepsilon,i}}(z_i). \end{aligned}$$

So, we obtain

$$(2.15) \quad \sum_{j=1}^k P_{\varepsilon,j}(x) - \varphi(x) = W_{\varepsilon,z_i,c_{\varepsilon,i}}(x) - c_{\varepsilon,i} + O(\varepsilon), \quad x \in B_{L\varepsilon}(z_i),$$

$$(2.16) \quad \sum_{j=1}^k V_{\varepsilon,Z,j}(x) - \varphi(x) = U_{\varepsilon,z_i,a_{\varepsilon,i}}(Z)(x) - a_{\varepsilon,i}(Z) + O\left(\frac{(\ln |\ln \varepsilon|)^2}{|\ln \varepsilon|^2}\right), \quad x \in B_{Ls_{\varepsilon,i}}(z_i).$$

From the above discussion, we see to prove Theorem ??, we just need to prove the following theorem



**Theorem 2.1.** *Let  $\varphi$  satisfy  $H$ , then there exists  $\varepsilon_0 > 0$ , such that Problem (??) has a solution  $u_\varepsilon$  with the form*

$$u_\varepsilon = \sum_{j=1}^k P_{\varepsilon,j} + \omega_\varepsilon \quad (N \geq 3), \quad \text{or} \quad u_\varepsilon = \sum_{j=1}^k V_{\varepsilon,Z_\varepsilon,j} + \omega_\varepsilon \quad (N = 2),$$

satisfying for  $j = 1, \dots, k$

$$\|\omega_\varepsilon\|_{L^\infty(\Omega)} = O(\varepsilon^{1-\theta}), \quad |z_{\varepsilon,j} - \bar{z}_j| = C_1 \varepsilon^{\frac{N-2}{N-1}}, \quad \text{dist}(z_{\varepsilon,j}, \partial\Omega) = C_2 \varepsilon^{\frac{N-2}{N-1}}, \quad (N \geq 3),$$

or

$$\|\omega_\varepsilon\|_{L^\infty(\Omega)} = O\left(\frac{(\ln|\ln\varepsilon|)^2}{|\ln\varepsilon|^2}\right), \quad |z_{\varepsilon,j} - \bar{z}_j| \leq C\left(\frac{\ln|\ln\varepsilon|}{|\ln\varepsilon|}\right), \quad \text{dist}(z_{\varepsilon,j}, \partial\Omega) \geq \frac{1}{|\ln\varepsilon|^\alpha}, \quad (N = 2),$$

where  $C_1, C_2, C$  and  $\alpha \geq 1$  are positive constants,  $\theta > 0$  is any small constant.

To complete this section, we give the following lemma which will be used in our estimates.

**Lemma 2.2.** *There is a large  $L > 0$  such that*

$$\sum_{j=1}^k P_{\varepsilon,j}(x) - \varphi(x) < 0, \quad x \in \Omega \setminus \bigcup_{j=1}^k B_{L\varepsilon}(z_j).$$

$$\sum_{j=1}^k V_{\varepsilon,Z_\varepsilon,j}(x) - \varphi(x) < 0, \quad x \in \Omega \setminus \bigcup_{j=1}^k B_{Ls_\varepsilon,j}(z_j).$$

*Proof.* The case  $N = 2$  can be proved as in [?].

For the case  $N \geq 3$ , we just need to choose  $L > 0$  such that

$$\max_{z \in \Omega} \varphi(z) R_{\varphi(z)}^{N-2} L^{2-N} \leq \frac{1}{2} \min_{z \in \Omega} \varphi(z).$$

□

### 3 The reduction

We only consider the case  $N \geq 3$  in this section since for the case  $N = 2$ , the argument is similar and some basic estimates are given in [?].

We recall that  $Z = (z_1, \dots, z_N)$  and  $z_j \in \Omega$  satisfies

$$(3.1) \quad |z_j - \bar{z}_j| < \delta, \quad d_j := d(z_j, \partial\Omega) \geq \gamma \varepsilon^{(N-2)/(N-1)},$$

where  $\gamma > 0$  is sufficiently small.

Let

$$F_{\varepsilon,Z} = \left\{ u \in L^\infty(\Omega) : \int_{\Omega} \frac{\partial P_{\varepsilon,j}}{\partial z_{j,h}} u = 0, j = 1, \dots, k, h = 1, \dots, N \right\},$$

and

$$E_{\varepsilon,Z} = \left\{ u \in W^{2,\infty}(\Omega) \cap H_0^1(\Omega) : \int_{\Omega} \Delta \left( \frac{\partial P_{\varepsilon,j}}{\partial z_{j,h}} \right) u = 0, j = 1, \dots, k, h = 1, \dots, N \right\}.$$

We define  $Q_\varepsilon$  to be the projection from  $L^\infty(\Omega)$  to  $F_{\varepsilon,Z}$  as follows:

$$Q_\varepsilon u = u - \sum_{j=1}^k \sum_{h=1}^N b_{j,h} \left( -\varepsilon^2 \Delta \frac{\partial P_{\varepsilon,j}}{\partial z_{j,h}} \right).$$

Hence the constants  $b_{j,h}$ ,  $j = 1, \dots, k$ ,  $h = 1, \dots, N$ , satisfy

$$(3.2) \quad \sum_{j=1}^k \sum_{h=1}^2 b_{j,h} \left( -\varepsilon^2 \int_{\Omega} \Delta \left( \frac{\partial P_{\varepsilon,j}}{\partial z_{j,h}} \right) \frac{\partial P_{\varepsilon,i}}{\partial z_{i,l}} \right) = \int_{\Omega} u \frac{\partial P_{\varepsilon,i}}{\partial z_{i,l}}.$$

Define

$$L_\varepsilon u = -\varepsilon^2 \Delta u - (p-1) \left( \sum_{j=1}^k P_{\varepsilon,j} - \varphi \right)_+^{p-2} u.$$

We have

**Lemma 3.1.** *There exist constants  $\rho_0 > 0$  and  $\varepsilon_0 > 0$ , such that for any  $\varepsilon \in (0, \varepsilon_0]$ ,  $Z$  satisfying (??),  $u \in E_{\varepsilon,Z}$  with  $Q_\varepsilon L_\varepsilon u = 0$  in  $\Omega \setminus \cup_{j=1}^k B_{L\varepsilon}(z_j)$ , then*

$$\|Q_\varepsilon L_\varepsilon u\|_{L^\infty(\Omega)} \geq \rho_0 \|u\|_{L^\infty(\Omega)}.$$

*Proof.* We will use  $\|\cdot\|_\infty$  to denote  $\|\cdot\|_{L^\infty(\Omega)}$ .

We argue by contradiction. Suppose that there are  $\varepsilon_n \rightarrow 0$ ,  $Z_n$  satisfying (??), and  $u_n \in E_{\varepsilon_n, Z_n}$  with  $Q_{\varepsilon_n} L_{\varepsilon_n} u_n = 0$  in  $\Omega \setminus \cup_{j=1}^k B_{L\varepsilon_n}(z_{j,n})$ , such that

$$\|Q_{\varepsilon_n} L_{\varepsilon_n} u_n\|_\infty \leq \frac{1}{n},$$

and  $\|u_n\|_\infty = 1$ .

Firstly, we estimate  $b_{j,h,n}$  in the following formula

$$(3.3) \quad Q_{\varepsilon_n} L_{\varepsilon_n} u_n = L_{\varepsilon_n} u_n - (p-1) \sum_{j=1}^k \sum_{h=1}^N b_{j,h,n} \left( W_{\varepsilon_n, z_{j,n}, c_{\varepsilon_n, j}} - c_{\varepsilon_n, j} \right)_+^{p-2} \left( \frac{\partial W_{\varepsilon_n, z_{j,n}, c_{\varepsilon_n, j}}}{\partial z_{j,h}} - \frac{\partial c_{\varepsilon_n, j}}{\partial z_{j,h}} \right).$$

For each fixed  $i$ , multiplying (??) by  $\frac{\partial P_{\varepsilon_n, i, n}}{\partial z_{i, l}}$ , noting that  $Q_{\varepsilon_n} L_{\varepsilon_n} u_n$  belongs to  $F_{\varepsilon_n, Z}$ , we obtain

$$\begin{aligned} & \int_{\Omega} u_n L_{\varepsilon_n} \frac{\partial P_{\varepsilon_n, i}}{\partial z_{i, l}} = \int_{\Omega} L_{\varepsilon_n} u_n \frac{\partial P_{\varepsilon_n, i}}{\partial z_{i, l}} \\ & = (p-1) \sum_{j=1}^k \sum_{h=1}^N b_{j, h, n} \int_{\Omega} (W_{\varepsilon_n, z_j, n, c_{\varepsilon_n, j}} - c_{\varepsilon_n, j})_+^{p-2} \left( \frac{\partial W_{\varepsilon_n, z_j, n, c_{\varepsilon_n, j}}}{\partial z_{j, h}} - \frac{\partial c_{\varepsilon_n, j}}{\partial z_{j, h}} \right) \frac{\partial P_{\varepsilon_n, i}}{\partial z_{i, l}}. \end{aligned}$$

By (??), we see

$$\begin{aligned} (3.4) \quad & \int_{\Omega} (W_{\varepsilon_n, z_j, n, c_{\varepsilon_n, j}} - c_{\varepsilon_n, j})_+^{p-2} \left( \frac{\partial W_{\varepsilon_n, z_j, n, c_{\varepsilon_n, j}}}{\partial z_{j, h}} - \frac{\partial c_{\varepsilon_n, j}}{\partial z_{j, h}} \right) \frac{\partial P_{\varepsilon_n, i}}{\partial z_{i, l}} \\ & = \int_{\Omega} (W_{\varepsilon_n, z_j, n, c_{\varepsilon_n, j}} - c_{\varepsilon_n, j})_+^{p-2} \frac{\partial W_{\varepsilon_n, z_j, n, c_{\varepsilon_n, j}}}{\partial z_{j, h}} \frac{\partial P_{\varepsilon_n, i}}{\partial z_{i, l}} + O(\varepsilon_n^{N-1}) \\ & = c_0(\delta_{jihl} + O(\varepsilon_n))\varepsilon_n^{N-2}, \end{aligned}$$

where  $\delta_{jihl} = 1$  if  $j = i$  and  $h = l$ , and  $\delta_{jihl} = 0$  if otherwise.

On the other hand, using (??) and Lemma ??, we obtain

$$\begin{aligned} & \int_{\Omega} \left( -\varepsilon_n^2 \Delta \frac{\partial P_{\varepsilon_n, i}}{\partial z_{i, l}} - (p-1) \left( \sum_{j=1}^k P_{\varepsilon_n, j} - \varphi \right)_+^{p-2} \frac{\partial P_{\varepsilon_n, i}}{\partial z_{i, l}} \right) u_n \\ & = (p-1) \int_{B_{L\varepsilon_n}(z_{i, n})} (W_{\varepsilon_n, z_{i, n}, c_{\varepsilon_n, i, n}} - c_{\varepsilon_n, i, n})_+^{p-2} \left( \frac{\partial W_{\varepsilon_n, z_{i, n}, c_{\varepsilon_n, i, n}}}{\partial z_{i, l}} - \frac{\partial c_{\varepsilon_n, i, n}}{\partial z_{i, h}} \right) u_n \\ & \quad - (p-1) \int_{B_{L\varepsilon_n}(z_{i, n})} (W_{\varepsilon_n, z_{i, n}, c_{\varepsilon_n, i, n}} - c_{\varepsilon_n, i, n} + O(\varepsilon))_+^{p-2} \frac{\partial P_{\varepsilon_n, i}}{\partial z_{i, l}} u_n \\ & = O(\varepsilon_n^N), \end{aligned}$$

which, together with (??) and (??), implies

$$b_{i, h, n} = O(\varepsilon_n^2).$$

Therefore,

$$\begin{aligned} & \sum_{j=1}^k \sum_{h=1}^2 b_{j, h, n} \left( -\varepsilon_n^2 \Delta \frac{\partial P_{\varepsilon_n, j}}{\partial z_{j, h}} \right) \\ & = (p-1) \sum_{j=1}^k \sum_{h=1}^2 b_{j, h, n} (W_{\varepsilon_n, z_j, n, c_{\varepsilon_n, j}} - c_{\varepsilon_n, j})_+^{p-2} \left( \frac{\partial W_{\varepsilon_n, z_j, n, c_{\varepsilon_n, j}}}{\partial z_{j, h}} - \frac{\partial c_{\varepsilon_n, j}}{\partial z_{j, h}} \right) \\ & = O\left( \sum_{j=1}^k \sum_{h=1}^2 |b_{j, h, n}| \varepsilon_n^{-1} \right) = O(\varepsilon_n). \end{aligned}$$

Thus, we obtain

$$L_{\varepsilon_n} u_n = Q_{\varepsilon_n} L_{\varepsilon_n} u_n + O(\varepsilon_n) = O\left(\frac{1}{n} + \varepsilon_n\right).$$

For any fixed  $i$ , define

$$\tilde{u}_{i,n}(y) = u_n(\varepsilon_n x + z_{i,n}).$$

Let

$$\tilde{L}_n u = -\Delta u - (p-1) \left( \sum_{j=1}^k P_{\varepsilon_n, j}(\varepsilon_n x + z_{i,n}) - \varphi(\varepsilon_n x + z_{i,n}) \right)_+^{p-2} u.$$

Then

$$\|\tilde{L}_n \tilde{u}_{i,n}\|_{\infty} = \|L_{\varepsilon_n} u_n\|_{\infty}.$$

As a result,

$$\tilde{L}_{\varepsilon_n} \tilde{u}_{i,n} = O\left(\frac{1}{n} + \varepsilon_n\right),$$

where  $\Omega_n = \{y : \varepsilon_n x + z_{i,n} \in \Omega\}$ .

Since  $\|\tilde{u}_{i,n}\|_{\infty} = 1$ , by the regularity theory on elliptic equations, we may assume that

$$\tilde{u}_{i,n} \rightarrow u_i, \quad \text{in } C_{loc}^1(\mathbb{R}^N).$$

It is easy to check that

$$\begin{aligned} & \left( \sum_{j=1}^k P_{\varepsilon_n, j}(\varepsilon_n x + z_{i,n}) - \varphi(\varepsilon_n x + z_{i,n}) \right)_+^{p-2} \\ &= \left( W_{\varepsilon_n, z_{i,n}, c_{\varepsilon_n, i}}(\varepsilon_n x + z_{i,n}) - c_{\varepsilon_n, i} + O(\varepsilon_n) \right)_+^{p-2} \\ &\rightarrow \left( W_{\varphi(z_0)} - \varphi(z_0) \right)_+^{p-2}, \end{aligned}$$

where  $z_0 \in \bar{\Omega}$ .

Hence,  $u_i$  satisfies the following equation

$$(3.5) \quad -\Delta u - (p-1)(W_{\varphi(z_0)} - \varphi(z_0))_+^{p-2} u = 0.$$

Now following the proof of [?], we can prove that any solution  $v$  to equation (??) must belong to  $\text{span}\left(\frac{\partial W_{\varphi(z_0)}}{\partial x_1}, \dots, \frac{\partial W_{\varphi(z_0)}}{\partial x_N}\right)$ . Hence

$$(3.6) \quad u_i = k_1^i \frac{\partial W_{\varphi(z_0)}}{\partial x_1} + \dots + k_N^i \frac{\partial W_{\varphi(z_0)}}{\partial x_N}.$$

Since

$$\int_{\Omega} \Delta \left( \frac{\partial P_{\varepsilon_n, i}}{\partial z_{i, h}} \right) u_n = 0,$$

we have

$$\int_{\mathbb{R}^N} (W_{\varphi(z_0)} - \varphi(z_0))_+^{p-2} \frac{\partial W_{\varphi(z_0)}}{\partial z_h} u_i = 0,$$

which, together with (??), gives  $u_i = 0$ . Thus,

$$u_n \rightarrow 0, \quad \text{in } C^1(B_{L\varepsilon_n}(z_{i, n})),$$

for any  $L > 0$ .

From Lemma ??, the assumption

$$Q_{\varepsilon_n} L_{\varepsilon_n} u_n = 0, \quad \text{in } \Omega \setminus \cup_{i=1}^k B_{L\varepsilon_n}(z_{i, n}).$$

gives

$$-\Delta u_n = 0, \quad y \in \Omega \setminus \cup_{i=1}^k B_{L\varepsilon_n}(z_{i, n}).$$

However,  $u_n = 0$  on  $\partial\Omega$  and  $u_n = o(1)$  on  $\partial B_{L\varepsilon_n}(z_{i, n})$ ,  $i = 1, \dots, k$ . So we have

$$u_n = o(1).$$

This is a contradiction. □

**Remark 3.2.** *Lemma ?? and the Fredholm alternative imply that  $Q_{\varepsilon} L_{\varepsilon}$  is one to one and onto from  $E_{\varepsilon, Z}$  to  $F_{\varepsilon, Z}$ .*

Now consider the equation

$$(3.7) \quad Q_{\varepsilon} L_{\varepsilon} \omega = Q_{\varepsilon} l_{\varepsilon} + Q_{\varepsilon} R_{\varepsilon}(\omega),$$

where

$$(3.8) \quad l_{\varepsilon} = \left( \sum_{j=1}^k P_{\varepsilon, j} - \varphi \right)_+^{p-1} - \sum_{j=1}^k (W_{\varepsilon, z_j, c_{\varepsilon, j}} - c_{\varepsilon, j})_+^{p-1},$$

and

$$(3.9) \quad R_{\varepsilon}(\omega) = \left( \sum_{j=1}^k P_{\varepsilon, j} - \varphi + \omega \right)_+^{p-1} - \left( \sum_{j=1}^k P_{\varepsilon, j} - \varphi \right)_+^{p-1} - (p-1) \left( \sum_{j=1}^k P_{\varepsilon, j} - \varphi \right)_+^{p-2} \omega.$$

By Remark ??, (??) can be rewritten as

$$(3.10) \quad \omega = G_{\varepsilon} \omega =: (Q_{\varepsilon} L_{\varepsilon})^{-1} Q_{\varepsilon} (l_{\varepsilon} + R_{\varepsilon}(\omega)).$$

The next proposition enable us to reduce the problem of finding a solution for (??) to a finite dimensional problem.

**Proposition 3.3.** *There is an  $\varepsilon_0 > 0$ , such that for any  $\varepsilon \in (0, \varepsilon_0]$  (??) has a unique solution  $\omega_\varepsilon$  with*

$$\|\omega_\varepsilon\|_\infty = O(\varepsilon^{1-\theta}),$$

where  $\theta$  is any small positive number.

*Proof.* Define

$$M = \tilde{E}_{\varepsilon,Z} \cap \{\|\omega\|_\infty \leq \varepsilon^{1-\theta}\},$$

where

$$\tilde{E}_{\varepsilon,Z} = \left\{ u \in L^\infty(\Omega) : \int_{\Omega} \Delta \left( \frac{\partial P_{\varepsilon,j}}{\partial z_{j,h}} \right) u = 0, \quad j = 1, \dots, k, \quad h = 1, \dots, N \right\}.$$

Then  $M$  is complete under the  $L^\infty$  norm and  $G_\varepsilon$  is a map from  $\tilde{E}_{\varepsilon,Z}$  to  $\tilde{E}_{\varepsilon,Z}$ . We claim that  $G_\varepsilon$  is a contraction map from  $M$  to  $M$ . Indeed, we only need to prove the following two facts.

1.  $G_\varepsilon$  maps from  $M$  into  $M$ .

For any  $\omega \in M$ , similar to Lemma ??, we can prove that for large  $L > 0$ ,

$$\left( \sum_{j=1}^k V_{\varepsilon,Z,j} - \varphi + \omega \right)_+ = 0 \quad \text{in } \Omega \setminus \cup_{j=1}^k B_{L\varepsilon}(z_j).$$

Note also that for any  $u \in L^\infty(\Omega)$ ,

$$Q_\varepsilon u = u \quad \text{in } \Omega \setminus \cup_{j=1}^k B_{L\varepsilon}(z_j).$$

Therefore, we find that for any  $\omega \in M$ ,

$$Q_\varepsilon l_\varepsilon + Q_\varepsilon R_\varepsilon(\omega) = l_\varepsilon + R_\varepsilon(\omega) = 0, \quad \text{in } \Omega \setminus \cup_{j=1}^k B_{L\varepsilon}(z_j).$$

So, applying Proposition ??, we see

$$\|(Q_\varepsilon L_\varepsilon)^{-1}(Q_\varepsilon l_\varepsilon + Q_\varepsilon R_\varepsilon(\omega))\|_\infty \leq C \|Q_\varepsilon l_\varepsilon + Q_\varepsilon R_\varepsilon(\omega)\|_\infty.$$

Thus, for any  $\omega \in M$ , we have

$$(3.11) \quad \begin{aligned} \|G_\varepsilon(\omega)\|_\infty &= \|(Q_\varepsilon L_\varepsilon)^{-1} Q_\varepsilon (l_\varepsilon + R_\varepsilon(\omega))\|_\infty \\ &\leq C \|Q_\varepsilon (l_\varepsilon + R_\varepsilon(\omega))\|_\infty. \end{aligned}$$

Using the argument in (??), we deduce from (??) that  $b_{j,h}$ , corresponding to  $u \in L^\infty(\Omega)$ , satisfies

$$|b_{j,h}| \leq C \varepsilon^{2-N} \sum_{i,l} \int_{\Omega} \left| \frac{\partial P_{\varepsilon,i}}{\partial z_{i,h}} \right| |u|.$$

Since

$$l_\varepsilon + R_\varepsilon(\omega) = 0, \quad \text{in } \Omega \setminus \cup_{j=1}^k B_{L\varepsilon}(z_j),$$

we find

$$\begin{aligned} |b_{j,h}| &\leq C\varepsilon^{2-N} \sum_{i,l} \left( \sum_{j=1}^k \int_{B_{L\varepsilon}(z_j)} \left| \frac{\partial P_{\varepsilon,i}}{\partial z_{i,l}} \right| |l_\varepsilon + R_\varepsilon(\omega)| \right) \\ &\leq C\varepsilon \|l_\varepsilon + R_\varepsilon(\omega)\|_\infty. \end{aligned}$$

As a result,

$$\begin{aligned} &\|Q_\varepsilon(l_\varepsilon + R_\varepsilon(\omega))\|_\infty \\ &\leq \|l_\varepsilon + R_\varepsilon(\omega)\|_\infty + C \sum_{j,h} |b_{j,h}| \| (W_{\varepsilon,z_j,c_{\varepsilon,j}} - c_{\varepsilon,j})_+^{p-2} \left( \frac{\partial W_{\varepsilon,z_j,c_{\varepsilon,j}}}{\partial z_{j,h}} - \frac{\partial c_{\varepsilon,j}}{\partial z_{j,h}} \right) \|_\infty \\ &\leq C(\|l_\varepsilon\|_\infty + \|R_\varepsilon(\omega)\|_\infty). \end{aligned}$$

It follows from (??) and (??) that

$$\|l_\varepsilon\|_\infty = O(\varepsilon) \sum_{j=1}^k \| (W_{\varepsilon,z_j,c_{\varepsilon,j}} - c_{\varepsilon,j})_+^{p-2} \|_\infty + O(\varepsilon^{N-1}) = O(\varepsilon).$$

For the estimate for  $\|R_\varepsilon(\omega)\|_\infty$ , noting the boundedness of  $W_{\varepsilon,z_j,c_{\varepsilon,j}} - c_{\varepsilon,j}$ , we see

$$\|R_\varepsilon(\omega)\|_\infty \leq C\|\omega\|_\infty^2.$$

As a consequence,

$$(3.12) \quad \|G(\omega)\|_\infty \leq C(\varepsilon + \|\omega\|_\infty^2) \leq \varepsilon^{1-\theta}.$$

Thus,  $G_\varepsilon$  is a map from  $M$  to  $M$ .

2.  $G_\varepsilon$  is a contraction.

In fact, for any  $\omega_i \in M$ ,  $i = 1, 2$ , we have

$$G_\varepsilon\omega_1 - G_\varepsilon\omega_2 = (Q_\varepsilon L_\varepsilon)^{-1} Q_\varepsilon (R_\varepsilon(\omega_1) - R_\varepsilon(\omega_2)).$$

Noting that

$$R_\varepsilon(\omega_1) = R_\varepsilon(\omega_2) = 0, \quad \text{in } \Omega \setminus \cup_{j=1}^k B_{L\varepsilon,j}(z_j),$$

we can deduce as the proof of Fact 1 that

$$\begin{aligned} \|G_\varepsilon\omega_1 - G_\varepsilon\omega_2\|_\infty &\leq C\|R_\varepsilon(\omega_1) - R_\varepsilon(\omega_2)\|_\infty \\ &\leq C(\|\omega_1\|_\infty + \|\omega_2\|_\infty)\|\omega_1 - \omega_2\|_\infty \\ &\leq C\varepsilon^{1-\theta}\|\omega_1 - \omega_2\|_\infty < \frac{1}{2}\|\omega_1 - \omega_2\|_\infty. \end{aligned}$$

Now we have proved that  $G_\varepsilon$  is a contraction map from  $M$  to  $M$ . By the contraction mapping theorem, there is an  $\omega_\varepsilon \in M$ , such that  $\omega_\varepsilon = G_\varepsilon\omega_\varepsilon$ . Moreover, it follows from (??) that

$$\|\omega_\varepsilon\|_\infty \leq \varepsilon^{1-\theta}.$$

□

## 4 Proof of Theorem ?? in case $N \geq 3$

In this section, we will choose  $Z$ , such that  $\sum_{j=1}^k P_{\varepsilon,j} + \omega_\varepsilon$  is a solution of (??), where  $\omega_\varepsilon$  be the solution to (??) obtained in Proposition ??.

Define

$$I(u) = \frac{\varepsilon^2}{2} \int_{\Omega} |Du|^2 - \frac{1}{p} \int_{\Omega} (u - \varphi)_+^p,$$

and

$$K(Z) = I\left(\sum_{j=1}^k P_{\varepsilon,j} + \omega_\varepsilon\right).$$

It is well known that if  $Z$  is a critical point of  $K(Z)$ , then  $\sum_{j=1}^k P_{\varepsilon,j} + \omega_\varepsilon$ , is a solution of (??).

**Lemma 4.1.**  $\omega_\varepsilon$  satisfies

$$\varepsilon^2 \int_{\Omega} |D\omega_\varepsilon|^2 \leq C\varepsilon^{N+2-2\theta}.$$

*Proof.*  $\omega_\varepsilon$  satisfies (??), so  $\omega_\varepsilon$  solves

$$-\varepsilon^2 \Delta \omega_\varepsilon = \left(\sum_{j=1}^k P_{\varepsilon,i} - \varphi + \omega_\varepsilon\right)_+^{p-1} - \sum_{j=1}^k \left(W_{\varepsilon,z_i,c_{\varepsilon,i}} - c_{\varepsilon,i}\right)_+^{p-1} + \sum_{j=1}^k \sum_{h=1}^N b_{j,h} \left(-\varepsilon^2 \Delta \frac{\partial P_{\varepsilon,j}}{\partial z_{j,h}}\right).$$

Hence, by (??), we see

$$\begin{aligned} \varepsilon^2 \int_{\Omega} |D\omega_\varepsilon|^2 &= \int_{\Omega} \left( \left(\sum_{j=1}^k P_{\varepsilon,i} - \varphi + \omega_\varepsilon\right)_+^{p-1} - \sum_{j=1}^k \left(W_{\varepsilon,z_i,c_{\varepsilon,i}} - c_{\varepsilon,i}\right)_+^{p-1} \right) \omega_\varepsilon \\ &\quad + \sum_{j=1}^k \sum_{h=1}^N b_{j,h} \left(-\varepsilon^2 \Delta \frac{\partial P_{\varepsilon,j}}{\partial z_{j,h}}\right) \\ &= (p-1) \int_{\Omega} \left(W_{\varepsilon,z_i,c_{\varepsilon,i}} - c_{\varepsilon,i}\right)_+^{p-2} \omega_\varepsilon^2 + O(\varepsilon^{N+1}) \|\omega_\varepsilon\|_\infty + O(\varepsilon^{N+3-3\theta}) \\ &= O(\varepsilon^{N+2-2\theta}). \end{aligned}$$

□

**Lemma 4.2.**

$$K(Z) = I\left(\sum_{j=1}^k P_{\varepsilon,j}\right) + O(\varepsilon^{N+2-2\theta}).$$

*Proof.* We have

$$K(Z) = \frac{\varepsilon^2}{2} \int_{\Omega} |D(\sum_{j=1}^k P_{\varepsilon,j})|^2 + \varepsilon^2 \sum_{j=1}^k \int_{\Omega} DP_{\varepsilon,j} D\omega_\varepsilon + \frac{\varepsilon^2}{2} \int_{\Omega} |D\omega_\varepsilon|^2$$



$$\begin{aligned}
& -\frac{1}{p} \int_{\Omega} \left( \sum_{j=1}^k P_{\varepsilon,j} - \varphi + \omega_{\varepsilon} \right)_+^p \\
&= I\left(\sum_{j=1}^k P_{\varepsilon,j}\right) + \frac{\varepsilon^2}{2} \int_{\Omega} |D\omega_{\varepsilon}|^2 + \varepsilon^2 \sum_{j=1}^k \int_{\Omega} DP_{\varepsilon,j} D\omega_{\varepsilon} - \int_{\Omega} \left( \sum_{j=1}^k P_{\varepsilon,j} - \varphi \right)_+^{p-1} \omega_{\varepsilon} \\
& -\frac{1}{p} \int_{\Omega} \left( \left( \sum_{j=1}^k P_{\varepsilon,j} - \varphi + \omega_{\varepsilon} \right)_+^p - \left( \sum_{j=1}^k P_{\varepsilon,j} - \varphi \right)_+^p - p \left( \sum_{j=1}^k P_{\varepsilon,j} - \varphi \right)_+^{p-1} \omega_{\varepsilon} \right).
\end{aligned}$$

Employing Lemma ?? and Proposition ??, we see

$$\begin{aligned}
& \int_{\Omega} \left( \left( \sum_{j=1}^k P_{\varepsilon,j} - \varphi + \omega_{\varepsilon} \right)_+^p - \left( \sum_{j=1}^k P_{\varepsilon,j} - \varphi \right)_+^p - p \left( \sum_{j=1}^k P_{\varepsilon,j} - \varphi \right)_+^{p-1} \omega_{\varepsilon} \right) \\
&= \sum_{i=1}^k \int_{B_{L\varepsilon}(z_i)} \left( \left( \sum_{j=1}^k P_{\varepsilon,j} - \varphi + \omega_{\varepsilon} \right)_+^p - \left( \sum_{j=1}^k P_{\varepsilon,j} - \varphi \right)_+^p - p \left( \sum_{j=1}^k P_{\varepsilon,j} - \varphi \right)_+^{p-1} \omega_{\varepsilon} \right) \\
&= O\left(\|\omega\|_{\infty}^2 \varepsilon^N\right) = O\left(\varepsilon^{N+2-2\theta}\right).
\end{aligned}$$

It follows from Lemma ?? and (??)

$$\begin{aligned}
& \varepsilon^2 \sum_{j=1}^k \int_{\Omega} DP_{\varepsilon,j} D\omega_{\varepsilon} - \int_{\Omega} \left( \sum_{j=1}^k P_{\varepsilon,j} - \varphi \right)_+^{p-1} \omega_{\varepsilon} \\
&= \int_{\Omega} \left( \sum_{j=1}^k \left( W_{\varepsilon,z_j,c_{\varepsilon,j}} - c_{\varepsilon,j} \right)_+^{p-1} \omega_{\varepsilon} - \left( \sum_{j=1}^k P_{\varepsilon,j} - \varphi \right)_+^{p-1} \omega_{\varepsilon} \right) \\
&= \int_{\cup_{i=1}^k B_{L\varepsilon}(z_i)} \left( \sum_{j=1}^k \left( W_{\varepsilon,z_j,c_{\varepsilon,j}} - c_{\varepsilon,j} \right)_+^{p-1} \omega_{\varepsilon} - \left( \sum_{j=1}^k P_{\varepsilon,j} - \varphi \right)_+^{p-1} \omega_{\varepsilon} \right) \\
&= O(\varepsilon) \|\omega_{\varepsilon}\|_{\infty} \sum_{j=1}^k \int_{B_{L\varepsilon}(z_j)} \left( W_{\varepsilon,z_j,c_{\varepsilon,j}} - c_{\varepsilon,j} \right)_+^{p-1} + O(\varepsilon^{2N-2} \|\omega\|_{\infty}) \\
& \quad + \|\omega_{\varepsilon}\|_{\infty}^2 \sum_{j=1}^k \int_{B_{L\varepsilon}(z_j)} \left( W_{\varepsilon,z_j,c_{\varepsilon,j}} - c_{\varepsilon,j} \right)_+^{p-2} + O(\varepsilon^{N+1} \|\omega\|_{\infty}) \\
&= O(\varepsilon^{N+2-2\theta}).
\end{aligned}$$

So we see

$$K(Z) = I\left(\sum_{j=1}^k P_{\varepsilon,j}\right) + O\left(\varepsilon^{N+2-2\theta}\right).$$

□

**Lemma 4.3.** *The function  $\phi$  defined by (??) satisfies*

$$\int_{B_1(0)} \phi^p = \frac{p|\partial B_1(0)|}{2N - pN + 2p} |\phi'(1)|^2, \quad \text{and} \quad \int_{B_1(0)} \phi^{p-1} = |\partial B_1(0)| |\phi'(1)|,$$

where  $|\partial B_1(0)|$  denotes the area of  $\partial B_1(0)$ .

*Proof.* We can easily check the following Pohozaev identity

$$\left(\frac{N}{p} - \frac{N-2}{2}\right) \int_{B_1(0)} \phi^p = \frac{1}{2} \int_{\partial B_1(0)} |D\phi|^2 (\nu \cdot x) ds.$$

As a result, we see

$$\int_{B_1(0)} \phi^p = \frac{p|\partial B_1(0)|}{2N - pN + 2p} |\phi'(1)|^2.$$

The other part is obvious, since

$$\int_{B_1(0)} \phi^{p-1} = \int_{B_1(0)} -\Delta \phi = - \int_{\partial B_1(0)} \frac{\partial \phi}{\partial \nu} ds = |\partial B_1(0)| |\phi'(1)|.$$

Hence, we complete the proof. □

Now we prove Theorem ??

**Proof of Theorem ??:**

Define

$$S = \{Z = (z_1, \dots, z_k) \in \Omega^k : |z_j - \bar{z}_j| \leq \delta, d_j := d(z_j, \partial\Omega) \geq \tau \varepsilon^{(N-2)/(N-1)}\},$$

where  $\tau$  will be determined later. Consider the problem

$$\min_{Z \in \bar{S}} K(Z).$$

There exists a minimizer  $Z_\varepsilon$  for  $K(Z)$  in  $\bar{S}$ . To complete the proof, it suffices to verify that  $Z_\varepsilon$  is an interior point of  $S$  and hence is a critical point of  $K(Z)$ .

From Lemma ??, we need to estimate  $I(\sum_{j=1}^k P_{\varepsilon,j})$ .

By (??),

$$c_{\varepsilon,i} R_{c_{\varepsilon,i}}^{N-2} = \varphi(z_i) R_{\varphi(z_i)}^{N-2} + O((\varepsilon^{N-2} h(z_i, z_i))^2) = \varphi(z_i) R_{\varphi(z_i)}^{N-2} + O(\varepsilon).$$

We have

$$\varepsilon^2 \int_{\Omega} |D \sum_{j=1}^k P_{\varepsilon,j}|^2 = \sum_{j=1}^k \sum_{i=1}^k \int_{\Omega} (W_{\varepsilon,z_j,c_{\varepsilon,j}} - c_{\varepsilon,j})_+^{p-1} P_{\varepsilon,i}$$

$$\begin{aligned}
&= \sum_{j=1}^k \sum_{i=1}^k \int_{\Omega} (W_{\varepsilon, z_j, c_{\varepsilon, j}} - c_{\varepsilon, j})_+^{p-1} (W_{\varepsilon, z_i, c_{\varepsilon, i}} - c_{\varepsilon, i} + \varphi(z_i) + O((\varepsilon^{N-2} h(z_i, z_i))^2)) \\
&= \sum_{j=1}^k \int_{\Omega} (W_{\varepsilon, z_j, c_{\varepsilon, j}} - c_{\varepsilon, j})_+^p + \sum_{j=1}^k \varphi(z_j) \int_{\Omega} (W_{\varepsilon, z_j, c_{\varepsilon, j}} - c_{\varepsilon, j})_+^{p-1} + O(\varepsilon^{N+1}) \\
&= \sum_{j=1}^k \int_{B_{R_{c_{\varepsilon, j}} \varepsilon}(z_j)} (W_{\varepsilon, z_j, c_{\varepsilon, j}} - c_{\varepsilon, j})_+^p + \sum_{j=1}^k \varphi(z_j) \int_{B_{R_{c_{\varepsilon, j}} \varepsilon}(z_j)} (W_{\varepsilon, z_j, c_{\varepsilon, j}} - c_{\varepsilon, j})_+^{p-1} + O(\varepsilon^{N+1}) \\
&= \sum_{j=1}^k R_{c_{\varepsilon, j}}^{N-2p/(p-2)} \varepsilon^N \int_{B_1(0)} \phi^p + \sum_{j=1}^k R_{c_{\varepsilon, j}}^{N-2(p-1)/(p-2)} \varphi(z_j) \varepsilon^N \int_{B_1(0)} \phi^{p-1} + O(\varepsilon^{N+1}).
\end{aligned}$$

Similarly, by Lemma ?? and (??), we see

$$\begin{aligned}
\int_{\Omega} \left( \sum_{j=1}^k P_{\varepsilon, j} - \varphi \right)_+^p &= \sum_{j=1}^k \int_{B_{L\varepsilon}(z_j)} (W_{\varepsilon, z_j, c_{\varepsilon, j}} - c_{\varepsilon, j} + O(\varepsilon))_+^p + O(\varepsilon^{2N-2}) \\
&= \sum_{j=1}^k \int_{B_{L\varepsilon}(z_j)} \left( R_{c_{\varepsilon, j}}^{-2/(p-2)} \phi\left(\frac{x - z_j}{R_{c_{\varepsilon, j}} \varepsilon}\right) + O(\varepsilon) \right)_+^p + O(\varepsilon^{2N-2}) \\
&= \sum_{j=1}^k R_{c_{\varepsilon, j}}^{N-2p/(p-2)} \varepsilon^N \int_{B_1(0)} \phi^p + O(\varepsilon^{N+1}).
\end{aligned}$$

Hence

$$\begin{aligned}
I\left(\sum_{j=1}^k P_{\varepsilon, j}\right) &= \frac{\varepsilon^2}{2} \int_{\Omega} \left| D \sum_{j=1}^k P_{\varepsilon, j} \right|^2 - \frac{1}{p} \int_{\Omega} \left( \sum_{j=1}^k P_{\varepsilon, j} - \varphi \right)_+^p \\
&= \left( \frac{1}{2} - \frac{1}{p} \right) \sum_{j=1}^k R_{c_{\varepsilon, j}}^{N-2p/(p-2)} \varepsilon^N \int_{B_1(0)} \phi^p \\
&\quad + \frac{1}{2} \sum_{j=1}^k R_{c_{\varepsilon, j}}^{N-2(p-1)/(p-2)} \varphi(z_j) \varepsilon^N \int_{B_1(0)} \phi^{p-1} + O(\varepsilon^{N+1}).
\end{aligned}$$

Using  $R_c = \left[ \frac{\phi'(1)}{(2-N)c} \right]^{\frac{p-2}{2}}$  and (??), we can check

$$\begin{aligned}
&R_{c_{\varepsilon, j}}^{N-2p/(p-2)} \\
&= \left( \frac{N-2}{|\phi'(1)|} \right)^{p-(p-2)N/2} \varphi(z_j)^{p-(p-2)N/2} \\
&\quad + \frac{2p+2N-pN}{2} \left( \frac{N-2}{|\phi'(1)|} \right)^{p-(p-2)N/2} \varphi(z_j)^{p-(p-2)N/2} R_{\varphi(z_j)}^{N-2} h(z_j, z_j) \varepsilon^{N-2} \\
&\quad + O((h(z_j, z_j) \varepsilon^{N-2})^2)
\end{aligned}$$

and

$$\begin{aligned}
& R_{c_{\varepsilon,j}}^{N-2(p-1)/(p-2)} \varphi(z_j) \\
= & \left( \frac{N-2}{|\phi'(1)|} \right)^{p-1-(p-2)N/2} \varphi(z_j)^{p-(p-2)N/2} \\
& + \frac{2p-2+2N-pN}{2} \left( \frac{N-2}{|\phi'(1)|} \right)^{p-1-(p-2)N/2} \varphi(z_j)^{p-(p-2)N/2} R_{\varphi(z_j)}^{N-2} h(z_j, z_j) \varepsilon^{N-2} \\
& + O((h(z_j, z_j) \varepsilon^{N-2})^2).
\end{aligned}$$

Now combing Lemmas ?? and ??, we have

$$\begin{aligned}
K(Z) &= I\left(\sum_{j=1}^k P_{\varepsilon,j}\right) + O(\varepsilon^{N+2-2\theta}) \\
&= \sum_{j=1}^k \varepsilon^N \left( \frac{N-2}{|\phi'(1)|} \right)^{p-1-(p-2)N/2} |\phi'(1)| |\partial B_1(0)| \varphi(z_j)^{p-(p-2)N/2} \times \\
&\quad \times \left( \frac{2}{2N-2p-pN} + \frac{1}{2} R_{\varphi(z_j)}^{N-2} h(z_j, z_j) \varepsilon^{N-2} \right) \\
&\quad + O(\varepsilon^{N+1}) + O(\varepsilon^{N+2-2\theta}).
\end{aligned}$$

Let  $Z_\varepsilon^* = (z_{\varepsilon,1}^*, \dots, z_{\varepsilon,k}^*) \in S$  be such that

$$|z_{\varepsilon,j} - \bar{z}_j| = d_j = \varepsilon^{\frac{N-2}{N-1}},$$

then

$$\varphi(z_{\varepsilon,j}^*) = \varphi(\bar{z}_j) + O(\varepsilon^{\frac{N-2}{N-1}}), \quad \varepsilon^{N-2} h(z_{\varepsilon,j}^*, z_{\varepsilon,j}^*) = O(\varepsilon^{\frac{N-2}{N-1}}).$$

As a consequence,

$$\begin{aligned}
K(Z_\varepsilon^*) &= \sum_{j=1}^k \frac{2\varepsilon^N}{2N-2p-pN} \left( \frac{N-2}{|\phi'(1)|} \right)^{p-1-\frac{(p-2)N}{2}} |\phi'(1)| |\partial B_1(0)| \varphi(\bar{z}_j)^{p-\frac{(p-2)N}{2}} \\
&\quad + O(\varepsilon^{\frac{N-2}{N-1}}).
\end{aligned}$$

But,

$$K(Z_\varepsilon) \leq K(Z_\varepsilon^*),$$

so, we deduce

$$\varphi(z_{\varepsilon,j}) - \varphi(\bar{z}_j) \leq C \varepsilon^{\frac{N-2}{N-1}}, \quad j = 1, \dots, k,$$

and

$$h(z_{\varepsilon,j}, z_{\varepsilon,j}) \leq C \varepsilon^{\frac{N-2}{N-1}}, \quad j = 1, \dots, k.$$

Hence, we see

$$|z_{\varepsilon,j} - \bar{z}_j| \leq C_1 \varepsilon^{\frac{N-2}{N-1}}, \quad \text{dist}(z_{\varepsilon,j}, \partial\Omega) \geq C_2 \varepsilon^{\frac{N-2}{N-1}}, \quad j = 1, \dots, k,$$

which implies

$$|z_{\varepsilon,j} - \bar{z}_j| = C_1 \varepsilon^{\frac{N-2}{N-1}}, \quad \text{dist}(z_{\varepsilon,j}, \partial\Omega) = C_2 \varepsilon^{\frac{N-2}{N-1}}, \quad j = 1, \dots, k,$$

where  $C_1$  and  $C_2$  are independent of  $\tau$ .

Therefore,  $Z_\varepsilon$  is a interior point of  $S$  if we choose  $\tau$  sufficiently small.

## 5 Proof of Theorem ?? in case $N = 2$

To investigate the case  $N = 2$ , we use  $V_{\varepsilon,Z,j}$ ,  $a_{\varepsilon,j}(Z)$  and  $U_{\varepsilon,z_j,a_{\varepsilon,j}(Z)}$  to replace  $P_{\varepsilon,j}$ ,  $c_{\varepsilon,j}$  and  $W_{\varepsilon,z_j,c_{\varepsilon,j}}$  in Section 3 respectively. and proceeding as we have done in the case  $N \geq 3$  (see also [?] for the details ), we can prove

**Proposition 5.1.** *There is an  $\varepsilon_0 > 0$ , such that for any  $\varepsilon \in (0, \varepsilon_0]$  (??) has a unique solution  $\omega_\varepsilon$  with*

$$\|\omega_\varepsilon\|_\infty = O\left(\frac{(\ln |\ln \varepsilon|)^2}{|\ln \varepsilon|^2}\right).$$

Moreover,

$$\varepsilon^2 \int_\Omega |D\omega|^2 = O\left(\frac{\varepsilon^2 (\ln |\ln \varepsilon|)^4}{|\ln \varepsilon|^4}\right).$$

Define

$$M(Z) = I\left(\sum_{j=1}^k P_{\varepsilon,j} + \omega_\varepsilon\right).$$

Then we have

**Lemma 5.2.**

$$M(Z) = I\left(\sum_{j=1}^k V_{\varepsilon,Z,j}\right) + O\left(\frac{\varepsilon^2 (\ln |\ln \varepsilon|)^4}{|\ln \varepsilon|^4}\right).$$

*Proof.* By definition,

$$\begin{aligned} M(Z) &= I\left(\sum_{j=1}^k V_{\varepsilon,Z,j}\right) + \frac{\varepsilon^2}{2} \int_\Omega |D\omega_\varepsilon|^2 + \varepsilon^2 \sum_{j=1}^k \int_\Omega DV_{\varepsilon,Z,j} D\omega_\varepsilon - \int_\Omega \left(\sum_{j=1}^k V_{\varepsilon,Z,j} - \varphi\right)_+^{p-1} \omega_\varepsilon \\ &\quad - \frac{1}{p} \int_\Omega \left( \left(\sum_{j=1}^k V_{\varepsilon,Z,j} - \varphi + \omega_\varepsilon\right)_+^p - \left(\sum_{j=1}^k V_{\varepsilon,Z,j} - \varphi\right)_+^p - p \left(\sum_{j=1}^k V_{\varepsilon,Z,j} - \varphi\right)_+^{p-1} \omega_\varepsilon \right). \end{aligned}$$

Employing (??), Lemma ?? and Proposition ??, we obtain

$$\begin{aligned}
& \int_{\Omega} \left( \left( \sum_{j=1}^k V_{\varepsilon,Z,j} - \varphi + \omega_{\varepsilon} \right)_+^p - \left( \sum_{j=1}^k V_{\varepsilon,Z,j} - \varphi \right)_+^p - p \left( \sum_{j=1}^k V_{\varepsilon,Z,j} - \varphi \right)_+^{p-1} \omega_{\varepsilon} \right) \\
&= \int_{\cup_{i=1}^k B_{L_{s_{\varepsilon,i}}}(z_i)} \left( \left( \sum_{j=1}^k V_{\varepsilon,Z,j} - \varphi + \omega_{\varepsilon} \right)_+^p - \left( \sum_{j=1}^k V_{\varepsilon,Z,j} - \varphi \right)_+^p - p \left( \sum_{j=1}^k V_{\varepsilon,Z,j} - \varphi \right)_+^{p-1} \omega_{\varepsilon} \right) \\
&= O\left(\|\omega\|_{\infty}^2 \varepsilon^2\right) = O\left(\frac{\varepsilon^2 (\ln |\ln \varepsilon|)^4}{|\ln \varepsilon|^4}\right),
\end{aligned}$$

and

$$\begin{aligned}
& \varepsilon^2 \sum_{j=1}^k \int_{\Omega} DV_{\varepsilon,Z,j} D\omega_{\varepsilon} - \int_{\Omega} \left( \sum_{j=1}^k V_{\varepsilon,Z,j} - \varphi \right)_+^{p-1} \omega_{\varepsilon} \\
&= \int_{\Omega} \left( \sum_{j=1}^k \left( U_{\varepsilon,z_j,a_{\varepsilon,j}}(Z) - a_{\varepsilon,j}(Z) \right)_+^{p-1} \omega_{\varepsilon} - \left( \sum_{j=1}^p V_{\varepsilon,Z,j} - \varphi \right)_+^{p-1} \omega_{\varepsilon} \right) \\
&= \int_{\cup_{i=1}^k B_{L_{s_{\varepsilon,i}}}(z_i)} \left( \sum_{j=1}^k \left( U_{\varepsilon,z_j,a_{\varepsilon,j}}(Z) - a_{\varepsilon,j}(Z) \right)_+^{p-1} \omega_{\varepsilon} - \left( \sum_{j=1}^p V_{\varepsilon,Z,j} - \varphi \right)_+^{p-1} \omega_{\varepsilon} \right) \\
&= O\left(\frac{(\ln |\ln \varepsilon|)^2}{|\ln \varepsilon|^2}\right) \|\omega_{\varepsilon}\|_{\infty} \sum_{j=1}^k \int_{B_{L_{s_{\varepsilon,j}}}(z_j)} \left( U_{\varepsilon,z_j,a_{\varepsilon,j}}(Z) - a_{\varepsilon,j}(Z) \right)_+^{p-1} \\
&\quad + \|\omega_{\varepsilon}\|_{\infty}^2 \sum_{j=1}^k \int_{B_{L_{s_{\varepsilon,p,j}}}(z_j)} \left( U_{\varepsilon,z_j,a_{\varepsilon,j}}(Z) - a_{\varepsilon,j}(Z) \right)_+^{p-2} \\
&= O\left(\frac{\varepsilon^2 (\ln |\ln \varepsilon|)^4}{|\ln \varepsilon|^4}\right).
\end{aligned}$$

So we see

$$M(Z) = I\left(\sum_{j=1}^k V_{\varepsilon,Z,j}\right) + O\left(\frac{\varepsilon^2 (\ln |\ln \varepsilon|)^4}{|\ln \varepsilon|^4}\right).$$

□

### Proof of Theorem ??:

Define

$$D = \left\{ Z = (z_1, \dots, z_k) \in \Omega^k : |z_j - \bar{z}_j| \leq \delta, d_j := d(z_j, \partial\Omega) \in \left( \frac{1}{|\ln \varepsilon|^{\tau_2}}, \frac{1}{|\ln \varepsilon|^{\tau_1}} \right) \right\},$$

where  $\tau_1$  and  $\tau_2$  will be determined later.

Consider the problem

$$\min_{Z \in D} M(Z).$$

There exists a minimizer  $Z_\varepsilon$  for  $M(Z)$  in  $\bar{D}$ . Now we verify that  $Z_\varepsilon$  is an interior point of  $D$  and hence is a critical point of  $M(Z)$ .

Similarly to the case  $N \geq 3$ , we have

$$\begin{aligned}
& \varepsilon^2 \int_{\Omega} |D \sum_{j=1}^k V_{\varepsilon, Z, j}|^2 = \sum_{j=1}^k \sum_{i=1}^k \int_{\Omega} (U_{\varepsilon, z_j, a_{\varepsilon, j}(Z)} - a_{\varepsilon, j}(Z))_+^{p-1} V_{\varepsilon, Z, i} \\
&= \sum_{j=1}^k \sum_{i=1}^k \int_{B_{s_{\varepsilon, j}}(z_j)} (U_{\varepsilon, z_j, a_{\varepsilon, j}(Z)} - a_{\varepsilon, j}(Z))_+^{p-1} (U_{\varepsilon, z_i, c_{\varepsilon, i}} - \frac{a_{\varepsilon, j}(Z)}{\ln R/s_{\varepsilon, i}} g(x, z_i)) \\
&= \frac{2\pi}{|\phi'(1)|^{p-2}} \left( \sum_{j=1}^k \left( \frac{a_{\varepsilon, j}(Z)}{\ln R/s_{\varepsilon, j}} \right)^{p-1} a_{\varepsilon, j}(Z) s_{\varepsilon, j}^2 - \sum_{j=1}^k \left( \frac{a_{\varepsilon, j}(Z)}{\ln R/s_{\varepsilon, j}} \right)^p s_{\varepsilon, j}^2 g(z_j, z_j) \right) \\
&\quad + O\left(\frac{\varepsilon^2}{|\ln \varepsilon|^2}\right),
\end{aligned}$$

and

$$\begin{aligned}
& \int_{\Omega} \left( \sum_{j=1}^k V_{\varepsilon, Z, j} - \varphi \right)_+^p \\
&= \sum_{j=1}^k \int_{B_{s_{\varepsilon, j}}(z_j)} \left( W_{\varepsilon, z_j, a_{\varepsilon, j}(Z)} - a_{\varepsilon, j}(Z) + O\left(\frac{(\ln |\ln \varepsilon|)^2}{|\ln \varepsilon|^2}\right) \right)_+^p + O\left(\frac{\varepsilon^2}{|\ln \varepsilon|^2}\right) = O\left(\frac{\varepsilon^2}{|\ln \varepsilon|^2}\right).
\end{aligned}$$

So, noting that

$$a_{\varepsilon, j}(Z) = \varphi(z_i) \left( 1 + \frac{g(z_j, z_j)}{\ln R/\varepsilon} + O\left(\frac{1}{|\ln \varepsilon|}\right) \right),$$

and

$$\varepsilon^{2/(p-2)} s_{\varepsilon, j}^{-2/(p-2)} \phi'(1) = \frac{a_{\varepsilon, i}}{\ln(s_{\varepsilon, j}/R)},$$

we obtain

$$\begin{aligned}
I\left(\sum_{j=1}^k V_{\varepsilon, Z, j}\right) &= \frac{\varepsilon^2}{2} \int_{\Omega} |D \sum_{j=1}^k V_{\varepsilon, Z, j}|^2 - \frac{1}{p} \int_{\Omega} \left( \sum_{j=1}^k V_{\varepsilon, Z, j} - \varphi \right)_+^p \\
&= 2\pi \sum_{j=1}^k \frac{\varphi^2(z_j) \varepsilon^2}{|\ln \varepsilon|} \left( 1 + (p-1) \frac{g(z_j, z_j)}{\ln R/\varepsilon} \right) + O\left(\frac{\varepsilon^2 \ln |\ln \varepsilon|}{|\ln \varepsilon|^2}\right).
\end{aligned}$$

As a result,

$$M(Z) = 2\pi \sum_{j=1}^k \frac{\varphi^2(z_j) \varepsilon^2}{|\ln \varepsilon|} \left( 1 + (p-1) \frac{g(z_j, z_j)}{\ln R/\varepsilon} \right) + O\left(\frac{\varepsilon^2 \ln |\ln \varepsilon|}{|\ln \varepsilon|^2}\right).$$

Let  $\tilde{Z}_\varepsilon = (\tilde{z}_{\varepsilon, 1}, \dots, \tilde{z}_{\varepsilon, k}) \in D$  be such that

$$|\tilde{z}_{\varepsilon, j} - \bar{z}_j| = d_j = \frac{1}{|\ln \varepsilon|^2},$$

then

$$\varphi(\tilde{z}_{\varepsilon,j}) = \varphi(\bar{z}_j) + O\left(\frac{1}{|\ln \varepsilon|^2}\right), \quad g(\tilde{z}_{\varepsilon,j}, \tilde{z}_{\varepsilon,j}) = O(\ln |\ln \varepsilon|), \quad j = 1, \dots, k.$$

As a consequence,

$$M(\tilde{Z}_\varepsilon) = 2\pi \sum_{j=1}^k \frac{\varphi^2(\bar{z}_j)\varepsilon^2}{|\ln \varepsilon|} + O\left(\frac{\varepsilon^2 \ln |\ln \varepsilon|}{|\ln \varepsilon|^2}\right).$$

Now, using the fact

$$M(Z_\varepsilon) \leq M(\tilde{Z}_\varepsilon),$$

we see

$$\varphi(z_{\varepsilon,j}) - \varphi(\bar{z}_j) \leq C\left(\frac{\ln |\ln \varepsilon|}{|\ln \varepsilon|}\right), \quad j = 1, \dots, k,$$

and

$$g(z_{\varepsilon,j}, z_{\varepsilon,j}) \leq C \ln |\ln \varepsilon|, \quad j = 1, \dots, k,$$

where  $C$  is independent of  $\tau_1$  and  $\tau_2$ .

Hence, we can check

$$|z_{\varepsilon,j} - \bar{z}_j| \leq C\left(\frac{\ln |\ln \varepsilon|}{|\ln \varepsilon|}\right), \quad \text{dist}(z_{\varepsilon,j}, \partial\Omega) \geq \frac{1}{|\ln \varepsilon|^C}, \quad j = 1, \dots, k.$$

Therefore,  $Z_\varepsilon$  is a interior point of  $D$  if we choose  $\tau_1$  to be sufficiently large and  $\tau_2$  sufficiently small in the definition of domain  $D$ .

## References

- [1] A. Ambrosetti, G. Mancini, On some free boundary problems, Recent contributions to nonlinear partial differential equations, Res. Notes in Math., 50, Pitman, Boston, Mass.-London, 1981, pp. 24–36.
- [2] A. Ambrosetti, M. Struwe, Existence of steady vortex rings in an ideal fluid, Arch. Rational Mech. Anal., 108(1989), 97–109.
- [3] A. Ambrosetti, J. Yang, Asymptotic behaviour in planar vortex theory, Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl., 1(1990), 285–291.
- [4] T. V. Badiani, Existence of steady symmetric vortex pairs on a planar domain with an obstacle, Math. Proc. Cambridge Philos. Soc., 123(1998), 365–384.
- [5] M. S. Berger, L. E. Fraenkel, Nonlinear desingularization in certain free-boundary problems, Comm. Math. Phys., 77(1980), 149–172.



- [6] L. Caffarelli, A. Friedman, Asymptotic estimates for the plasma problem, *Duke Math. J.*, 47(1980), 705–742.
- [7] F. Flucher, J. Wei, Asymptotic shape and location of small cores in elliptic free-boundary problems, *Math. Z.*, 228(1998), 683–703.
- [8] E. N. Dancer, S. Yan, The Lazer-McKenna conjecture and a free boundary problem in two dimensions, *J. Lond. Math. Soc.*, 78(2008), 639–662.
- [9] L. E. Fraenkel, M. S. Berger, A global theory of steady vortex rings in an ideal fluid, *Acta Math.*, 132(1974), 13–51.
- [10] G. Li, S. Yan, J. Yang, An elliptic problem related to planar vortex pairs, *SIAM J. Math. Anal.*, 36(2005), 1444–1460.
- [11] W. M. Ni, On the existence of global vortex rings, *J. Anal. Math.*, 37(1980), 208–247.
- [12] J. Norbury, Steady planar vortex pairs in an ideal fluid, *Comm. Pure Appl. Math.*, 28(1975), 679–700.
- [13] O. Rey, A multiplicity result for a variational problem with lack of compactness, *Nonlinear Anal.*, 13(1989), 1241–1249.
- [14] B. Turkington, On steady vortex flow in two dimensions. I, II. *Comm. Partial Differential Equations*, 8(1983), 999–1071.
- [15] J. Wei, D. Ye, F. Zhou, Analysis of boundary bubbling solutions for an anisotropic Emden-Fowler equation,
- [16] J. Yang, Existence and asymptotic behavior in planar vortex theory, *Math. Models Methods Appl. Sci.*, 1(1991), 461–475.
- [17] J. Yang, Global vortex rings and asymptotic behaviour, *Nonlinear Anal.*, 25(1995), 531–546.