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A Note on the Positive Solutions of an Inhomogeneous Elliptic Equation on Rn

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A note on the positive solutions of an inhomogeneous elliptic Equation on $\mathbb{R}^n$ *

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Abstract: This paper is contributed to the elliptic equation

$$\Delta u + K(|x|)u^p + \mu f(|x|) = 0 \tag{0.1}$$

where $p > 1, x \in \mathbb{R}^n, n \geq 3, \Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$ and $\mu \geq 0$ is a constant. We study the structure of positive radial solutions of (0.1) and obtain the uniqueness of solution decaying faster than r^{-m} at ∞ if μ is small enough under some assumptions on K and f, where m is the slow decay rate.

Key words and phrases: structure of solutions, positive solutions, elliptic equation.

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1. Introduction

In this paper, we study the elliptic equation

$$\Delta u + K(x)u^p + \mu f(x) = 0 \quad \text{ in } \mathbb{R}^n, \tag*{$(1.1)_\mu$}$$

where $p > 1$, $\mu \geq 0$ is a constant, $x \in \mathbb{R}^n, n \geq 3, \Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$ is the n-dimension Laplacian and $0 \leq K(|x|), 0 \leq f \not\equiv 0$ are given local Hölder continuous functions in $\mathbb{R}^n \backslash \{0\}$.

For the physical reasons, we consider the positive radial solutions of $(1.1)_\mu$ when $K(x) = K(r)$, $f(x) = f(r)$, where $r = |x|$. Eq. $(1.1)_\mu$ then reduces to

$$u'' + \frac{n-1}{r}u' + K(r)u^p + \mu f(r) = 0, \quad r > 0. \tag{1.2}$$

For the same reasons, the regular solutions that have finite limits at $r = 0$, are particularly interesting, which lead us to consider the initial value problem

$$\begin{cases} u'' + \frac{n-1}{r}u' + K(r)u^p + \mu f(r) = 0 & r > 0, \\ u(0) = \alpha > 0. \end{cases} \tag{1.3}$$

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We use $u_\alpha = u(r,\alpha)$ to denote the solution of Eq. (1.3).

The hypotheses of $K(x)$ are divided into two cases: the fast decay case and the slow decay case. For the fast decay case, we refer to [13], [18], [20] and [21] for results of Eq. $(1.1)_0$. In this paper, we will focus on the slow decay case, i.e., $K(r) \geq Cr^l$, for some $l > -2$ and r large.

First, let us introduce a collection of hypotheses on $K(x)$:

(K.1). $K(x) = k_\infty |x|^l + O(|x|^{-d})$ as $|x| \to +\infty$ for some constants $k_\infty > 0$, $l > -2$ and $d > n - \lambda_2 - m(p+1)$, where λ_2 is defined by (1.5) and m is defined by (1.4).

(K.2). $K(x) = O(|x|^\tau)$ at $|x| = 0$ for some $\tau > -2$.

(K.3). $K(r)$ is locally Lipschitz continuous and $\frac{d}{dr}(r^{-l}K(r)) \leq 0$ for a.e. $r > 0$.

Also, we introduce the following notations, which will be used throughout this paper:

$$\begin{aligned} m &\equiv \frac{l+2}{p-1}, \quad b_0 \equiv n-2-2m, \\ L &\equiv [m(n-2-m)]^{\frac{1}{p-1}}, \quad c_0 \equiv (p-1)L^{p-1}, \end{aligned} \tag{1.4}$$

$$p_c = \begin{cases} \dfrac{(n-2)^2 - 2(l+2)(n+l) + 2(l+2)\sqrt{(n+l)^2-(n-2)^2}}{(n-2)(n-10-4l)} & n > 10+4l, \\ \infty \qquad 3 \leq n \leq 10+4l\ . \end{cases}$$

Particularly, when $l = 0$ we have ,

$$p_c = \begin{cases} \dfrac{(n-2)^2 - 4n + 4\sqrt{n^2-(n-2)^2}}{(n-2)(n-10)} & n > 10, \\ \infty \qquad 3 \leq n \leq 10, \end{cases}$$

which was first introduced in [17]. Note that we have $m > 0$ and $b_0 > 0$ when $p > \frac{n+2l+2}{n-2}$ and $l > -2$.

Consider the equation

$$\lambda^2 + b_0\lambda + c_0 = 0, \tag{1.5}$$

here b_0 and c_0 are as in (1.4). When $p > p_c$, (1.5) has two negative roots $-\lambda_2 < -\lambda_1 < 0$ and $b_0 > \lambda_2$.

Now let us state some hypotheses on $f(x)$:

(f.1). $f(x) = O(|x|^\kappa)$ as $|x| \to 0$ for some $\kappa > -2$.

(f.2). $f(x) = O(|x|^{-q})$ as $|x| \to \infty$, for some constant $q > n - m - \lambda_2$.

There are many results about the existence and nonexistence of the positive solutions for problem $(1.1)_\mu$. For the homogeneous case, i.e.,

$$\Delta u + K(|x|)u^p = 0, \quad x \in \mathbb{R}^n \tag{1.6}$$

Ni and Yasutani showed that (1.6) has one positive solution $u(r) > 0$ satisfying $u(0) = \alpha$ for every $\alpha > 0$ in [21] and later the solutions are proved having slow decay in [19] and [3]. For the inhomogeneous case, G. Bernard obtained the existence result for $0 \leq f \leq \frac{p-1}{[p(1+|x|^2)]^{\frac{p}{p-1}}} L^p$ when $K(x) \equiv 1$ in [8] and Bae and Ni obtained the nonexistence result (see Theorem 1 in [7]) and the infinite multiplicity result (see Theorem 2 in [7]). Other recent results along this line include [1] and [2] etc. Especially, Bae, Chang and Pahk obtained the existence of infinitely many positive solutions for Eq. $(1.1)_\mu$. The main result of Bae, Chang and Pahk ([6]) can be stated in the following theorem (where f is allowed to change signs):

Theorem A Let $p \geq p_c$, Assume that $K(x)$ satisfies (K.1), (K.2) and f satisfies (f.1), (f.2) with $-(1+|x|^{mp})f(x) \leq min_{|z|=|x|}K(z)$. Then there exists $\mu_* > 0$ such that for every $\mu \in [0, \mu_*)$, Eq. $(1.1)_\mu$ possesses infinitely many positive entire solutions with the asymptotic behavior $\dfrac{L}{k_\infty^{\frac{1}{p-1}}}|x|^{-m} at\ \infty.$

We say the solution u of Eq. (1.2) is a slowly decaying solution if $u(r) \sim cr^{-m}$ at ∞ for some constant $c > 0$. (here $A(r) \sim B(r)$ at ∞ means that $\lim_{r\to\infty} \frac{A(r)}{B(r)} = 1$) and fast decaying if $u(r) = O(r^{2-n})$ at ∞. A natural and interesting question concerning Eq. (1.2) is: do two slow decay solutions with different initial values intersect each other, or, in other words, do the slow decay solutions of Eq. (1.2) have monotonicity property? It is known that the monotone property of the solutions of Eq. (1.2) has important implications, like stability, etc (see [10, 11, 15, 16, 19]).

For the homogeneous equation, for example (1.6), it is shown by Wang ([22]), Ni and Yosutani ([21]) that for small p, any two positive solutions intersect each other. Wang also showed that for large p, the solutions of (1.6) posses monotone property for a class of K, and gave explicitly the lower bound of the p value. Then Gui ([14]) extended the result to a more general class of $K(x)$. Liu, Li and Deng ([19]) studied the monotonicity of solutions of (1.6) with respect to the initial data α and got a sharp estimate p_c on the exponent p under some general conditions imposed on $K(x)$ (see Theorem 1 in [19]). Later, Bae and Chang ([3] [5]) extended the monotonicity results from C^1 condition on K in ([19]) to monotone condition on K (see Theorem 1.1 in [5] and Theorem 1.2 in [3]).

It was known that, for every $\alpha > 0$, the solution of (1.3) with $f \equiv 0$, is positive under the hypotheses (K.1), (K.2) and (K.3) for $p > \frac{n+2l+2}{n-2}$. But when $f \not\equiv 0$, solutions of (1.3) with sufficiently small initial values will have finite zeros. In [11], we show that there is a constant α_*, such that for any $\alpha > \alpha_*$, the solution of (1.3) is positive and have the following structures:

Theorem B Suppose that $K(r)$ satisfies (K.1), (K.2) and (K.3), f satisfies (f.1) and (f.2). Let $u(r,\alpha)$ be the solution of (1.3). Denote $\boldsymbol{A} = \{\alpha > 0, u(r,\alpha)$ is a positive solution of (1.3) for all $r \geq 0\}$ and $\boldsymbol{S} = \{\alpha > 0, u(r,\alpha)$ is a positive solution of (1.3) for all $r \geq 0$ and is slow decay$\}$. Define $\alpha_* = \alpha(K,\mu) \equiv \inf\{\alpha \in \boldsymbol{A}\} > 0$, $\alpha_{**} = \inf\{\alpha \in \boldsymbol{S}\}$, then $0 < \alpha_* \leq \alpha_{**}$ and

(i) if $p > p_c$, then $\alpha_{**} < \infty$ for $\mu \in [0, \mu_*)$, and $\boldsymbol{A} = [\alpha_*, \infty)$, $\boldsymbol{S} = (\alpha_{**}, \infty)$ and $u_\alpha(r)$ and $u_\beta(r)$ can not intersect each other for any $\alpha_* \leq \alpha < \beta$, i.e. $0 < u_\alpha(r) < u_\beta(r)$.

(ii) if $\dfrac{n+2+2l}{n-2} < p < p_c$ and $u_\alpha(r)$ and $u_\beta(r)$ are slow decay solutions of (1.2), then they will intersect infinity many times.

Theorem A establishes the existence of the slow decay solutions for problem (1.2) and meanwhile, Theorem B indicates that there may be a gap between α_* and α_{**} in which the solutions of (1.3) decay faster than the slow rate m. So there is a natural question for Eq. (1.2): when does $\alpha_* = \alpha_{**}$ hold? The purpose of this paper is to prove that problem (1.3) has exactly one positive solution which decays faster than the slow decay solutions under some assumptions on K and f if $p > p_c$.

The following are the main results:

Theorem 1.1. Assume that f satisfies (f.1) and K satisfy (K.1) and (K.2), and also $f = \sigma r^{-q}[1 + o(1)]$ at ∞ for some constant $\sigma > 0$. Then, there exist a $\mu_\star > 0$ such that

(i) there exists a positive solution to Eq. (1.2) satisfying $\lim_{r\to\infty} r^{q-2}u(r) = \mu\sigma/[(n-q)(q-2)]$ for all $\mu \in (0, \mu_\star)$ if $p > (n+l)/(n-2)$ and $m+2 < q < n$.

(ii) there exists a positive solution to Eq. (1.2) satisfying $\lim_{r\to\infty} \frac{r^{n-2}u(r)}{\log r} = \mu\sigma/(n-2)$ for all $\mu \in (0, \mu_\star)$ if $p > (n+2+2l)/(n-2)$ and $q = n$.

(iii) there exists a positive solution to Eq. (1.2) satisfying $\lim_{r\to\infty} r^{n-2}u(r) = C$, where $C > 0$ is a constant, for all $\mu \in (0, \mu_\star)$ if $p > (n+2+2l)/(n-2)$ and $q > n$.

Theorem 1.2. Let $p > p_c$. Then, there exist at most one radial positive solutions of (1.2) satisfying $\lim_{r\to\infty} r^m u(r) = 0$ if $f = \sigma r^{-q}[1 + 0(1)]$ at ∞ for some constants $q > n - m - \lambda_2$ and $\sigma > 0$, K satisfies (K.1) ,(K.2) and (K.3) with $d > \lambda_2 = \ell$.

Combining Theorem 1.1, Theorem 1.2 and theorem B, we easily conclude the following corollary:

Corollary 1.3. Let $p > p_c$, f and K are as in Theorem B with $d > \lambda_2 - \ell$ and α_* and α_{**} are defined as in Theorem B. Then, $\alpha_* = \alpha_{**}$. Furthermore, u_{α_*} decays like r^{2-q} when $m + 2 < q < n$, u_{α_*} decays like $r^{2-q}\log r$ when $q = n$, and u_{α_*} decays like r^{2-n} when $q > n$ if μ is small enough and $f = \sigma r^{-q}[1 + 0(1)]$ at ∞.

In this paper, only the regular positive solutions are studied. And for the existence of singular positive radial solutions for Eq. $(1.1)_1$, we refer the readers to Theorem 1.6 in [4] and Theorem 1.6 and 1.8 in [11]

This paper is organized as follows. In Section 2, we prove Theorem 1.1. In Section 3, Theorem 1.2 is proved.

2. Proof of Theorem 1.1.

In this section, some Lemmas are given, based on which, the Theorem 1.1 can be proved. Throughout this section (f.1) and (K.2) are assumed.

First, we give the following Lemma, which can be found in [4] and [12].

Lemma 2.1 Suppose $K(r) = O(r^l)$ for some $l > -2$, $f(r) = \sigma r^{-q}[1 + o(1)]$ at ∞, and $p > (n+l)/(n-2)$, $q > (2p+l)/(p-1)$. Let u be a positive radial solution of $(1.1)_1$ satisfying $\lim_{r\to\infty} r^m u(r) = 0$. Then we have

$$\lim_{r\to\infty} r^{q-2}u(r) = \frac{\sigma}{(n-q)(q-2)} \qquad \text{if } m+2 < q < n; \tag{2.1}$$

$$\lim_{r\to\infty} \frac{r^{n-2}u(r)}{\log r} = \frac{\sigma}{n-2} \qquad \text{if } q = n; \tag{2.2}$$

and

$$\lim_{r\to\infty} r^{n-2}u(r) = C \quad \text{for some } C > 0 \qquad \text{if } \ q > n. \tag{2.3}$$

Next, we give the following a priori estimate:

Lemma 2.2. Let u be a positive solution of Eq. $(1.1)_1$ with $u(\infty) = 0$, the following equality holds

$$\int_{|x-y|>R} \frac{(Ku^p + f)(y)}{|x-y|^{n-2}} dy + \frac{1}{R^{n-2}} \int_{|x-y|<R} (Ku^p + f)(y) dy = (n-2)\omega_n \bar{u}(x; R) \qquad \text{for } R > 0,$$

where ω_n is the area of the unit sphere in $\mathbb{R}^n$ and $\bar{u}(x; r)$ is the spherical mean of u on a ball centered at x with radius r.

Proof: Denote $\bar{w}(r)$ be the spherical mean of $w \in C(\mathbb{R}^n)$ on a ball centered at x with radius r , i.e.,

$$\bar{w}(r) = \frac{1}{\omega_n r^{n-1}} \int_{|x-y|=r} w(y) dS \qquad \text{for } r > 0,$$

where dS is the surface measure. Then we have $\overline{\Delta w} = \Delta \bar{w}$.

Let u be a solution of Eq. $(1.1)_1$. Taking the spherical mean on both hand sides of Eq. $(1.1)_1$, we have

$$\overline{\Delta u} + \overline{(Ku^p + f)} = 0 \qquad \text{for } \ r > 0.$$

Hence

$$(r^{n-1}\bar{u}')' + r^{n-1}\overline{(Ku^p + f)} = 0, \qquad r > 0.$$

Integrating this equation from 0 to r yields

$$\bar{u}'(r) + \frac{1}{r^{n-1}} \int_0^r s^{n-1} \overline{(Ku^p + f)}(s) ds = 0.$$

Integrating the above equation from r to R, where $0 < r < R$, we obtain

$$\bar{u}(R) - \bar{u}(r) + \int_r^R \frac{1}{r^{n-1}} \int_0^t s^{n-1} \overline{(Ku^p + f)}(s) ds dt = 0.$$

Applying Fubini's theorem, this equality becomes

$$\begin{aligned} &\bar{u}(R) - \bar{u}(r) + \int_0^r \int_r^R \frac{s^{n-1}}{t^{n-1}} \overline{(Ku^p + f)}(s) dt ds \\ &+ \int_r^R \int_s^R \frac{s^{n-1}}{t^{n-1}} \overline{(Ku^p + f)}(s) dt ds = 0. \end{aligned}$$

Thus

$$\begin{aligned} \bar{u}(R) - \bar{u}(r) \quad &= \int_0^r \frac{s^{n-1}}{n-2} \overline{(Ku^p + f)}(s) \left(\frac{1}{r^{n-2}} - \frac{1}{R^{n-2}} \right) ds \\ &+ \int_r^R \frac{s^{n-1}}{n-2} \overline{(Ku^p + f)}(s) \left(\frac{1}{s^{n-2}} - \frac{1}{R^{n-2}} \right) ds. \end{aligned}$$

Taking the limit as $R \to \infty$ and using the monotone convergence theorem yields that

$$\bar{u}(r) = \frac{1}{r^{n-2}} \int_0^r \frac{s^{n-1}}{(n-2)} \overline{(Ku^p + f)}(s) ds + \int_r^\infty \frac{s}{n-2} \overline{(Ku^p + f)}(s) ds.$$

This is equivalent to

$$(n-2)\omega_n \bar{u}(r) = \int_{|x-y|>r} \frac{(Ku^p + f)(y)}{|x-y|^{n-2}} dy + \frac{1}{r^{n-2}} \int_{|x-y|<r} (Ku^p + f)(y) dy,$$

which gives the required equality.

Lemma 2.3. Let u be a solution of Eq. $(1.1)_1$. If $f(x) \geq a|x|^{-q}$ at ∞ for some $a > 0$, then $u(x) \geq C|x|^{2-q}$ for some $C > 0$.

Proof: Let $R \to 0$ on both hand sides of the equality in Lemma 2.2, we have

$$\int_{\mathbb{R}^n} \frac{(Ku^p + f)(y)}{|x-y|^{n-2}} dy = (n-2)\omega_n \bar{u}(x; 0).$$

However, $\bar{u}(x; 0) = u(x)$, $K \geq 0$ and $u > 0$ yield that

$$u(x) \geq \frac{1}{(n-2)\omega_n} \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-2}} dy.$$

Since $f(x) \sim \sigma |x|^{-q}$ at ∞, we have $f(x) \geq \varepsilon\sigma/|x|^q$ for some small $\varepsilon > 0$ at ∞. By the similar argument as the proof of Lemma 2.6 in [18], we conclude that at ∞

$$\frac{1}{(n-2)\omega_n} \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-2} dy} \geq \begin{cases} \frac{C}{|x|^{q-2}} & \text{if } 2 < q < n, \\ \frac{C \log |x|}{|x|^{n-2}} & \text{if } q = n, \\ \frac{C}{|x|^{n-2}} & \text{if } q > n \end{cases}$$

for some constant $C > 0$. Hence, at ∞

$$u(x) \geq \begin{cases} \frac{C}{|x|^{q-2}} & \text{if } 2 < q < n, \\ \frac{C \log |x|}{|x|^{n-2}} & \text{if } q = n, \\ \frac{C}{|x|^{n-2}} & \text{if } q > n. \end{cases}$$

■

Now, we are ready to establish the existence of positive solutions of $(1.1)_\mu$ with fast decay. We intend to apply the well-known super- and sub-solution method, which is based on the following Lemma (see Theorem 2.10 in [20]).

Lemma 2.4. Suppose that $\phi(x)$ is a super-solution of $(1.1)_\mu$ and $\psi(x)$ is a sub-solution of $(1.1)_\mu$, where $K(x)$ and $f(x)$ are locally Hölder continuous functions in $\mathbb{R}^n \backslash \{0\}$, and $\phi \geq \psi$ in $\mathbb{R}^n$. Then $(1.1)_\mu$ possesses a solution u satisfying $\psi \leq u \leq \phi$ in $\mathbb{R}^n$.

Suppose that f is locally Hölder continuous in $\mathbb{R}^n/\{0\}$. For convenience, we denote

$$\Gamma * f(x) = \frac{1}{(n-2)\omega_n} \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-2}} dy.$$

The following Lemma shows the existence of fast decay positive solution of $(1.1)_\mu$ and the positive solution of $(1.1)_\mu$ decaying between the fast decay and the slow decay.

Lemma 2.5. Let $K(r) = O(r^l)$ for some $l > -2, p > \frac{n+2+2l}{n-2}$, f satisfies (f.1) and $f \sim \sigma |x|^{-q}$ at ∞ for some $\sigma > 0$, where $q > m+2$. Then, Eq. $(1.1)_\mu$ possesses a solution $u(x)$ satisfying the following inequality if μ is small enough:

(I) when $m+2 < q \leq n+2+l$, $0 < u(x) < 2\mu\Gamma * (K(x)\chi_{B_0(1)} + f)$;

(II) when $q > n+2+l$, $0 < u(x) < 2\mu\Gamma * \left(K(x)\chi_{B_0(1)} + f + \frac{1}{1+|x|^{n+2+l}}\right)$, where $\chi_{B_0(1)}$ is the characteristic function of the unit ball $B_0(1)$.

Proof: (I) Let $w = \Gamma * (K(x)\chi_{B_0(1)} + f)$. It is easy to very that w is a solution of the following equation:

$$\Delta w + K(x)\chi_{B_0(1)} + f = 0, \qquad x \in \mathbb{R}^n.$$

Denote $w_1 = 2\mu w$, we have

$$\begin{aligned} &\Delta w_1 + K(x) w_1^p + \mu f \\ &= -2\mu(K(x)\chi_{B_0(1)} + f) + K(x)(2\mu w)^p + \mu f \\ &= -\mu f - 2\mu K(x)(\chi_{B_0(1)} - 2^{p-1}\mu^{p-1} w^p) \end{aligned}$$

Obviously, when μ is small enough, $\chi_{B_0(1)} - 2^{p-1}\mu^{p-1}w^p > 0$ for $|x| \le 1$. Then, we have

$$\Delta w_1 + K(x)w_1^p + \mu f \le 0 \quad \text{for } |x| \le 1 \tag{2.4}$$

if $\mu > 0$ is small enough.

On the other hand, for large $|x|$, we have $f \sim \sigma|x|^{-q},\ K(x) = O(|x|^l)$. To make the proof more clear, we divide the argument into the following three cases:

Case 1^0: The case when $m + 2 < q < n$. By Lemma 2.3 and Lemma 2.1 in [9], we have $w \sim c|x|^{2-q}$ at ∞. then there exist $c_1,\ c_2,\ c_3 > 0$ such that

$$\begin{aligned}&\Delta w_1 + K(x)w_1^p + \mu f\\ &\le -\mu[f - 2^p\mu^{p-1}K(x)w^p)]\\ &\le -\mu[c_1|x|^{-q} - 2^p\mu^{p-1}c_2|x|^l(c_3|x|^{2-q})^p] \quad \text{for } |x| > 1.\end{aligned}$$

Since $c_1 - 2^p\mu^{p-1}c_2c_3^p > 0$ if μ is small enough and $-q \ge l + (2-q)p \Leftrightarrow q > m+2$, we conclude that $\Delta w_1 + K(x)w_1^p + \mu f \le 0$ for $|x| > 1$ if $\mu > 0$ is small enough.

Case 2^0: The case when $q = n$. In this case, $w \sim c|x|^{2-n}\ln r$, then we have

$$\begin{aligned}&\Delta w_1 + K(x)w_1^p + \mu f\\ &\le -\mu[c_1|x|^{-n} - 2^p\mu^{p-1}c_2|x|^l(c_3|x|^{2-n}\ln r)^p] \quad \text{for } |x| > 1.\end{aligned}$$

Since $-n > l + (2-n)p \Leftrightarrow p > \frac{n+2+2l}{n-2}$, we conclude that $\Delta w_1 + K(x)w_1^p + \mu f \le 0$ for $|x| > 1$ if $\mu > 0$ is small enough.

Case 3^0: The case when $n < q < n + 2 + l$. In this case, $w \sim c|x|^{2-n}$, then we have

$$\begin{aligned}&\Delta w_1 + K(x)w_1^p + \mu f\\ &\le -\mu[c_1|x|^{-q} - 2^p\mu^{p-1}c_2|x|^l(c_3|x|^{2-n})^p] \quad \text{for } |x| > 1.\end{aligned}$$

Since $-(n+2+l) > l + (2-n)p \Leftrightarrow p > \frac{n+2+2l}{n-2}$, and hence $-q > l + (2-n)p$, we conclude that $\Delta w_1 + K(x)w_1^p + \mu f \le 0$ for $|x| > 1$ if $\mu > 0$ is small enough.

Combining (2.4) and the cases $1^0 - 3^0$, we can choose a $\mu > 0$ small enough such that

$$\Delta w_1 + K(x)w_1^p + \mu f \le 0, \quad x \in \mathbb{R}^n.$$

Then, $w_1 = 2\mu w$ is a super-solution of Eq. $(1.1)_\mu$. Obviously $v = 0$ is a sub-solution of Eq. $(1.1)_\mu$ and $v \le w_1$. By Lemma 2.4, there exists a solution u satisfying

$$0 \le u(x) \le w_1, \quad x \in \mathbb{R}^n.$$

The Maximum Principle implies that $u > 0$.

Now, we are going to prove (II). Similarly, let $w = \Gamma * (K(x)\chi_{B_0(1)} + f + \frac{1}{1+|x|^{n+2+l}})$. Then w satisfies the following equation:

$$\Delta w + K(x)\chi_{B_0(1)} + f + \frac{1}{1+|x|^{n+2+l}} = 0, \quad x \in \mathbb{R}^n.$$

Let $w_1 = 2\mu w$, then

$$\begin{aligned}&\Delta w_1 + K(x)w_1^p + \mu f\\ &= -2\mu\left(K(x)\chi_{B_0(1)} + f + \frac{1}{1+|x|^{n+2+l}}\right) + K(x)(2\mu w)^p + \mu f\\ &= -\mu f - \frac{2\mu}{1+|x|^{n+2+l}} - 2\mu K(x)(\chi_{B_0(1)} - 2^{p-1}\mu^{p-1}w^p).\end{aligned}$$

Since $-(n+2+l) > l+(2-n)p \Leftrightarrow p > \frac{n+2+2l}{n-2}$, by the similar argument as the proof of (I), we can easily verify that w_1 is a super-solution of $(1.1)_\mu$ if μ is small enough. It is known that $v=0$ is always a sub-solution of $(1.1)_\mu$ and $v \le w_1$. Then the conclusion of (II) follows from Lemma 2.4 and the Maximum Principle.

Proof of Theorem 1.1: In fact, for every case, in the proof of Lemma 2.5, define

$$\mu_\star = \{\mu \mid 2\mu w \text{ is a super- solution of Eq. (1.2)}\},$$

where w is defined as in the proof of Lemma 2.5. Then, for every $\mu \in (0, \mu_\star)$, $2\mu w$ is a super-solution of (1.2).

(i): When $m+2 < q < n$, by Lemma 2.5(I) and Lemma 2.1 in [9], for every $\mu \in (0, \mu_\star)$, there exists a solution u of Eq. (1.2) such that $u(r) \le C_1 r^{2-q}$ at ∞ for some $C_1 > 0$. On the other hand, by Lemma 2.3, there exist a $C_2 > 0$, $C_1 > C_2$ such that $u(r) \ge C_2 r^{2-q}$. Then, there exists a solution of Eq. (1.2) decay like r^{2-q} at ∞. By Lemma 2.1 and $q-2 > m$, we have that $\lim_{r\to\infty} r^{q-2}u(r) = \mu d/[(n-q)(q-2)]$.

(ii): When $q = n$, by similar argument as in (i), for every $\mu \in (0, \mu_\star)$, there exists a solution u of Eq. (1.2) such that $u(r) \le C_1 r^{2-n}\log r$ at ∞ for some $C_1 > 0$. On the other hand, by Lemma 2.3, there exist a $C_2 > 0$, $C_1 > C_2$ such that $u(r) \ge C_2 r^{2-n}\log r$. Then, there exists a solution of Eq. (1.2) decay like $r^{2-n}\log r$ at ∞. By Lemma 2.1, $\lim_{r\to\infty} \frac{r^{n-2}u(r)}{\log r} = \mu d/(n-2)$.

(iii): When $q > n$, by Lemma 2.5 and Lemma 2.1 in [9], for every $\mu \in (0, \mu_\star)$, there exists a solution u of Eq. (1.2) such that $u(r) \le C_1 r^{2-n}$ at ∞ for some $C_1 > 0$. On the other hand, by Lemma 2.3, there exist a $C_2 > 0$, $C_1 > C_2$ such that $u(r) \ge C_2 r^{2-n}$. The conclusion (iii) follows from Lemma 2.1.

3. Proof Theorem 1.2

In this section, we start with two lemmas, based on which our Theorem 1.2 will be proved. For their proofs, we refer the readers to the proofs of Theorem 5.1 in [19] and Lemma 2.20 in [15].

Lemma 3.1 Suppose (K.1), (K.3) and $p > p_c$, and there exists $\gamma > \lambda_2$ such that

$$(r^{-l}K(r) - k_\infty) = O\left(\frac{1}{r^\gamma}\right) \quad \text{at } r = \infty\ . \tag{3.1}$$

Let $\bar u$ be a positive radial slow decay solution of (1.6). Then $\bar u$ has the following expansion at $r = \infty$:

$$\bar u(r) = \begin{cases} \dfrac{L}{k_\infty^{\frac{1}{p-1}} r^m} + \dfrac{a_1}{r^{m+\lambda_1}} + \dfrac{a_2}{r^{m+2\lambda_1}} + \cdots + \dfrac{b_1}{r^{m+\lambda_2}} + \cdots + O\left(\dfrac{1}{r^{n-2+\varepsilon}}\right) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if } \lambda_2 \ne \Lambda\lambda_1, \\ \dfrac{L}{k_\infty^{\frac{1}{p-1}} r^m} + \dfrac{a_1}{r^{m+\lambda_1}} + \dfrac{a_2}{r^{m+2\lambda_1}} + \cdots + \dfrac{c_1 \log r}{r^{m+\Lambda\lambda_1}} + \dfrac{b_1}{r^{m+\lambda_2}} \\ \qquad\qquad\qquad + \cdots + O\left(\frac{1}{r^{n-2+\varepsilon}}\right) \qquad\quad \text{if } \lambda_2 = \Lambda\lambda_1, \end{cases} \tag{3.2}$$

where $a_i,\ b_i,\ i = 1, 2, \cdots$, are (solution dependent) constants.

Lemma 3.2 Suppose w_2 be a positive radial super-solution of $\Delta w + K(x)w = 0$ in B_R and w_1 is a radial sub-solution of the same equation in B_R with $w_1(0) > 0$. Then

$$w_1(r) \ge \frac{w_1(0)}{w_2(0)} w_2(r)$$

for all $0 \le r \le R$. Moreover

$$w_1(R) > \frac{w_1(0)}{w_2(0)} w_2(R)$$

if one of the functions is not a solution of the equation.

Proof of Theorem 1.2: Let u_β and u_η be the solutions of (1.3) with the initial value $u_\beta(0) = \beta$ and $u_\eta(0) = \eta$ respectively and $\beta \ge \eta$, $\bar{u}_\alpha$ and $\bar{u}_\beta$ be the solutions of (1.6) with the initial value $\bar{u}_\alpha(0) = \alpha$ and $\bar{u}_\beta(0) = \beta$ respectively and $\alpha > \beta$. We are going to prove that $\beta = \eta$ if both $\lim_{r\to\infty} r^m u_\beta(r) = 0$ and $\lim_{r\to\infty} r^m u_\eta(r) = 0$ hold.

First it can be shown that $u_\beta \le \bar{u}_\beta$ (see Lemma 2.2 in [11]).

Suppose by contradiction that $\beta > \eta$. Denote $w_1 = u_\beta - u_\eta$, then we have

$$\Delta w_1 + K_1 w_1 = 0, \quad x \in \mathbb{R}^n,$$

where $K_1 = K(r)\dfrac{u_\beta^p - u_\eta^p}{u_\beta - u_\eta} \le pK(r)u_\beta^{p-1} \le pK(r)\bar{u}_\beta^{p-1}$.

Denote $w_2 = \bar{u}_\alpha - \bar{u}_\beta$. Similarly, we have

$$\Delta w_2 + K_2 w_2 = 0, \quad x \in \mathbb{R}^n,$$

where $K_2 = K(r)\dfrac{\bar{u}_\alpha^p - \bar{u}_\beta^p}{\bar{u}_\alpha - \bar{u}_\beta} \ge pK(r)\bar{u}_\beta^{p-1}$. So w_1 and w_2 are sub and super-solutions of

$$\Delta w + pK(r)\bar{u}_\beta^{p-1} w = 0, \quad x \in \mathbb{R}^n.$$

From Lemma 3.2, we deduce that for any $R > 0$,

$$w_1(r) \ge \frac{w_1(0)}{w_2(0)} w_2(r) \qquad \text{for all } \ 0 \le r \le R. \tag{3.3}$$

On the other hand, from Proposition 3.1 in [5], Lemma 4.3 in [10] and Lemma 3.1, we have

$$w_2 \ge cr^{-(m+\lambda_1)} \quad \text{at } \infty \tag{3.4}$$

for some $c > 0$ since we assume $(K.1)$ with $d > \lambda_2 - \ell$ and $p > p_c$. Now we are going to estimate $w_1(r)$ as $r \to \infty$ if both $\lim_{r\to\infty} r^m u_\beta(r) = 0$ and $\lim_{r\to\infty} r^m u_\eta(r) = 0$ hold. In fact, we have $q > n - m - \lambda_2 = m + 2 + \lambda_1 > m + 2$ and hence $q > m + 2 = \frac{2p+l}{p-1}$. Now Lemma 2.1 can be applied to yield the following three cases for w_1 at ∞:

Case 1: $w_1 = u_\beta - u_\eta = o(r^{2-q})$ if $m + 2 < q < n$.

Case 2: $w_1 = u_\beta - u_\eta = o(r^{2-n}\log r)$ if $q = n$.

Case 3: $w_1(r) = u_\beta - u_\eta = O(r^{2-n})$ if $q > n$.

By using the fact $p > p_c$ and (1.5), $b_0 = n - 2 - 2m = \lambda_1 + \lambda_2$, i.e., $n - m - \lambda_2 = m + 2 + \lambda_1$. And also by $(f.2)$, we have that $q > n - m - \lambda_2$, hence $q - 2 > m + \lambda_1$. Especially when $q = n$, $n - 2 > m + \lambda_1$. Comparing w_1 and w_2 at ∞ ((3.4) and the three cases above), there exists a constant $R_1 > 0$ large enough such that $w_1(r) < \frac{w_1(0)}{w_2(0)} w_2(r)$ for $r > R_1$, which contradicts the inequality (3.3). This completes the proof of Theorem 1.2.

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