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## ASYMPTOTIC BEHAVIOR OF LINEARIZED VISCOELASTIC FLOW PROBLEM

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**ABSTRACT.** In this article, we provide some asymptotic behaviors of linearized viscoelastic flows in a general two-dimensional domain with certain parameters small and the time variable large.

**1. Introduction.** Viscoelastic fluid flows are governed by Oldroyd's mathematical model. Such a model (see [11, 12]) may be defined by the rheological relation

$$k_0 \Sigma + k_1 \frac{\partial \Sigma}{\partial t} = \eta_0 \xi + \eta_1 \frac{\partial \xi}{\partial t}, \quad k_1 \Sigma(x, 0) = \eta_1 \xi(x, 0). \quad (1)$$

Here  $\Sigma$  is the deviator of the stress tensor and  $\xi$  is the strain tensor. Namely,  $\xi$  is an  $m \times m$  matrix with components

$$\xi_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right),$$

where  $u = u(x, t) = (u_1(x, t), \dots, u_m(x, t))$  is the velocity of the fluid motion and  $k_0, k_1, \eta_0, \eta_1$  are positive constants, while  $m$ , the space dimension, may take the values 2, 3.

Relation (1) and the motion equation in the Cauchy form leads us to the following viscoelastic flow equations:

$$\left\{ \begin{array}{l} u_t - \varepsilon \Delta u + (u \cdot \nabla) u - \rho e^{-\delta t} \int_0^t e^{\delta \tau} \Delta u \, d\tau + \nabla p = f, \\ \operatorname{div} u = 0 \text{ in } \Omega \times R^+, \\ u = 0 \text{ on } \partial \Omega \times R^+, \\ u(x, 0) = u_0 \text{ in } \Omega, \\ (p, 1) = \int_{\Omega} p(x, t) \, dx = 0, \end{array} \right. \quad (2)$$

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where  $u_t = \frac{\partial u}{\partial t}$  and

$$\varepsilon = \frac{\eta_1}{k_1} > 0, \quad \delta = \frac{k_0}{k_1} > 0, \quad \rho = \frac{1}{k_1^2}(\eta_0 k_1 - k_0 \eta_1), \quad \varepsilon_0 = \frac{\eta_0}{k_0} = \varepsilon + \frac{\rho}{\delta} > 0,$$

$\Omega$  is an open bounded domain of points  $x = (x_1, \dots, x_m)$  in  $R^m$  with smooth boundary  $\partial\Omega$ ,  $p = p(x, t)$  is the pressure of the fluid,  $f = f(x, t)$  is a prescribed external force, and  $u_0 = u_0(x)$  is the initial velocity. The last condition  $(p, 1) = 0$  in (2) is introduced for the uniqueness of the pressure  $p$ . The system (2) reduces to the Navier-Stokes flow equations if  $\frac{\rho}{\delta} = 0$ .

The flow equations (2) have been investigated by Oskolkov and Kotsiolis [9], where Ladyzhenskaja's methods were applied (see [10]). These theoretical investigations were continued in the articles of Agranovich and Sobolevskii [1, 2, 3], Sobolevskii [16, 17], Orlov and Sobolevskii [13] and He et al. [6, 7]. The existence and uniqueness of the solution of problem (2), local in time for  $m = 3$  and global in time for  $m = 2$ , were established in [1, 2, 3, 13]. Furthermore, some numerical methods of solving (2) have been investigated by Cannon et al. [4], Pani et al. [14, 15] and He et al. [6, 7].

We consider here the linearized viscoelastic flow equations

$$\begin{cases} u_t - \varepsilon \Delta u - \rho e^{-\delta t} \int_0^t e^{\delta \tau} \Delta u \, d\tau + \nabla p = f, \\ \operatorname{div} u = 0 \text{ in } \Omega \times R^+, \\ u = 0 \text{ on } \partial\Omega \times R^+, \quad u(x, 0) = u_0 \text{ in } \Omega, \quad (p, 1) = 0, \end{cases} \quad (3)$$

where  $\Omega$  is an open bounded domain of points  $x = (x_1, x_2)$  in  $R^2$  with smooth boundary  $\partial\Omega \in C^2$ , or  $\Omega$  is a convex polygonal domain.

The asymptotic behavior of wall-bounded flows at large Reynolds number or small viscosity is one of the most intriguing and largely open problems both in fluid mechanics and in mathematical analysis. On the one hand the nonlinear term tends to mix the modes and propagate energy among them. On the other hand it is known that important phenomena occur in a thin region near boundary called the boundary layer, in particular the generation of vortices which propagate in the fluid and drive the flow. As a preliminary step towards the understanding of the asymptotic behavior of the solutions to the viscoelastic flow equations in a general two-dimensional domain with nonslip boundary condition at small viscosity (large Reynolds number), we study here the asymptotic behavior of the solutions to the linearized viscoelastic flow problem; our results are an extension of the result provided by Temam and Wang [20]. For the asymptotic behavior of the solutions to the linearized Navier-Stokes flow equations and full Navier-Stokes flow equations, Temam and Wang have produced a lot of beautiful results: the reader may refer to [18, 19, 20, 21, 22, 23].

Here, we are interested in the asymptotic behavior of the linearized viscoelastic flow equations (3) if  $f(x, t)$  approaches a steady-state force  $\bar{f}(x)$  as  $t \rightarrow \infty$ . So the corresponding flow equations are the steady-state linearized Navier-Stokes flow equations:

$$\begin{cases} -\varepsilon_0 \Delta \bar{u} + \nabla \bar{p} = \bar{f}, \quad \operatorname{div} \bar{u} = 0 \text{ in } \Omega, \\ \bar{u} = 0 \text{ on } \partial\Omega, \quad (\bar{p}, 1) = 0, \end{cases} \quad (4)$$

where  $\bar{f}(x) = \lim_{t \rightarrow \infty} f(x, t)$  in the  $L^2$ -norm. Next, we consider the equations obtained by passing to the limit, as  $\frac{\varepsilon}{\delta} \rightarrow 0$  and  $\varepsilon_0 = \varepsilon + \frac{\varepsilon}{\delta} \rightarrow 0$ , respectively, in the flow equations (3). The corresponding flow equations are the linearized Navier-Stokes flow equations

$$\begin{cases} u_t^{\varepsilon_0} - \varepsilon_0 \Delta u^{\varepsilon_0} + \nabla p^{\varepsilon_0} = f, \operatorname{div} u^{\varepsilon_0} = 0 \text{ in } \Omega \times R^+ \\ u^{\varepsilon_0} = 0 \text{ on } \partial\Omega \times R^+, u^{\varepsilon_0}(x, 0) = u_0(x) \text{ in } \Omega, (p^{\varepsilon_0}, 1) = 0, \end{cases} \quad (5)$$

and the linearized Euler flow equations

$$\begin{cases} u_t^0 + \nabla p^0 = f, \operatorname{div} u^0 = 0 \text{ in } \Omega \times R^+, \\ u^0 \cdot n = 0 \text{ on } \partial\Omega \times R^+, u^0(x, 0) = u_0(x) \text{ in } \Omega, (p^0, 1) = 0, \end{cases} \quad (6)$$

where  $n$  is the unit outer normal of  $\partial\Omega$ . Hereafter, for convenience, we assume that  $\rho \geq 0$  and write  $u(x, t)$  and  $\bar{u}(x)$  as  $u(t)$  and  $\bar{u}$ , respectively. In order to consider the convergence of the flow equations (3) to the flow equations (6), we need to introduce the following boundary layer flow equations:

$$\begin{cases} \theta_t^{\varepsilon_0} - \varepsilon_0 \Delta \theta^{\varepsilon_0} + \nabla \eta^{\varepsilon_0} = 0, \operatorname{div} \theta^{\varepsilon_0} = 0 \text{ in } \Omega \times R^+, \\ \theta^{\varepsilon_0}|_{\partial\Omega} = -u^0 \text{ on } \partial\Omega, \\ \theta^{\varepsilon_0}(0) = 0. \end{cases} \quad (7)$$

For further discussion of the boundary layer flows, the reader may refer to Temam and Wang [20].

The mathematical setting of the flow equations (3)–(7) needs the following Hilbert spaces:

$$X = H_0^1(\Omega)^2, \quad M = L_0^2(\Omega) = \left\{ q \in L^2(\Omega) ; \int_{\Omega} q dx = 0 \right\}.$$

The spaces  $L^2(\Omega)^l$ ,  $l = 1, 2, 4$ , are endowed with the  $L^2$ -scalar product and  $L^2$ -norm denoted by  $(\cdot, \cdot)$  and  $|\cdot|$ . The space  $X$  is equipped with its usual scalar product and norm

$$((u, v)) = (\nabla u, \nabla v), \quad \|u\| = ((u, u))^{1/2}.$$

Moreover, we denote the norm of the Sobolev space  $H^i(\Omega)$  or  $H^i(\Omega)^2$  by  $\|\cdot\|_i$  with  $i = 1, 2$ . The subspaces  $V$  and  $H$  of  $X$  and  $L^2(\Omega)^2$  are defined by

$$\begin{aligned} V &= \left\{ v \in X ; \operatorname{div} v = 0 \text{ in } \Omega \right\}, \\ H &= \left\{ v \in L^2(\Omega)^2 ; \operatorname{div} v = 0 \text{ in } \Omega \text{ and } v \cdot n = 0 \text{ on } \partial\Omega \right\}. \end{aligned}$$

Also, we use often the weighted Banach space

$$L^{(\gamma, \beta)}(R^+; F) = \left\{ v(t) \in L_{loc}^{\gamma}(R^+; F) ; \sup_{t \geq 0} e^{-\beta t} \int_0^t e^{\beta s} \|v\|_F^{\gamma} ds < \infty \right\},$$

with the norm

$$\|v\|_{L^{(\gamma, \beta)}(R^+; F)} = \sup_{t \geq 0} \left( e^{-\beta t} \int_0^t e^{\beta s} \|v\|_F^{\gamma} ds \right)^{1/\gamma},$$

for some real numbers  $\gamma \geq 1$ ,  $\beta \geq 0$  and Hilbert space  $F$ . For some fixed  $\gamma \geq 1$ , there hold the inclusions

$$L^{(\gamma, \beta_2)}(R^+; F) \subset L^{(\gamma, \beta_1)}(R^+; F), \quad \forall 0 \leq \beta_2 \leq \beta_1, \quad L^{(\gamma, 0)}(R^+; F) = L^{\gamma}(R^+; F),$$

since  $e^{(\beta_1 - \beta_2)s} \leq e^{(\beta_1 - \beta_2)t}$  for the range of  $0 \leq s \leq t$ .

In this paper, we prove the following main results on the asymptotic behaviors of the flow equations (3). For the sake of simplicity, we always assume that  $\rho \geq 0$ .

**Theorem 1.1.** *Assume that  $\alpha \geq 0$ ,  $0 \leq \beta < \min\{\varepsilon\lambda_1, \delta\}$ , and  $u_0 \in H$ , and let  $f \in L^\infty(R^+; L^2(\Omega)^2)$ .*

(i) *If  $f$  satisfies*

$$\lim_{t \rightarrow \infty} e^{\beta t} t^\alpha |f(t) - \bar{f}|^2 = 0, \quad (8)$$

then

$$\lim_{t \rightarrow \infty} e^{\beta t} t^\alpha (|u_t(t)|^2 + \|u(t) - \bar{u}\|^2 + \|p(t) - \bar{p}\|_1^2) = 0. \quad (9)$$

(ii) *If  $u_0 \in H^1(\Omega)^2 \cap H$ ,  $f \in L^{(1, \beta)}(R^+; X)$ , then*

$$\left\{ \begin{array}{l} \|u - u^{\varepsilon_0}\|_{L^\infty(R^+; H^1)} \leq \kappa \varepsilon_0^{-\frac{1}{2}} \left(\frac{\rho}{\delta}\right)^{\frac{1}{2}}, \\ \|u_t - u_t^{\varepsilon_0}\|_{L^{(2, \varepsilon_0 \lambda_1)}(R^+; L^2)} + \varepsilon_0 \|u - u^{\varepsilon_0}\|_{L^{(2, \varepsilon_0 \lambda_1)}(R^+; H^2)} \\ \leq \kappa \left(\frac{\rho}{\delta}\right)^{\frac{1}{2}}, \\ \|p - p^{\varepsilon_0}\|_{L^{(2, \varepsilon_0 \lambda_1)}(R^+; H^1)} \leq \kappa \left(\frac{\rho}{\delta}\right)^{\frac{1}{2}} \end{array} \right. \quad (10)$$

for all  $t \geq 0$ .

(iii) *Finally, if  $\Omega$ ,  $u_0 \in H^1(\Omega)^2 \cap H$ ,  $f \in L^\infty(R^+; L^2(\Omega)^2) \cap L^{(1, \beta)}(R^+; X)$  are sufficiently smooth that  $\|u_t^0(t)\|_1 \leq \kappa$  for all  $t \geq 0$ , then*

$$\left\{ \begin{array}{l} |u(t) - u^0(t)| \leq \kappa t^{\frac{1}{2}} e^t \varepsilon_0^{\frac{1}{4}}, \\ \left| \int_0^t (p - p^0) ds \right| \leq \kappa t^{\frac{1}{2}} e^t \varepsilon_0^{\frac{1}{4}} \end{array} \right. \quad (11)$$

for all  $t \geq 0$ .

Hereafter,  $\kappa$  is used to denote a general constant depending on the data  $(u_0, f, \Omega)$ ,  $\|\cdot\|_1$  denotes the norm on  $H^1(\Omega)$ , and  $\lambda_1$  is the minimal eigenvalue of the Laplace operator  $-\Delta$  such that

$$|v|^2 \leq \lambda_1^{-1} \|v\|^2 \quad \forall v \in X. \quad (12)$$

The estimate (11) above is an extension of the result provided by Temam and Wang [20].

**2. Convergence to the steady-state flow.** This section is devoted to deriving the asymptotic behavior of the steady-state linearized Navier-Stokes flow equations (3) if  $f \rightarrow \bar{f}$  as  $t \rightarrow \infty$ . Before deriving the asymptotic behavior, let us first provide the following lemma.

**Lemma 2.1.** *Assume that  $\alpha \geq 0$ ,  $\beta \geq 0$ , and  $u \in L^1_{loc}(R^+; X)$ . Then, for all  $t \geq t_0$ ,*

$$\begin{aligned} & 2 \int_{t_0}^t \sigma^\alpha(s) J(s; u, e^{\beta s} u(s)) ds \\ &= \rho \sigma^\alpha(t) e^{-(2\delta-\beta)t} \left\| \int_{t_0}^t e^{\delta\tau} u d\tau \right\|^2 - \rho \sigma^\alpha(t_0) e^{-(2\delta-\beta)t_0} \left\| \int_0^{t_0} e^{\delta\tau} u d\tau \right\|^2 \\ & \quad + \rho \int_{t_0}^t G_{\alpha,\beta}(s) e^{-(2\delta-\beta)s} \left\| \int_0^s e^{\delta\tau} u d\tau \right\|^2 ds, \end{aligned} \quad (13)$$

where

$$\begin{aligned} \sigma(t) &= \max\{1, t\}, \quad \sigma^0(t) = 1, \quad J(t; u, v) = \rho e^{-\delta t} \left( \left( \int_0^t e^{\delta\tau} u d\tau, v \right) \right), \\ G_{\alpha,\beta}(t) &= (2\delta - \beta) \sigma^\alpha(t) - \sigma_t^\alpha(t). \end{aligned}$$

*Proof.* This proof is carried out by integration by parts in a straightforward manner and is hence omitted here.  $\square$

Now, we introduce the continuous bilinear form  $d(\cdot, \cdot)$  on  $X \times M$ :

$$d(v, q) = -(v, \nabla q) = (q, \operatorname{div} v) \quad \forall v \in X, q \in M.$$

Once we apply the Agmon, Douglis and Nirenberg theory, we obtain

$$\|q\|_1 \leq c |\nabla q| \quad \forall q \in H^1(\Omega) \cap M, \quad (14)$$

where  $c > 0$  is used to denote a general positive constant depending on  $\Omega$ .

The equations (4) imply that the steady-state solution  $(\bar{u}, \bar{p})$  satisfies

$$\begin{cases} \bar{u}_t - \varepsilon \Delta \bar{u} - \rho e^{-\delta t} \int_0^t e^{\delta\tau} \Delta \bar{u} d\tau + \nabla \bar{p} = \bar{f} + \frac{\rho}{\delta} e^{-\delta t} \Delta \bar{u} \\ \quad \text{in } \Omega \times R^+, \\ \bar{u} \in V, \bar{p} \in M, \bar{u}(0) = \bar{u}(x) \in V. \end{cases} \quad (15)$$

The flow equations (3) and (15) imply that  $(z, r) = (u - \bar{u}, p - \bar{p})$  satisfies

$$\begin{cases} z_t - \varepsilon \Delta z - \rho e^{-\delta t} \int_0^t e^{\delta\tau} \Delta z d\tau + \nabla r = g \quad \text{in } \Omega \times R^+, \\ z(t) \in V, r(t) \in M \text{ for } t > 0, z(0) = z_0 = u_0 - \bar{u} \in H, \end{cases} \quad (16)$$

where

$$g(t) = f(t) - \bar{f} - \frac{\rho}{\delta} e^{-\delta t} \Delta \bar{u}.$$

So (16) leads to the following variational formulation of  $(z, r)$ :

$$(z_t, v) + \varepsilon((z, v)) + J(t; z, v) - d(v, r) + d(z, q) = (g, v) \quad \forall (v, q) \in (X, M), \quad (17)$$

$$z(0) = z_0 \in H. \quad (18)$$

With the form assumed for  $\Omega$ , problem (4) admits a unique solution  $(\bar{u}, \bar{p})$  which satisfies

$$\varepsilon_0 \|\bar{u}\|_2 + \|\bar{p}\|_1 \leq c \|\bar{f}\|; \quad (19)$$

moreover, letting  $P$  be the Leray-Hopf projection, the following inequalities hold:

$$\begin{aligned} \|Pu\| &\leq c\|u\| \quad \forall u \in X, \\ \|u\|_2 &\leq c|P\Delta u| \quad \forall u \in H^2(\Omega)^2 \cap V. \end{aligned} \quad (20)$$

For the details of these facts, the reader may refer to Heywood and Rannacher [8].

From (8) and (19), it follows that

$$\lim_{t \rightarrow \infty} e^{\beta t} t^\alpha |g(t)|^2 = 0, \quad 0 \leq \beta < \min\{\varepsilon\lambda_1, \delta\}. \quad (21)$$

**Theorem 2.2.** *Assume that  $\alpha \geq 0$ ,  $0 \leq \beta < \min\{\varepsilon\lambda_1, \delta\}$ , and  $u_0 \in H$ , and let  $f \in L^\infty(R^+; L^2(\Omega)^2)$  satisfy (8). Then*

$$\lim_{t \rightarrow \infty} e^{\beta t} t^\alpha (|u_t(t)|^2 + \|u(t) - \bar{u}\|^2 + \|p(t) - \bar{p}\|_1^2) = 0. \quad (22)$$

*Proof.* First, fixing  $\beta < \beta_0 < \min\{\varepsilon\lambda_1, \delta\}$  and taking  $(v, q) = 2e^{\beta_0 t}(z, r)$  in (17), using (12) and noting

$$2|(g, z)| \leq 2|g||z| \leq \varepsilon\|z\|^2 + \varepsilon^{-1}\lambda_1^{-1}|g|^2, \quad \varepsilon\lambda_1 - \beta_0 > 0, \quad (23)$$

we obtain

$$\frac{d}{dt}(e^{\beta_0 t}|z|^2) + 2J(t, z, e^{\beta_0 t}z(t)) \leq \varepsilon^{-1}\lambda_1^{-1}e^{\beta_0 t}|g|^2. \quad (24)$$

Integrating (24) from 0 to  $t$  and using Lemma 2.1 with  $(\alpha, \beta) = (0, \beta_0)$  and  $G_{0, \beta_0} \geq 0$ , we obtain, after a final multiplication by  $e^{-\beta_0 t}$ ,

$$\begin{aligned} |z(t)|^2 + \rho e^{-2\delta t} \left\| \int_0^t e^{\delta\tau} z \, d\tau \right\|^2 \\ \leq e^{-\beta_0 t} |z_0|^2 + \varepsilon^{-1}\lambda_1^{-1}\beta_0^{-1} \sup_{t \geq 0} |g(t)|^2 \quad \forall t \geq 0. \end{aligned} \quad (25)$$

Next, by taking  $(v, q) = 2\sigma^\alpha(t)e^{\beta_0 t}(z, r)$  in (17), we obtain

$$\begin{aligned} \frac{d}{dt}(\sigma^\alpha(t)e^{\beta_0 t}|z|^2) + 2\varepsilon\sigma^\alpha(t)e^{\beta_0 t}\|z\|^2 + 2\rho\sigma^\alpha(t)J(t, z, e^{\beta_0 t}z(t)) \\ \leq (\beta_0\sigma^\alpha(t) + \sigma_t^\alpha(t))e^{\beta_0 t}|z|^2 + 2\sigma^\alpha(t)e^{\beta_0 t}|(g, z)|. \end{aligned} \quad (26)$$

Integrating (26) from  $t_0$  to  $t$  and using Lemma 2.1 with  $\beta = \beta_0$ , we obtain

$$\begin{aligned} \sigma^\alpha(t)e^{\beta_0 t}|z(t)|^2 + 2\varepsilon \int_{t_0}^t \sigma^\alpha(s)e^{\beta_0 s}\|z\|^2 ds + \rho\sigma^\alpha(t)e^{-(2\delta-\beta_0)t} \left\| \int_0^t e^{\delta\tau} z \, d\tau \right\|^2 \\ + \rho \int_{t_0}^t G_{\alpha, \beta_0}(s)e^{-(2\delta-\beta_0)s} \left\| \int_0^s e^{\delta\tau} z \, d\tau \right\|^2 ds \\ \leq \sigma^\alpha(t_0)e^{\beta_0 t_0}|z(t_0)|^2 + \rho\sigma^\alpha(t_0)e^{-(2\delta-\beta_0)t_0} \left\| \int_0^{t_0} e^{\delta\tau} z \, d\tau \right\|^2 \\ + \int_{t_0}^t (\beta_0\sigma^\alpha(s) + \sigma_t^\alpha(s))e^{\beta_0 s}|z|^2 ds \\ + 2 \int_{t_0}^t \sigma^\alpha(s)e^{\beta_0 s}|(g, z)| ds. \end{aligned} \quad (27)$$

Using (23) and taking  $t_0$  such that

$$G_{(\alpha, \beta_0)}(t) \geq 0, \quad (\varepsilon\lambda_1 - \beta_0)\sigma^\alpha(t) - \sigma_t^\alpha(t) \geq 0 \quad \forall t \geq t_0, \quad (28)$$

we then derive from (27) that

$$\begin{aligned} & \sigma^\alpha(t)e^{\beta_0 t}|z(t)|^2 + \rho\sigma^\alpha(t)e^{-(2\delta-\beta_0)t} \left\| \int_0^t e^{\delta\tau} z \, d\tau \right\|^2 \\ & \leq \sigma^\alpha(t_0)e^{\beta_0 t_0}|z(t_0)|^2 + \rho\sigma^\alpha(t_0)e^{-(2\delta-\beta_0)t_0} \left\| \int_0^{t_0} e^{\delta\tau} z \, d\tau \right\|^2 \\ & \quad + \frac{1}{\varepsilon\lambda_1} \int_{t_0}^t \sigma^\alpha(s)e^{\beta_0 s}|g|^2 ds \end{aligned} \quad (29)$$

and

$$\begin{aligned} & \sigma^\alpha(t)e^{\beta_0 t}|z(t)|^2 + \varepsilon \int_0^t \sigma^\alpha(s)e^{\beta_0 s}\|z\|^2 ds + \rho\sigma^\alpha(t)e^{-(2\delta-\beta_0)t} \left\| \int_0^t e^{\delta\tau} z \, d\tau \right\|^2 \\ & \leq \sigma^\alpha(t_0)e^{\beta_0 t_0}|z(t_0)|^2 + \rho\sigma^\alpha(t_0)e^{-(2\delta-\beta_0)t_0} \left\| \int_0^{t_0} e^{\delta\tau} z \, d\tau \right\|^2 \\ & \quad + \int_0^t (\beta_0\sigma^\alpha(s) + \sigma_t^\alpha(s))e^{\beta_0 s}|z|^2 ds + \frac{1}{\varepsilon\lambda_1} \int_0^t \sigma^\alpha(s)e^{\beta_0 s}|g|^2 ds \end{aligned} \quad (30)$$

for all  $t \geq 0$ .

Multiplying (29) and (30), respectively, by  $e^{-(\beta_0-\beta)t}$ , passing to the limit  $t \rightarrow \infty$ , and using (25), we get

$$\lim_{t \rightarrow \infty} \sigma^\alpha(t)e^{\beta t}|z(t)|^2 + \rho \lim_{t \rightarrow \infty} \sigma^\alpha(t)e^{-(2\delta-\beta)t} \left\| \int_0^t e^{\delta\tau} z \, d\tau \right\|^2 = 0, \quad (31)$$

$$\lim_{t \rightarrow \infty} \sigma^\alpha(t)e^{\beta t}\|z(t)\|^2 = 0, \quad (32)$$

where the L'Hôpital rule is used. By (32), there exists  $t_1 \geq t_0$  such that

$$\sigma^\alpha(t)e^{\beta t}\|z(t)\|^2 \leq 1 \quad \forall t \geq t_1. \quad (33)$$

Moreover, we derive from (17) that

$$|z_t|^2 + \varepsilon \frac{d}{dt} \|z\|^2 + 2\rho \left( \left( \int_0^t e^{\delta\tau} z \, d\tau, e^{-\delta t} z_t \right) \right) \leq |g|^2, \quad \forall t \geq 0. \quad (34)$$

Multiplying (34) by  $\sigma^\alpha(t)e^{\beta_0 t}$  yields

$$\begin{aligned} & \sigma^\alpha(t)e^{\beta_0 t}|z_t|^2 + \varepsilon \frac{d}{dt} (\sigma^\alpha(t)e^{\beta_0 t}\|z\|^2) \\ & \quad + 2\rho e^{-\delta t} \left( \left( \int_0^t e^{\delta\tau} z \, d\tau, \frac{d}{dt} (\sigma^\alpha(t)e^{\beta_0 t} z) \right) \right) \\ & \quad - 2(\beta_0\sigma^\alpha(t) + \sigma_t^\alpha(t))J(t, z, e^{\beta_0 t} z(t)) \\ & \leq \varepsilon(\beta_0\sigma^\alpha(t) + \sigma_t^\alpha(t))e^{\beta_0 t}\|z\|^2 + \sigma^\alpha(t)e^{\beta_0 t}|g|^2 \quad \forall t \geq 0. \end{aligned} \quad (35)$$



Integrating (35) from  $t_1$  to  $t$ , we obtain

$$\begin{aligned}
& \int_{t_1}^t \sigma^\alpha(s) e^{\beta_0 s} |z_t|^2 ds + \varepsilon \sigma^\alpha(t) e^{\beta_0 t} \|z(t)\|^2 \\
& \quad + 2\rho \int_{t_1}^t e^{-\delta t} \left( \left( \int_0^t e^{\delta \tau} z d\tau, \frac{d}{dt}(\sigma^\alpha(t) e^{\beta_0 t} z) \right) \right) ds \\
& \leq 2 \int_{t_1}^t (\beta_0 \sigma^\alpha(s) + \sigma_t^\alpha(s)) J(s, z, e^{\beta_0 s} z(s)) ds \\
& \quad + \varepsilon \sigma^\alpha(t_1) e^{\beta_0 t_1} \|z(t_1)\|^2 + \varepsilon \int_{t_1}^t (\beta_0 \sigma^\alpha(s) + \sigma_t^\alpha(s)) e^{\beta_0 s} \|z\|^2 ds \\
& \quad + \int_{t_1}^t \sigma^\alpha(s) e^{\beta_0 s} |g|^2 ds
\end{aligned} \tag{36}$$

for all  $t \geq 0$ . From Lemma 2.1,

$$\begin{aligned}
2\beta_0 \int_{t_1}^t \sigma^\alpha(s) J(s; z, e^{\beta_0 s} z(s)) ds & \leq \beta_0 \rho \sigma^\alpha(t) e^{-(2\delta - \beta_0)t} \left\| \int_{t_1}^t e^{\delta \tau} z d\tau \right\|^2 \\
& + \beta_0 \rho ((2\delta - \beta_0) + \alpha t_1^{-1}) \int_{t_1}^t \sigma^\alpha(s) e^{-(2\delta - \beta_0)s} \left\| \int_0^s e^{\delta \tau} z d\tau \right\|^2 ds.
\end{aligned} \tag{37}$$

Using integration by parts, we have

$$\begin{aligned}
I & = 2\rho \int_{t_1}^t e^{-\delta s} \left( \left( \int_0^s e^{\delta \tau} z d\tau, \frac{d}{ds}(\sigma^\alpha(s) e^{\beta_0 s} z) \right) \right) ds \\
& = 2\rho e^{-\delta t} \left( \left( \int_0^t e^{\delta \tau} z d\tau, \sigma^\alpha(t) e^{\beta_0 t} z(t) \right) \right) \\
& \quad - 2\rho e^{-\delta t_1} \left( \left( \int_0^{t_1} e^{\delta \tau} z d\tau, \sigma^\alpha(t_1) e^{\beta_0 t_1} z(t_1) \right) \right) \\
& \quad + 2\rho \delta \int_{t_1}^t e^{-\delta s} \left( \left( \int_0^s e^{\delta \tau} z d\tau, \sigma^\alpha(s) e^{\beta_0 s} z \right) \right) ds \\
& \quad - 2\rho \int_{t_1}^t \sigma^\alpha(s) e^{\beta_0 s} \|z\|^2 ds.
\end{aligned}$$

Hence,

$$\begin{aligned}
|I| & \leq \frac{\varepsilon}{4} \sigma^\alpha(t) e^{\beta_0 t} \|z(t)\|^2 + \frac{4}{\varepsilon} \rho^2 e^{-(2\delta - \beta_0)t} \left\| \int_0^t e^{\delta \tau} z d\tau \right\|^2 \\
& \quad + \frac{\varepsilon}{4} \sigma^\alpha(t_1) e^{\beta_0 t_1} \|z(t_1)\|^2 + \frac{4}{\varepsilon} \rho^2 e^{-(2\delta - \beta_0)t_1} \left\| \int_0^{t_1} e^{\delta \tau} z d\tau \right\|^2 \\
& \quad + \frac{\varepsilon}{4} \int_{t_1}^t \sigma^\alpha(s) e^{\beta_0 s} \|z\|^2 ds \\
& \quad + \frac{4}{\varepsilon} \rho^2 \delta^2 \int_{t_1}^t \sigma^\alpha(s) e^{-(2\delta - \beta_0)s} \left\| \int_0^s e^{\delta \tau} z d\tau \right\|^2 ds \\
& \quad + 2\rho \int_{t_1}^t \sigma^\alpha(s) e^{\beta_0 s} \|z\|^2 ds.
\end{aligned} \tag{38}$$

Combining (36) with (37)–(38) yields

$$\begin{aligned}
 & \int_{t_1}^t \sigma^\alpha(s) e^{\beta_0 s} |z_t|^2 ds + \varepsilon \sigma^\alpha(t) e^{\beta_0 t} \|z(t)\|^2 \\
 & \leq c \sigma^\alpha(t_1) e^{\beta_0 t_1} \|z(t_1)\|^2 + c \int_{t_1}^t \sigma^\alpha(s) e^{\beta_0 s} \|z\|^2 ds \\
 & \quad + c \sigma^\alpha(t_1) e^{-(2\delta - \beta_0)t_1} \left\| \int_0^{t_1} e^{\delta \tau} z d\tau \right\|^2 \\
 & \quad + c \sigma^\alpha(t) e^{-(2\delta - \beta_0)t} \left\| \int_0^t e^{\delta \tau} z d\tau \right\|^2 \\
 & \quad + c \int_{t_1}^t \sigma^\alpha(s) e^{-(2\delta - \beta_0)s} \left\| \int_0^s e^{\delta \tau} z d\tau \right\|^2 ds \\
 & \quad + \int_{t_1}^t \sigma^\alpha(s) e^{\beta_0 s} |g|^2 ds
 \end{aligned} \tag{39}$$

for all  $t \geq 0$ . Multiplying (39) by  $e^{-(\beta_0 - \beta)t}$ , passing to the limit  $t \rightarrow \infty$ , and using (21) and (31)–(32) and the L'Hôpital rule, we obtain

$$\lim_{t \rightarrow \infty} t^\alpha e^{\beta t} |z_t(t)|^2 = 0. \tag{40}$$

In the same way, we find from (16), (20), and (40) that

$$\begin{aligned}
 & \lim_{t \rightarrow \infty} t^\alpha e^{\beta t} \left| \varepsilon \Delta z(t) + \rho e^{-\delta t} \int_0^t e^{\delta \tau} \Delta z d\tau \right|^2 \\
 & \leq c \lim_{t \rightarrow \infty} t^\alpha e^{\beta t} (|z_t(t)|^2 + |g(t)|^2) = 0.
 \end{aligned} \tag{41}$$

Finally, we derive from (14) and (16) that

$$\begin{aligned}
 & \sigma^\alpha(t) e^{\beta t} \|r(t)\|_1^2 \\
 & \leq c \sigma^\alpha(t) e^{\beta t} \left( |z_t(t)|^2 + |g(t)|^2 + \left| \varepsilon \Delta z(t) + e^{-\delta t} \int_0^t e^{\delta \tau} \Delta z d\tau \right|^2 \right).
 \end{aligned}$$

Passing to the limit  $t \rightarrow \infty$  in the above estimate and using (21) and (40)–(41), we get

$$\lim_{t \rightarrow \infty} t^\alpha e^{\beta t} \|r(t)\|_1^2 = 0. \tag{42}$$

Combining (32) with (40)–(42) yields (22). The proof is now complete.  $\square$

**3. Convergence to the linearized Navier-Stokes flow equations.** In this section we are concerned with the asymptotic behavior, as the physical constant  $\rho/\delta$  approaches zero, of the flows governed by (3). Here we assume that  $\varepsilon$  is fixed.

**Theorem 3.1.** *Assume that  $u_0 \in H^1(\Omega)^2 \cap H$  and  $f \in L^{1,\beta}(R^+; X)$  with  $0 \leq \beta < \min\{\varepsilon\lambda_1, \delta\}$ . Let  $(u, p)$  and  $(u^{\varepsilon_0}, p^{\varepsilon_0})$  be the solutions of (3) and (5), respectively. Then*

$$\begin{cases} \|u(t) - u^{\varepsilon_0}(t)\|^2 \leq \kappa \varepsilon_0^{-1} \frac{\rho}{\delta}, \\ e^{-\varepsilon_0 \lambda_1 t} \int_0^t e^{\varepsilon_0 \lambda_1 s} (|u_t - u_t^{\varepsilon_0}|^2 + \varepsilon_0^2 \|u - u^{\varepsilon_0}\|_2^2 + \|p - p^{\varepsilon_0}\|_1^2) ds \leq \kappa \frac{\rho}{\delta} \end{cases} \tag{43}$$

for all  $t \geq 0$ .

*Proof.* By taking the scalar product of (3) with  $2e^{\beta t}u$  and  $-2e^{\beta t}P\Delta u$ , respectively, and using (12) and the inequality  $\beta \leq \varepsilon\lambda_1$ , we obtain

$$\begin{aligned} & \frac{d}{dt}(e^{\beta t}|u|^2) + \varepsilon e^{\beta_0 t}\|u\|^2 + \rho e^{-(2\delta-\beta)t} \frac{d}{dt} \left\| \int_0^t e^{\delta\tau} u \, d\tau \right\|^2 \\ & \leq 2e^{\beta t}|f||u|, \end{aligned} \quad (44)$$

$$\begin{aligned} & \frac{d}{dt}(e^{\beta t}\|u\|^2) + 2\varepsilon e^{\beta t}|P\Delta u|^2 + \rho e^{-(2\delta-\beta)t} \frac{d}{dt} \left| P\Delta \int_0^t e^{\delta\tau} u \, d\tau \right|^2 \\ & \leq 2e^{\beta t}\|Pf\|\|u\| + \varepsilon\lambda_1 e^{\beta t}\|u\|^2. \end{aligned} \quad (45)$$

Integrating (44) and (45) from 0 to  $t$ , one finds, after a final multiplication by  $e^{-\beta t}$ , that

$$\begin{aligned} & |u(t)|^2 + \varepsilon e^{-\beta t} \int_0^t e^{\beta s}\|u\|^2 ds + \rho e^{-2\delta t} \left\| \int_0^t e^{\delta\tau} u \, d\tau \right\|^2 \\ & \quad + \rho(2\delta - \beta)e^{-\beta t} \int_0^t e^{-(2\delta-\beta)s} \left\| \int_0^t e^{\delta\tau} u \, d\tau \right\|^2 ds \\ & \leq |u_0|^2 + 2e^{-\beta t} \int_0^t e^{\beta s}|f||u| \, ds \quad \forall t \geq 0, \end{aligned} \quad (46)$$

$$\begin{aligned} & \|u(t)\|^2 + \varepsilon e^{-\beta t} \int_0^t e^{\beta s}|P\Delta u|^2 ds + \rho e^{-2\delta t} \left| P\Delta \int_0^t e^{\delta\tau} u \, d\tau \right|^2 \\ & \quad + \rho(2\delta - \beta)e^{-\beta t} \int_0^t e^{-(2\delta-\beta)s} \left| P\Delta \int_0^t e^{\delta\tau} u \, d\tau \right|^2 ds \\ & \leq \|u_0\|^2 + 2e^{-\beta t} \int_0^t e^{\beta s}\|Pf\|\|u\| \, ds \quad \forall t \geq 0. \end{aligned} \quad (47)$$

It follows from (46) that

$$\sup_{t \geq 0} |u(t)|^2 \leq 2|u_0|^2 + 4\|f\|_{L^1(\cdot, \beta)}^2. \quad (48)$$

Using (20) and (48) in (46) and (47), we see that

$$\begin{aligned} & \|u(t)\|^2 + \varepsilon e^{-\beta t} \int_0^t e^{\beta s}|\Delta u|^2 ds + \rho e^{-2\delta t} \left| \Delta \int_0^t e^{\delta\tau} u \, d\tau \right|^2 \\ & \quad + \rho(2\delta - \beta)e^{-\beta t} \int_0^t e^{-(2\delta-\beta)s} \left| \Delta \int_0^s e^{\delta\tau} u \, d\tau \right|^2 ds \leq \kappa \end{aligned} \quad (49)$$

for all  $t \geq 0$ .

Moreover, we derive from (3) and (5) that  $(z, r) = (u - u^{\varepsilon_0}, p - p^{\varepsilon_0})$  satisfies

$$\begin{cases} z_t - \varepsilon_0 \Delta z + \nabla r = -\frac{\rho}{\delta} \Delta u + \rho e^{-\delta t} \int_0^t e^{\delta\tau} \Delta u \, d\tau & \text{in } \Omega \times \mathbb{R}^+, \\ z(t) \in V, r(t) \in M \text{ for } t > 0, z(0) = 0. \end{cases} \quad (50)$$

So (50) leads to the following variational formulation of  $(z, r)$ :

$$\begin{aligned} & (z_t, v) + \varepsilon_0((z, v)) - d(v, r) + d(z, q) \\ & = \frac{\rho}{\delta}((u, v)) - \rho e^{-\delta t} \left( \left( \int_0^t e^{\delta\tau} u \, d\tau, v \right) \right) \end{aligned} \quad (51)$$

for all  $(v, q) \in (X, M)$  with  $z(0) = 0$ .

By taking  $(v, q) = 2e^{\varepsilon_0 \lambda_1 t}(z, r)$  in (51) and using (12), we obtain

$$\begin{aligned} & \frac{d}{dt}(e^{\varepsilon_0 \lambda_1 t}|z|^2) + \frac{\varepsilon_0}{2}e^{\varepsilon_0 \lambda_1 t}\|z\|^2 \\ & \leq \frac{4}{\varepsilon_0} \left(\frac{\rho}{\delta}\right)^2 e^{\varepsilon_0 \lambda_1 t}\|u\|^2 + \frac{4}{\varepsilon_0}\rho^2 e^{-(2\delta - \varepsilon_0 \lambda_1)t} \left\| \int_0^t e^{\delta\tau} u \, d\tau \right\|^2. \end{aligned} \quad (52)$$

Integrating (52) from 0 to  $t$  and using (49), we obtain, after a final multiplication by  $e^{-\varepsilon_0 \lambda_1 t}$ , that

$$|z(t)|^2 + \varepsilon_0 e^{-\varepsilon_0 \lambda_1 t} \int_0^t e^{\varepsilon_0 \lambda_1 s} \|z\|^2 ds \leq \kappa \varepsilon_0^{-1} \frac{\rho}{\delta} \quad \forall t \geq 0. \quad (53)$$

Moreover, we infer from (51) that

$$\begin{aligned} & e^{\varepsilon_0 \lambda_1 t}|z_t|^2 + \varepsilon_0 \frac{d}{dt}(e^{\varepsilon_0 \lambda_1 t}\|z\|^2) \\ & \leq \varepsilon_0^2 \lambda_1 e^{\varepsilon_0 \lambda_1 t}\|z\|^2 + 2 \left(\frac{\rho}{\delta}\right)^2 e^{\varepsilon_0 \lambda_1 t} |\Delta u|^2 \\ & \quad + 2\rho^2 e^{-(2\delta - \varepsilon_0 \lambda_1)t} \left| \int_0^t e^{\delta\tau} \Delta u \, d\tau \right|^2. \end{aligned} \quad (54)$$

Integrating (54) from 0 to  $t$  and using (49) and (53), we obtain, after a final multiplication by  $e^{-\varepsilon_0 \lambda_1 t}$ , that

$$\varepsilon_0 \|z(t)\|^2 + e^{-\varepsilon_0 \lambda_1 t} \int_0^t e^{\varepsilon_0 \lambda_1 s} |z_t|^2 ds \leq \kappa \frac{\rho}{\delta} \quad \forall t \geq 0. \quad (55)$$

In the same way, we find from (50), (20) and (55) that

$$|\varepsilon_0 P \Delta z(t)|^2 \leq c \left(\frac{\rho}{\delta}\right)^2 |P \Delta u|^2 + c |z_t|^2 + c \rho^2 e^{-2\delta t} \left| \int_0^t e^{\delta\tau} P \Delta u \, d\tau \right|^2, \quad (56)$$

which with (49) and (55) yields

$$\varepsilon_0^2 e^{-\varepsilon_0 \lambda_1 t} \int_0^t e^{\varepsilon_0 \lambda_1 s} |P \Delta z|^2 ds \leq \kappa \frac{\rho}{\delta}, \quad (57)$$

Finally, we derive from (14) and (50) that

$$\begin{aligned} e^{\varepsilon_0 \lambda_1 t} \|r(t)\|_1^2 & \leq c e^{\varepsilon_0 \lambda_1 t} \left( |z_t(t)|^2 + \varepsilon_0^2 |\Delta z(t)|^2 + \left(\frac{\rho}{\delta}\right)^2 |\Delta u|^2 \right) \\ & \quad + c \rho^2 e^{-(2\delta - \varepsilon_0 \lambda_1)t} \left| \int_0^t e^{\delta\tau} \Delta u \, d\tau \right|^2. \end{aligned} \quad (58)$$

Integrating (58) from 0 to  $t$  and using (20), (49), (55), and (57), one finds, after a final multiplication by  $e^{-\varepsilon_0 \lambda_1 t}$ , that

$$e^{-\varepsilon_0 \lambda_1 t} \int_0^t e^{\varepsilon_0 \lambda_1 s} \|r\|_1^2 ds \leq \kappa \frac{\rho}{\delta} \quad \forall t \geq 0. \quad (59)$$

Combining (59) with (55) and (53) yields (43). The proof is now complete.  $\square$

**4. Convergence to the linearized Euler flow equations.** In this section we are concerned with the asymptotic behavior, as the viscosity coefficient  $\varepsilon_0 \rightarrow 0$ , of the flows governed by the linearized viscoelastic flow equations (3).

**Theorem 4.1.** *Assume that  $\Omega, u_0 \in H^1(\Omega)^2 \cap H$  and let  $f \in L^\infty(R^+; L^2(\Omega)^2) \cap L^{(1,\beta)}(R^+; X)$  be sufficiently smooth that*

$$\|u_t^0(t)\|_1 + \|u^0(t)\|_1 \leq \kappa \quad \forall t \geq 0. \quad (60)$$

Then

$$|u(t) - u^0(t) - \theta^{\varepsilon_0}(t)| + \varepsilon_0^{1/2} \left( \int_0^t \|u - u^0 - \theta^{\varepsilon_0}\|^2 ds \right)^{1/2} \leq \kappa t^{1/2} \varepsilon_0^{1/2}, \quad (61)$$

$$\left| \int_0^t (p - p^0 - \eta^{\varepsilon_0}) ds \right| \leq \kappa t^{1/2} \varepsilon_0^{1/2} (1 + t^{1/2} \varepsilon_0^{1/2}), \quad (62)$$

for all  $t \geq 0$ .

*Proof.* Setting  $(w, r) = (u - u^0 - \theta^{\varepsilon_0}, p - p^0 - \eta^{\varepsilon_0})$ , we derive from (3), (6)–(7) that

$$\begin{cases} w_t - \varepsilon_0 \Delta w + \nabla r = \varepsilon_0 \Delta u^0 - \frac{\rho}{\delta} \Delta u + \rho e^{-\delta t} \int_0^t e^{\delta \tau} \Delta u d\tau, \\ w \in V, r \in M \quad \forall t > 0, \quad w(0) = 0. \end{cases} \quad (63)$$

Taking the scalar product of (63) with  $2w$  and using (49) and (60) yields

$$\begin{aligned} \frac{d}{dt} |w(t)|^2 + \varepsilon_0 \|w(t)\|^2 &\leq -\varepsilon_0 \|w(t)\|^2 + \varepsilon_0 \|u^0(t)\| \|w(t)\| \\ &\quad + \frac{\rho}{\delta} \left( \|u(t)\| + \sup_{0 \leq s \leq t} \|u(s)\| \right) \|w(t)\| \\ &\leq \kappa \varepsilon_0. \end{aligned} \quad (64)$$

Integrating (64) from 0 to  $t$ , we obtain (61).

Moreover, we infer from (12), (63) and the inf-sup condition [5],

$$|q| \leq c \sup_{v \in X} \frac{d(v, q)}{\|v\|} \quad \forall q \in M, \quad (65)$$

that

$$\begin{aligned} \left| \int_0^t r ds \right| &\leq c |w(t)| + c \varepsilon_0 \int_0^t \|w\| ds + c \varepsilon_0 \int_0^t \|u^0\| ds \\ &\quad + c \frac{\rho}{\delta} \left( \int_0^t \|u\| ds + t \sup_{t \geq 0} \|u(t)\| \right). \end{aligned} \quad (66)$$

Using (60)–(61) and (49) in (66), we obtain (62). The proof is now complete.  $\square$

**Theorem 4.2.** *Under the assumptions of Theorem 4.1,*

$$|\theta^{\varepsilon_0}(t)| + \varepsilon_0^{1/2} \left( \int_0^t \|\theta^{\varepsilon_0}\|^2 ds \right)^{1/2} \leq \kappa t^{1/2} e^t \varepsilon_0^{1/4} \quad \forall t \geq 0, \quad (67)$$

$$\left| \int_0^t \eta^{\varepsilon_0} ds \right| \leq \kappa (t^{1/2} + t) e^t \varepsilon_0^{1/4} \quad \forall t \geq 0. \quad (68)$$

*Proof.* First, by using the fundamental formula of calculus, the Cauchy-Schwarz inequality, and location, we are easily able to prove that

$$\|\phi\|_{L^2(\partial\Omega)}^2 \leq c|\phi|\|\phi\|_1 \quad \forall \phi \in H^1(\Omega) \quad (69)$$

(see [19]).

Taking the  $L^2$ -scalar product of the first equation in (7) with  $-2P\Delta\theta^{\varepsilon_0}$  and  $2\theta^{\varepsilon_0}$ , respectively, and using (20) and (69), we derive

$$\begin{aligned} \frac{d}{dt}\|\theta^{\varepsilon_0}\|^2 + 2\varepsilon_0|P\Delta\theta^{\varepsilon_0}|^2 &\leq 2\left|\int_{\partial\Omega} u_t^0 \cdot \nabla\theta^{\varepsilon_0} \cdot n \, ds_x\right| \\ &\leq 2\|u_t^0\|_1\|\theta^{\varepsilon_0}\|^{1/2}|P\Delta\theta^{\varepsilon_0}|^{1/2} \\ &\leq \varepsilon_0|P\Delta\theta^{\varepsilon_0}|^2 + \|\theta^{\varepsilon_0}\|^2 + c\varepsilon_0^{-1/2}\|u_t^0\|_1^2 \quad \forall t \geq 0, \end{aligned} \quad (70)$$

$$\begin{aligned} \frac{d}{dt}|\theta^{\varepsilon_0}|^2 + 2\varepsilon_0\|\theta^{\varepsilon_0}\|^2 &\leq 2\varepsilon_0\left|\int_{\partial\Omega} \theta^{\varepsilon_0} \cdot \nabla\theta^{\varepsilon_0} \cdot n \, ds_x\right| \\ &\leq 2\varepsilon_0\|\theta^{\varepsilon_0}\|_{L^2(\partial\Omega)^2}\|\nabla\theta^{\varepsilon_0}\|_{L^2(\partial\Omega)^4} \\ &\leq 2\varepsilon_0|\theta^{\varepsilon_0}|^{1/2}\|\theta^{\varepsilon_0}\|_1|P\Delta\theta^{\varepsilon_0}|^{1/2} \\ &\leq \varepsilon_0\|\theta^{\varepsilon_0}\|_1^2 + |\theta^{\varepsilon_0}|^2 + c\varepsilon_0^2|P\Delta\theta^{\varepsilon_0}|^2 \quad \forall t \geq 0. \end{aligned} \quad (71)$$

Integrating (70) and (71) from 0 to  $t$ , respectively, and using (60) and the Gronwall lemma, we obtain

$$\|\theta^{\varepsilon_0}(t)\|^2 + \varepsilon_0 \int_0^t |P\Delta\theta^{\varepsilon_0}|^2 \, ds \leq \kappa t e^t \varepsilon_0^{-1/2} \quad \forall t \geq 0, \quad (72)$$

$$\begin{aligned} |\theta^{\varepsilon_0}(t)|^2 + \varepsilon_0 \int_0^t \|\theta^{\varepsilon_0}\|^2 \, ds &\leq c e^t \varepsilon_0^2 \int_0^t |P\Delta\theta^{\varepsilon_0}|^2 \, ds \\ &\leq \kappa t e^{2t} \varepsilon_0^{1/2} \quad \forall t \geq 0. \end{aligned} \quad (73)$$

Finally, by integrating the first equation in (7) with respect to  $t$ , using (65) and (12), we obtain

$$\left|\int_0^t \eta^{\varepsilon_0}(s) \, ds\right| \leq c|\theta^{\varepsilon_0}(t)| + \varepsilon_0 \int_0^t \|\theta^{\varepsilon_0}(s)\|_0 \, ds \leq \kappa(t^{1/2} + t)e^t \varepsilon_0^{1/4} \quad (74)$$

for all  $t \geq 0$ . With (73) and (74), we have completed the proof of Theorem 4.2.  $\square$

Combining Theorem 4.1 with Theorem 4.2 yields the following convergence result.

**Theorem 4.3.** *Under the assumptions of Theorem 4.1, the inequalities*

$$|u(t) - u^0(t)| \leq \kappa t^{1/2} e^t \varepsilon_0^{1/4}, \quad (75)$$

$$\left|\int_0^t (p - p^0) \, ds\right| \leq \kappa(t^{1/2} + t)e^t \varepsilon_0^{1/4} \quad (76)$$

hold for all  $t \geq 0$ .

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## REFERENCES

- [1] Yu. Ya. Agranovich and P. E. Sobolevskii, 1989, *Investigation of a mathematical model of a viscoelastic fluid*, (Russian) Dokl. Akad. Nauk Ukrain. SSR Ser. A, **86**(10)(1989),3–6.
- [2] Yu. Ya. Agranovich and P. E. Sobolevskii, *Motion of a nonlinear viscoelastic fluid*, (Russian) Dokl. Akad. Nauk SSSR, **314**(1990), 521–525; translation in Soviet Math. Dokl., **42**(1991), 474–478.
- [3] Yu. Ya. Agranovich and P. E. Sobolevskii, *Motion of nonlinear visco-elastic fluid*, Nonlinear Anal. **32** (1998), 755–760.
- [4] J. R. Cannon, R. E. Ewing, Y. N. He and Y. N. Lin, *A modified nonlinear Galerkin method for the viscoelastic fluid motion equations*, Internat. J. Engrg. Sci., **37**(1999), 1643–1662.
- [5] V. Girault and P. A. Raviart, “Finite Element Approximation of the Navier-Stokes Equations”, Springer-Verlag, Berlin-New York, 1979.
- [6] Y. N. He, Y. P. Lin, S.S. P. Shen and R. Tait, *On the convergence of viscoelastic fluid flows to a steady state*, Adv. Differential Equations, **7**(2002), 717–742.
- [7] Y. N. He, Y. P. Lin, S. S. P. Shen, W. W. Sun and R.Tait, *Finite element approximation for the viscoelastic fluid motion problem*, J. Comput. Appl. Math., **155**(2003),201–222.
- [8] J. G. Heywood and R. Rannacher, *Finite element approximation of the nonstationary Navier-Stokes problem, I, Regularity of solutions and second-order error estimates for spatial discretization*, SIAM J. Numer. Anal., **19**(1982), 275–311.
- [9] A. A. Kotsiolis and A. P. Oskolkov, *Solvability of the fundamental initial-boundary value problem for the equations of motion of Oldroyd fluids on  $(0, \infty)$  and the behavior of its solution as  $t \rightarrow +\infty$* , (Russian) Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI), **150**(1986); Voprosy Kvant. Teor. Polya i Statist. Fiz., **6**, 48–52, 219; translation in J. Soviet Math., **46**(1989), 1595–1598.
- [10] O. A. Ladyzenskaja, 1970, “Mathematical questions of the dynamics of a viscous incompressible fluid (Russian), Second revised and supplemented edition”, Izdat. “Nauka”, Moscow 1970.
- [11] J. G. Oldroyd, *Non-Newtonian flow of liquids and solids*, in “Rheology, Theory and Applications, Vol.1” (eds. F. R. Eirich), Academic Press, New York, 1956, 653–682.
- [12] J. G. Oldroyd, “Reologia” Mir, Moscow, 1962.
- [13] V. P. Orlov and P. E. Sobolevskii, *On mathematical models of a viscoelasticity with a memory*, Differential Integral Equations, **4**(1991), 103–115.
- [14] A. K. Pani and J. Y. Yuan, *Semidiscrete finite element Galerkin approximations to the equations of motion arising in the Oldroyd model*, IMA J. Numer. Anal., **25**(2005), 750–782.
- [15] A. K. Pani, J. Y. Yuan and P. D. Damázio, *On a linearized backward Euler method for the equations of motion of Oldroyd fluids of order one*, SIAM J. Numer. Anal., **44**(2006), 804–825.
- [16] P. E. Sobolevskii, *Stabilization of viscoelastic fluid motion (Oldroyd’s mathematical model)*, Differential Integral Equations, **7**(1994), 1597–1612.
- [17] P. E. Sobolevskii, *Asymptotics of stable viscoelastic fluid motion (Oldroyd’s mathematical model)*, Math. Nachr., **177**(1996), 281–305.
- [18] R. Temam and X. Wang, *Asymptotic analysis of the linearized Navier-Stokes equations in a channel*, Differential Integral Equations, **8**(1995), 1591–1618.
- [19] R. Temam and X. Wang, *Asymptotic analysis of Oseen type equations in a channel at small viscosity*, Indiana Univ. Math. J., **45**(1996), 863–916.
- [20] R. Temam and X. Wang, *Asymptotic analysis of the linearized Navier-Stokes equations in a general 2D domain*, Asymptot. Anal., **14**(1997), 293–321.
- [21] R. Temam and X. Wang, *On the behavior of the solutions of the Navier-Stokes equations at vanishing viscosity*, Ann. Scuola Norm. Sup. Pisa Cl.Sci. (4), **25**(1997), 807–828.
- [22] R. Temam and X. Wang, *Boundary layers associated with incompressible Navier-Stokes equations: the noncharacteristic boundary case*, J. Differential Equations, **179**(2002), 647–686.
- [23] X. Wang, *A Kato type theorem on zero viscosity limit of Navier-Stokes flows*, Indiana Univ. Math. J., **50** (2001), 223–241.

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