5-2002

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July 12, 1999

Abstract

In this paper, we study the existence and nonexistence of multiple positive solutions for problem

$$\begin{cases} \Delta u + K(x)u^p = 0 & \text{in } \Omega. \\ u > 0 & \text{in } \Omega, \quad u \in H^1_{\text{loc}}(\Omega) \cap C(\overline{\Omega}). \\ u\big|_{\partial\Omega} = 0, & u \to \mu > 0 \quad \text{as } |x| \to \infty \end{cases}$$

where $\Omega = \mathbb{R}^N \setminus \omega$ is an exterior domain in $\mathbb{R}^N$, $\omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary and $N > 2$. $\mu \geq 0$, $p > 1$ are some given constants. $K(x)$ satisfies: $K(x) \in C^\alpha_{\text{loc}}(\Omega)$ and $\exists C, \epsilon, M > 0$ such that, $|K(x)| \leq C|x|^l$ for any $|x| \geq M$, with

*Research supported by the Natural Science Foundation of China

$l \leq -2-\epsilon$. Some existence and nonexistence of multiple solutions have been discussed under different assumptions on K.

Key words and phrases: multiple solutions, critical exponents, elliptic equations.

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AMS Classification: 35J10 35J20 35J60 35J65

1 Introduction

In this paper, we study the existence of multiple solutions for problem

$$\begin{cases} \Delta u + K(x)u^p = 0 & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \quad u \in H^1_{\rm loc}(\Omega) \cap C(\overline{\Omega}). \\ u\big|_{\partial\Omega} = 0, \end{cases} \tag{1.1}$$

with the boundary condition $u \to \mu > 0 \quad$ as $|x| \to \infty$, where $\Omega = \mathbb{R}^N \setminus \omega$ is an exterior domain in $\mathbb{R}^N$, $\omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary and $N > 2$. $p > 1$ is a given constants. $K(x)$ satisfies:

(H_1) $K(x) \in C^\alpha_{\rm loc}(\Omega)$, $K \not\equiv 0$ and $\exists C, \epsilon, M > 0$ such that, $|K(x)| \leq C|x|^l$ for any $|x| \geq M$,

with $l \leq -2-\epsilon$.

Such a problem occur in various branches of mathematical physis and Geometry. For $K(x) \equiv |x|^l$, $\Omega = R^N$ equation (1.1) is known as Lane-Emden equation, sometimes it is also referred to as the Emden-Fowler equation in astrophysics, where u represents the density of a single star. When $p = \frac{N+2}{N-2}$, $\Omega = \mathbb{R}^N$ and $n \geq 3$, equation (1.1) is called the conformal scalar curvature equation in $\mathbb{R}^N$. Let g be the usual metric in $\mathbb{R}^N$, the problem of finding a metric g_1 which is conformal to g (i.e. $g_1 = u^{\frac{4}{N-2}}g$, for some positive function u with scalar curvature $\tilde{K}$ is equivalent to that to find a positive solution of (1.1) with $K = \frac{N-2}{4(N-1)}\tilde{K}$. For a detail overview on (1.1), we refer readers to the papers [N2], [LN1], [Z] and the references therein.

Equations like (1.1) has been studied by many mathematicians ([B], [CZ], [CL1-2], [DL1-2], [DLZ], [DN1-2], [Es], [G], [GE], [JPY], [KL], [LY1-2], [Lio], [WW], [Y], [YY], [ZC]). Ni ([N1]), Kenig and Ni ([KN]) proved existence theorems for (1.1) under the condition (H_1). It is shown in ([N1]) that if K is nonnegative with $K \geq Cr^l$ for some $l > (N-2)(p-1)-2$ at infinity, or if K is nonpositive with $-K \geq Cr^l$ for some $l > -2$ at infinity then (1.1) possesses no positive solutions, where $C > 0$. Lin ([Lin]) proved the existence for (1.1) under the condition that $|K| \leq \frac{\phi(|x|)}{|x|^2}$ at infinity with $\int^\infty \frac{\phi(r)}{r} dr < \infty$. Lin in [Lin] also proved a nonexistence result when K is nonpositive with $-K \geq Cr^{-2}$ at infinity. Other nonexistence results are given in [BLY] and [LN1]. In case of that $|K| \leq Cr^{(N-2)(p-1)-2-\varepsilon}$ at infinity for some positive constants C and ε, the existence and asymptotics of positive solutions are studied by many authors, here we only metion the results of, for example, Ni, Yosutani [NY] ,[LN1], LN2] and Li [L2]. In the fast decay case $|K| \leq Cr^l$, $l < -2$, Ni showed that (1.1) possesses infinitely many positive solutions which are bounded from below by positive constants (see [N1] and [LN1]). Li and Ni ([LN1]) showed that, for positive bounded solution of (1.1), the limit $u_\infty = \lim_{x\to\infty} u(x)$ always exists for any $\varepsilon > 0$, furthermore, if $u_\infty = 0$, then

$$u(x) \leq \begin{cases} C|x|^{2-N} & \text{if } p > \frac{N+l}{N-2}, \\ C_\varepsilon |x|^{\frac{(1-\varepsilon)(l+2)}{1-p}} & \text{if } p \leq \frac{N+l}{N-2}, \end{cases}$$

and if $u_\infty > 0$, then

$$|u - u_\infty| \leq \begin{cases} C|x|^{2-N} & \text{if } l < -N, \\ C|x|^{2-N}\log|x| & \text{if } l = -N, \\ C|x|^{2+l} & \text{if } -N < l < -2, \end{cases}$$

at ∞. These results are refined in [LN2] and [L2].

Recently, Zhao (see [Z]) studied the following problem:

$$\begin{cases} \Delta u + K(x)f(u) = 0 & \text{in } \Omega \\ u > 0 & \text{in } \Omega, \quad u \in H^1_{\text{loc}}(\Omega) \cap C(\overline{\Omega}) \\ u\big|_{\partial\Omega} = 0, & u \to \mu > 0 \quad \text{as } |x| \to \infty. \end{cases} \tag{1.1$_\mu$}$$

The existence of one positive solution of problem $(1.1)_\mu$ when f is superlinear at 0 was obtained with some assumptions (Green-tight function) on $K(x)$ for small $\mu > 0$. A naturnal and interesting problem is that how many solutions can be obtained for a given $\mu > 0$. There seems to have been little progress in this direction. The purpose of this paper is to discuss the existence and nonexistence of multiple solutions for problem $(1.1)_\mu$ for a given $\mu > 0$. The main results of this paper can be included in the following theorems:

Theorem 1.1. *Suppose (H_1). Let $h(x)$ be a positive harmonic function in Ω satisfying*

$$h(x)\Big|_{\partial\Omega} = 0\,, \qquad \lim_{|x|\to\infty} h(x) = 1.$$

Then

(i) *If $K(x) \le 0$, then for any $\mu > 0$, there exists a unique solution u_μ of $(1.1)_\mu$. In addition, $u_\mu \le \mu h$ on Ω and u_μ is increasing with respect to μ.*

(ii) *If $K(x) \ge 0$, $\exists \mu^* \in (0, -\infty)$ and $\mu^* < +\infty$, such that for $\mu > \mu^*$ there does not exist a solution of $(1.1)_\mu$; and for $\mu \in (0,\ \mu^*)$, there exists a minimal solution u_μ of $(1.1)_\mu$. In addition, u_μ is increasing with respect to μ, $u_\mu \ge \mu h$ in Ω and, as $\mu \to \mu^*$, u_μ increase to u_{μ^*} the minimal solution of $(1.1)_{\mu^*}$, and u_{μ^*} is unique.*

(iii) *If $K(x)$ change sign, we can find a $\mu^* \in (0, +\infty)$ such that problem $(1.1)_\mu$ possesses at least one solution for all $\mu \in (0, \mu^*)$.*

Theorem 1.2. *Suppose that $p = \frac{N+2}{N-2}$, H_1), $0 \le K(x) \in L^1(\Omega)$, and*

(H_2) $K(x) > 0$ in a neighborhood V of some point $x_0 \in \Omega$ such that

$$K(x_0) = \sup_{x\in\Omega} K(x)$$

and $K(x) = K(x_0) + 0(|x - x_0|^2)$ near x_0. Then problem $(1.1)_\mu$ possesses at least two solutions u_μ and U_μ with $u_\mu < U_\mu$ if $\mu \in (0,\ \mu^*)$, where μ^* is given by Theorem 1.1.

This paper is organized as follows: We first give some Lemmas in Section 2, which will be used in the proof of Theorem 1.1. Then the existence and nonexistence of minimal solution for problem $(1.1)_\mu$ is given in Section 3 by the standard barrier method. Finally, the existence of the second solution for $(1.1)_\mu$ is given in Section 4 by using the variational method.

2 Preliminaries

In this Section, we will prove some Lemmas which will be used in the proof of Theorem 1.1.

Lemma 2.1. *Let f be a locally Hölder continuous function on Ω with the following decay property*

$$|f(x)| \le C|x|^{\ell} \qquad at\ \infty \tag{2.1}$$

with $C > 0$, $\ell < -2$, and w be the Newtonian potential of f, i.e.

$$w(x) = \int_{\Omega} G(x,y)f(y)dy\ .$$

where $G(x,y)$ is the Green function for Ω corresponding to the Laplacian $-\Delta$. Then $w(x)$ is well-defined and at ∞ we have

$$|w(x)| \le \begin{cases} C|x|^{2-N} & \text{if } \ell < -N \\ C|x|^{2-N}\ln|x| & \text{if } \ell = -N \\ C|x|^{2+\ell} & \text{if } -N < \ell < -2 \end{cases} \tag{2.2}$$

Proof. This Lemma may be proved by standard arguments. We include a proof here for the sake of completeness.

From the definition of Green function, we can easily deduce that

$$G(x,y) \le \frac{C_N}{|x-y|^{N-2}}. \tag{2.3}$$

where $C_N = (N(N-2)\omega_N)^{-1}$ and ω_N is the volume of the unit ball in $\mathbb{R}^N$. Using this fact and (2.1) we can find a constant $C > 0$ such that

$$|w(x)| \le C\int_{\Omega} \frac{1}{|x-y|^{N-2}(1+|y|^{-\ell})}dy\ . \tag{2.4}$$

Thus $w(x)$ is well-defined. Next we decompose the integral (2.4) as follows.

$$\begin{aligned} |w(x)| &\le \left(\int_{|y-x|\le\frac{|x|}{2}} + \int_{\frac{|x|}{2}\le|y-x|\le 2|x|} + \int_{2|x|\le|y-x|}\right) \frac{C}{|y-x|^{N-2}(1+|y|^{-\ell})}dy \\ &\equiv I_1 + I_2 + I_3 \end{aligned}$$

where I_1, I_2 and I_3 are defined by the last equality. Same as [LN2] we can conclude that

$$I_1 \le \frac{C}{|x|^{-\ell}} \int_0^{\frac{|x|}{2}} \frac{1}{r^{N-2}} r^{N-1} dr = C|x|^{2+\ell},$$

$$I_2 \le C \int_{|x|/2}^{2|x|} \frac{1}{r^{N-2} r^{-\ell}} r^{N-1} dr = C|x|^{2+\ell},$$

$$I_3 \le \begin{cases} C|x|^{2-N} & \text{if } N+\ell < 0, \\ C|x|^{2-N}(\ell n|x| + 1) & \text{if } N+\ell = 0, \\ C|x|^{2-N}(1+|x|^{N+\ell}) & \text{if } N+\ell > 0, \end{cases}$$

Now, it is easy to see that (2.2) holds. □

Lemma 2.2. *Under the assumption of Lemma 2.1, suppose v is a solution of*

$$\begin{cases} -\Delta v = f(x) & in\ \Omega, \\ v\big|_{\partial\Omega} = 0 \quad \lim_{|x|\to\infty} v(x) = \mu. \end{cases} \tag{2.5}$$

Then

$$v = \mu h(x) + \int_\Omega G(x,y) f(y) dy \tag{2.6}$$

where $h(x)$ is the positive harmonic function in Ω satisfying

$$h(x)\Big|_{\partial\Omega} = 0\,, \qquad \lim_{|x|\to\infty} h(x) = 1 \tag{2.7}$$

and $G(x,y)$ is the Green functions for Ω corresponding to Δ.

Proof. From [Z] and Lemma 2.1, we can deduce that $h(x)$ exists with $0 < h < 1$ in Ω and the integral in (2.6) is well-defined. Set $w(x) = \int_\Omega G(x,y) f(y) dy$. For an arbitrary but fixed point $z \in \Omega$, choose R large enough such that $R > |z|$ and $\omega \subset B_R(0)$. Now we define

$$w_1(x) = \int_{\Omega \cap B_R(0)} G(x,y) f(y) dy,$$

$$w_2(x) = \int_{\mathbb{R}^N \setminus B_R(0)} G(x,y) f(y) dy.$$

Then it is standard that

$$\Delta w_1(z) + f(z) = 0 \text{ and } \Delta w_2(z) = 0.$$

Since $w = w_1 + w_2$ we have

$$\Delta w + f = 0 \quad \text{in } \Omega. \tag{2.8}$$

By Lemma 2.1 and the property of Green functions we have

$$w\Big|_{\partial\Omega} = 0, \quad \lim_{|x|\to\infty} w(x) = 0\ . \tag{2.9}$$

Therefore

$$\begin{cases} \Delta(v-w) = 0, \\ (v-w)\big|_{\partial\Omega} = 0,\ \lim_{|x|\to\infty}(v-w) = \mu. \end{cases}$$

By the uniqueness of the above problem, we have

$$v - w = \mu h(x)\ .$$

This gives (2.6). □

Theorem 2.3. *Suppose* (H_1) *and let* u *be a bounded solution of (1.1)*$_\mu$*, then*

$$|u(x) - \mu h(x)| \le \begin{cases} C|x|^{2-N} \text{ at } \infty & \text{if } \ell < -N \\ C|x|^{2-N}\ell n|x| \text{ at } \infty & \text{if } \ell = -N \\ C|x|^{2+\ell} \text{ at } \infty & \text{if } -N < \ell < -2 \end{cases}$$

where $h(x)$ *is the unique solution of (2.7).*

The proof of the above theorem can come directly from Lemma 2.1 and Lemma 2.2.

Lemma 2.4. *Suppose* (H_1) *with* $l = -\frac{N+2}{2} - \epsilon$ *for some* $\epsilon > 0$, $K(x) \ge 0$, $K(x) \not\equiv 0$ *and* u_μ *be the solution of (1.1)*$_\mu$*. Then*

$$u_\mu(x) - \mu h(x) \in \mathcal{D}_0^{1,2}(\Omega)$$

where $h(x)$ *is the unique solution of (2.7) and* $\mathcal{D}_0^{1,2}(\Omega)$ *is a Sobolev's space defining as the completion of* $C_0^\infty(\Omega)$ *in the norm* $\int_\Omega |\nabla u|^2 dx = \|u\|^2$.

Proof. From Theorem 2.3, (2.7), and $(1.1)_\mu$ we can easily conclude that

$$\begin{cases} \triangle(u_\mu - \mu h(x)) + K(x)u_\mu^p = 0 \\ (u_\mu - \mu h(x))|_{\partial\Omega} = 0, \ \ \lim_{|x|\to\infty}(u_\mu - \mu h(x)) = 0 \end{cases}$$

and

$$-\int_{\Omega} \triangle(u_\mu - \mu h(x))(u_\mu - \mu h(x))dx = \int_{\Omega} |\nabla(u_\mu - \mu h(x))|^2 dx.$$

Thus

$$\begin{aligned}
&\int_{\Omega} |\nabla(u_\mu - \mu h(x))|^2 dx = \int_{\Omega} K(x) u_\mu^p (u_\mu - \mu h(x))dx \\
&= \int_{\Omega\cap B_R} K(x) u_\mu^p (u_\mu - \mu h(x))dx + \int_{R^N\setminus B_R} K(x) u_\mu^p (u_\mu - \mu h(x))dx \\
&\leq C + C_1 \int_R^\infty r^l s(r) r^{N-1} dr \\
&\leq +\infty
\end{aligned}$$

if $l < -\frac{N+2}{2}$. Here

$$s(r) = \begin{cases} |r|^{2-N} & if \;\; l < -N, \\ |r|^{2-N} \ln |r|, & if \;\; l = -N, \\ |r|^{2+l}, & if \;\; -N < l < -2. \end{cases}$$

Remark 2.1. The conclusion of Lemma 2.4 still remain true if we replace the assumption $l = -\frac{N+2}{2} - \epsilon$ by $S(|x|)K(x) \in L^1$ near ∞.

□

3 Existence of minimal solution

In this Section, we will give a complete proof of Theorem 1.1 by the standard barrier method.

Lemma 3.1. *Suppose (H_1) and $K(x) \geq 0, K(x) \not\equiv 0$, then there exists a constant $0 < \mu^* < \infty$ such that problem $(1.1)_\mu$ possesses a minimal solution for all $\mu \in (0, \mu^*)$ and no solution for problem $(1.1)_\mu$ for $\mu > \mu^*$.*

Proof. First of all, we prove that problem $(1.1)_\mu$ has a minimal solution if μ is small enough.

In fact, let $\varphi(x) = h(x) + \int_\Omega G(x,y)K(y)dy$. From Lemma 2.2, $\varphi(x)$ is a solution of

$$\begin{cases} -\Delta\varphi = K(x) & \text{in } \Omega \\ \varphi\big|_{\partial\Omega} = 0, \; \lim_{|x|\to\infty} \varphi(x) = 1 \end{cases} \tag{3.1}$$

Denoting $\varphi_\mu(x) = \mu\varphi(x)$, we have $\varphi_\mu(x) \geq \mu h(x)$ because $K(x) \geq 0$ in Ω. Then

$$\begin{cases} -\Delta\varphi_\mu - K(x)\varphi_\mu^p = K(x)(\mu - (\mu\varphi)^p) \geq 0 \\ \varphi_\mu\big|_{\partial\Omega} = 0, \ \lim_{|x|\to\infty} \phi_\mu = \mu \end{cases}$$

if μ is small enough. So $\bar{u} = \mu\varphi$ is a supersolution of $(1.1)_\mu$ if μ is small enough. It is easy to check that $\underline{\mathrm{u}} = \mu h(x)$ is a subsolution of $(1.1)_\mu$ for all $\mu > 0$ and all positive supersolution of $(1.1)_\mu$ must be larger than or equal to μh. The method of sub and supersolution yields our first claim.

Next, we set

$$\mu^* = \sup\{\mu > 0, \mid \text{problem } (1.1)_\mu \text{ possesses at least one solution } \} \tag{3.2}$$

so that $\mu^* > 0$. For any $\mu \in (0, \mu^*)$, from the definition of μ^*, we can find an $\bar{\mu} > \mu$ such that problem $(1.1)_{\bar{\mu}}$ possesses a solution $\mu_{\bar{\mu}}$ and hence $u_{\bar{\mu}}$ is a supersolution of $(1.1)\ _\mu$. It is easy to verify that $\underline{\mathrm{u}}_\mu = \mu h$ is a subsolution of $(1.1)_\mu$ for all $\mu > 0$ and all positive supersolution of $(1.1)_\mu$ must be larger than or equal to μh. Using monotone interation we can get the minimal solution u_μ for all $\mu \in (0, \mu^*)$.

Now, we are going to prove that $\mu^* < +\infty$. In fact, if u_μ solves $(1.1)_\mu$, since $u_\mu \geq \mu h$ we have

$$\begin{cases} -\triangle(u_\mu - \mu h(x)) = -\Delta u_\mu = K(x)(u_\mu)^{p-1} \geq K(x)(\mu h)^{p-1}(u_\mu - \mu h(x)) & in \ \ \Omega \\ (u_\mu - \mu h(x)) > 0 & in \ \ \Omega, \\ (u_\mu - \mu h(x)) \in \mathcal{D}_0^{1,2}(\Omega) \end{cases}$$

Thus the first eigenvalue of $-\Delta - K(x)(\mu h)^{p-1}$ on $\mathcal{D}^{1,2}(\Omega)$ is positive and this is impossible for μ large.

From the definition of μ^* we know that there is no solution for problem $(1.1)_\mu$ if $\mu > \mu^*$. □

Lemma 3.2. *Suppose H_1) with $l = -\frac{N+2}{2} - \epsilon$ and $K(x) \geq 0$, $K(x) \not\equiv 0$. u_μ be the minimal solution of $(1.1)_\mu$ for $\mu \in (0, \mu^*)$. Then the minimizing problem*

$$\sigma_\mu = \inf\left\{\int_\Omega |\nabla w|^2 dx \ \ | \ \ w \in \mathcal{D}_0^{1,2}(\Omega), \ \int_\Omega pK(x)u_\mu^{p-1}w^2 dx = 1\right\} \tag{3.3}$$

can be attained by a function $\psi_\mu > 0$ which satisfies the equation

$$\begin{cases} -\Delta w = \sigma p K(x) u_\mu^{p-1} w & \text{in } \Omega \\ w \in \mathcal{D}_0^{1,2}(\Omega) \end{cases} \tag{3.4$_\mu$}$$

with $\sigma = \sigma_\mu$. Furthermore, $\sigma_\mu > 1$ for all $\mu \in (0, \mu^*)$.

Proof. We first prove that the functional $\int_\Omega pKu_\mu^{p-1}w^2dx$ is weakly sequentially compact. In fact, let $\{w_n\}$ is a bounded sequence in $\mathcal{D}_0^{1,2}(\Omega)$ with weak limit $w \in \mathcal{D}_0^1(\Omega)$, the boundedness of K and u_μ in Ω and the use of Hölder inequality in a ball B_R for a large R, and $B'_R = \mathbb{R}^N \backslash B_R$ give

$$\begin{aligned} &\int_\Omega K u_\mu^{p-1} |w_n - w|^2 dx \\ &\le C_1 \int_{B_R \cap \Omega} |w_n - w|^2 dx + C \left(\int_{B'_R} |w_n - w|^{\frac{2N}{N-2}} dx \right)^{\frac{N-2}{N}} \left(\int_{B'_R} K(x)^{\frac{N}{2}} dx \right)^{\frac{2}{N}} \end{aligned}$$

where C, C_1 are positive constants, independent of w_n, w. It follows from the compactness of the embedding $\mathcal{D}_0^{1,2}(\Omega \cap B_R) \hookrightarrow L^2(\Omega \cap B_R)$ and assumption (H$_1$) we have

$$\begin{aligned} \int_\Omega K u_\mu^{p-1} (w_n - w)^2 dx &\le C_1 \int_{B_R \cap \Omega} |w_n - w|^2 dx + C \int_R^\infty r^{-(2+\epsilon)\cdot \frac{N}{2}} r^{N-1} dx \\ &\le \frac{\epsilon_1}{2} + \frac{\epsilon_1}{2} = \epsilon_1 \end{aligned}$$

for any $\epsilon_1 > 0$ if R and n are large enough. This gives us that the functional $\int_\Omega pKu_\mu^{p-1}w_n^2dx$ is weakly sequentially compact. Consequently standard minimization procedure shows that σ_μ is attained by a function $\psi_\mu \ge 0$, $\psi_\mu \in \mathcal{D}_0^{1,2}(\Omega)$, satisfying $(3.4)_\mu$ with $\sigma = \sigma_\mu$. By assumption (H$_1$) we deduce $\sigma_\mu pK(x)u_\mu^{p-1}(x)|x|^\delta \in L^q(\Omega)$ for some $\delta > 0$ and $q > \frac{N}{2}$. Therefore a result of Egnel [E] implies that ψ_μ is bounded in Ω and $\psi_\mu = 0(|x|^{2-N})$ as $|x| \to \infty$ and standard Hölder estimates then imply that $\psi_\mu \in C^{1,\alpha}_{\text{loc}}(\Omega)$ for all $0 < \alpha < 1$.

Next, we prove $\sigma_\mu > 1$. In fact, for $\mu < \bar{\mu}$, $\mu, \bar{\mu} \in (0, \mu^*)$ problem $(1.1)_\mu$ and $(1.1)_{\bar{\mu}}$ have a minimal solution u_μ and $u_{\bar{\mu}}$ respectively. Because $u_{\bar{\mu}}$ is a supersolution of $(1.1)_\mu$, we have

$u_\mu \le u_{\bar\mu}$. Set $v_{\bar\mu} = u_{\bar\mu} - \bar\mu h$, $v_\mu = u_\mu - \mu h$. From Lemma 2.5 we have

$$\begin{cases} -\Delta v_{\bar\mu} = K(x)(v_{\bar\mu} + \bar\mu h)^p & v_{\bar\mu} > 0 \text{ in } \Omega \\ v_{\bar\mu}\Big|_{\partial\Omega} = 0,\ \lim_{|x|\to\infty} v_{\bar\mu}(x) = 0 & \text{and } v_{\bar\mu} \in \mathcal{D}_0^{1,2}(\Omega) \end{cases}$$

$$\begin{cases} -\Delta v_{\mu} = K(x)(v_{\mu} + \mu h)^p & v_{\mu} > 0 \text{ in } \Omega \\ v_{\mu}\Big|_{\partial\Omega} = 0,\ \lim_{|x|\to\infty} v_{\mu}(x) = 0 & \text{and } v_{\mu} \in \mathcal{D}_0^{1,2}(\Omega) \end{cases}$$

and

$$-\Delta(v_{\bar\mu} - v_\mu) = K(x)[(v_{\bar\mu} + \bar\mu h)^p - (v_\mu + \mu h)^p] = K(x)(u_{\bar\mu}^p - u_\mu^p) \ge 0.$$

Maximum principle gives us that

$$v_{\bar\mu} - v_\mu > 0 \quad \text{ in } \Omega\,. \tag{3.5}$$

Furthermore,

$$\begin{cases} -\Delta(v_{\bar\mu} - v_\mu) = K(x)(u_{\bar\mu}^p - u_\mu^p) \ge K(x)pu_\mu^{p-1}(v_{\bar\mu} - v_\mu + (\bar\mu - \mu)h) \\ (v_{\bar\mu} - v_\mu) \in \mathcal{D}_0^{1,2}(\Omega)\,. \end{cases} \tag{3.6}$$

On the other hand,

$$\begin{cases} -\Delta\psi_\mu = \sigma_\mu K(x)pu_\mu^{p-1}\psi_\mu & \psi_\mu \ge 0 \text{ in } \Omega \\ \psi_\mu \in \mathcal{D}_0^{1,2}(\Omega)\,. \end{cases} \tag{3.7}$$

Multiplying (3.6) by ψ_μ and (3.7) by $w \equiv u_{\bar\mu} - u_\mu$ we deduce

$$\int_\Omega \nabla w \nabla \psi_\mu dx \ge \int_\Omega pK(x)u_\mu^{p-1}(w + (\bar\mu - \mu)h)\psi_\mu dx$$

and

$$\int_\Omega \nabla\psi_\mu \nabla w dx = \sigma_\mu p \int_\Omega K(x)u_\mu^{p-1}\psi_\mu w\, dx\,.$$

Thus

$$\begin{aligned} \sigma_\mu p \int_\Omega K(x)u_\mu^{p-1}\psi_\mu w\, dx &\ge \int_\Omega pK(x)u_\mu^{p-1}w\psi_\mu + p(\bar\mu - \mu)\int_\Omega K(x)u_\mu^{p-2}h\psi_\mu\, dx \\ &> \int_\Omega pK(x)u_\mu^{p-1}w\psi_\mu dx \end{aligned}$$

which gives $\sigma_\mu > 1$. $\square$

Lemma 3.3. *Suppose (H_1), $K(x) \geq 0$, $K(x) \not\equiv 0$ and $K(x) \in L^1(\Omega)$. Then there exists a constant $C > 0$ independent of μ such that*

$$\|u_\mu - \mu h\|_{\mathcal{D}_0^{1,2}(\Omega)} \leq C \quad \text{for all } \mu \in (0, \mu^*)$$

where u_μ is the minimal solution of $(1.1)_\mu$ and h is the unique solution of (2.7).

Proof. Set $v_\mu = u_\mu - \mu h$. From Lemma 2.4 we have

$$\begin{cases} -\Delta v_\mu = K(x)(v_\mu + \mu h)^p, \\ v_\mu \in \mathcal{D}_0^{1,2}(\Omega) \end{cases} . \tag{3.8}$$

From Lemma 3.2 and (3.8) we deduce

$$\int_\Omega |\nabla v_\mu|^2 dx = \int_\Omega K(x)(v_\mu + \mu h)^p v_\mu dx \tag{3.9}$$

$$\int_\Omega |\nabla v_\mu|^2 dx \geq \sigma_\mu p \int_\Omega K(x)(v_\mu + \mu h)^{p-1} v_\mu^2 dx \tag{3.10}$$

and hence

$$\begin{aligned} \sigma_\mu p \int_\Omega K(x)(v_\mu + \mu h)^{p-1} v_\mu^2 dx &\leq \int_\Omega K(x)(v_\mu + \mu h)^p v_\mu dx \\ &\leq \int_\Omega K(x)(v_\mu + \mu h)^{p-1} v_\mu^2 dx + \int_\Omega K(x)(v_\mu + \mu h)^{p-1} \mu h v_\mu \, dx \, . \end{aligned}$$

So, for any $\epsilon > 0$,

$$\begin{aligned} (p-1) \int_\Omega K(x)(v_\mu + \mu h)^{p-1} v_\mu^2 dx &\leq \int_\Omega K(x) \mu h (v_\mu + \mu h)^{p-1} v_\mu dx \\ &\leq C \int_\Omega (K(x) v_\mu^p + K(x) v_\mu) dx \\ &\leq C \left(\int_\Omega K(x) dx \right)^{\frac{1}{p+1}} \left(\int_\Omega K v_\mu^{p+1} dx \right)^{\frac{p}{p+1}} \\ &\quad + C \left(\int_\Omega K(x) dx \right)^{\frac{p}{p+1}} \left(\int_\Omega K(x) v_\mu^{p+1} dx \right)^{\frac{1}{p+1}} \\ &\leq C_\epsilon \int_\Omega K(x) dx + \epsilon \int_\Omega K(x) v_\mu^{p+1} dx \end{aligned}$$

by Hölder's inequality and Young's inequality. Taking $\epsilon > 0$ small enough we deduce

$$\int_\Omega K(x) v_\mu^{p+1} dx \leq C \int_\Omega K(x) dx \leq C_1. \tag{3.11}$$

From (3.9), (3.10) we also have

$$\int_\Omega |\nabla v_\mu|^2 dx \leq \frac{1}{p}\int_\Omega |\nabla v_\mu|^2 dx + \int_\Omega K(x)\mu h(v_\mu + \mu h)^{p-1} v_\mu dx$$

and hence

$$\begin{aligned}\left(1-\frac{1}{p}\right)\int_\Omega |\nabla v_\mu|^2 dx &\leq C|\mu^*|^p \|h\|_\infty^p \int_\Omega K(x) v_\mu dx + C\mu^* \|h\|_\infty \int_\Omega K(x) v_\mu^p dx \\ &\leq C\left(\int_\Omega K(x)dx\right)^{\frac{1}{p+1}} \left(\int_\Omega K(x) v_\mu^{p+1} dx\right)^{\frac{p}{p+1}} \\ &\quad + C\left(\int_\Omega K(x)dx\right)^{\frac{p}{p+1}} \left(\int_\Omega K(x) v_\mu^{p+1} dx\right)^{\frac{1}{p+1}} \\ &\leq C\end{aligned}$$

because of (3.11) and that $K(x) \in L^1(\Omega)$. □

Lemma 3.4. *Let $h(x)$ be the solution of (2.7) and suppose H_1), then for any $\mu > 0$, there exists a unique solution u_μ of $(1.1)_\mu$ if $K(x) \leq 0$. In addition, $u_\mu \leq \mu h$ on Ω and u_μ is increasing in μ.*

Proof. We remark that μh is a supersolution of $(1.1)_\mu$ which satisfies

$$\begin{cases} -\Delta(\mu h) - K(x)(\mu h)^p \geq -\mu \Delta h = 0 \quad \text{in } \Omega \\ \mu h\big|_{\partial\Omega} = 0, \ \lim_{|x|\to\infty} \mu h(x) = \mu \\ \mu h > 0 \quad \text{in } \Omega \end{cases} \tag{3.12}$$

Next, let $\psi(x) = \int_\Omega G(x, y)|K(y)|dy$; from Lemma 2.2, $\psi(x)$ is the positive solution of

$$\begin{cases} -\Delta v = |K(x)| \\ v\big|_{\partial\Omega} = 0, \ v(x) \to 0 \quad \text{as } |x| \to \infty \end{cases} \tag{3.13}$$

we set $\underline{u} = (\mu h - \lambda\psi)^+$ for some $\lambda > 0$. We then have by standard results

$$-\Delta \underline{u} \leq -\lambda \big|K(x)\big|_{\{\underline{u} \geq 0\}} \leq K(x)\underline{u}^p \text{ on } \Omega$$

if λ is chosen such that

$$\underline{u}^p \leq (\mu h)^p \leq \lambda.$$

where $h(x)$ is the solution of (2.7). Thus $\underline{u}$ is a nontrivial subsolution satisfies $\underline{u} \le \mu h$ and the existence part is complete.

The various uniqueness and comparison results are deduced from the following claim.

Let $v, w \in H^1_{\text{loc}}(\Omega) \cap C_b(\Omega)$ satisfy

$$-\Delta v + |K(x)|v^p \le 0 \quad \text{in } \Omega \quad v \ge 0 \text{ in } \Omega \quad \lim_{|x|\to\infty} v \le \mu, v\big|_{\partial\Omega} = 0,$$
$$-\Delta w + |K(x)|w^p \ge 0 \quad \text{in } \Omega \quad w \ge 0 \text{ in } \Omega \quad \lim_{|x|\to\infty} w \ge \mu, w\big|_{\partial\Omega} = 0,$$

then $v \le w$ on Ω.

Indeed, for all $\epsilon > 0$, we may find R large enough such that

$$v \le (1+\epsilon)w \equiv w_\epsilon \quad \text{for } |x| \ge R$$

since we have on $B_R \cap \Omega$

$$\begin{aligned} &-\Delta(w_\epsilon - v) + p|K(x)|w_\epsilon^{p-1}(w_\epsilon - v) \\ &\ge -\Delta(w_\epsilon - v) + |K(x)|(w_\epsilon^p - v^p) \\ &= -\Delta w_\epsilon + |K(x)|w_\epsilon^p - (-\Delta v + |K(x)|v^p) \ge 0\,. \end{aligned}$$

Since the first eigenvalue of $-\Delta + p|K(x)|w_\epsilon^{p-1}$ is positive (on $H^1_0(\Omega \cap B_R)$ we deduce $w_\epsilon \ge v$ in Ω. Let $\epsilon \to 0$ we obtain our claim. Using the above claim we can easily deduce the uniqueness and that $u_\mu \le \mu h$ for all $\mu > 0$ and $u_{\mu_1} \le u_{\mu_2}$ if $\mu_1 \le \mu_2$. □

Lemma 3.5. *Suppose (H_1), if $K(x)$ change sign, we can find a positive constant μ^* such that problem $(1.1)_\mu$ possesses at least one solution.*

Proof. Consider problem

$$\begin{cases} -\Delta v = K(x)(v+\mu h)^p, \quad v > 0 & \text{in } \Omega, \\ v\big|_{\partial\Omega} = 0\,, \lim_{|x|\to\infty} v(x) = 0. \end{cases} \tag{3.14}$$

From Lemma 3.1, we can find a positive constant μ^* such that problem

$$\begin{cases} -\Delta v = K^+(x)(v+\mu h)^p & \text{in } \Omega, \\ v\big|_{\partial\Omega} = 0\,,\ \lim_{|x|\to\infty} v = 0,\ v > 0 & \text{in } \Omega \end{cases}$$

possess a minimal solution $\bar{v}$ for all $\mu \in (0, \mu^*)$. From Lemma 3.4, problem

$$\begin{cases} -\Delta v = -K^-(x)(v+\mu h)^p & \text{in } \Omega, \\ v\big|_{\partial\Omega} = 0\,,\ \lim_{|x|\to\infty} v = 0,\ v<0 & \text{in } \Omega. \end{cases}$$

possesses a unique solution $\underline{v}$ for all $\mu > 0$. Then $\bar{v}$ is a supersolution of $(3.14)_\mu$ and $\underline{v}$ is a subsolution of $(3.14)_\mu$. Furthermore $v = \bar{v} - \underline{v}$ satisfies

$$\begin{cases} -\Delta v = K^+(x)(\bar{v}+\mu h)^p + K^-(x)(\underline{v}+\mu h)^p] \geq 0, \\ v\big|_{\partial\Omega} = 0,\ \lim_{|x|\to\infty} v = 0, \end{cases}$$

maximum principle implies that $v > 0$. The existence of solution for $(3.14)_\mu$ with $K(x)$ change sign come from the method of super-subsolution. Suppose v_μ be the solution of $(3.14)_\mu$, then $u_\mu = v_\mu + \mu h$ is a solution of $(1.1)_\mu$ with $0 < \underline{v} + \mu h < u_\mu < \bar{v} + \mu h$. □

Theorem 3.6. *Suppose (H_1). Let h be the solution of (2.10), then*

(i) *If $K(x) \leq 0$, for any $\mu > 0$, there exists a unique solution u_μ of $(1.1)_\mu$. In addition, $u_\mu \leq \mu h$ on Ω and u_μ is increasing in μ.*

(ii) *If $K(x) \geq 0$, $\exists \mu^* \in (0, \infty]$ and $\mu^* < +\infty$ if $K(x) \not\equiv 0$, such that for $\mu > \mu^*$ there does not exist a solution of $(1.1)_\mu$ and for $\mu \in (0,\ \mu^*)$, there exists a minimal solution u_μ of $(1.1)_\mu$. In addition, u_μ is increasing in μ, $u_\mu \geq \mu h$ in Ω. Finally, if*

$$K(x) \in L^1(\Omega), \tag{3.15}$$

then as $\mu \to \mu^*$, u_μ increase to u_{μ^*} the minimal solution of $(1.1)_{\mu^*}$, and u_{μ^*} is unique.

(iii) *If $K(x)$ change sign, we can find a $\mu^* \in (0, +\infty)$ such that problem $(1.1)_\mu$ possesses at least one solution for all $\mu \in (0, \mu^*)$.*

Proof. From the above lemmas, we only have to prove that problem $(1.1)_{\mu^*}$ has a unique solution under the assumption (3.15). Denote the corresponding solution of $(1.1)_\mu$ by u_μ. Let $v_\mu = u_\mu - \mu h$. From assumption (3.15) and Lemma 3.3, we know $v_\mu \in \mathcal{D}_0^{1,2}(\Omega)$ and

$$\|v_\mu\|_{\mathcal{D}_0^{1,2}(\Omega)} \leq C < +\infty \text{ for all } \mu \in (0, \mu^*)$$

where C is a positive constant independent of μ. We claim that

$$\int_\Omega v_\mu^q dx \leq C < \infty \tag{3.16}$$

for all $q \geq \frac{2N}{N-2}$, where C is some positive constant independent of N. First of all, we consider $p \in \left(1, \frac{N+2}{N-2}\right)$, the subcritical case. We adapt the argument due to Brezis and Kato [BK] to deduce the above claim. In fact, v_μ is a solution of

$$\begin{cases} -\Delta v_\mu = K(x)(v_\mu + \mu h)^p \\ v_\mu \in \mathcal{D}_0^{1,2}(\Omega) \quad v_\mu > 0 \qquad\qquad \text{in } \Omega \end{cases} \tag*{$(3.17)_\mu$}$$

Let $i > 1$, multiplying $(3.17)_\mu$ by v_μ^i and integrating by parts we obtain

$$4i(1+i)^{-2} \int_\Omega |\nabla v_\mu^{\frac{1}{2}(1+i)}|^2 dx = \int_\Omega K(x)(v_\mu + \mu h)^p v_\mu^i dx\,.$$

We refer by Hölder's and Young's inequalities that

$$\begin{aligned}
\int_\Omega K(x)(v_\mu + \mu_\mu h)^p v_\mu^i dx &\leq C \int_\Omega K(x)(v_\mu^p + (\mu h)^p) v_\mu^i dx \\
&\leq C \int_\Omega K(x) v_\mu^{p+i} dx + C \int_\Omega K(x) v_\mu^i dx \\
&\leq C \int_\Omega K(x) v_\mu^{p+i} dx + C \left(\int_\Omega |K(x)| dx\right)^{\frac{p}{p+i}} \left(\int_\Omega K(x) v_\mu^{p+i} dx\right)^{\frac{i}{p+i}} \\
&\leq C \int_\Omega K(x) v_\mu^{p+i} dx + C \\
&\leq C \int_\Omega v_\mu^{p+i} dx + C\,.
\end{aligned}$$

Thus

$$4i(1+i)^{-2} \int_\Omega |\nabla v_\mu^{\frac{1}{2}(1+i)}|^2 dx \leq C \int_\Omega v_\mu^{p+i} dx + C\,. \tag{3.18}$$

Let $\epsilon > 0$, be arbitrary, then for $i \geq \frac{2N}{N-2} - p > 1$, we have

$$t^{p+i} \leq \epsilon t^{i + \frac{N+2}{N-2}} + C_\epsilon t^{\frac{2N}{N-2}}. \tag{3.19}$$

for all $\epsilon > 0$ and $t \geq 0$, because $i + \frac{N+2}{N-2} > p + i \geq \frac{2N}{N-2}$. Applying the Sobolev's inequality and (3.17)—(3.19) we have

$$\begin{aligned}\left(\int_\Omega v_\mu^q dx\right)^{\frac{N-2}{N}} &= \left(\int_\Omega \left(v_\mu^{\frac{1}{2}(1+i)}\right)^{\frac{2N}{N-2}} dx\right)^{\frac{N-2}{N}} \\ &\le C\int_\Omega v_\mu^{p+i} dx + C \\ &\le C\epsilon \int_\Omega v_\mu^{i+\frac{N+2}{N-2}} dx + C_\epsilon \int_\Omega v_\mu^{\frac{2N}{N-2}} dx + C \\ &= C\epsilon \int_\Omega v_\mu^{\frac{(N-2)q}{N}} \cdot v_\mu^{\frac{4}{N-2}} dx + C \\ &\le C\epsilon \left(\int_\Omega v_\mu^q dx\right)^{\frac{N-2}{N}} \left(\int_\Omega v_\mu^{\frac{2N}{N-2}} dx\right)^{\frac{2}{N}} + C\end{aligned}$$

with $q = \frac{N(1+i)}{N-2}$. From Lemma 3.3 and Sobolev inequality we deduce $\{v_\mu\}$ is bounded in $L^q(\Omega)$ for large $q > 1$ if we choose ϵ small enough.

Now, we are going to deal with the case when $p = \frac{N+2}{N-2}$. Our method is a combination of ideas found in papers of Brezis and Kato [BK] and Egnell [E]. For $j \ge 1$, define $\varphi_j(t) = t^j$, $t \ge 0$ and $\psi_j(t) = \int_0^t [\varphi_j'(s)]^2 ds = \frac{j^2}{2j-1} t^{2j-1}$. Let $\mu \in (0, \mu^*)$ and v_μ be the corresponding minimal solution of $(3.17)_\mu$. From Lemma 3.2 we have

$$\int_\Omega \nabla v_\mu \nabla v dx \ge p \int_\Omega K(x)(v_\mu + \mu h)^{p-1} v\, dx \tag{3.20}$$

for all $v \in \mathcal{D}_0^{1,2}(\Omega)$. By Theorem 2.3, Lemma 2.4 and Remark 2.1 we know $\varphi_j(v_\mu) \in \mathcal{D}_0^{1,2}(\Omega)$. We may choose $v = \varphi_j(v_\mu)$ in (3.20) to obtain

$$\int_\Omega |\varphi_j'(v_\mu)|^2 |\nabla v_\mu|^2 dx \ge p \int_\Omega K(x)(v_\mu + \mu h)^{p-1} \varphi_j^2(v_\mu) dx\,. \tag{3.21}$$

Since v_μ is a solution of $(3.17)_\mu$ and $\psi_j(v_\mu) \in \mathcal{D}_0^{1,2}(\Omega)$, we also have

$$\int_\Omega \psi_j'(v_\mu)|\nabla v_\mu|^2 dx = \int_\Omega K(x)(v_\mu + \mu h)^p \psi_j(v_\mu) dx\,. \tag{3.22}$$

From (3.22), (3.21) we obtain

$$p\int_\Omega K(x)(v_\mu + \mu h)^{p-1} v_\mu^{2j} \le \frac{j^2}{2j-1}\left[\int_\Omega K(x)(v_\mu + \mu h)^{p-1} v_\mu^{2j} + \int_\Omega K(x)(v_\mu + \mu h)^{p-1} \mu h v_\mu^{2j-1}\right] \tag{3.23}$$

since $\frac{j^2}{2j-1} \ge 1$ and is increasing in j, we may choose $j > 1$ sufficiently close to 1 such that

$\frac{j^2}{2j-1} < p$ for $j \le j_0$. Set $\alpha(j,p) = p - \frac{j^2}{2j-1} > 0$. Then (3.23) gives

$$\begin{aligned}\alpha(j,\ p)\int_\Omega K(x)v^{p+2j-1}dx &\le \alpha(j,\ p)\int_\Omega K(x)(v_\mu+\mu h)^{p-1}v_\mu^{2j}dx \\ &\le \frac{j^2}{2j-1}\int_\Omega K(x)(v_\mu+\mu h)^{p-1}\mu h v_\mu^{2j-1}dx \\ &\le \frac{Cj^2}{2j-1}\left[\int_\Omega K(x)v_\mu^{p+2j-2}\mu h\,dx + \int_\Omega K(x)(\mu h)^p v_\mu^{2j-1}dx\right] \\ &\le C\left[\int_\Omega K(x)v_\mu^{p+2j-2}dx + \int_\Omega K(x)v_\mu^{2j-1}dx\right].\end{aligned}$$

because $\mu < \mu^*, K(x) \le C$. Since

$$\begin{aligned}\int_\Omega K(x)v_\mu^{p+2j-2}dx &\le C\left[\int_\Omega K(x)dx\right]^{\frac{1}{p+2j-1}}\left[\int_\Omega K(x)v_\mu^{p+2j-1}dx\right]^{\frac{p+2j-2}{p+2j-1}} \\ &\le C\int_\Omega K(x)dx + \frac{\delta}{2}\int_\Omega K(x)v_\mu^{p+2j-1}dx\end{aligned}$$

for all $\delta > 0$ and similarily,

$$\int_\Omega K(x)v_\mu^{2j-1}dx \le C\int_\Omega K(x)dx + \frac{\delta}{2}\int_\Omega K(x)v_\mu^{p+2j-1}dx$$

for all $\delta > 0$, we can deduce

$$(\alpha(j,\ p)-\delta)\int_\Omega K(x)v_\mu^{p+2j-1}dx \le C_\delta\int_\Omega K(x)dx\ .$$

From the assumption of $K(x) \in L^1(\Omega)$, we have

$$\int_\Omega K(x)v_\mu^{p+2j-1}dx \le C_\delta \tag{3.23*}$$

for $j \in (1, j_0]$ and $C > 0$ independent of $\mu \in (0, \mu^*)$ if we take δ small enough. This shows that (3.16) holds for all $q \in [\frac{2N}{N-2},\ p+2j_0-1]$. To establish (3.16) for all $q \ge \frac{2N}{N-2}$ we use ideas in Brezis and Kato [BK]. Set $q_0 = \frac{2N}{N-2}$, $\delta = p+2j_0-1-\frac{2N}{N-2} > 0$. Multiplication of (3.17) by $v_\mu^{q_0-1}$, integration by parts and simple application of Hölder's inequality and

Young's inequality yield

$$
\begin{aligned}
(q_0-1)q_0^{-2}\int|\nabla v_\mu^{\frac{q_0}{2}}|^2dx &= \int_\Omega K(x)(v_\mu+\mu h)^p v_\mu^{q_0-1}dx \\
&\le C\int_\Omega K(x)v_\mu^{p+q_0-1}dx + C\int_\Omega K(x)v_\mu^{q_0-1}dx \\
&\le C\int_\Omega K(x)v_\mu^{p+q_0-1}dx \\
&+ C\left(\int_\Omega K(x)dx\right)^{\frac{p}{p+q_0-1}}\left(\int_\Omega K(x)v_\mu^{p+q_0-1}dx\right)^{\frac{q_0-1}{p+q_0-1}} \\
&\le C\int_\Omega K(x)v_\mu^{p+q_0-1}dx + C\int_\Omega K(x)dx
\end{aligned}
$$

which gives us

$$
(q_0-1)q_0^{-2}\int|\nabla v_\mu^{\frac{q_0}{2}}|^2dx \le C\int_\Omega K(x)v_\mu^{p+q_0-1}dx + C\int_\Omega K(x)dx \tag{3.24}
$$

where C is a positive constant independent of μ.

For any given $\epsilon > 0$ we can find a positive constant C_ϵ such that

$$
v_\mu^{p-1+q_0} \le \epsilon v^{p-1+q_0+\frac{2\delta}{N}} + C_\epsilon v_\mu^{q_0} .
$$

This can be easily verified by the fact that $q_0 < p-1+q_0 < p-1+q_0+\frac{2\delta}{N}$. Therefore, it follows from Hölder inequality, Sobolev's inequality and (3.23)* with $j = j_0$ that

$$
\begin{aligned}
\int_\Omega K(x)v_\mu^{p+q_0-1}dx &\le \epsilon\int_\Omega K(x)v_\mu^{p-1+q_0+\frac{2\delta}{N}}dx + C_\epsilon C \\
&\le \epsilon\left(\int_\Omega K(x)(v_\mu^{q_0})^{\frac{p+1}{2}}dx\right)^{\frac{2}{p+1}}\left(\int_\Omega K(x)v_\mu^{p+2j_0-1}dx\right)^{\frac{2}{N}} + C_\epsilon \\
&\le \epsilon C\int_\Omega (v_\mu^{\frac{q_0}{2}})^{p+1}dx + C_\epsilon C \\
&\le \epsilon C\int_\Omega |\nabla v_\mu^{\frac{q_0}{2}}|^2dx + C_\epsilon C.
\end{aligned}
$$

which gives us

$$
\int_\Omega K(x)v_\mu^{p+q_0-1}dx \le \epsilon C\int_\Omega |\nabla v_\mu^{\frac{q_0}{2}}|^2dx + C_\epsilon C. \tag{3.25}
$$

It follows from (3.24) and (3.25), with ϵ sufficiently small, that

$$
\int_\Omega |\nabla v_\mu^{\frac{q_0}{2}}|^2dx \le C \tag{3.26}
$$

for some constant C, independent of μ, and by Sobolev's inequality we have

$$\int_\Omega v_\mu^{\frac{q_0}{2}\cdot q_0} dx \le C \ .$$

The desired inequality (3.16) then follows easily by iteration. Set $q_1 = \frac{q_0^2}{2}$ and $q_k = \frac{q_{k-1}^2}{2}$.

Denote $g_\mu(x) = K(x)(v_\mu(x) + \mu h(x))^p$. From the above proof we deduce $g_\mu(x) \in L^q(\Omega)$ for all $q \ge p+1$ and

$$\begin{aligned}\int_\Omega |g_\mu(x)|^q dx &\le C\int_\Omega K(x)^q v_\mu(x)^{pq} dx + C\int_\Omega K(x)^q(\mu h(x))^{pq} dx \\ &\le C|K(x)|_\infty^{q-1}\int_\Omega K(x) v_\mu^{pq} dx + C\mu^*|h(x)|_\infty^q \int_\Omega K(x)^q dx \\ &\le C\end{aligned}$$

for all $\mu \in (0, \mu^*)$.

We employ a classical a priori estimate to obtain

$$\|v_\mu\|_{\infty, B_R(x)\cap\Omega} \le C_R(\|v_\mu\|_{p+1, B_{2R}(x)\cap\Omega} + \|g_\mu\|_{q, B_{2R}(0)\cap\Omega}]$$

for solution of $-\Delta v = g_\mu(x)$, where $B_R(x)$ is a ball of radius R and centre x, and C_R is a constant independent of μ and x. Hölder estimates in $B_R \cap \Omega$ then shows that

$$\|v_\mu\|_{C^{1,\alpha}(B_R\cap\Omega)} \le C_R$$

for some constant C_R, independing of μ. A simple diagonalization argument and the Ascoli-Arzela theorem may be employed to show that for a subsequence $\mu_n \to \mu^*$, $v_{\mu_n}, |\nabla v_{\mu_n}|$ converge uniformly on each compact subset of Ω, to a function $v_{\mu^*} \in \mathcal{D}_0^{1,2}(\Omega)$. It follows that

$$\int_\Omega \nabla v \cdot \nabla v_{\mu^*} dx = \int_\Omega K(x)(v_{\mu^*} + \mu^* h)^p v \, dx$$

for all $v \in C_0^\infty(\Omega)$ and therefore v_{μ^*} is a nonnegative weak solution of $(3.17)_\mu$. Thus $u_{\mu^*} = v_{\mu^*} + \mu^* h$ is a solution of $(1.1)_{\mu^*}$.

Finally, we prove that u_{μ^*} is unique. In fact, from the definition we can easily deduce that $\sigma_{\mu^*} = 1$ by applying the implicit function theorem to the function $F : \mathcal{D}_0^{1,2}(\Omega) \hookrightarrow \mathcal{D}_0^{1,2}(\Omega)$ with

$$F(u) = -\Delta u - K(x)(u + \mu h)^p \qquad u \in \mathcal{D}_0^{1,2}(\Omega) \ .$$

If there exists another solution $\bar{u}_{\mu^*} \geq u_{\mu^*}$ for problem $(1.1)_{\mu^*}$, set $\bar{v}_{\mu^*} = \bar{u}_{\mu^*} - \mu^* h$, $v_{\mu^*} = u_{\mu^*} - \mu^* h$. We have from $(3.17)_\mu$

$$\begin{aligned} -\Delta(\bar{v}_{\mu^*} - v_{\mu^*}) &= K(x)[(\bar{v}_{\mu^*} + \mu^* h)^p - (v_{\mu^*} + \mu^* h)^p] \\ &= K(x)[p(v_{\mu^*} + \mu^* h)^{p-1}(\bar{v}_{\mu^*} - v_{\mu^*}) \\ &\quad + p(p-1)(v_{\mu^*} + \theta(\bar{v}_{\mu^*} - v_{\mu^*}) + \mu^* h)^{p-2}(\bar{v}_{\mu^*} - v_{\mu^*})] \end{aligned}$$

for some $\theta(x) \in [0,1]$. From Lemma 3.2 and the above equality, we deduce

$$\begin{aligned} \sigma(\mu^*) \int_\Omega p(v_{\mu^*} + \mu^* h)^{p-1}(\bar{v}_{\mu^*} - v_{\mu^*})\psi_{\mu^*} dx &= \int_\Omega \nabla \psi_{\mu^*} \nabla(\bar{v}_{\mu^*} - v_\mu) dx \\ = \int_\Omega p(v_{\mu^*} + \mu^* h)^{p-1}(\bar{v}_{\mu^*} - v_{\mu^*})\psi_{\mu^*} dx & \\ + \int_\Omega p(p-1)(v_{\mu^*} + \theta(\bar{v}_{\mu^*} - v_{\mu^*}) + \mu^* h)^{p-2}(\bar{v}_{\mu^*} - v_{\mu^*})^2 \psi_{\mu^*} dx & \\ \text{i.e.} \qquad (\sigma(\mu^*) - 1) \int_\Omega p(v_{\mu^*} + \mu^* h)^{p-1}(\bar{v}_{\mu^*} - v_{\mu^*})\psi_{\mu^*} dx & \\ = \int_\Omega p(p-1)(v_{\mu^*} + \theta(\bar{v}_{\mu^*} - v_{\mu^*}) + \mu^* h)^{p-2}(\bar{v}_{\mu^*} - v_{\mu^*})^2 \psi_{\mu^*} dx & \end{aligned}$$

we can obtain that $\bar{v}_{\mu^*} \equiv v_{\mu^*}$ from $\sigma(\mu^*) = 1$. □

4 The existence of second solution

For $\mu \in (0, \mu^*)$, let u_μ be the first solution of $(1.1)_\mu$ and consider the problem

$$\begin{cases} -\Delta v = K(x)((v + u_\mu)^p - u_\mu^p) & \text{in } \Omega \\ v \in \mathcal{D}_0^1(\Omega), \quad v > 0 & \text{in } \Omega. \end{cases} \tag{4.1$_\mu$}$$

It is clear that $U_\mu = v_\mu + u_\mu$ is a solution of $(1.1)_\mu$ if v_μ is a solution of $(4.1)_\mu$. Consider the energy functional J_μ defined by

$$J_\mu(v) = \int_\Omega \frac{1}{2}|\nabla v|^2 - K(x) \left[\frac{1}{p+1}(u_\mu + v^+)^{p+1} - \frac{1}{p+1}u_\mu^{p+1} - u_\mu^p v^+\right] dx .$$

Standard procedure from the calculus of variations shows that J_μ is well defined in $\mathcal{D}_0^1(\Omega)$ with continuous Freehet derivative given by

$$J'_\mu(v)\varphi = \int_\Omega [\nabla v \nabla \varphi - K(x)((u_\mu + v^+)^p - u_\mu^p)]\varphi \, dx \quad \varphi \in \mathcal{D}_0^{1,2}(\Omega)$$

A critical point v of J_μ is a weak solution of the equation

$$-\Delta v = K(x)[(u_\mu + v^+)^p - u_\mu^p] \quad v \in \mathcal{D}_0^1(\Omega)$$

and if $v > 0$ in $\mathbb{R}^N$, then v is a solution of $(4.1)_\mu$.

The following Lemma comes from the fact that

$$\lim_{s\to 0} \frac{(u_\mu + s)^p - u_\mu^p - pu_\mu^{p-1}s}{s} = 0$$

and

$$\lim_{s\to\infty} \frac{(u_\mu + s)^p - u_\mu^p - pu_\mu^{p-1}s}{s^p} = 1.$$

Lemma 4.1. *For any $\epsilon > 0$, there exist a $C_\epsilon > 0$ such that*

$$(u_\mu + s)^p - u_\mu^p - pu_\mu^{p-1}s \le \epsilon u_\mu^{p-1}s + C_\epsilon s^p$$

for all $s \ge 0$.

Lemma 4.2. *Suppose (H_1) with $\ell = -\frac{N+2}{2} - \epsilon$. There exists two constant $\alpha > 0$, $\rho > 0$ such that*

$$J_\mu(v) \ge \alpha > 0, \quad \text{for } v \in \mathcal{D}_0^1(\Omega), \quad \|v\| = \rho\ .$$

Proof. Lemma 4.1 implies that

$$\begin{aligned}
J_\mu(v) &= \frac{1}{2}\int_\Omega (\nabla v|^2 dx - \frac{p}{2}\int_\Omega K(x)u_\mu^{p-1}(v^+)^2 dx \\
&\quad - \int_\Omega \int_0^{v^+} K(x)[(u_\mu + s)^p - u_\mu^p - pu_\mu^{p-1}s] ds\, dx \\
&\ge \frac{1}{2}\int_\Omega (\nabla v|^2 - pK(x)u_\mu^{p-1}(v^+)^2 dx \\
&\quad - \int_\Omega K(x)\left(\frac{\epsilon}{2}u_\mu^{p-1}(v^+)^2 + C_\epsilon \frac{(v^+)^{p+1}}{p+1}\right) dx\ .
\end{aligned}$$

Furthermore, from the definition of σ_μ in Lemma 3.2, we have

$$\int_\Omega |\nabla v|^2 dx \ge \sigma_\mu p \int_\Omega K(x)u_\mu^{p-1}(v^+)^2 dx$$

and, therefore, by $\sigma_\mu > 1$ we obtain by choosing ϵ small enough

$$\begin{aligned}
J_\mu(v) &\ge \frac{1}{2\sigma_\mu}(\sigma_\mu - 1 - \epsilon)\int_\Omega |\nabla v|^2 dx - \frac{C_\epsilon}{p+1}\int_\Omega K(x)v^{p+1} dx \\
&\ge \frac{1}{4\sigma_\mu}(\sigma_\mu - 1)\int_\Omega |\nabla v|^2 dx - C\left[\int_\Omega |\nabla v|^2 dx\right]^{p+1} \\
&= \frac{1}{4\sigma_\mu}(\sigma_\mu - 1)\|v\|^2 - C\|v\|^{p+1}
\end{aligned}$$

and the conclusion in Lemma 4.2 follows. ∎

Lemma 4.3. *Suppose (H_1) with $\ell = -\frac{N+2}{2} - \epsilon$. Then there exist $0 < \psi_0 \in \mathcal{D}_0^1(\Omega)$ and $R_0 > 0$ such that*

$$J_\mu(R\psi_0) < 0$$

for $R \geq R_0$.

Proof. Let $h(x,s) = K(x)((u_\mu + s)^p - u_\mu^p - s^p)$, since $u_\mu(x)$ is bounded in Ω, it is easy to check that

$$\lim_{s\to 0} \frac{h(x,s)}{s} \leq M$$
$$\lim_{s\to\infty} \frac{h(x,s)}{s^p} = 0$$

uniformly in $x \in \Omega$, where $M > 0$ is some constant independent of x. Therefore, for any $\epsilon > 0$, there is a constant $C_\epsilon > 0$ such that

$$h(x,s) \leq \epsilon s^p + C_\epsilon s \ .$$

Now, choose a nonzero function $\psi_0 \in C_0^\infty(\Omega)$ such that $\psi_0 \geq 0$ and $K(x) \geq k_0 > 0$ on the support of ψ_0. Then

$$J_\mu(R\psi_0) \leq \frac{1}{2}R^2\|\psi_0\|^2 - \frac{R^{p+1}}{p+1}\int_\Omega K(x)\psi_0^{p+1}dx + C_\epsilon R^2 \int_\Omega K\psi_0^2 dx + \epsilon R^{p+1}\int_\Omega K\psi_0^{p+1}dx \ .$$

It is then clear from the choice of ψ_0, that for ϵ sufficiently small there is $R_0 > 0$ such that

$$J_\mu(R\psi_0) < 0 \text{ for all } R \geq R_0 \ .$$

This completes the proof of Lemma 4.3, with R_0 and ψ_0 as above. □

In order to use mountain pass Lemma [BN] to obtain the solution of $(4.1)_\mu$, we suppose moreover (H_2).

Set

$$\Gamma = \{\gamma \in C([0,1],\ \mathcal{D}_0^{1,2}(\Omega)),\ \gamma(0) = 0,\ \gamma(1) = R_0\psi_0\},$$

where ψ_0 is given by Lemma 4.2. We exploit the fact that the critical equation

$$-\Delta u = u^{\frac{N+2}{N-2}} \qquad \text{in } \mathbb{R}^N$$

has the positive radial solution

$$u_\epsilon(x) = k\left[\frac{\epsilon}{\epsilon^2 + |x - x_0|^2}\right]^{\frac{N-2}{2}}$$

with $k = (N(N-2))^{\frac{N-2}{4}}$ for any $\epsilon > 0$, $x \in \mathbb{R}^N$. Furthermore,

$$\int_{R^N} |\nabla u_\epsilon|^2 dx = \int_{R^N} u_\epsilon^{p+1} dx = S^{N/2},$$

and for some positive constant c depending only on N $cu_\epsilon(x)$ attains the infimum for the variational problem

$$S = \inf\left\{\|u\|^2 \mid \int_{\mathbb{R}^N} u^{p+1} dx = 1 \quad u \in \mathcal{D}_0^{1,2}(\mathbb{R}^N)\right\}.$$

Let $R > 0$ be small enough that $B_{2R}(x_0) \in V$. Let ψ be a piecewise smooth function with support in B_{2R} such that $\psi(x) \equiv 1$ in $B_R(x_0)$, $0 \le \psi(x) \le 1$ in $B_{2R}(x_0)$ and $|\nabla\psi(x)| \le \frac{1}{R}$. Define

$$w_\epsilon(x) = \psi(x) u_\epsilon(x)$$

and

$$v_\epsilon(x) = w_\epsilon(x)\left[\int_\Omega K(x) w_\epsilon^{p+1} dx\right]^{\frac{-1}{p+1}}.$$

The proof of the following Lemma follows the same lines as in [BK].

Lemma 4.4. *If assumptions (H$_1$) - (H$_2$) holds and* $p = \frac{N+2}{N-2}$, *then there exist some positive constant* $\epsilon > 0$ *and* $t_0 > 0$ *such that*

$$J_\mu(t_0 v_\epsilon) < 0$$

and

$$0 < \sup_{t \ge 0} J_\mu(t v_\epsilon) < \frac{1}{N} S^{\frac{N}{2}} (\|K\|_{L^\infty})^{\frac{2-N}{2}}.$$

Proof. Since $\frac{\partial u_\epsilon}{\partial \gamma} \le 0$, we have

$$\int_{B_R} |\nabla w_\epsilon|^2 dx = \int_{B_R} |\nabla u_\epsilon|^2 dx \le \int_{B_R} u_\epsilon^{p+1} dx$$

and by the assumption (H_2) we also have

$$K(x_0)\int_{B_R} u_\epsilon^{p+1}dx \le \int_{B_R} K(x)u_\epsilon^{p+1}dx + 0(\epsilon^2)\ .$$

Simple calculations also show that

$$\int_{\mathbb{R}^N\setminus B_R} u_\epsilon^{p+1}dx = 0(\epsilon^N)$$
$$A_\epsilon \equiv \int_{R^N\setminus B_R} |\nabla w_\epsilon|^2 dx = 0(\epsilon^{N-2})$$

as $\epsilon \to 0$ and

$$S = \left[\int_{\mathbb{R}^N} u_\epsilon^{p+1}dx\right]^{\frac{2}{N}}\ .$$

Therefore, we have

$$\begin{aligned}\int_{\mathbb{R}^N} |\nabla w_\epsilon|^2 dx &= \int_{B_R} |\nabla w_\epsilon|^2 dx + A_\epsilon \\ &\le \int_{B_R} u_\epsilon^{p+1}dx + A_\epsilon \\ &\le S\left[\int_{B_R} u_\epsilon^{p+1}dx\right]^{\frac{2}{p+1}} + A_\epsilon \\ &\le S\|K\|_\infty^{-\frac{2}{p+1}}\left[\int_{B_R} K(x)w_\epsilon^{p+1}dx\right]^{\frac{2}{p+1}} + 0(\epsilon^2) + 0(\epsilon^{N-2})\ .\end{aligned}$$

Set $V_\epsilon \equiv \int_{\mathbb{R}^N} |\nabla v_\epsilon|^2 dx$, since for small $\epsilon > 0$, say $\epsilon \le \epsilon_0$, it is easy to see that

$$\int_{B_R} K(x)w_\epsilon^{p+1}dx \ge C_{\epsilon_0}$$

for some positive constant C_{ϵ_0}, the definition of V_ϵ and the last two inequalities imply that

$$V_\epsilon \le S(\|K\|_\infty)^{\frac{2}{p+1}} + 0(\epsilon^2) + 0(\epsilon^{N-2})\ .$$

We consider now $J_\mu(v)$

$$\begin{aligned}J_\mu(v) &= \frac{1}{2}\int_\Omega |\nabla v|^2 dx - \frac{1}{p+1}\int_\Omega K(x)[(u_\mu+v^+)^{p+1} - u_\mu^{p+1}]dx + \int_\Omega K(x)u_\mu^p v^+ dx \\ &= \frac{1}{2}\int_\Omega |\nabla v|^2 dx - \int_\Omega K(x)\int_0^{v^+}((u_\mu+s)^p - u_\mu^p)ds\,dx \\ &= \frac{1}{2}\int_\Omega |\nabla v|^2 dx - \frac{1}{p+1}\int_\Omega K(x)(v^+)^{p+1}dx \\ &- \int_\Omega K(x)\int_0^{v^+}[(u_\mu+s)^p - u_\mu^p - s^p]ds\,dx\ .\end{aligned}$$

Set $F(x,\ v) = K(x)\int_0^{v^+}((u_\mu + s)^p - u_\mu^p - s^p)ds$, then

$$J_\mu(tv_\epsilon) = \frac{1}{2}t^2 V_\epsilon - \frac{1}{p+1}t^{p+1} - \int_\Omega F(x, tv_\epsilon)dx \ .$$

Clearly, $\lim_{t\to\infty} J_\mu(tv_\epsilon) = -\infty$ for all $\epsilon > 0$, hence $\sup_{t\geq 0} J_\mu(tv_\epsilon)$ is attained by some $t_\epsilon \geq 0$, we may assume $t_\epsilon > 0$ for $\epsilon > 0$, otherwise there would be nothing to prove.

It follows from $\frac{d}{dt}J_\mu(tv_\epsilon)\big|_{t=t_\epsilon} = 0$ and the monotonicity of F in v that

$$t_\epsilon \leq V_\epsilon^{\frac{1}{p-1}} \leq C_0 \text{ for all } \epsilon > 0 \ ,$$

where C_0 is some positive constant independent of ϵ. By the monotonicity property of $\frac{1}{2}t^2 V_\epsilon - \frac{1}{p+1}t^{p+1}$ on the interval $(0, V_\epsilon^{\frac{1}{p-1}}]$ we then have

$$\sup_{t\geq 0} J_\mu(tv_\epsilon) = J_\mu(t_\epsilon v_\epsilon) \leq \frac{1}{N}V_\epsilon^{\frac{N}{2}} - \int_{B_{2R}} F(x, t_\epsilon v_\epsilon)dx \ .$$

The estimate on V_ϵ and the above inequality imply that

$$\sup_{t\geq 0} J_\mu(tv_\epsilon) \leq \frac{1}{N}S^{\frac{N}{2}}(\|k\|_\infty)^{\frac{2-N}{2}} - \int_{B_{2R}} F(x, tv_\epsilon)dx + 0(\epsilon^L) \ ,$$

where $L = \min(N-2, 2)$. The conclusion will follows if we can show that

$$\lim_{\epsilon\to 0} \epsilon^{-L} \int_{B_{2R}} F(x, t_\epsilon v_\epsilon)dx = +\infty \ .$$

First we claim that

$$\lim_{\epsilon\to 0} t_\epsilon > 0 \ .$$

Indeed, by $\frac{d}{dt}J_\mu(tv_\epsilon)\big|_{t=t_\epsilon} = 0$ we have

$$V_\epsilon - t_\epsilon^{p-1} - t_\epsilon^{-1}\int_\Omega K(x)[(u_\mu + t_\epsilon v_\epsilon)^p - u_\mu^p - t_\epsilon^p v_\epsilon^p]v_\epsilon dx = 0 \ .$$

We show that

$$\lim_{\epsilon\to 0} t_\epsilon^{-1}\int_\Omega K(x)[(u_\mu + t_\epsilon v_\epsilon)^p - u_\mu^p - t_\epsilon^p v_\epsilon^p]v_\epsilon dx = 0 \ .$$

This will follows by the same procedure as in [BK] (p465–466) by observing first that for all $\delta > 0$, $\exists C_\delta > 0$ such that

$$|f(x,\ u)| \equiv |(u_\mu + u)^p - u_\mu^p - u^p| \leq \delta u^p + C_\delta u \ ,$$

for all $u > 0$. This follows easily from the boundedness of u_μ. Indeed for $u \geq \frac{1}{\delta}$, we have

$$|f(x,u)| = u^p p \int_0^{\frac{u_\mu}{u}} ((s+1)^{p-1} - s^{p-1})ds \leq C_\delta u^p$$

for some constant C. For $u \leq \frac{1}{\delta}$ we have

$$\begin{aligned} |f(x,u)| &\leq \left| \frac{(u_\mu + u)^p - u_\mu^p}{u} \right| u + u^p \\ &\leq p(u_\mu + u)^p u + \left(\frac{1}{\delta}\right)^{p-1} \\ &\leq C_\delta u \ . \end{aligned}$$

It then follows that a positive constant C, independent of ϵ, exists such that

$$\sup_{t \geq 0} J_\mu(tv_\epsilon) \leq \frac{1}{N} S^{\frac{N}{2}} (\|K\|_\infty)^{\frac{2-N}{2}} - \int_{B_{2R}} F(x,\ Cv_\epsilon)dx + 0(\epsilon^L)$$

for sufficiently small $\epsilon > 0$. A change of variables yields

$$\lim_{\epsilon \to 0^+} \epsilon^{-L} \int_{B_{2R}} F(x,\, Cv_\epsilon)dx = +\infty$$

as in [BN]. □

Lemma 4.5. *Assume H_2) and H_1). Suppose moreever $0 \not\equiv K(x) \geq 0$ and $K(x) \in L^1(\Omega)$. Then problem (4.1)$_\mu$ has at least two solution for each $\mu \in (0, \mu^*)$ if $p = \frac{N+2}{N-2}$.*

Proof. The conditions for the mountain pass Lemma [BN] are satisfied by Lemma 4.2, 4.3. Hence there is a sequence $\{v_n\} \subset \mathcal{D}_0^1(\Omega)$ such that $J_\mu(v_n) \to c$ and $J'_\mu(v_n) \to 0$ in $\mathcal{D}_0^1(\Omega)$ as $n \to \infty$, where

$$c = \inf_{\nu \in \Gamma} \sup_{u \in \nu} J_\mu(u) \ .$$

Thus

$$J_\mu(v_n) = \frac{1}{2} \int_\Omega |\nabla v_n|^2 dx - \int_\Omega \left[\frac{1}{p+1} K(x)(u_\mu + v_n^+)^{p+1} - \frac{1}{p+1} u_\mu^{p+1} - u_\mu^p v_n^+ \right] dx = c + o(1), \tag{4.2}$$

and

$$J'_\mu(v_n)\psi = \int_\Omega \nabla v_n \nabla \psi dx - \int_\Omega K((u_\mu + v_n^+)^p - u_\mu^p)\psi dx = o(1)\|\psi\| \tag{4.3}$$

as $n \to \infty$ and $\psi \in \mathcal{D}_0^1(\Omega)$. Choose $\frac{1}{p+1} < \theta < \frac{1}{2}$ and $\psi = v_n$. It follows from (4.2), (4.3) that

$$\begin{aligned} c + o(1) &\geq \frac{1}{2}\int_\Omega |\nabla v_n|^2 dx - \frac{1}{p+1}\int_\Omega K(u_\mu + v_n^+)^{p+1} dx \\ &= \left(\frac{1}{2} - \theta\right)\|v_n\|^2 + \theta(||v_n||^2 - \int_\Omega K[(u_\mu + v_n^+)^p - u_\mu^p] v_n dx) \\ &+ \left(\theta - \frac{1}{p+1}\right)\int_\Omega K(x)(u_\mu + v_n^+)^p v_n^+ dx - \theta \int_\Omega K u_\mu^p v_n^+ dx \\ &- \frac{1}{p+1}\int_\Omega K u_\mu (u_\mu + v_n^+)^p dx \\ &= \left(\frac{1}{2} - \theta\right)\|v_n\|^2 + o(1)\|v_n\| \\ &+ \left(\theta - \frac{1}{p+1}\right)\int_\Omega K(x)(v_n^+ - \tau u_\mu)(u_\mu + v_n^+)^p dx \\ &- \theta \int_\Omega K(x) u_\mu^p v_n^+ dx \ , \end{aligned}$$

where $\tau = (p+1)^{-1}(\theta - (p+1)^{-1})^{-1}$. Notice that we have used the obvious equality

$$\int_\Omega K(x)((u_n + v_n^+)^p - u_n^p) v_n dx = \int_\Omega K[(u_\mu + v_n^+)^p - u_\mu^p] v_n^+ dx \ .$$

Using Hölder's and Sobolev's inequality we have

$$\begin{aligned} \int_\Omega K u_\mu^p v_n^+ dx &\leq \|u_\mu\|_\infty^p \int_\Omega K(x) v_n^+ dx \\ &\leq C\left(\int_\Omega K(x)^2 dx\right)^{\frac{1}{2}}\left(\int v_n^{+2} dx\right)^{\frac{1}{2}} \\ &\leq C_1 \|v_n^+\| \end{aligned}$$

for some constant $C > 0$.

Because $g(x) = (s - \tau u_\mu)(u_n + s)^p$ gets its minimum at

$$s = \frac{p\tau - 1}{1 + p}$$

we have

$$\begin{aligned} c + o(1) &\geq \left(\frac{1}{2} - \theta\right)\|v_n\|^2 + o(1)\|v_n\| \\ &- \frac{2(p(1+\tau))^p(\theta(p+1) - 1)}{(p+1)^{p+2}}\int_\Omega K(x) u_\mu^{p+1} dx \\ &- \theta C_1 \|v_n\| \end{aligned}$$

since $\|u_\mu\|_{L^\infty}$ is bounded, $K(x) \in L^1(\Omega)$. From the above inequality we can deduce $\{v_n\}$ is bounded in $\mathcal{D}_0^1(\Omega)$. Standard embedding theorem then show that $\{v_n\}$ has a subsequence, still denoted by $\{v_n\}$ for which

$$\begin{aligned} &v_n \rightharpoonup v \text{ weakly in } \mathcal{D}_0^1(\Omega) \\ &v_n \to v \text{ a.e. in } \Omega \\ &v_n \rightharpoonup v \text{ weakly in } L^{p+1}(\Omega)\ . \end{aligned}$$

It follows from (4.2) and (4.3) that v is a weak solution of

$$-\Delta v = K(x)[(u_\mu + v^+)^p - u_\mu^p] \qquad v \in \mathcal{D}_0^1(\Omega)\ .$$

Furthermore, (4.3) with $\psi = v^-$ implies that $\int_\Omega |\nabla v^-|^2 dx = 0$ and therefore $\int_\Omega |v^-|^{p+1} dx = 0$, by Sobolev embedding. This shows that $v \geq 0$ a.e. in Ω, we show next that $v \not\equiv 0$.

Consider the sequence $\{w_n\}$, $w_n = v_n - v$, for a subsequence of $\{w_n\}$, denoted the same way, we define

$$\ell = \lim_{n\to\infty} \|w_n\|^2\ .$$

If $\ell = 0$, the continuity of J_μ on $\mathcal{D}_0^1(\Omega)$ implies that

$$0 < \alpha \leq c = \lim_{n\to\infty} J_\mu(v_n) = J_\mu(v) \qquad \text{and hence } v \not\equiv 0\ .$$

If $\ell > 0$, we proceed as follows. Using (4.3) with $\psi = v_n$, the boundedness of $\|v_n\|$, the weak convergence of v_n to v in $L^{p+1}(\Omega)$ and the fact that $u_\mu \in L^\infty(\Omega)$, $K(x) \in L^{\frac{p+1}{p}}$ we obtain

$$\begin{aligned} &\int_\Omega K(x)(u_\mu + v_n)^p u_\mu dx \to \int_\Omega K(x)(u_\mu + v)^p u_\mu\, dx \\ &\int_\Omega K(x) u_\mu^p v_n dx \to \int_\Omega k u_\mu^p v\, dx \qquad . \end{aligned}$$

We have

$$\int_\Omega |\nabla v_n|^2 dx - \int_\Omega K(x)(u_\mu + v_n^+)^{p+1} dx + \int_\Omega K(u_\mu + v)^p u_\mu dx + \int_\Omega K(x) u_\mu^p v\, dx = o(1)\ . \quad (4.4)$$

Using a lemma of Bresis and Lieb [BL] and (4.4) we obtain

$$\begin{aligned} \int_\Omega |\nabla w_n|^2 dx + \int_\Omega |\nabla v|^2 dx - \int_\Omega K(x)(v_n^+ - v)^{p+1} dx \\ = \int_\Omega K(x)((u_\mu + v)^p - u_\mu^p) v\, dx + o(1)\ . \end{aligned}$$

Since v is a solution of problem $(4.1)_\mu$, we have

$$\int_\Omega |\nabla w_n|^2 dx = \int_\Omega K(x)(v^+ - v)^{p+1} dx + o(1)\,. \tag{4.5}$$

Using (4.3) and Bresis-Lieb Lemma [BL] we also have

$$\begin{aligned} o(1) + c &= \frac{1}{2}\int_\Omega |\nabla w_n|^2 dx + \frac{1}{2}\int_\Omega |\nabla v|^2 dx - \frac{1}{p+1}\int_\Omega K(x) w_n^{p+1} dx \\ &\quad - \frac{1}{p+1}\int_\Omega K(x)(u_\mu + v)^{p+1} dx + \frac{1}{p+1}\int_\Omega K(x) u_\mu^{p+1} dx + \int_\Omega K(x) u_\mu^p v\, dx \\ &= \frac{1}{2}\int_\Omega |\nabla w_n|^2 dx - \frac{1}{p+1}\int_\Omega K(v_n^+ - v)^{p+1} dx + J_\mu(v)\ . \end{aligned}$$

which gives us

$$o(1) + c = \frac{1}{2}\int_\Omega |\nabla w_n|^2 dx - \frac{1}{p+1}\int_\Omega K(v_n^+ - v)^{p+1} dx + J_\mu(v)\ . \tag{4.6}$$

It follows from (4.5) and (4.6) that

$$c = \frac{1}{N}\ell + J_\mu(v)\ .$$

We also have by Sobolev's inequality and (4.5) that

$$\begin{aligned} \int_\Omega |\nabla w_n|^2 dx &\geq S\left(\int_\Omega |w_n|^{p+1} dx\right)^{\frac{2}{p+1}} \\ &\geq S\left(\int_\Omega |v_n^+ - v|^{p+1} dx\right)^{\frac{2}{p+1}} \\ &\geq \left(\frac{1}{\sup_\Omega K(x)}\right)^{\frac{2}{p+1}} S\left(\int_\Omega K(x)(|v_n^+ - v|^{p+1}) dx\right)^{\frac{2}{p+1}} \\ &= \left(\frac{1}{\sup_\Omega K}\right)^{\frac{2}{p+1}} S(\|w_n\|^2 + o(1))^{\frac{1}{p+1}} \end{aligned}$$

which gives in the limit, as $n \to \infty$, the inequality

$$\ell \geq [\sup_\Omega K(x)]^{-\frac{2}{p+1}} S \ell^{\frac{2}{p+1}} \tag{4.8}$$

since $\ell > 0$, (4.7) and (4.8) give

$$c \geq \frac{1}{N}(\sup_\Omega K(x))^{\frac{2-N}{2}} S^{\frac{N}{2}} + J_\mu(v) \tag{4.9}$$

which implies from Lemma 4.4 that $J_\mu(v) < 0$, thus $v \not\equiv 0$. □

From the above lemmas we conclude the theorem 1.2.

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