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Abstract. In this paper, we present an approach for measuring inconsistency in a knowledge base. We first define the degree of inconsistency using a four-valued semantics for the description logic $\mathcal{ALC}$. Then an ordering over knowledge bases is given by considering their inconsistency degrees. Our measure of inconsistency can provide important information for inconsistency handling.

1 Introduction

Inconsistency has often been viewed as erroneous information in a knowledge base, but this is not necessarily the best perspective on the problem. The study of inconsistency handling in Artificial Intelligence indeed has a long tradition, and corresponding results are recently being transferred to description logics which are a family of decidable subsets of first-order logic.

There are mainly two classes of approaches to dealing with inconsistent description logic based knowledge bases. The first class of approaches is to circumvent the inconsistency problem by applying a non-standard reasoning method to obtain meaningful answers [1, 2] – i.e. to ignore the inconsistency in this manner. The second class of approaches to deal with logical contradictions is to resolve logical modeling errors whenever a logical problem is encountered [3, 4].

However, given an inconsistent knowledge base, it is not always clear which approach should be taken to deal with the inconsistency. Another problem is that when resolving inconsistency, there are often several alternative solutions and it would be helpful to have some extra information (such as an ordering on elements of the knowledge base) to decide which solution is the best one. It has been shown that analyzing inconsistency is helpful to decide how to act on inconsistency [5], i.e. whether to ignore it or to resolve it. Furthermore, measuring inconsistency in a *knowledge base* in classical logic can provide some context information which can be used to resolve inconsistency [6–8].

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There are mainly two classes of inconsistency measures in classical logic. The first class of measures is defined by the number of formulas which are responsible for an inconsistency, i.e. a knowledge base in propositional logic is more inconsistent if more logical formulas are required to produce the inconsistency [9]. The second class considers the propositions in the language which are affected by the inconsistency. In this case, a knowledge base in propositional logic is more inconsistent if more propositional variables are affected by the inconsistency [6, 10]. The approaches belonging to the second class are often based on some paraconsistent semantics because we can still find models for inconsistent knowledge bases using paraconsistent logics.

Most of the work on measuring inconsistency is concerned with knowledge bases in propositional logic. In [11], the authors generalized the work on measuring inconsistency in quasi-classical logic to the first-order case with restriction to prenex conjunctive form (PCNF) since all first-order theories can be translated into PCNF. However, it is still not clear how to properly perform PCNF on description logics (DLs) while maintaining DLs structures.

The main contributions of this paper are summarized as follows:

- We present an approach for measuring inconsistency of a DL knowledge base.
- We define domain-dependent inconsistency for a consistent knowledge base. This makes it possible to measure the inconsistency degree of a consistent DL knowledge bases with respect to a domain.
- An ordering is given which provides a way to order all knowledge bases according to their inconsistency degree. With respect to such an ordering, consistent knowledge bases are always less inconsistent than all inconsistent knowledge bases.

At the same time, there are potential applications for inconsistency measures for knowledge bases, as they provide evidences for reliability of knowledge bases when an inconsistency occurs. In a scenario where knowledge bases are merged together, we can give higher priority to knowledge bases which are less inconsistent. When resolving inconsistency in the merged knowledge base, we can delete or weaken some axioms from the knowledge base with lower priority.

In this paper, we propose an approach for measuring inconsistency in description logic based knowledge bases. We first define the degree of inconsistency using a four-valued semantics for description logic $\mathcal{ALC}$. By analyzing the degree of inconsistency of a knowledge base, we can either resolve inconsistency if the degree is high (e.g. greater than 0.7) or ignore it otherwise. After that, an ordering over inconsistent knowledge bases is given by considering their inconsistency degrees.

This paper is organized as follows. We first provide some basic notions for Description Logics in Section 2. Then, the concept of domain-dependent (in)consistency is defined in Section 3. Our measure of inconsistency is then given in Section 4. Finally, we discuss related work and conclude the paper in Section 5.

2 Preliminaries

2.1 The Description Logic $\mathcal{ALC}$

We briefly review the terminology of the description logic $\mathcal{ALC}$ and its relation with first order logic FOL. For comprehensive background reading, please refer to [12, 13].

Table 1. Syntax and semantics of $\mathcal{ALC}$ and translation from $\mathcal{ALC}$ to FOL

Constructor	**Syntax**	$\Phi(C,x)$	**Semantics**
atomic concept A	A	$A(x)$	$A^I \subseteq \Delta^I$
abstract role R_A	R	$R(x,y)$	$R^I \subseteq \Delta^I \times \Delta^I$
individuals I	o	o	$o^I \in \Delta^I$
conjunction	$C_1 \sqcap C_2$	$\Phi(C_1,x) \cap \Phi(C_2,x)$	$C^I \cap D^I$
disjunction	$C_1 \sqcup C_2$	$\Phi(C_1,x) \cup \Phi(C_2,x)$	$C^I \cup D^I$
negation	$\neg C$	$\neg A(x)$	$\Delta^I \setminus C^I$
exists restriction	$\exists R.C$	$\exists y R(x,y) \wedge \Phi(C,y)$	$\{x \mid \exists y, (x,y) \in R^I$ and $y \in C^I\}$
value restriction	$\forall R.C$	$\forall y R(x,y) \rightarrow \Phi(C,y)$	$\{x \mid \forall y, (x,y) \in R^I$ implies $y \in C^I\}$

Axiom Name	**Syntax**	$\Phi(\cdot)$	**Semantics**
concept inclusion	$C_1 \sqsubseteq C_2$	$\forall x, \Phi(C_1,x) \rightarrow \Phi(C_2,x)$	$C_1^I \subseteq C_2^I$
concept assertion	$C(a)$	$\Phi(C,a)$	$a^I \in C^I$
role assertion	$R(a,b)$	R(a,b)	$(a^I, b^I) \in R^I$

Corresponding to monadic predicates, dyadic predicates, and functional constants, **concept, role,** and **individual** are fundamental notions of description logics. We assume that we are given a set of concept names (i.e., atomic unitary predicates), a set of role names (i.e., atomic binary predicates) and a set of individuals (i.e., functional constants). Complex concepts (complex monadic formulae) in $\mathcal{ALC}$ can be formed from these inductively as follows.

1. All atomic concept are concepts;
2. If C, D are concepts, then $C \sqcup D$, $C \sqcap D$, and $\neg C$ are concepts;
3. If C is a concept and R is a role, then $\forall R.C$ and $\exists R.C$ are concepts.

For example, suppose *Doctor, Man* are the given atomic concepts, *hasChild* is an atomic role, and *lucy, bill* are two individuals. Then, *Doctor* $\sqcap$ *Man* is a complex concept representing male doctors; the complex concept $\forall hasChild.Doctor$ means the concept representing things whose children are all doctors, and $\exists hasChild.\top$ is a complex concept corresponding to the set of individuals who have at least one child. Then $(Doctor \sqcap Man)(bill)$ means that *bill* is a male doctor, and $hasChild(lucy, bill)$ means that *bill* is a child of $lucy$.

The formal definition of the semantics of $\mathcal{ALC}$ is given by means of interpretations $I = (\Delta^I, \cdot^I)$ consisting of a non-empty domain Δ^I and a mapping $\cdot^I$ satisfying the conditions in Table 1 whose third column is the translation $\Phi(C,x)$ ([14, 13]) from every concept C to a first-order formula, where a, b are constant symbols, y is a fresh variable symbol and x is either the constant symbol a or a variable symbol. The unique name assumption is adapted by Description Logics.

An $\mathcal{ALC}$ knowledge base (or knowledge base for simplicity) consists of a set of assertions, called the $ABox$, and a set of inclusion axioms, calld the $TBox$. Assertions are of the form $C(a)$ or $R(a,b)$, where a, b are individuals and C and R are concepts and roles, respectively. Inclusion axioms are of the form $C \sqsubseteq D$, where C and D are

Table 2. Semantics of $\mathcal{ALC}4$ Concepts

Constructor Syntax	**Semantics**
A	$A^I = \langle P, N \rangle$, where $P, N \subseteq \Delta^I$
R	$R^I = \langle R_P, R_N \rangle$, where $R_P, R_N \subseteq \Delta^I \times \Delta^I$
o	$o^I \in \Delta^I$
$C_1 \sqcap C_2$	$\langle P_1 \cap P_2, N_1 \cup N_2 \rangle$, if $C_i^I = \langle P_i, N_i \rangle$ for $i = 1, 2$
$C_1 \sqcup C_2$	$\langle P_1 \cup P_2, N_1 \cap N_2 \rangle$, if $C_i^I = \langle P_i, N_i \rangle$ for $i = 1, 2$
$\neg C$	$(\neg C)^I = \langle N, P \rangle$, if $C^I = \langle P, N \rangle$
$\exists R.C$	$\langle \{x \mid \exists y, (x, y) \in \mathit{proj}^+(R^I)$ and $y \in \mathit{proj}^+(C^I)\}$, $\{x \mid \forall y, (x, y) \in \mathit{proj}^+(R^I)$ implies $y \in \mathit{proj}^-(C^I)\}\rangle$
$\forall R.C$	$\langle \{x \mid \forall y, (x, y) \in \mathit{proj}^+(R^I)$ implies $y \in \mathit{proj}^+(C^I)\}$, $\{x \mid \exists y, (x, y) \in \mathit{proj}^+(R^I)$ and $y \in \mathit{proj}^-(C^I)\}\rangle$

concepts. The translation $\Phi(\cdot)$ from an axiom to a FOL formula is also given in the third column of Table 1. Informally, an assertion $C(a)$ means that the individual a is an instance of the concept C, and an assertion $R(a, b)$ means that the individual a is related with the individual b via the property R. The inclusion axiom $C \sqsubseteq D$ means that each individual of C is an individual of D.

An interpretation satisfies an $\mathcal{ALC}$ knowledge base (i.e. is a model of the knowledge base) iff it satisfies each axiom in both the $ABox$ and the $TBox$. An $\mathcal{ALC}$ knowledge base is called satisfiable (unsatisfiable) iff there exists (does not exist) such a model. In $\mathcal{ALC}$, reasoning tasks, i.e. the derivation of logical consequences, can be reduced to satisfiability checking of ontologies [12, 15].

From the translations from $\mathcal{ALC}$ axioms to FOL formulae shown in Table 1, $\mathcal{ALC}$ is a subset of FOL, which is proven decidable [12].

2.2 Four-valued Semantics for $\mathcal{ALC}$

We consider the four-valued semantics for $\mathcal{ALC}$ given in [2]. Semantically, four-valued interpretations map individuals to elements of the domain of the interpretation, as usual. For concepts, however, to allow for reasoning with inconsistencies, a four-valued interpretation over a domain Δ^I assigns to each concept C a pair $\langle P, N \rangle$ of (not necessarily disjoint) subsets of Δ^I. Intuitively, P is the set of elements known to belong to the extension of C, while N is the set of elements known to be not contained in the extension of C. P and N are not necessarily disjoint and mutually complemental with respect to the domain.

Formally, a four-valued interpretation is a pair $I = (\Delta^I, \cdot^I)$ with Δ^I as domain, where $\cdot^I$ is a function assigning elements of Δ^I to individuals, and subsets of $(\Delta^I)^2$ to concepts, such that the conditions in Table 2 are satisfied, where functions $\mathit{proj}^+(\cdot)$ and $\mathit{proj}^-(\cdot)$ are defined by $\mathit{proj}^+\langle P, N \rangle = P$ and $\mathit{proj}^-\langle P, N \rangle = N$.

The idea of four-valued semantics is based on the idea of having four truth values, instead of the classical two. The four truth values stand for *true, false, unknown* and *contradictory*. We use the symbols $t, f, \ddot{\bot}, \ddot{\top}$, respectively, for these truth values, and

Table 3. Semantics of inclusion axioms in $\mathcal{ALC}4$

Axiom Name	Syntax	Semantics
material inclusion	$C_1 \mapsto C_2$	$\Delta^I \setminus proj^-(C_1^I) \subseteq proj^+(C_2^I)$
internal inclusion	$C_1 \sqsubset C_2$	$proj^+(C_1^I) \subseteq proj^+(C_2^I)$
strong inclusion	$C_1 \to C_2$	$proj^+(C_1^I) \subseteq proj^+(C_2^I)$ and $proj^-(C_2^I) \subseteq proj^-(C_1^I)$
concept assertion	$C(a)$	$a^I \in proj^+(C^I)$
role assertion	$R(a,b)$	$(a^I, b^I) \in proj^+(R^I)$

the set of these four truth values is denoted by FOUR. The correspondence between truth values from FOUR and concept extensions are defined as follows:

Definition 1 *For instances $a \in \Delta^I$ and concept names C,*

- $C^I(a) = t$, *iff* $a^I \in proj^+(C^I)$ *and* $a^I \notin proj^-(C^I)$,
- $C^I(a) = f$, *iff* $a^I \notin proj^+(C^I)$ *and* $a^I \in proj^-(C^I)$,
- $C^I(a) = \ddot{\top}$, *iff* $a^I \in proj^+(C^I)$ *and* $a^I \in proj^-(C^I)$,
- $C^I(a) = \ddot{\bot}$, *iff* $a^I \notin proj^+(C^I)$ *and* $a^I \notin proj^-(C^I)$.

The correspondence between FOUR and role extensions can be defined in a similar way.

Obviously, for the semantics defined above, we ensure that a number of useful equivalences from classical DLs, such as the double negation law and the de Morgan Laws, hold.

The increase of truth values for four-valued semantics allows for several ways to define meaningful notions of four-valued implication. Indeed, there are three major notions of implication in the literature [16]. Corresponding to them, we have three ways to explain class inclusions in $\mathcal{ALC}$: the material inclusion axiom, the internal inclusion axiom, and the strong inclusion axiom, denoted as $C \mapsto D$, $C \sqsubset D$, and $C \to D$, respectively, to distinguish from classical class inclusion $C \sqsubseteq D$. The semantics of the three different types of inclusion axioms is formally defined in Table 3 (together with the semantics of concept assertions).

These three class inclusion axioms provide knowledge base engineers with a flexible way to define different knowledge bases according to their different semantics [2]. However, when 4-valued models are used to measure inconsistency, we will point out that in Section 4.1 only one of them, the material inclusion, is proper. This is also a reason why other 4-valued description logics [17, 18] are not suitable for measuring inconsistency.

We say that a four-valued interpretation I satisfies a four-valued knowledge base KB (i.e. is a model of it) iff it satisfies each assertion and each inclusion axiom in KB. A knowledge base KB is 4-valued satisfiable (unsatisfiable) iff there exists (does not exist) such a model.

3 Domain-dependent Inconsistency

In this section, we define a domain-dependent inconsistency in DLs. We first recall the notion of inconsistency in DLs.

Definition 2 *A knowledge base KB is classically inconsistent iff KB has no classical model. A knowledge base which is classically inconsistent is called an inconsistent knowledge base. Otherwise, it is called a consistent knowledge base.*

According to Definition 2, *KB* is inconsistent iff it has no classical model. However, given a knowledge base which is consistent, it may be ”inconsistent” for a domain.

Example 3 *Given a knowledge base KB* $= \{\mathcal{T}, \mathcal{A}\}$*, where* $\mathcal{T} = \{A \sqsubseteq \exists R.\neg A\}$ *and* $\mathcal{A} = \{A(a)\}$*, KB is consistent because* KB *has a classical model* $I = \langle \Delta^I, \cdot^I \rangle$*, where* $\Delta^I = \{a, b\}$ *and* $A^I = \{a\}, R^I = \{(a, b)\}$*. However, KB has no classical model with respect to the domain* $\{a\}$*.*

We have the following definition of domain-dependent inconsistency.

Definition 4 *For a given domain* $\mathcal{D}$*, we call KB domain-dependently inconsistent with respect to* $\mathcal{D}$*, denoted* $\mathcal{D}$*-inconsistent, if KB has no classical model with respect to* $\mathcal{D}$*. Otherwise it is called* $\mathcal{D}$*-consistent.*

Example 5 *(Example 3 continued) Consider two domains* $\Delta^{I_1} = \{a\}$ *and* $\Delta^{I_2} = \{a, b\}$*. It is easy to check that KB is* Δ^{I_1}*-inconsistent, but* Δ^{I_2}*-consistent.*

Given another knowledge base *KB*$' = \{A \sqsubseteq \exists R.A, A(a)\}$, *KB*$'$ is both Δ^{I_1}-consistent and Δ^{I_2}-consistent. Therefore, *KB* is “more inconsistent” than *KB*$'$. In the settings where only finite domains are considered, such as databases, the concept of domain-dependent (in)consistency can provide us with an approach to distinguish the extent of inconsistency of two logically consistent knowledge bases.

We give an important property of domain-dependent (in)consistency.

Proposition 6 *An* $\mathcal{ALC}$ *knowledge base KB is consistent, if and only if there exists a positive integer* N*, such that for any finite domain* $\mathcal{D}$ *whose cardinality is greater than* N *(i.e.* $|\mathcal{D}| \geq N$*), KB is* $\mathcal{D}$*-consistent.*

The proposition holds because of the finite model property of $\mathcal{ALC}$ and the fact that it is equality-free. It says that for a consistent knowledge base in $\mathcal{ALC}$, it will be domain-dependently consistent after the domain's cardinality becomes greater than a finite positive integer. Obviously, this property does not hold for other DLs in general and neither for FOL theories which do not have the finite model property or are not equality-free.

4 Inconsistency Measure

In this section, we measure inconsistency of an $\mathcal{ALC}$ knowledge base using four-valued models. In section 4.1 we discuss which kind of semantics of class inclusions as defined in Table 3 is appropriate to be used for measuring inconsistency. In section 4.2, we define the (domain-dependent) inconsistency degree of a knowledge base and study specially the properties of the inconsistency degree for $\mathcal{ALC}$ knowledge bases. Finally, in section 4.3, we give an ordering on knowledge bases based on the inconsistency degrees.

4.1 The Choice of Class Inclusion Axioms

Without explicit declaration, if a class inclusion axiom is expressed in the form $C \sqsubseteq D$, its semantics is the classical semantics as defined in Table 1. If it is in the form of $C \mapsto D, C \sqsubset D$, or $C \rightarrow D$, it is interpreted under the four-valued semantics as defined in Table 3.

Example 7 *Consider $\mathcal{T} = \{A \sqcup \neg A \sqsubseteq A \sqcap \neg A\}$ which is a TBox of an inconsistent knowledge base. Based on four-valued semantics, we have the following three ways to interpret the subsumption: $\mathcal{T}_1 = \{A \sqcup \neg A \mapsto A \sqcap \neg A\}$, $\mathcal{T}_2 = \{A \sqcup \neg A \sqsubset A \sqcap \neg A\}$, and $\mathcal{T}_3 = \{A \sqcup \neg A \rightarrow A \sqcap \neg A\}$, respectively. Now consider the following two 4-valued interpretations:*

$$I_1 = (\Delta^{I_1}, \cdot^{I_1}) : \; A^{I_1} = \langle \Delta^{I_1}, \Delta^{I_1} \rangle$$
$$I_2 = (\Delta^{I_2}, \cdot^{I_2}) : \; A^{I_2} = \langle \emptyset, \emptyset \rangle$$

According to Table 3, $\mathcal{T}_1$ has a unique 4-valued model I_1, while $\mathcal{T}_2$ and $\mathcal{T}_3$ both have I_1 and I_2 as 4-valued models.

In the above example, the difference between I_1 and I_2 is that I_1 assigns contradiction to the concept A, while I_2 assigns nothing to a contradictory value, though knowledge base $\mathcal{T}$ is inconsistent. Therefore, if we interpret a subsumption of an inconsistent knowledge base as internal or strong class inclusion axiom, there may exist a 4-valued model which does not assign contradiction to any concept or role name. We give a proposition which shows an important property of material inclusion. We first introduce some denotations.

Definition 8 *Let I be a four-valued model of KB with domain Δ^I, and let $\mathcal{L}_{KB}$ be the set of atomic concepts and roles occurring in KB. The* inconsistency set *of I for KB, written ConflictOnto(I, KB), is defined as follows:*

$$\mathit{ConflictOnto}(I, \mathit{KB}) = \mathit{ConflictConcepts}(I, \mathit{KB}) \cup \mathit{ConflictRoles}(I, \mathit{KB}),$$

where ConflictConcepts(I, KB) $= \{A(a) \mid A^I(a) = \ddot{\top}, A \in \mathcal{L}_{KB}, a \in \Delta^I\}$, and ConflictRoles$(I, KB) = \{R(a_1, a_2) \mid R^I(a_1, a_2) = \ddot{\top}, R \in \mathcal{L}_{KB}, a_1, a_2 \in \Delta^I\}$.

Intuitively, *ConflictOnto*(I, KB) is the set of conflicting atomic individual assertions.

Proposition 9 *Given an $\mathcal{ALC}$ knowledge base KB=$(\mathcal{T}, \mathcal{A})$, KB is inconsistent if and only if ConflictOnto$(I, KB) \neq \emptyset$ for very 4-valued model I of KB, provided that all class inclusion axioms in $\mathcal{T}$ are explained as material inclusions.*

According to Proposition 9 and the counterexample 7, it is more desirable to interpret class inclusion by material inclusion. So, in the rest of this section, we choose only the semantics of material inclusion as the 4-valued semantics of class inclusion. That is, other semantics are not used to measure inconsistency of an an knowledge base, though they are used to reason with an knowledge base in [2].

4.2 Inconsistency Degree

In this section, we give formal definitions of the inconsistency degree of an inconsistent knowledge base and the domain-dependent inconsistency degree for a consistent knowledge base.To do this, we need the following notions.

Definition 10 *For the knowledge base KB and a 4-valued interpretation I,*

$$\mathit{GroundOnto}(I, \mathit{KB}) = \mathit{GroundConcepts}(I, \mathit{KB}) \cup \mathit{GroundRoles}(I, \mathit{KB}),$$

where $\mathit{GroundConcepts}(I, \mathit{KB}) = \{A(a) \mid a \in \Delta^I, A \in \mathcal{L}_{\mathit{KB}}\}$, $\mathit{GroundRoles}(I, \mathit{KB}) = \{R(a_1, a_2) \mid a_1, a_2 \in \Delta^I, R \in \mathcal{L}_{\mathit{KB}}\}$.

Intuitively, GroundOnto(I, KB) is the collection of all atomic individual assertions.

In order to define the degree of inconsistency, we consider only interpretations with finite domains. This is reasonable in practical cases because only a finite number of individuals can be represented or would be used. This is also reasonable from the theoretical aspect because $\mathcal{ALC}$ has the finite model property — that is, if a knowledge base is consistent and within the expressivity of $\mathcal{ALC}$, then it has a classical model whose domain is finite.

Definition 11 *The* inconsistency degree *of a knowledge base w.r.t. a model $I \in \mathcal{M}4(\mathit{KB})$, denote $\mathit{Inc}_I(\mathit{KB})$, is a value in $[0, 1]$ calculated in the following way:*

$$\mathit{Inc}_I(\mathit{KB}) = \frac{|\mathit{ConflictOnto}(I, \mathit{KB})|}{|\mathit{GroundOnto}(I, \mathit{KB})|}$$

That is, the inconsistency degree of *KB* w.r.t. I is the ratio of the number of conflicting atomic individual assertions divided by the amount of all possible atomic individual assertions of *KB* w.r.t. I. It measures to what extent a given knowledge base contains inconsistency w.r.t. I.

Example 12 *Consider knowledge base $\mathit{KB}_1 = (\mathcal{T}, \mathcal{A})$, where $\mathcal{T} = \{A \sqsubseteq B \sqcap \neg B\}$, $\mathcal{A} = \{A(a)\}$. A 4-valued model of KB_1 is as follows: $I_1 = (\Delta^{I_1}, \cdot^{I_1})$, where $\Delta^{I_1} = \{a\}$, $A^{I_1}(a) = t$, and $B^{I_1}(a) = \ddot{\top}$. For this model, $\mathit{GroundOnto}(I_1, \mathit{KB}_1) = \{A(a), B(a)\}$, and $B(a)$ is the unique element in $\mathit{ConflictOnto}(I_1, \mathit{KB}_1)$. Therefore, $\mathit{Inc}_{I_1}(\mathit{KB}_1) = \frac{1}{2}$.*

In [11], it has been shown that for a fixed domain, not all the models need to be considered to define an inconsistency measure because some of them may overestimate the degree of inconsistency. Let us go back to Example 12.

Example 13 *(Example 12 Continued) Consider another 4-valued model of KB_1: $I_2 = (\Delta^{I_2}, \cdot^{I_2})$, where $\Delta^{I_2} = \{a\}$, $A^{I_2}(a) = \ddot{\top}$, $B^{I_2}(a) = \ddot{\top}$. I_1 and I_2 share the same domain. Since $|\mathit{ConflictOnto}(I_2, \mathit{KB}_1)| = |\{B(a), A(a)\}| = 2$, we have $I_1 \leq_{Incons} I_2$ by Definition 14. This is because $\cdot^{I_2}$ assigns $contradiction$ to $A(a)$. However, $A(a)$ is not necessary a conflicting axiom in four-valued semantics. Therefore, we conclude that $Inc_{I_2}(\mathit{KB})$ overestimates the degree of inconsistency of KB_1.*

We next define a partial ordering on $\mathcal{M}4(\mathit{KB})$ such that the minimal elements w.r.t. it can be used to define the inconsistency measure for *KB*.

Definition 14 *(Model ordering w.r.t. inconsistency) Let I_1 and I_2 be two four-valued models of a knowledge base KB such that $|\Delta_1^I| = |\Delta_2^I|$. We say the inconsistency of I_1 is less than or equal to I_2, written $I_1 \leq_{Incons} I_2$, if and only if $Inc_{I_1}(KB) \leq Inc_{I_2}(KB)$.*

The condition $|\Delta^{I_1}| = |\Delta^{I_2}|$ in this definition just reflects the perspective that only models with the same cardinality of domain are comparative. As usual, $I_1 <_{Incons} I_2$ denotes $I_1 \leq_{Incons} I_2$ and $I_2 \not\leq_{Incons} I_1$, and $I_1 \equiv_{Incons} I_2$ denotes $I_1 \leq_{Incons} I_2$ and $I_2 \leq_{Incons} I_1$. $I_1 \leq_{Incons} I_2$ means that I_1 is more consistent than I_2.

The model ordering w.r.t. inconsistency is used to define preferred models.

Definition 15 *Let KB be a DL-based knowledge base and $n(n \geq 1)$ be a given cardinality. The* preferred models w.r.t $\leq_{Incons}$ of size n*, written $PreferModel_n(KB)$, are defined as follows:*

$$PreferModel_n(KB) = \{I \mid |\Delta^I| = n; \forall I' \in \mathcal{M}4(KB), |\Delta^{I'}| = n \text{ implies } I \leq_{Incons} I'\}$$

That is, $PreferModel_n(KB)$ is the set of all models of size n which are minimal with respect to $\leq_{Incons}$.

From Definition 14 and Definition 15, it is easy to see that for the preferred models I_1 and I_2 with a same cardinality, inconsistency degrees of the knowledge base $w.r.t$ them are equal. That is, $Inc_{I_1}(KB) = Inc_{I_2}(KB)$, which means I_1 and I_2 have the same amount of contradictory atomic assertions, though the elements of their domains may be quite different.

For simplicity, we say an interpretation is *well-sized* if and only if the cardinality of its domain is equal to or greater than the number of individuals in *KB*. Because of the unique name assumption of the DL $\mathcal{ALC}$, an interpretation can be a model only if it is well-sized. Moreover, the following theorem asserts the existence of preferred models among the well-sized interpretations.

Theorem 16 *For any given $\mathcal{ALC}$ knowledge base KB, preferred models among well-sized interpretations always exist.*

Above we have considered the inconsistency degrees of knowledge bases with respect to four-valued models, especially with respect to preferred models. Now we define an integrated inconsistency degree of a knowledge base allowing for different domains.

Definition 17 *Given a knowledge base KB and an arbitrary cardinality $n(n \geq 1)$, let I_n be an arbitrary model in $PreferModels_n(KB)$. The inconsistency degree sequence of KB, called $OntoInc(KB)$, is defined as $\langle r_1, r_2, ..., r_n, ...\rangle$, where $r_n = Inc(I_n, KB)$ if I_n is well-sized. and let $r_n = *$ otherwise.*

In Definition 17, we use $*$ as a kind of null value. Given a domain with size n, we have the following three cases: (1) if $r_n = *$, it means that the knowledge knowledge base has no 4-valued models with size n; (2)if $r_n = 0$, it means that KB has a classical model among its 4-valued models. (3) if $r_n > 0 (\neq *)$, it means KB has no classical models but has 4-valued models.

From Theorem 16 and the unique name assumption of $\mathcal{ALC}$ the following property holds obviously.

Proposition 18 *Assume KB is a knowledge base and $OntoInc(KB) = \langle r_1, r_2, ...\rangle$, and N is the number of individuals of KB. Then*

$$r_i \begin{cases} = * & \text{if } 0 < i < N, \\ \geq 0 (\neq *) & \text{if } i \geq N. \end{cases}$$

This proposition shows that for a knowledge base, its inconsistency measure cannot be a meaningless sequence — that is, each element is the null value $*$. Moreover, the non-null values in the sequence start just from the position which equals the number of individuals in the knowledge base, and remains greater than zero in the latter positions of the sequence for inconsistent knowledge bases and becomes zero after n becomes large enough for consistent $\mathcal{ALC}$ knowledge bases.

Example 19 *(Example 7 continued) Obviously, for any four-valued model $I = \langle \Delta^I, \cdot^I \rangle$ of $\mathcal{T}$, A must be assigned to $\langle \Delta^I, \Delta^I \rangle$, therefore OntoInc$(KB) = \{1, 1, ...\}$.*

Example 20 *(Example 12 continued) Each preferred model I of KB$_1$ must satisfy that (1) it assigns one and only one individual assertion in $\{B(a), A(a)\}$ to the contradictory truth value $\ddot{\top}$ — that is, $B^I(a) = \ddot{\top}$ and $A(a) = t$, or $B^I(a) = t$ and $A(a) = \ddot{\top}$; (2) it assigns other grounded assertions to truth values among the set $\{t, f, \ddot{\bot}\}$. So $|ConflictOnto(I, KB)| = 1$. Consequently, $OntoInc(KB_1) = \{\frac{1}{2}, \frac{1}{4}, ..., \frac{1}{2n}, ...\}$.*

Example 21 *(Example 5 continued) It is easy to check that OntoInc$(KB') = \langle 1, 0, 0, ...\rangle$, while OntoInc$(KB'') = \langle 0, 0, ...\rangle$. By definition 22, KB$'$ has more domain-dependent inconsistency than KB$''$, that is, KB$'' \prec_{Incons}$ KB$'$.*

4.3 Ordering Knowledge Bases with respect to their Inconsistent Degrees

In this section, we define an ordering over all knowledge bases inspired by [19].

Definition 22 *Given two knowledge bases KB$_1$ and KB$_2$, assume $OntoInc(KB_1) = \langle r_1, r_2, ...\rangle$ and $OntoInc(KB_2) = \langle r'_1, r'_2, ...\rangle$. We say KB$_1$ is strictly less inconsistent than KB$_2$, written KB$_1 \prec_{Incons}$ KB$_2$, iff one of the following conditions holds:*

1. *$N_1 < N_2$*
2. *$N_1 = N_2$, $r_n \leq r'_n (\forall n \geq K)$, and there exists $n_0 \geq K$ such that $r_n < r'_n$*

where $N_1 = min\{i : r_i = 0\}, N_2 = min\{i : r'_i = 0\}, K = min\{i : r_i \neq *, r'_i \neq *\}$.

N_1 (or N_2) is the first position from which the elements of the sequence $\langle r_1, r_2, ...\rangle$ (or $\langle r'_1, r'_2, ...\rangle$) become 0. For an inconsistent knowledge base, the position is infinite, so we denote $N_1 = \infty$ which is strictly greater than any finite number. So does N_2. According to Proposition 18, $\prec_{Incons}$ is well-defined. Moreover, an equality of two knowledge bases can be defined by *KB*$_1 =_{Incons}$ *KB*$_2$ if and only if for all $n \geq K, r_n = r'_n$.

By Definition 22 and Proposition 6, obviously, all consistent knowledge bases are less inconsistent than any inconsistent knowledge base.

For consistent knowledge bases, according to Definition 22, one knowledge base is less domain-inconsistent than the other if and only if 0 begins at an earlier position in its inconsistency degree sequence than that in the inconsistency degree sequence of the other knowledge base.

For the ordering among inconsistent knowledge bases, we only compare the values from the position at which both sequences have non-null values, according to Definition 22. This is because there exist infinite elements of sequences of their inconsistency degree which are non-null, and non-zero. These elements together are to reflect the useful information about the inconsistency of knowledge bases.

Example 23 *(Example 20 continued) Suppose $KB_2 = \{A \sqsubseteq B \sqcap \neg B, A \sqsubseteq C, A(a)\}$. In its preferred models, the individual assertions related to C are not involved with the contradictory truth value, so $OntoInc(KB_2) = \{\frac{1}{3}, \frac{1}{6}, ..., \frac{1}{3n}, ...\}$. By definition 22, $KB_2 \prec_{Incons} KB_1$, which means that KB_2 is less inconsistent than KB_2.*

Example 24 *(Example 19, 21, 23 continued)*

$$KB'' \prec_{Incons} KB' \prec_{Incons} KB_2 \prec_{Incons} KB_1 \prec_{Incons} KB.$$

5 Related Work and Conclusion

This paper provides a way to distinguish description logic based knowledge bases considering their different inconsistency degrees.

In the literature, there are basically two other works on defining four-valued semantics for description logics [20, 18]. However, their definitions of class inclusion axioms are actually the same as the internal inclusion defined in Table 3, so that their approaches are not suitable for measuring inconsistency according to our analysis in Section 4.1.

Our work is closely related to the work of inconsistency measuring given in [11], where Quasi-Classical models (QC logic [21]) are used as the underlying semantics. In this paper, we use four-valued models for description logics as the underlying semantics. This is because QC logic needs to translate each formula in the theory into prenex conjunctive normal form (PCNF). This is not practical, especially for a large knowledge base, because it may be quite time consuming and users probably do not like their knowledge bases to be modified syntactically. In this paper, we can see that four-valued models also provide us with a novel way to distinguish knowledge bases with different inconsistency degrees.

It is apparent that the inconsistency measure defined by our approach can be used to compute each axiom's contribution to inconsistency of a whole knowledge base by adapting the method proposed in [8], thereby providing important information for resolving inconsistency in a knowledge base. Moreover, we find that four-valued models may provide us with some way to quantify also the incompleteness degree of knowledge bases because of the additional truth value $\ddot{\bot}$ with respect to three-valued semantics, which is among our future work.

In [11], every set of formulae definitely has at least one QC model because neither the constant predicate t (tautology) nor the constant predicate f (false) is contained in the language. However, corresponding to t and f, the top concept $\top$ and bottom concept $\bot$ are two basic concept constructors for $\mathcal{ALC}$. Due to space limitation, we presume that the ontologies do not use $\top$ and $\bot$ as concept constructors. The discussion of the inconsistency measure for an arbitrary inconsistent ontology will be left as future work.

For an implementation of our approach, the key point is to compute the number of conflicting assertions in a preferred model with respect to any given finite domain. We are currently working on the algorithm, which will be presented in a future paper.

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