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Stability of a Semilinear Cauchy Problem *

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Abstract

This paper is contributed to the Cauchy problem

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta u + K(|x|)u^p & \text{in } \mathbb{R}^n \times (0, T), \\ u(x, 0) = \varphi(x) & \text{in } \mathbb{R}^n, \end{cases} \quad (0.1)$$

with initial function $\varphi \neq 0$. The stability and instability of the positive radical steady states, which are positive solutions of

$$\Delta u + K(|x|)u^p = 0, \quad (0.2)$$

has been discussed with different assumption on $K(x)$ and φ under the norm

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$$\|\psi\|_\lambda = \sup_{x \in \mathbb{R}^n} |(1 + |x|^\lambda)\psi(x)| \quad (0.3)$$

where φ and ψ are some non-negative continuous functions in \mathbb{R}^n , and λ is a real number.

Key words and phrases: stability, Cauchy problem, asymptotic stability.

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1 Introduction

In this paper, we study the stability and instability of the positive radical steady states, which are positive solutions of

$$\Delta u + K(|x|)u^p = 0, \quad (1.1)$$

of the following Cauchy problem:

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta u + K(|x|)u^p & \text{in } \mathbb{R}^n \times (0, T), \\ u(x, 0) = \varphi(x) & \text{in } \mathbb{R}^n. \end{cases} \quad (1.2)$$

Where $p > 1$, $x \in \mathbb{R}^n$, $\Delta = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$ is the n -dimensional Laplacian, $T > 0$, and $\varphi \not\equiv 0$ is a bounded nonnegative continuous function in \mathbb{R}^n .

We often assume that $K \in C^\alpha(\mathbb{R}^n \setminus 0)$ for some fixed $\alpha \in (0, 1)$, so that the solutions of (1.1) are classical on $0 < |x| < \infty$. However, at $x = 0$, when K is “bad”, usually we can not expect the solutions to be differentiable, or even continuous owing to the singularity of K at $x = 0$. Let u be a solution of (1.1), the singular point $x = 0$ of (1.1) is called a *removable singular point* of $u(x)$ if $u(0) \equiv \lim_{x \rightarrow 0} u(x)$ exists, otherwise $x = 0$ is called a *nonremovable singular point*. It is showed by Ni and Yotsutani ([NY]) that when $x = 0$ is a removable singular point of a regular solution, the existence of the derivatives of the solution depends on the “blow-up” rate of K at $x = 0$ ([NY] Propersition 4.4)

Let $u \in C^2(\mathbb{R}^n \setminus 0)$ be a solution of (1.1). If $x = 0$ is a removable singular point of u , then u is said to be a regular solution of (1.1). If $x = 0$ is a nonremovable singular point of u , then u is said to be a irregular solution of (1.1).

For the physical reasons, we consider the positive radial solutions of (1.1), when $K = K(r)$, where $r = |x|$, equation (1.1) reduces to

$$u'' + \frac{n-1}{r}u' + K(r)u^p = 0, \quad r > 0. \quad (1.3)$$

For the same reasons, the regular solutions, that have limits at $r = 0$, are particularly interest, this leads us to consider the initial problem

$$\begin{cases} u'' + \frac{n-1}{r}u' + K(r)u^p = 0 & r > 0, \\ u(0) = \alpha > 0. \end{cases} \quad (1.4)$$

In this paper, we use notation $u_\alpha = u(r; \alpha)$ to denote the solution of (1.4).

Equation (1.1) was studied by many mathematicians. It is showed ([N1] and [Lin]) that if $|K| \geq Cr^{(n-2)(p-1)-2}$ at infinity for some constant $C > 0$, then (1.1) possesses no positive solutions. In case of that $|K| \leq Cr^{(n-2)(p-1)-2-\varepsilon}$ at infinity for some positive constants C and ε , the existence and asymptotics of positive solutions are studied by many authors, here we only metioned the results of, for example, W.-M. Ni, S Yosutani [NY] and Y. Li [L]. In the fast decay case $|K| \leq Cr^l$, $l < -2$, Ni showed that (1.1) possesses infinitely many positive solutions which are bounded from below by positive constants (see [N1] and [Lin]). Li and Ni ([LN]) showed that, for positive solution of (1.1), the limit $u_\infty = \lim_{x \rightarrow \infty} u(x)$ always exists for any $\varepsilon > 0$, furthermore, if $u_\infty = 0$, then

$$u(x) \leq \begin{cases} C|x|^{2-n} & \text{if } p > \frac{n+l}{n-2}, \\ C_\varepsilon|x|^{\frac{(1-\varepsilon)(l+2)}{1-p}} & \text{if } p \leq \frac{n+l}{n-2}, \end{cases}$$

where C_ε is a constant depending on ε ; and if $u_\infty > 0$, then

$$|u - u_\infty| \leq \begin{cases} C|x|^{2-n} & \text{if } l < -n, \\ C|x|^{2-n} \log|x| & \text{if } l = -n, \\ C|x|^{2+l} & \text{if } -n < l < -2, \end{cases}$$

at ∞ . Li refined these results and gave the limit u_∞ explicitly (see [L] or Theorem A in this section.)

In this paper, we will focus us on the slow decay case, i.e., $K(r) \geq Cr^l$, for some $l > -2$ and r large.

First, let us introduce a collection of hypotheses on K .

(K.1). $K(r) > 0$ in $r > 0$ and $\lim_{r \rightarrow \infty} r^{-l}K(r) = k_\infty > 0$, where $l > -2$,

(K.1'). $K(r) > 0$ in $r > 0$ and $\lim_{r \rightarrow 0^+} r^{-l}K(r) = k_0 > 0$, where $l > -2$,

(K.2). $K(r)$ is differentiable and $[\frac{d}{dr}(r^{-l}K(r))]^+ \in L^1$, $r > 0$,

(K.3). $K(r)$ is differentiable and $[\frac{d}{dr}(r^{-l}K(r))]^- \in L^1$, $r > 0$,

(K.4). $K(r)$ is differentiable and $\frac{d}{dr}(r^{-l}K(r)) \leq 0$, $r > 0$.

Also we introduce the following notations, those will be used throughout this paper:

$$m \equiv \frac{l+2}{p-1}, \quad b_0 \equiv n-2-2m$$

$$L \equiv [m(n-2-m)]^{\frac{1}{p-1}}, \quad c_0 \equiv (p-1)L^{p-1} \quad (1.5)$$

It is easy to see that in the slow decay case $l > -2$, when $p > \frac{n+2l+2}{n-2}$, we have $m > 0$ and $b_0 > 0$.

There are many results about the existence and nonexistence of the positive solutions for problem (1.4). Ni and Yasutani showed that (1.4) has exactly one solution $u(r) > 0$, and $u(0) = \alpha$, for every $\alpha > 0$, under the assumption that (K.1'), (K.4) and $m \leq (n-2)/2$ (see Theorem 6 in [NY]). In the case of that $m > (n-2)/2$, if $r^{-l}K(r)$ has a positive limit at $r = 0^+$, then there exists $\alpha_1 > 0$ such that for every $\alpha \geq \alpha_1$, equation (1.4) has no entire positive solution with initial value α . This is the result of Theorem 2 in [NY]. Under such sense $m = \frac{n-2}{2}$ is a critical index to the problem (1.4). The existence of positive solutions is also established in [DN] and [LN]. Under various assumptions on K , uniqueness of positive solutions of (1.4) is obtained in [KL] and [YY].

The folowing theorem is obtained by Li, it gives an accurate description on the asymptotic behavior of positive solutions of (1.1).

Theorem A. (Theorem 1, [L]) *Let u be a positive radial solution of (1.1). Assume that K satisfies*

- (i) (K.1) and (K.2), if $0 < m < (n - 2)/2$, or
- (ii) (K.1) and (K.3), if $(n - 2)/2 < m < n - 2$.

Then

$$\lim_{r \rightarrow \infty} r^m u(r) = u_\infty \equiv \begin{cases} \frac{L}{k_\infty^{\frac{1}{p-1}}}, & \text{or} \\ 0. \end{cases}$$

Furthermore, if $u_\infty = 0$, then

$$\lim_{r \rightarrow \infty} r^{n-2} u(r)$$

exists and is finite and positive.

Remark 1.1 . When $l = -2$, a result similar to Theorem A holds (See [LN] and [L].)

A natural and interesting question concerning equation (1.4) is: do two solutions with different initial values intersect each other, or, in other words, do the solutions of (1.4) have monotone property? It is known that the monotone property of the solutions of (1.4) has some important complications, like stability, etc.

It is showed by Wang([W]), Ni and Yosutani ([NY]) that for small p , any two positive solutions intersect each other. Wang also showed that for large p , the solutions of (1.4) possess monotone property for a special class of K , and gave explicitly the lower bound of the p value.

In case of $K(r) = r^l$, $l > -2$, the following function

$$U_s(r) = Lr^{-m} \tag{1.6}$$

is a singular solution of equation (1.3) with $K(r) = r^l$. To state Wang's result, we define constant p_c by

$$p_c = \begin{cases} \frac{(n-2)^2 - 2(l+2)(n+l) + 2(l+2)\sqrt{(n+l)^2 - (n-2)^2}}{(n-2)(n-10-4l)} & n > 10 + 4l, \\ \infty & 3 \leq n \leq 10 + 4l, \end{cases}$$

particularly, when $l = 0$ we have,

$$p_c = \begin{cases} \frac{(n-2)^2 - 4n + 4\sqrt{n^2 - (n-2)^2}}{(n-2)(n-10)} & n > 10 \\ \infty & 3 \leq n \leq 10. \end{cases}$$

We have

Theorem B. ([W] Proposition 3.7.(iii), (iv)) *Let $u_\alpha(r)$ be the solution of (1.4) with $K(r) = r^l$. Then we have*

(i) *when $(n+2+2l)/(n-2) < p < p_c$, the graph of $u_\alpha(r)$ oscillates around that of $U_s(r)$ for every $\alpha > 0$,*

(ii) *when $p \geq p_c$, the graph of u_α does not intersect that of U_s (i.e., $u(r) < U_s(r)$) for every $\alpha > 0$. Furthermore, $u_\alpha(r)$ is increasing with respect to α .*

For large p , Theorem B is extended to a more general class of $K(x)$ by Gui ([G] Lemma 3.1).

Recently, we studied the monotonicity of solutions of (1.4) with respect to the initial data α and got a sharp estimate p_c on the exponent p under more general condition imposed on $K(x)$. More exactly, we have [DLL]

Theorem C. *Suppose that $K(r)$ satisfies (K.1), (K.1') and (K.4). Let $u_\alpha(r) = u_\alpha(r, \alpha)$ and $u_\beta(r) = u_\beta(r, \beta)$ be two positive solutions of equation (1.4) with $u_\alpha(0) = \alpha$, $u_\beta(0) = \beta$, and $0 < \alpha < \beta$. Then*

(i) *when $p > p_c$, $u_\alpha(r)$ and $u_\beta(r)$ can not intersect each other, i.e., $u_\alpha(r) < u_\beta(r)$.*

(ii) *when $(n+2+2l)/(n-2) < p < p_c$, $u_\alpha(r)$ and $u_\beta(r)$ will intersect infinity many times,*

We also studied the singular solutions of equation (1.3), which blow up at $r = 0$, and gave a general uniqueness theorem.

For the stability and instability of the positive radial steady states, which are positive solutions of (1.4), of the Cauchy problem (1.2) with initial function $\varphi \not\equiv 0$, It seems that the first general result is given by Fujita ([F]). It is showed that for $p > \frac{n+2}{n}$, the solution $u(x, t; \varphi)$ of (1.2) exists globally in time with sufficient small φ , and for $1 < p < \frac{n+2}{n}$, $u(x, t; \varphi)$ blows up in finite time for any $\varphi \geq 0, \varphi \not\equiv 0$. Thus the trivial steady state $u_0 \equiv 0$ is unstable in any proper topology for $1 < p < \frac{n+2}{n}$. In the case of that $p > \frac{n+2}{n}$, we are also interested in the topology in which the attraction domain of $u_0 \equiv 0$ is depicted.

In case of $K \equiv 1$, for the global existence of $u(x, t; \varphi)$, the condition given by Fujita on φ is that it is bounded by $\varepsilon e^{-|x|^2}$ for some small ε ; Weissler ([We]) studied the problem in L^p -space and his condition on φ can be interpreted as to that φ is bounded by $\varepsilon(1+|x|)^{-\gamma}$ for some constants $\gamma > \frac{2}{p-1}$ and ε small enough; Lee and Ni improved this condition to that φ has decay rate of $C|x|^{-\frac{2}{p-1}}$ at ∞ , where C is a positive constant (see [LeN].) A delicate study of the stability of positive steady state u_α of (1.2), which is the a solution of (1.4), is given by Gui, Ni and Wang in [GNW]. To describe the stability, they introduced the following norm:

$$\|\psi\|_\lambda = \sup_{x \in \mathbb{R}^n} |(1+|x|^\lambda)\psi(x)| \quad (1.7)$$

where ψ is a non-negative continuous function in \mathbb{R}^n , and λ is a real number.

Definition 1.1 . We say that a steady state u_α of (1.2) is stable with respect to some norm $\|\cdot\|_\lambda$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that, for $\|\varphi - u_\alpha\|_\lambda < \delta$, we have $\|u(\cdot, t; \varphi) - u_\alpha\|_\lambda < \varepsilon$ for all $t > 0$; u_α is said to be weakly asymptotically stable with respect to $\|\cdot\|_\lambda$ if u_α is stable with respect to $\|\cdot\|_\lambda$ and there exists $\delta > 0$ such that, for $\|\varphi - u_\alpha\|_\lambda < \delta$, we have $\|u(\cdot, t; \varphi) - u_\alpha\|_{\lambda'} \rightarrow 0$ as $t \rightarrow \infty$ for all $\lambda' < \lambda$.

Consider the quadratic equation

$$\lambda^2 + b_0\lambda + c_0 = 0. \quad (1.8)$$

Here b_0 and c_0 are as in (1.5). When $p > p_c$, (1.8) has two negative roots $-\lambda_2 < -\lambda_1 < 0$. For $K \equiv 1$, it is showed in [GNW] that the steady states of (1.2) are stable with respect to norm $\|\cdot\|_{m+\lambda_1}$, and weakly asymptotic stable with respect to norm $\|\cdot\|_{m+\lambda_2}$, here m is defined in (1.5). With the topology introduced in (1.7), we prove the stability and asymptotic stability of its steady states for a more general class of K . This is an extension of the results obtained by Gui, Ni and Wang [GNW]. Since our K is not homogeneous and hence steady states can not be obtained by scaling, some key techniques in [GNW] do not apply to our case. To overcome this, we construct super and subsolutions in a different way and give some very delicate estimates on them. All of these much depend on the properties (Theorem C) and the asymptotic expansions at infinite of the positive solutions of (1.4). Our main theorems are stated as follow.

Theorem 1. *Suppose that K satisfies (K.1), (K.1'), and (K.4), $p_c > p > \frac{n+2+2l}{n-2}$. Then the following conclusions hold:*

- (i) *if $\varphi \leq u_\alpha$ and $\varphi \not\equiv 0$ for some $\alpha > 0$, then $\|u(\cdot, t; \varphi)\|_{L^\infty(\mathbb{R}^n)} \rightarrow 0$ as $t \rightarrow \infty$;*
- (ii) *if $\varphi \geq u_\alpha$ and $\varphi \not\equiv u_\alpha$ for some $\alpha > 0$, then the solution $u(\cdot, t; \varphi)$ blows up in finite time.*

Theorem 2. *Suppose that K satisfies (K.1), (K.1') and (K.4), $p > p_c$. Then any positive steady state u_α of (1.2) is:*

- (i) *stable with respect to the norm $\|\cdot\|_{m+\lambda_1}$;*
- (ii) *weakly asymptotically stable with respect to the norm $\|\cdot\|_{m+\lambda_2}$.*

This paper is organized as follows: In Section 2, we introduce a estimate on the solutions of (1.4). The asymptotic expansions of solutions of (1.4) are given in Section 3, based on which we prove the stability and asymptotic stability of the steady state of (1.2) in Section 4.

2 Preliminaries

In this section, we will introduce an estimate on the solutions of equation (1.4) which will be used in the proof of our main Theorems. The idea is due to Wang ([W]).

Without any particular statement, all solutions appearing in this and the following sections are regular ones. First, let us introduce the following transformation, which will be used frequently in this and later sections.

Lemma 2.1 . Suppose that u is a positive solution of (1.3). Let $r = e^t$, $t \in (-\infty, +\infty)$ and $v(t) = r^q u(r)$, then v satisfies

$$v'' + (n - 2 - 2q)v' - q(n - 2 - q)v + K(e^t)e^{(q+2-pq)t}v^p = 0. \quad (2.1)$$

This Lemma can be proved by straight forward calculations, we omit it here.

Lemma 2.2 . Suppose that $K(r)$ satisfies (K.1), (K.1') and (K.4). Let $u(r)$ be the positive solution of (1.3). If $p \geq p_c$, then $r^m u(r)$ is strictly increasing in r and

$$(r^{-l}K(r))(r^m u(r))^{p-1} < L^{p-1}. \quad (2.2)$$

Proof: Let $q = m$ in Lemma 2.1, then we have that

$$v'' + b_0 v' - L^{p-1} v + k(t)v^p = 0, \quad (2.3)$$

here $k(t) = e^{-lt}K(e^t)$, $v = e^{mt}u(e^t)$ and m , b_0 , and L are as in (1.5). We need only to show that $k(t)v^{p-1} < L^{p-1}$. By (K.1) (K.1') and (K.4), $\lim_{t \rightarrow \infty} k(t) = k_\infty$, $\lim_{t \rightarrow -\infty} k(t) = k_0 > 0$ and $k'(t) \leq 0$ for $t \in \mathbb{R}$. Since $v(t) > 0$ for $t \in \mathbb{R}$, and $\lim_{t \rightarrow -\infty} v(t) = 0^+$, we have $k(t)v^{p-1} < L^{p-1}$ at the neighborhood of $t = -\infty$.

On the contrary, suppose that there exists $t \in \mathbb{R}$, such that $k(t)v^{p-1} \geq L^{p-1}$. Let

$$T = \min\{t \in \mathbb{R} \mid k(t)v^{p-1} \geq L^{p-1}\}. \quad (2.4)$$

then $T > -\infty$, $k(t)v^{p-1} < L^{p-1}$ for $t < T$ and $k(T)v^{p-1}(T) = L^{p-1}$. From (2.3) we have that

$$v'' + b_0 v' > 0 \quad (2.5)$$

for all $t < T$. This implies $e^{b_0 t}v'$ is strictly increasing on $(-\infty, T)$. By Propersition 4.1.(b) in [NY] and the facts that both $r^{-l}K(r)$ and $u(r)$ are bounded, from equation (1.3) we have that

$$\begin{aligned} u'(r) &= - \int_0^r \left(\frac{s}{r}\right)^{n-1} K(s)u^p(s)ds \\ &= O\left(r^{l+1}\right) \end{aligned}$$

at $r = 0^+$. Hence $v'(t) = mr^m u(r) + r^{m+1}u'(r)$, $r = e^t$, goes to zero as $t \rightarrow -\infty$. Since $b_0 > 0$, we have that $e^{b_0 t}v'(t) \rightarrow 0$ as $t \rightarrow -\infty$, and $v'(t) > 0$ for $t \in (-\infty, T)$. Let $q(v) = v'(t) > 0$ for $(0, \frac{L}{k(T)^{\frac{1}{p-1}}})$, then $q(v) > 0$, $q(v) \rightarrow 0^+$ as $v \rightarrow 0^+$, and satisfies

$$\frac{dq}{dv} = -b_0 + \frac{L^{p-1}v - k(t)v^p}{q}. \quad (2.6)$$

Therefore in the $q - v$ plane the line $q = \mu\left(\frac{L}{(k(T))^{p-1}} - v\right)$ must intersect the graph of $q(v)$ for every $\mu > 0$. Let $(v_\mu, q(v_\mu))$ be the intersection with the smallest v -coordinate for each $\mu > 0$, then we have $(dq/dv) \geq -\mu$, moreover, the following holds at $(v_\mu, q(v_\mu))$

$$\frac{dq}{dv}(v_\mu) = -b_0 + \frac{L^{p-1}v_\mu - k(t)v_\mu^{p-1}}{\mu\left(\frac{L}{k(T)^{p-1}} - v_\mu\right)}. \quad (2.7)$$

Since $k'(t) \leq 0$, it follows that $k(t) \geq k(T)$ for $t \leq T$, and by mean value theorem, there exists $\tilde{v}_\mu \in (v_\mu, L^{p-1}/k(T))$ such that

$$\begin{aligned} -\mu &\leq -b_0 + \frac{k(T)v_\mu\left(\frac{L^{p-1}}{k(T)} - v_\mu^{p-1}\right)}{\mu\left(\frac{L}{k(T)^{p-1}} - v_\mu\right)} \\ &= -b_0 + \frac{(p-1)k(T)v_\mu\tilde{v}_\mu^{p-2}}{\mu} \\ &< -b_0 + \frac{(p-1)L^{p-1}}{\mu}. \end{aligned}$$

with $\tilde{v}_\mu \in (v_\mu, L^{p-1}/k(T))$. So we have

$$\mu^2 - b_0\mu + c_0 > 0 \quad (2.8)$$

holding for all $\mu > 0$, so the determinant of the quadratic form (2.8) must be negative, i.e., $b_0^2 - 4(p-1)L^{p-1} < 0$. By direct calculations, (2.8) holds if and only if $p < p_c$. The contradiction shows $k(t)v^{p-1} < L^{p-1}$ for all $t \in \mathbb{R}^1$, hence (2.2) holds. Consequently, (2.5) holds for all $t \in \mathbb{R}$. Multiplying (2.5) by e^{b_0t} and integrating over $(-\infty, t)$, we get $v'(t) > 0$ for $t \in \mathbb{R}$, hence $r^m u(r)$ is strictly increasing. \square

Remark 2.1 . By Theorem A we know that $\lim_{r \rightarrow \infty} r^m u(r)$ exists and is either $\frac{L}{k_\infty^{p-1}}$ or 0. Since $r^m u(r)$ is strictly increasing by Lemma 2.2, it follows that the limit is $\frac{L}{k_\infty^{p-1}}$.

3 Asymptotic Expansion At Infinity

In this section, we will extend the expansion results for $K \equiv 1$ obtained by Gui, Ni and Wang in [GNW] to our more general K assumed in Theorem C. The techniques are first developed by Li in [L].

Let u be a solution of (1.4). By Theorem A we know that $\lim_{r \rightarrow \infty} r^m u(r) = u_\infty \equiv \frac{L}{k_\infty^{\frac{1}{p-1}}}$. Let $w(t) = r^m u(r) - u_\infty$, $r = e^t$ then w satisfies

$$w'' + b_0 w' - L^{p-1}(u_\infty + w) + k(t)(u_\infty + w)^p = 0. \quad (3.1)$$

where $k(t)$ is given by lemma 2.2, b_0 , L are given by (1.5). Let $g(\tau) = (u_\infty + \tau)^p - u_\infty^p - pu_\infty^{p-1}\tau$, then g has expansion

$$g(w) = d_2 w^2 + \cdots + d_M w^M + O(w^{M+1}) \quad (3.2)$$

at $w = 0$ for any positive integer $M \geq 2$. Where d_i , $i = 2, \dots, M$, depend only on p, n , and l , and $d_2 = \frac{p(p-1)}{2} u_\infty^{p-2} > 0$. Let $\varphi(t) = u_\infty^p(k(t) - k_\infty)$, $d_1 = pu_\infty^{p-1}$, $\tilde{g}(w) = d_1 w + g(w)$. Denote

$$G(t, w) = \varphi(t) + (k(t) - k_\infty)\tilde{g}(w) + k_\infty g(w),$$

then (3.1) becomes

$$w'' + b_0 w' + c_0 w + G(t, w) = 0 \quad (3.3)$$

Since $p > p_c$, the characteristic equation of (3.3) has two negative roots

$$-\lambda_2 < -\lambda_1 < 0.$$

Furthermore, we assume that there exists some positive constant $\gamma > 0$ such that

$$\varphi(t) = u_\infty^p(k(t) - k_\infty) = O(e^{-\gamma t}) \quad \text{at } t = \infty. \quad (3.4)$$

Recall that $w'(t) = mr^m u(r) + r^{m+1} u'(r)$, $r = e^t$, and that

$$r^{m+1} u'(r) = -\frac{1}{r^{n-m-2}} \int_0^r s^{n-1} K(s) u^p(s) ds.$$

By (K.1) and Theorem A(i) we have that

$$r^{m+1}u'(r) = O(1),$$

as $r \rightarrow \infty$. Thus $w'(t)$ is bounded at $t = \infty$.

Let $G_1(t) = \int_0^t g(s)ds$. Multiplying (3.3) by w' and integrating from t to $T > t$,

$$\begin{aligned} w'^2(T) + 2b_0 \int_t^T w'^2 ds + c_0 w^2(T) + 2 \int_t^T w' \varphi(t) \left[1 + \frac{1}{u_\infty^p} \tilde{g}(w) \right] ds \\ + 2k_\infty G_1(w(T)) = w'^2 + c_0 w^2 + 2k_\infty G_1(w). \end{aligned}$$

Let T (a sequence if necessary) go to ∞ , we have

$$\int_t^\infty w'^2 ds \leq C(w'^2 + w^2 + G_1(w) + \int_t^\infty |\varphi| ds) \quad (3.5)$$

for some constant C . Hence $w'^2 \in L^1(T, \infty)$.

Multiplying (3.3) by w and integrating from t to $T > t$,

$$\begin{aligned} ww'(T) + \frac{b_0}{2} w^2 + c_0 \int_t^T w^2 ds + \int_t^T wG(s, w) ds \\ = ww'(t) + \int_t^T w'^2 ds + \frac{b_0}{2} w^2. \end{aligned}$$

Since $w \rightarrow 0$ as $t \rightarrow \infty$, there exists $T_0 > 0$ that $k_\infty w g(w) < \frac{c_0}{2} w^2$ when $t > T_0$. Letting T (again, taking a sequence if necessary) go to ∞ , we have that

$$\int_t^\infty w^2 ds \leq C(w^2 + \int_t^\infty w'^2 ds + \int_t^\infty |\varphi| ds) \quad (3.6)$$

for some constant C . Combining (3.5) and (3.6) we conclude that for any positive integer i ,

$$\int_{t_i}^\infty \dots \int_{t_1}^\infty w'(s)^2 ds \dots dt_{i-1} < \infty.$$

This is equivalent to

$$t^i w'^2 \in L^1(T, \infty). \quad (3.7)$$

A direct consequence of (3.7) is that $w' \in L^1(T, \infty)$ by letting $i = 2$ and using Hölder inequality.

We will deal with the cases of $\gamma \leq \lambda_1$ and $\gamma > \lambda_2$ separately.

Case 1° $0 < \gamma \leq \lambda_1$.

Let $R(t) = e^{(\gamma-\varepsilon)t}w(t)$, $\varepsilon \in (0, \gamma)$. Then $R(t)$ is a solution of the following equation

$$R'' + (b_0 - 2(\gamma - \varepsilon))R' + b(\gamma, \varepsilon)R + e^{(\gamma-\varepsilon)t}G(t, w) = 0.$$

Where $b(\gamma, \varepsilon) \equiv \gamma^2 - b_0\gamma + c_0 + \varepsilon(b_0 + \varepsilon - 2\gamma) > 0$. Multiplying above equation by $2R'$ and integrating over (T, t)

$$\begin{aligned} R'^2(t) + 2(b_0 - 2(\gamma - \varepsilon)) \int_T^t R'^2 ds + b(\gamma, \varepsilon)R^2(t) + \\ 2 \int_T^t e^{(\gamma-\varepsilon)s} R'(s)G(s, w(s)) ds = R'^2(T) + b(\gamma, \varepsilon)R^2(T). \end{aligned} \quad (3.8)$$

Since $w'(t)$ is bounded at $t = \infty$, if choose $\varepsilon > \frac{\gamma}{2}$, then $e^{(\gamma-\varepsilon)t}R'(t)[\varphi(t) + (k(t) - k_\infty)\tilde{g}(w)] \in L^1(T, \infty)$. Hence, integrating by parts, from (3.8) we conclude

$$b(\gamma, \varepsilon)R^2(t) + \frac{g(w)}{w}R^2(t) - \int_T^t \frac{d}{ds} \left(\frac{g(w(s))}{w(s)} \right) R^2(s) ds \leq C(T)$$

for some constant $C(T)$.

We claim for large T , we have

$$R^2(t) \leq \frac{2}{b(\gamma, \varepsilon)}C(T)$$

holding uniformly for all $t > T$.

In fact, since $w' \in L^1(0, \infty)$, there exists T_0 such that if $t > T_0$, then $\frac{g(w)}{w} > 0$ (since $d_2 > 0$) and

$$\int_T^\infty \left| \frac{d}{ds} \left(\frac{g(w(s))}{w(s)} \right) \right| ds < \frac{b(\gamma, \varepsilon)}{2}$$

for $T > T_0$.

On the contrary, if there exists $t_0 > T > T_0$ such that $R^2(t) < R^2(t_0) = \frac{2}{b(\gamma, \varepsilon)}C(T)$ for $T_0 < t < t_0$, then we get

$$\left(b(\gamma, \varepsilon) + \frac{g(w)}{w} - \int_T^\infty \left| \frac{d}{ds} \left(\frac{g(w(s))}{w(s)} \right) \right| ds \right) R^2(t_0) \leq C(T).$$

From this we derive

$$R^2(t_0) < \frac{2}{b(\gamma, \varepsilon)}C(T),$$

which is a contradiction. It follows that R is bounded at $t = \infty$, and from (3.8) we get

$$|w| + |w'| = O(e^{-(\gamma-\varepsilon)t}). \quad (3.9)$$

Since w is a solution of (3.3), we can write w as follows (see [H])

$$w(t) = ae^{-\lambda_1 t} + be^{-\lambda_2 t} + \frac{1}{\lambda_2 - \lambda_1} \int_T^t (e^{\lambda_2(s-t)} - e^{\lambda_1(s-t)}) G(s, w(s)) ds \quad (3.10)$$

for some constants a and b . By the definition of G , we get

$$w(t) = ae^{-\lambda_1 t} + be^{-\lambda_2 t} + \psi_1(t, T) + \frac{1}{\lambda_2 - \lambda_1} \int_T^t (e^{\lambda_2(s-t)} - e^{\lambda_1(s-t)}) [(k(s) - k_\infty)\tilde{g}(w) + k_\infty g(w)] ds, \quad (3.11)$$

where

$$\begin{aligned} \psi_1(t, T) &= \frac{1}{\lambda_2 - \lambda_1} \int_T^t (e^{\lambda_2(s-t)} - e^{\lambda_1(s-t)}) \varphi(s) ds \\ &= \begin{cases} O(e^{-\gamma t}) & \text{if } \gamma < \lambda_1, \\ O(te^{-\gamma t}) & \text{if } \gamma = \lambda_1. \end{cases} \end{aligned}$$

Bringing (3.9) into (3.11) we get

$$w(t) = O(e^{-2(\gamma-\varepsilon)t}).$$

Now, we take $\varepsilon \in (\frac{\gamma}{2}, \frac{3\gamma}{4})$, and let $\theta = 3\gamma - 4\varepsilon$, then $E_1(t, w) \equiv G(t, w) - \varphi(t) = (k(t) - k_\infty)\tilde{g}(w) + k_\infty g(w) = O(e^{-(\gamma+\theta)t})$. Without loss of generality, we assume that $\theta \notin \text{span}\{\gamma, \lambda_1, \lambda_2\}$ over \mathbb{Z} . Thus, from (3.11) we get

$$w(t) = \begin{cases} \psi_1(t, T) + O(e^{-(\gamma+\theta)t}) & \text{if } \lambda_1 > \gamma + \theta, \\ \psi_1(t, T) + a_1 e^{-\lambda_1 t} + O(e^{-(\gamma+\theta)t}) & \text{if } \lambda_1 < \gamma + \theta < \lambda_2, \\ \psi_1(t, T) + a_1 e^{-\lambda_1 t} + b_1 e^{-\lambda_2 t} + O(e^{-(\gamma+\theta)t}) & \text{if } \lambda_2 < \gamma + \theta. \end{cases} \quad (3.12)$$

It is worthy of noting that while dealing with the calculations above, we break up the integrals $\int_T^t e^{\lambda_i s} E_1(s, w) ds$ into two parts $\int_T^\infty - \int_t^\infty$, once $\lambda_i < \theta + \gamma$, and

$$a_1 = a - \frac{1}{\lambda_2 - \lambda_1} \int_T^\infty e^{\lambda_1 s} E_1(s, w) ds$$

and

$$b_1 = b + \frac{1}{\lambda_2 - \lambda_1} \int_T^\infty e^{\lambda_2 s} E_1(s, w) ds.$$

Before giving the general expansion form of w at $t = \infty$, we carry our calculations one more to make the process more clear. For example, we deal with the case $\lambda_1 > \gamma + \theta$. Define $E_2(t, w) \equiv E_1(t, w) - (d_1(k(t) - k_\infty)\psi_1(t, T) + d_2 k_\infty \psi_1^2) = O(e^{-(2\gamma+\theta)t})$. Bring (3.12) into (3.11) we have

$$w(t) = ae^{-\lambda_1 t} + be^{-\lambda_2 t} + \psi_1(t, T) + \psi_2(t, T) + \frac{1}{\lambda_2 - \lambda_1} \int_T^t (e^{\lambda_2(s-t)} - e^{\lambda_1(s-t)}) E_2(s, w) ds$$

$$= \begin{cases} \psi_1(t, T) + \psi_2(t, T) + O(e^{-(2\gamma+\theta)t}), & \text{if } \lambda_1 > 2\gamma + \theta, \\ \psi_1(t, T) + \psi_2(t, T) + a_1 e^{-\lambda_1 t} + O(e^{-(2\gamma+\theta)t}), & \text{if } \lambda_1 < 2\gamma + \theta < \lambda_2, \\ \psi_1(t, T) + \psi_2(t, T) + a_1 e^{-\lambda_1 t} + b_1 e^{-\lambda_2 t} + O(e^{-(2\gamma+\theta)t}), & \text{if } \lambda_2 < 2\gamma + \theta. \end{cases}$$

Where

$$\psi_2(t, T) = \frac{1}{\lambda_2 - \lambda_1} \int_T^t (e^{\lambda_2(s-t)} - e^{\lambda_1(s-t)}) (d_1(k(s) - k_\infty)\psi_1(s, T) + d_2 k_\infty \psi_1^2) ds,$$

$$a_1 = a - \frac{1}{\lambda_2 - \lambda_1} \int_T^\infty e^{\lambda_1 s} E_2(s, w) ds,$$

and

$$b_1 = b + \frac{1}{\lambda_2 - \lambda_1} \int_T^\infty e^{\lambda_2 s} E_2(s, w) ds.$$

Suppose that, $k_i, i = 1, 2$, are the positive integers, such that, $(k_i - 1)\gamma < \lambda_i \leq k_i\gamma$. For such k_i , we can choose θ by adjusting ε in such way that $(k_i - 1)\gamma < \lambda_i < k_i\gamma + \theta$.

Generally, by calculations similar to the previous, we have the following expansion after the k_2 th iteration

$$w(t) = \psi_1(t, T) + \cdots + \psi_{k_2}(t, T) + \sum_{\substack{j \in I_i \\ i < (k_2\gamma + \theta)/\lambda_1}} a_{ij}(t)e^{-i\lambda_1 t} + b_1 e^{-\lambda_2 t} + O(e^{-(k_2\gamma + \theta)}). \quad (3.13)$$

Where

$$I_i = \{j \in N \mid j\gamma + i\lambda_1 < k_2\gamma + \theta\},$$

and, $a_{10}(t) = a_1$ and b_1 are constants. If $\lambda_1 \neq k_1\gamma$, $\lambda_2 \neq k_2\gamma$, $\psi_i(t, T) = O(e^{-i\gamma t})$, and $a_{ij}(t) = O(e^{-j\gamma t})$ depending only on a_1 and ψ_1 ; if $\lambda_1 = k_1\gamma$, $\lambda_2 < k_2\gamma$, then $\psi_i(t, T) = O(e^{-i\gamma t})$, for $i < k_1$, and $\psi_i(t, T) = O(te^{-i\gamma t})$ for $k_1 \leq i < k_2$, and $a_{ij}(t) = O(e^{-j\gamma t})$ for $j < k_1$, $a_{ij}(t) = O(te^{-j\gamma t})$ for $k_1 \leq j \leq k_2$; if $\lambda_1 = k_1\gamma$, $\lambda_2 = k_2\gamma$, then $\psi_i(t, T) = O(e^{-i\gamma t})$ when $i < k_1$, $\psi_i(t, T) = O(te^{-i\gamma t})$ when $k_1 \leq i < k_2$, $\psi_{k_2}(t, T) = O(t^2 e^{-i\gamma t})$, a_{ij} are the same as in the case $\lambda_1 = k_1\gamma$, $\lambda_2 < k_2\gamma$.

It is easy to see that all the coefficients of the terms before $b_1 e^{-\lambda_2 t}$ are determined once a_1 is fixed. Keeping this procedure and back to our original variable r , without discriminations we use the same notations as in (3.13), u has expansion of the following form

$$u(r) = \frac{1}{r^m} \sum_{i=1}^{k_2} \psi_i(r) + \sum_{\substack{j \in I_i \\ i < (k_2\gamma + \theta)/\lambda_1}} \frac{a_{ij}(r)}{r^{m+i\lambda_1}} + \frac{b_1}{r^{m+\lambda_2}} + \cdots + O(r^{-(n-2+\varepsilon)}) \quad (3.14)$$

at $r = \infty$ for some $\varepsilon > 0$, where $a_{10}(r) \equiv a_1$ and b_1 are constants.

Case 2° $\gamma > \lambda_1$.

For simplicity, we assume that $\gamma > \lambda_2$. Let $R(t) = e^{(\lambda_1 - \varepsilon)t} w(t)$, $\varepsilon \in (0, \lambda)$. Then $R(t)$ is a solution of the following equation

$$R'' + (b_0 - 2(\lambda_1 - \varepsilon))R' + \varepsilon(b_0 + \varepsilon - 2\lambda_1)R + e^{(\lambda_1 - \varepsilon)t} G(t, w) = 0.$$

Where $G(t, w)$ is defined in (3.3). Similar to *Case 1°*, we have that

$$|w| + |w'| = O(e^{-(\lambda_1 - \varepsilon)t}).$$

Again, using formular (3.10) we have that

$$\begin{aligned}
w(t) &= ae^{-\lambda_1 t} + be^{-\lambda_2 t} + \psi_0(t, T) \\
&\quad \frac{1}{\lambda_2 - \lambda_1} \int_T^t (e^{\lambda_2(s-t)} - e^{\lambda_1(s-t)}) [(k(s) - k_\infty)\tilde{g}(w) + k_\infty g(w)] ds,
\end{aligned} \tag{3.15}$$

where

$$\begin{aligned}
\psi_0(t, T) &= \frac{1}{\lambda_2 - \lambda_1} \int_T^t (e^{\lambda_2(s-t)} - e^{\lambda_1(s-t)}) \varphi(s) ds \\
&= \frac{1}{\lambda_2 - \lambda_1} \left(\int_T^\infty - \int_t^\infty \right) \\
&= \tilde{a}_1 e^{-\lambda_1 t} + \tilde{b}_1 e^{-\lambda_2 t} - \frac{1}{\lambda_2 - \lambda_1} \int_t^\infty (e^{\lambda_2(s-t)} - e^{\lambda_1(s-t)}) \varphi(s) ds \\
&\equiv \tilde{a}_1 e^{-\lambda_1 t} + \tilde{b}_1 e^{-\lambda_2 t} + \psi_1(t),
\end{aligned}$$

and $\psi_1(t) = O(e^{-\gamma t})$ at $t = \infty$. Let $\theta = 3\lambda_1 - 4\epsilon$. Again, we assume θ can not be expressed as a linear combination of γ , λ_1 and λ_2 over \mathbb{Z} . Then

$$w(t) = \begin{cases} a_1 e^{-\lambda_1 t} + O(e^{-(\lambda_1 + \theta)t}), & \text{if } \lambda_2 > \lambda_1 + \theta \\ a_1 e^{-\lambda_1 t} + b_1 e^{-\lambda_2 t} + O(e^{-(\lambda_1 + \theta)t}), & \text{if } \lambda_2 < \lambda_1 + \theta < \gamma \\ a_1 e^{-\lambda_1 t} + b_1 e^{-\lambda_2 t} + \psi_1(t, T) + O(e^{-(\lambda_1 + \theta)t}), & \text{if } \gamma < \lambda_1 + \theta. \end{cases} \tag{3.16}$$

where $a_1 = a + \tilde{a}_1$, $b_1 = b + \tilde{b}_1$ and

$$\tilde{a}_1 = \frac{-1}{\lambda_2 - \lambda_1} \int_T^\infty e^{\lambda_1 s} \varphi(s) ds, \quad \tilde{b}_1 = \frac{1}{\lambda_2 - \lambda_1} \int_T^\infty e^{\lambda_2 s} \varphi(s) ds.$$

To make our expansion more clear, we repeat the iteration one more time. Consider the case $\lambda_1 + \theta < \lambda_2 < 2\lambda_1 + \theta$. Putting (3.16) into (3.15), we obtain

$$w(t) = \begin{cases} a_1 e^{-\lambda_1 t} + a_2 e^{-2\lambda_1 t} + b_1 e^{-\lambda_2 t} + O(e^{-(2\lambda_1 + \theta)t}) & \text{if } \lambda_2 \neq 2\lambda_1 \\ a_1 e^{-\lambda_1 t} + a_2 e^{-2\lambda_1 t} + c_1 t e^{-2\lambda_1 t} + b_1 e^{-\lambda_2 t} + O(e^{-(2\lambda_1 + \theta)t}) & \text{if } \lambda_2 = 2\lambda_1. \end{cases} \tag{3.17}$$

Keep doing in this way and back to the old variable r , like (3.14) we obtain

$$u(r) = \begin{cases} \frac{L}{k_\infty^{\frac{1}{p-1}} r^m} + \frac{a_1}{r^{m+\lambda_1}} + \frac{a_2}{r^{m+2\lambda_1}} + \dots + \frac{b_1}{r^{m+\lambda_2}} + \dots + O\left(\frac{1}{r^{n-2+\varepsilon}}\right), \\ \hspace{15em} \text{if } \lambda_2 \neq \Lambda\lambda_1 \\ \\ \frac{L}{k_\infty^{\frac{1}{p-1}} r^m} + \frac{a_1}{r^{m+\lambda_1}} + \frac{a_2}{r^{m+2\lambda_1}} + \dots + \frac{c_1 \log r}{r^{m+\Lambda\lambda_1}} + \frac{b_1}{r^{m+\lambda_2}} \\ \hspace{15em} + \dots + O\left(\frac{1}{r^{n-2+\varepsilon}}\right), \hspace{2em} \text{if } \lambda_2 = \Lambda\lambda_1 \end{cases} \quad (3.18)$$

for some positive integer $\Lambda > 1$. From the above calculations and discussions it follows

Theorem 3.1 . Suppose (K.1), (K.4) and $p > p_c$, and there exists $\gamma > 0$ such that

$$r^{-l}K(r) - k_\infty = O\left(\frac{1}{r^\gamma}\right) \text{ at } r = \infty.$$

Let u be a solution of (1.4) satisfying $\lim_{r \rightarrow \infty} r^m u(r) = \frac{L}{k_\infty^{\frac{1}{p-1}}}$. Then u has an expansion at $r = \infty$, which, in particular, is (3.14) if $\gamma \leq \lambda_1$, or (3.18) if $\gamma > \lambda_2$.

Remark 3.1 . For the case that $\gamma \in (\lambda_1, \lambda_2)$, by a similar argument the solution u has an expansion which consisting some mixed terms between $a_1 r^{-(m+\lambda_1)}$ and $b_1 r^{-(m+\lambda_2)}$, which are generated by $a_1 r^{-(m+\lambda_1)}$ and $\varphi(\log(r))$. For a given solution u , $a_1 r^{-(m+\lambda_1)}$ and $b_1 r^{-(m+\lambda_2)}$ are the two independent terms in its expansion at infinity. This fact will be made more clear in the following section.

4 Stability and Asymptotic Stability

This section is devoted to the stability and asymptotic stability of the Cauchy problem (1.2). For $K \equiv 1$, it was showed by Fujita in [F] that the solution $u(x, t; \varphi)$ of (1.2) blows up in finite time for $1 < p < \frac{n+2}{n}$, and for $p > \frac{n+2}{n}$, $u(x, t; \varphi)$ exists globally in time for sufficient small φ . Thus, the trivial steady state $u_0 \equiv 0$ is unstable in any proper topology for $1 < p < \frac{n+2}{n}$. For large p , say, $p > \frac{n+2}{n}$, we are caring about the domain of attraction for $u_0 \equiv 0$.

The condition given by Fujita on φ is that it is bounded by $\varepsilon e^{-|x|^2}$ for some small ε ; Weissler (see [We]) improved this condition to that φ has polynomial decay at $|x| = \infty$; the exact decay power $|x|^{-\frac{2}{p-1}}$ is given by Lee and Ni in [LeN], and also by Wang in [W].

The stability of positive steady state u_α of (1.2), which is a solution of (1.4), is studied by Gui, Ni and Wang in [GNW] for the case $K \equiv 1$ under the norm (1.7).

It is showed in [GNW] that the positive steady states of (1.2) are stable with respect to norm $\|\cdot\|_{m+\lambda_1}$, and asymptotic stable with respect to norm $\|\cdot\|_{m+\lambda_2}$. The main purpose of this section is to extend the results of Gui, Ni and Wang to general $K(x)$ which satisfies (K.1), (K.1') and (K.4).

The key ingredient in the proof of Theorem 2 is a comparason principle originating with Gui, see [G] and [GNW].

Definition 4.1 . A function v is said to be a super-solution of equation

$$\Delta u + f(x, u) = 0 \tag{4.1}$$

in an open set $\Omega \subset \mathbb{R}^n$ if $\Delta v + f(x, v) \leq 0$ in Ω ; and v is said to be a sub-solution if $\Delta v + f(x, v) \geq 0$ in Ω .

For $f(x, u) = K(x)u$, $\Omega = B_R$ in \mathbb{R}^n , equation (4.1) becomes

$$\Delta u + K(x)u = 0 \tag{4.2}$$

and we have Gui's lemma as follows

Lemma 4.1 . Suppose w_1 is a positive radial super-solution of (4.2) in B_R and w_2 is a radial sub-solution of (4.2) in B_R with $w_2(0) > 0$. Then

$$w_2(r) \geq \frac{w_2(0)}{w_1(0)} w_1(r) \tag{4.3}$$

for all $0 \leq r \leq R$. Moreover

$$w_2(R) > \frac{w_2(0)}{w_1(0)} w_1(R) \tag{4.4}$$

if one of the functions is not a solution of (4.2).

See Lemma 2.20 in [GNW] for the proof of Lemma 4.1.

Definition 4.2 . A function u is called a continuous weak super-solution of (1.2) if

- (i) u is continuous on $\Omega_T = \mathbb{R}^n \times [0, T)$ and $u(\cdot, 0) \geq \varphi$;
- (ii) u satisfies

$$\int_{\mathbb{R}^n} u(x, t) \eta(x, t) dx \Big|_0^{T'} \geq \int_0^{T'} \int_{\mathbb{R}^n} [u(x, s)(\Delta \eta + \eta_t)(x, t) + \eta(x, s) f(u(x, s))] dx dt \quad (4.5)$$

for all $T' \in [0, T)$ and $0 \leq \eta(x, t) \in C^{2,1}(\mathbb{R}^n \times (0, T))$ with $\text{supp}(\eta(\cdot, t))$ being compact in \mathbb{R}^n for $t \in [0, T]$. Similarly, a continuous weak sub-solution is defined by reversing the inequalities in (i) and (4.5).

Proposition 4.1 . Suppose K satisfies (K.1) and (K.4) in (R, ∞) for some large R . Then

- (i) if \bar{u} and \underline{u} are bounded continuous weak super-sub-solutions of (1.2) respectively, then $\bar{u} \geq \underline{u}$ on $\mathbb{R}^n \times (0, T)$, and (1.2) has a unique solution u satisfying $\bar{u} \geq u(x, t; \varphi) \geq \underline{u}$ and $u \in C^{2,1}(\mathbb{R}^n \setminus \{0\}) \times (0, T)$ if $-2 < l < 0$, $u \in C^{2,1}(\mathbb{R}^n \times (0, T))$ if $l \geq 0$;
- (ii) if the initial value φ in (1.2) is a bounded continuous super(sub)-solution of the elliptic equation (1.1) in \mathbb{R}^n , then the solution $u(x, t; \varphi)$ of (1.2) is strictly decreasing (increasing) in t as long as it exists provided φ is not a steady state of (1.2).
- (iii) if φ is radially symmetric, so is $u(x, t; \varphi)$ in x -variable.

All these results can be found in [W]. Part (i) is the consequence of Lemma 1.2 if $l \geq 0$, Theorem 2.4(i) if $-2 < l < 0$; part (ii) can be proved the same argument in Theorem 2.4(ii) if $-2 < l < 0$, or Lemma 2.6(ii) and the strong maximum principle if $l \geq 0$; part (iii) can be proved by Theorem 2.3 if $-2 < l < 0$, Lemma 2.6 if $l \geq 0$.

By the same argument in [W], we can show the Proposition 2.28 in [W] holds for equation (1.2), thus we have

Proposition 4.2 . Suppose that $p_c > p > \frac{n+2+2l}{n-2}$. Then

(i) if $\varphi \leq \psi$ in \mathbb{R}^n , where ψ is a radial continuous super-solution but not a solution of (1.1), then the solution of (1.2) exists globally in time with $u \leq \psi$ and $\|u(\cdot, t; \varphi)\|_{L^\infty(\mathbb{R}^n)} \rightarrow 0$ as $t \rightarrow \infty$.

(ii) if $\varphi \geq \psi$ in \mathbb{R}^n , where ψ is a radial continuous sub-solution but not a solution of (1.1), then the solution $u(x, t; \varphi)$ of (1.2) blows up in finite time.

For the purpose of construction of super-sub-solutions, which will be used in the proof of Theorem 2, we need the following existence result

Theorem 4.1 . Suppose that $m < \frac{n-2}{2}$ (i.e. $p > \frac{n+2+2l}{n-2}$) and H is a radial smooth function which satisfies

$$K + H > 0, \quad \left(\frac{n}{p+1} - \frac{n-2}{2} \right) (K + H) + \frac{r}{p+1} (K + H)' \leq 0 \quad (4.6)$$

in \mathbb{R}^n . Then for each $\beta > 0$ the Cauchy problem

$$\begin{cases} v'' + \frac{n-1}{r}v' + (K + H)v^p = 0 \\ v(0) = \beta \end{cases} \quad (4.7)$$

always has a positive solution v_β in $[0, \infty)$.

The proof of Theorem 4.1 rests in the following Pohozaev type identity.

Consider the initial problem

$$v'' + \frac{n-1}{r}v' + f(r, v^+) = 0, \quad v(0) = \beta > 0. \quad (4.8)$$

Where $f(r, v) \in C((0, \infty) \times [0, \infty))$, for every $r \in (0, \infty)$, $f(r, v)$ is locally Lipschitz continuous in $v \in (0, \infty)$, and for every $M, R > 0$, $r \sup_{0 \leq v \leq M} f(r, v) \in L^1(0, R)$ and

$$r \sup \left\{ \frac{|f(r, v_2) - f(r, v_1)|}{|v_2 - v_1|} : 0 \leq v_1 < v_2 \leq M \right\} \in L^1(0, R); \quad f(r, v) \geq 0 \text{ in } (0, \infty) \times [0, \infty).$$

Then the following results hold.

Proposition 4.3 . (i) There exists a unique solution of (4.8), which is continuous and nonincreasing in $[0, \infty)$.

(ii) Let v be the unique solution of (4.8). Then we have

$$\begin{aligned} & \frac{n-2}{2}R^{n-1}v'(R)v(R) + \frac{1}{2}R^n v'^2(R) + R^n F(R, v(R)) \\ &= \int_0^R \left\{ nF(r, v(r)) - \frac{n-2}{2}vf(r, v^+) + rF_r(r, v(r)) \right\} r^{n-1} dr \end{aligned} \quad (4.9)$$

for any $R > 0$, where $F(r, u) = \int_0^u f(r, z^+) dz$.

Proof: The proof of part (i) comes from the Proposition 4.2 in [NY], and the proof of part (ii) is the result of Proposition 4.3 in [NY].

Proof of Theorem 4.1: Specially let $f(r, v) = (K + H)v^p$. By part (i) of Proposition 4.3, (4.7) has a unique solution v , with $v(0) = \beta > 0$. On the contrary, if there exists $R > 0$, such that $v(r) > 0$ for $r < R$, and $v(R) = 0$. By part (ii) of Proposition 4.3 we have

$$\frac{1}{2}R^2 v'(R)^2 = \int_0^R \left\{ \left(\frac{n}{p+1} - \frac{n-2}{2} \right) (K + H) + \frac{r}{p+1} (K + H)' \right\} v^{p+1} r^{n-1} dr.$$

Since $v'(r) < 0$, by Hopf's boundary lemma we derive a contradiction. Thus v_β , the solution of (4.7), is entire positive in $[0, \infty)$.

Q.E.D.

In applications, we usually choose $H(r) \equiv h(r)r^l$, where $h(\cdot)$ is a bounded continuous function having support set in a small ball cetered at origin, furthermore, we may assume that $|h(\cdot)|$ decreases in r . For example, let $\tilde{k}(r) \equiv K(r)r^{-l}$, then (4.6) is equivalent to

$$\frac{n+l}{p+1} - \frac{n-2}{2} + \frac{r}{p+1} \frac{\tilde{k}' + h'}{\tilde{k} + h} \leq 0. \quad (4.10)$$

Since our $p > \frac{n+2+2l}{n-2}$, (4.10) is a very relaxed condition on H , which is guarranted if h is "small" and "smooth". By choosing $h \geq (\leq) 0$, we get s super(sub)-solution of (1.4) for $\alpha = \beta$.

Proof of Theorem 1: The proof of Theorem 1 is the same with that of Theorem 1.14 in [GNW] by employing Theorem C(i), Lemma 4.1 and Proposition 4.2. Here we omit the detail.

Q.E.D.

We divide the proof of Theorem 2 into two parts. First we show that the solutions of (1.4) are stable with the norm $\|\cdot\|_{m+\lambda_1}$, then the asymptotic stability with the norm $\|\cdot\|_{m+\lambda_2}$.

Lemma 4.2 . Suppose $p > p_c$. Let u_α and u_β are the solution of (1.4) with initial values $u_\alpha(0) = \alpha$, $u_\beta(0) = \beta$. Then

$$\lim_{\beta \rightarrow \alpha} \|u_\beta - u_\alpha\|_{m+\lambda_1} = 0$$

Proof: Let $w = r^{m+\lambda_1}(u_\beta - u_\alpha)$. Then w satisfies

$$\begin{aligned} w'' + \frac{n-1-2(m+\lambda_1)}{r}w' &= r^{m+\lambda_1}\Delta(u_\beta - u_\alpha) + (m+\lambda_1)(n-2-m-\lambda_1)r^{m+\lambda_1-2}(u_\beta - u_\alpha) \\ &= \left(-K(r)\frac{u_\beta^p - u_\alpha^p}{u_\beta - u_\alpha}r^2 + L^{p-1} + b_0\lambda_1 - \lambda_1^2\right)r^{m+\lambda_1-2}(u_\beta - u_\alpha) \\ &= \left(pL^{p-1} - K(r)\frac{u_\beta^p - u_\alpha^p}{u_\beta - u_\alpha}r^2\right)r^{m+\lambda_1-2}(u_\beta - u_\alpha). \end{aligned}$$

Denote by ψ the right hand side. Choosing $|\beta - \alpha| < \frac{\alpha}{2}$, then by Theorem C and Theorem 3.1, $|u_\beta - u_\alpha| < |u_{\frac{3\alpha}{2}} - u_{\frac{\alpha}{2}}| = O(r^{-(m+\lambda_1)})$ at $r = \infty$. To estimate the first factor of the right hand side, one recalls Lemma 2.2 that

$$pL^{p-1} - K(r)\frac{u_\beta^p - u_\alpha^p}{u_\beta - u_\alpha}r^2 > pL^{p-1} - pK(r)u_{\frac{3\alpha}{2}}^{p-1}r^2 > 0.$$

On the other hand, by (K.4), Lemma 2.2 and the expansion results in section 3 (Theorem 3.1), we have

$$\begin{aligned} pL^{p-1} - K(r)\frac{u_\beta^p - u_\alpha^p}{u_\beta - u_\alpha}r^2 &< pL^{p-1} - pK(r)u_{\frac{\alpha}{2}}^{p-1}r^2 \\ &\leq pk_\infty(u_\infty^{p-1} - (r^m u_{\frac{\alpha}{2}})^{p-1}) \\ &\leq p(p-1)k_\infty u_\infty^{p-2} |u_\infty - r^m u_{\frac{\alpha}{2}}| \\ &= O(r^{-\min(\lambda_1, \gamma)}) \end{aligned}$$

where γ is defined by (3.4) in section 3. Hence $\psi = O(r^{-2-\min(\lambda_1, \gamma)})$ at $r = \infty$, and

$$w'' + \frac{n-1-2(m+\lambda_1)}{r}w' - \psi = 0.$$

Recall the fact that $n-2-2(m+\lambda_1) = b_0 - 2\lambda_1 > 0$. Multiplying above equation by $r^{n-2-2(m+\lambda_1)}$ and integrating over $(0, r)$, we have

$$w'(r) = \int_0^r \left(\frac{s}{r}\right)^{n-2-2(m+\lambda_1)} \psi(s) ds.$$

Integrating again over $(0, r)$ and exchanging integrals order, for any $\varepsilon > 0$

$$\begin{aligned} w(r) &= \frac{1}{n-2-2(m+\lambda_1)} \int_0^r \left(1 - \left(\frac{s}{r}\right)^{n-2-2(m+\lambda_1)}\right) s\psi(s) ds \\ &= \frac{1}{n-2-2(m+\lambda_1)} \left(\int_0^{R_\varepsilon} + \int_{R_\varepsilon}^r \right) \quad \text{for } 0 < R_\varepsilon < r. \end{aligned}$$

Since $r\psi(\cdot) \in L^1(0, \infty)$, there exists R_ε , which is independent of β when $\beta > \frac{\alpha}{2}$, such that

$$\begin{aligned} \left| \int_{R_\varepsilon}^r s\psi(s) ds \right| &\leq \int_{R_\varepsilon}^\infty |s\psi(s)| ds \\ &< (n-2-2(m+\lambda_1)) \frac{\varepsilon}{2}. \end{aligned}$$

For such fixed R_ε , choose β close α enough that we have $\int_0^{R_\varepsilon} s\psi(s) ds < \frac{n-2-2(m+\lambda_1)}{2} \varepsilon$. Therefore

$$w(r) < \varepsilon.$$

Which completes the proof.

Q.E.D.

Now let u_α be the solution of (1.4), $a_{1,\alpha}$ denote the coefficient of the term $r^{-(m+\lambda_1)}$ in the expansion of u_α in Theorem 3.1 ($a_{1,\alpha}$ is the a_1 there). Thus Lemma 4.2 shows that $a_{1,\alpha}$ is continuous in α . More than that, actually $a_{1,\alpha}$ is the very first character of the solutions of (1.4) in the following sense

Lemma 4.3 . Suppose $p > p_c$. Let u_α and u_β are two solutions of (1.4) with $\beta > \alpha > 0$. Then $a_{1,\beta} > a_{1,\alpha}$.

Proof: Since $u_\beta > u_\alpha$, by Theorem C and Theorem 3.1 we have

$$u_\beta - u_\alpha = \frac{a_{1,\beta} - a_{1,\alpha}}{r^{m+\lambda_1}} + O\left(\frac{1}{r^{m+\lambda_1+\varepsilon'}}\right)$$

at $r = \infty$ for some $\varepsilon' > 0$. Whence we know $a_{1,\beta} \geq a_{1,\alpha}$. Suppose that $a_{1,\beta} = a_{1,\alpha}$. Let $0 < \varepsilon < \varepsilon'$, $w = r^{m+\lambda_1+\varepsilon}(u_\beta - u_\alpha)$, then $w = 0$ at both $r = 0$ and ∞ , and w satisfies

$$\Delta w - \frac{2(m + \lambda_1 + \varepsilon)}{r} w' + \left[(\lambda_1 + \varepsilon)^2 - b_0(\lambda_1 + \varepsilon) - L^{p-1} + K(r) \frac{u_\beta^p - u_\alpha^p}{u_\beta - u_\alpha} r^2 \right] r^{-2} w = 0$$

By Lemma 2.2, $K(r) \frac{u_\beta^p - u_\alpha^p}{u_\beta - u_\alpha} r^2 < pL^{p-1}$. If we choose $0 < \varepsilon < \min(\varepsilon', \lambda_2 - \lambda_1)$, then the coefficient of w is negative; thus by maximum principle it follows $w(r) \leq 0$ in $r \in [0, \infty)$, which is a contradiction.

Q.E.D.

Now we are ready to show that the steady state u_α is stable with respect to norm $\|\cdot\|_{m+\lambda_1}$.

Proof of Theorem 2(i):

The proof is similar to that of the first part of Theorem 1.15 in [GNW]. Here we give the sketch.

For any given $\varepsilon > 0$, there exists $\eta \in (0, \frac{\alpha}{2})$ such that $\|u_{\alpha \pm \eta} - u_\alpha\|_{m+\lambda_1} < \varepsilon$ by Lemma 4.2. For such ε , by Theorem C and Lemma 4.3, there exists R_ε such that

$$r^{m+\lambda_1}(u_{\alpha+\eta} - u_\alpha) > \frac{a_{1,\alpha+\eta} - a_{1,\alpha}}{2} > 0$$

and

$$r^{m+\lambda_1}(u_\alpha - u_{\alpha-\eta}) > \frac{a_{1,\alpha} - a_{1,\alpha-\eta}}{2} > 0$$

in $[R_\varepsilon, \infty)$. Choosing $\delta_1 = \frac{1}{2} \min(a_{1,\alpha+\eta} - a_{1,\alpha}, a_{1,\alpha} - a_{1,\alpha-\eta})$, then if $\|\varphi - u_\alpha\|_{m+\lambda_1} < \delta_1$, we have

$$\begin{aligned} r^{m+\lambda_1}(u_{\alpha+\eta} - \varphi) &\geq r^{m+\lambda_1}(u_{\alpha+\eta} - u_\alpha) - r^{m+\lambda_1} |\varphi - u_\alpha| \\ &> \frac{a_{1,\alpha+\eta} - a_{1,\alpha}}{2} - \delta_1 \\ &\geq 0 \end{aligned}$$

and

$$\begin{aligned}
r^{m+\lambda_1}(\varphi - u_{\alpha-\eta}) &\geq r^{m+\lambda_1}(u_\alpha - u_{\alpha-\eta}) - r^{m+\lambda_1} |\varphi - u_\alpha| \\
&> \frac{a_{1,\alpha} - a_{1,\alpha-\eta}}{2} - \delta_1 \\
&\geq 0
\end{aligned}$$

in $[R_\varepsilon, \infty)$. On the other hand, since $u_{\alpha+\eta} > u_\alpha > u_{\alpha-\eta}$ on $[0, R_\varepsilon]$, therefore there exists $\delta_2 > 0$ such that $u_{\alpha+\eta} > \varphi > u_{\alpha-\eta}$ if $\|\varphi - u_\alpha\|_{m+\lambda_1} < \delta_2$. Let $\delta = \min(\delta_1, \delta_2)$, then $u_{\alpha+\eta} > \varphi > u_{\alpha-\eta}$ provided $\|\varphi - u_\alpha\|_{m+\lambda_1} < \delta$. From Proposition 4.1(i) the solution $u(\cdot, t; \varphi)$ of (1.2) satisfies $u_{\alpha+\eta} > u(\cdot, t; \varphi) > u_{\alpha-\eta}$, thus $\|u(\cdot, t; \varphi) - u_\alpha\|_{m+\lambda_1} < \varepsilon$.

Q.E.D.

Suppose H is a function which is small and nonnegative, and $\pm H$ satisfies condition (4.6) in Theorem 4.1, then the following equation(s)

$$\begin{cases} u'' + \frac{n-1}{r}u' + (K \pm H)u^p = 0 \\ u(0) = \beta \end{cases} \quad (4.11)$$

have a unique positive solution, denoted by \bar{u}_β and \underline{u}_β respectively.

Lemma 4.4 . $\bar{u}_\beta \leq u_\beta$.

Proof: To show $u_\beta \geq \bar{u}_\beta$, we only need to show that $u_\delta > \bar{u}_\beta$ for any $\delta > \beta$. Let $w_1 = u_\delta - \bar{u}_\beta$. If the conclusion does not hold, then there exists R such that $w_1 > 0$ in $[0, R)$, $w_1(R) = 0$, and

$$\Delta w_1 + K_1(r)w_1 \geq 0 \quad \text{in } B_R,$$

where $K_1 = K(r)\frac{u_\delta^p - \bar{u}_\beta^p}{u_\delta - \bar{u}_\beta} < pK(r)u_\delta^{p-1}$. On the other hand, letting $w_2 = u_{2\delta} - u_\delta$, then we have

$$\Delta w_2 + K_2(r)w_2 = 0 \quad \text{in } \mathbb{R}^n$$

with $K_2 = K(r)\frac{u_{2\delta}^p - u_\delta^p}{u_{2\delta} - u_\delta} > pK(r)u_\delta^{p-1}$. By Lemma 4.1 it follows $w_1(R) > 0$, which is a contradiction.

Q.E.D.

We have a comparison lemma of the following version.

Lemma 4.5 . Suppose H and H_1 are nonnegative and satisfy (4.6), and $H \geq H_1$. Let \bar{v}_β denote the solution of (4.11) by replacing H by H_1 . Then for small H we have $\bar{v}_\beta \geq \bar{u}_\beta$.

Proof: It is sufficient to show that $\bar{v}_\delta \geq \bar{u}_\beta$ for any $\delta > \beta$. Let $w_1 = \bar{v}_\delta - \bar{u}_\beta$. If there exists $R > 0$ such that $w_1(r) > 0$ for $r \in [0, R)$, and $w_1(R) = 0$. Then

$$\Delta w_1 + K_1(r)w_1 \geq 0 \quad \text{in } B_R,$$

where $K_1 = (K + H) \frac{\bar{v}_\delta^p - \bar{u}_\beta^p}{\bar{v}_\delta - \bar{u}_\beta} < p(K + H)u_\delta^{p-1}$ by Lemma 4.4. Now let $w_2 = u_{3\delta} - u_{2\delta}$, then w_2 satisfies

$$\Delta w_2 + K_2(r)w_2 = 0$$

with $K_2 = K \frac{u_{3\delta}^p - u_{2\delta}^p}{u_{3\delta} - u_{2\delta}} > pKu_{2\delta}^{p-1}$. If we choosing H such that $\text{supp}(H) \subset B_1$, where B_1 is the unit ball, and $H < K[\left(\frac{u_{2\delta}}{u_\delta}\right)^{p-1} - 1]$, then we have $K_1 < K_2$. From Lemma 4.1 it follows $w_1(R) > 0$. The contradiction completes the proof.

Q.E.D.

Let u_α be the solution of (1.4). Now we are ready to construct a pair of super- and sub-solutions by which u_α is bounded.

Lemma 4.6 . For any $\beta > \alpha > \gamma$, there exists a small nonnegative function $H_1 \not\equiv 0$, such that $\bar{u}_\beta > u_\alpha > \underline{u}_\gamma$.

Proof: We prove the left inequality first. Let H be as in Lemma 4.5 and \bar{v}_β be the solution of (4.11). Let H_1 be a nonnegative function such that $H_1 \leq H$, and \bar{u}_β the solution of (4.11) with H being replaced by H_1 . Then by Lemma 4.5 we know that $\bar{u}_\beta \geq \bar{v}_\beta$. Let $w_1 = \bar{u}_\beta - u_\alpha$. Suppose there exists a $R > 0$ such that $w_1 > 0$ in $[0, R)$ and $w_1(R) = 0$. Denote $R_1 \equiv \sup\{R > 0 \mid \bar{v}_\beta(R) > \frac{\beta+\alpha}{2}\}$. It is easy to see that $R_1 < R$ and $w_1(r) > \frac{\beta+\alpha}{2}$ for $r < R_1$. We first choose H_1 that $\text{supp}(H_1) \subset B_{R_1}$, then w_1 satisfies

$$\Delta w_1 + K_1(r)w_1 \geq 0 \quad \text{in } B_R,$$

where $K_1 = K \frac{\bar{u}_\beta^p - u_\alpha^p}{\bar{u}_\beta - u_\alpha} + H_1 \frac{\bar{u}_\beta^p}{\bar{u}_\beta - u_\alpha} < pKu_\beta^{p-1} + \frac{2H_1}{\beta-\alpha}u_\beta^p$ by Lemma 4.4. Now Let $w_2 = u_{3\beta} - u_{2\beta}$, then w_2 satisfies

$$\Delta w_2 + K_2(r)w_2 = 0$$

with $K_2 = K \frac{u_{3\beta}^p - u_{2\beta}^p}{u_{3\beta} - u_{2\beta}} > pKu_{2\beta}^{p-1}$. Secondly, if we choose $H_1 < \frac{p(\beta-\alpha)}{2}K[(\frac{u_{2\beta}}{u_\beta})^{p-1} - 1]$, then $K_1 < K_2$. Thus by Lemma 4.1 $w_1(R) > 0$ and we get a contradiction.

We prove the right inequality in the similar way. Let $w_3 = u_\alpha - \underline{u}_\gamma$. If there exists $R > 0$ such that $w_3(r) > 0$ in $[0, R)$ and $w_3(R) = 0$. Define $R_2 = \sup\{R > 0 \mid u_\alpha > \frac{\alpha+\gamma}{2}\}$, then $R_2 < R$. Let $\text{supp}(H_1) \subset B_{R_2}$, then w_3 satisfies

$$\Delta w_3 + K_3(r)w_3 \geq 0 \quad \text{in } B_R,$$

where $K_3 = K \frac{u_\alpha^p - \underline{u}_\gamma^p}{u_\alpha - \underline{u}_\gamma} + H_1 \frac{\underline{u}_\gamma^p}{u_\alpha - \underline{u}_\gamma} < pKu_\alpha^{p-1} + \frac{2H_1}{\alpha-\gamma}u_\alpha^p$. Let $w_4 = u_{3\alpha} - u_{2\alpha}$, then w_4 satisfies

$$\Delta w_4 + K_4(r)w_4 = 0$$

with $K_4 = K \frac{u_{3\alpha}^p - u_{2\alpha}^p}{u_{3\alpha} - u_{2\alpha}} > pKu_{2\alpha}^{p-1}$. If we let $H_1 < \frac{p(\alpha-\gamma)}{2}K[(\frac{u_{2\alpha}}{u_\alpha})^{p-1} - 1]$, then $K_3 < K_4$, again by Lemma 4.1 we get $w_3(R) > 0$. It is easy to get conditions on H_1 , under which the Lemma holds. This completes the proof.

Q.E.D.

For fixed H_1 that is given in the proof of Lemma 4.6, we define

$$\beta_0 = \min\{\beta > \alpha \mid \bar{u}_\beta \geq u_\alpha\}, \quad (4.12)$$

$$\gamma_0 = \max\{\gamma < \alpha \mid \underline{u}_\gamma \leq u_\alpha\}. \quad (4.13)$$

Then, \bar{u}_{β_0} and \underline{u}_{γ_0} are a pair of super-sub-solutions of (1.4). From the proof of Lemmas 4.6, it is easy to see that

$$\bar{u}_{\beta_0} > u_\alpha > \underline{u}_{\gamma_0}.$$

Let \bar{a}_{1,β_0} , $a_{1,\alpha}$ and $\underline{a}_{1,\gamma_0}$ denote the coefficients of term $r^{-(m+\lambda_1)}$ respecting to \bar{u}_{β_0} , u_α and \underline{u}_{γ_0} in the expansions of Theorem 3.1, therefore we have $\bar{a}_{1,\beta_0} \geq a_{1,\alpha} \geq \underline{a}_{1,\gamma_0}$. In fact we have the following

Lemma 4.7 . Suppose $p > p_c$. Then for fixed H_1 which is given in the proof of Lemma 4.6, we have

$$\bar{a}_{1,\beta_0} = a_{1,\alpha} = \underline{a}_{1,\gamma_0}.$$

Proof: We only give the proof of the first equality concerning to the super-solution. It can be handled in a similar way for the sub-solution.

Suppose that $\bar{a}_{1,\beta_0} > a_{1,\alpha}$. It follows $\| \bar{u}_{\beta_0} - u_\alpha \|_{m+\lambda_1} \geq \frac{1}{2}(\bar{a}_{1,\beta_0} - a_{1,\alpha}) > 0$ for r large enough. Using a similar argument of Lemma 4.2 one can show that for each fixed $\beta > 0$, $\lim_{\delta \rightarrow 0} \| \bar{u}_{\beta+\delta} - \bar{u}_\beta \|_{m+\lambda_1} = 0$. Thus there exist $\delta_1 > 0$ and $R > 0$ such that $\| \bar{u}_{\beta_0 \pm \delta} - \bar{u}_{\beta_0} \|_{m+\lambda_1} < \frac{1}{4}(\bar{a}_{1,\beta_0} - a_{1,\alpha})$ when $0 < \delta < \delta_1$, and $r^{m+\lambda_1} | \bar{u}_{\beta_0} - u_\alpha | > \frac{1}{2}(\bar{a}_{1,\beta_0} - a_{1,\alpha})$ when $r > R$. Hence, if $r > R$,

$$\begin{aligned} r^{m+\lambda_1}(\bar{u}_{\beta_0-\delta} - u_\alpha) &= r^{m+\lambda_1}(\bar{u}_{\beta_0-\delta} - \bar{u}_{\beta_0}) + r^{m+\lambda_1}(\bar{u}_{\beta_0} - u_\alpha) \\ &> \frac{1}{4}(\bar{a}_{1,\beta_0} - a_{1,\alpha}). \end{aligned}$$

On the other hand, since $\bar{u}_{\beta_0} > u_\alpha$, there exists $\delta_2 > 0$, such that $\bar{u}_{\beta_0 \pm \delta} > u_\alpha$ on $[0, R]$ for $0 < \delta < \delta_2$. Let $\delta = \frac{1}{2} \min(\delta_1, \delta_2)$, then $\bar{u}_{\beta_0-\delta} > u_\alpha$ in $[0, R]$. Which contradicts to our definition of β_0 .

Q.E.D.

From Theorem C, Lemma 4.3 and Lemma 4.7 we have the following consequence

Corollary 4.1 . u_α is the only solution of equation (1.4) between \underline{u}_{γ_0} and \bar{u}_{β_0} .

Proposition 4.4 . Suppose $p > p_c$. Let u_α be the solution of (1.4). Then there exist a sequence of super-solutions $\bar{u}^{(1)} > \bar{u}^{(2)} > \dots > u_\alpha$, and a sequence of sub-solutions $\underline{u}^{(1)} < \underline{u}^{(2)} < \dots < u_\alpha$, such that u_α is the only solution of (1.4) in the ordered interval $\underline{u}^{(\tau)} < u_\alpha < \bar{u}^{(\tau)}$, $\tau = 1, 2, \dots$, and, moreover

$$\lim_{\tau \rightarrow \infty} \bar{u}^{(\tau)} = u_\alpha = \lim_{\tau \rightarrow \infty} \underline{u}^{(\tau)} \quad (4.14)$$

Proof: Let $\bar{u}^{(1)} = \bar{u}_{\beta_0}$. H_1 is given in the proof of Lemma 4.6. By Corollary 4.1, there is no solution between u_α and $\bar{u}^{(1)}$. Consider equation

$$u'' + \frac{n-1}{r}u' + (K + \frac{H_1}{\tau})u^p = 0 \quad \text{in } \mathbb{R}^n, \quad \tau = 1, 2, \dots \quad (4.15)$$

For $\tau = 2$, $\{u_\alpha, \bar{u}^{(1)}\}$ is a pair of sub-super-solutions of (4.15). By a similar argument to Theorem 2.10 in [N1], it follows that (4.15) has a radial solution, denoted by $\bar{u}^{(2)}$, satisfies

$\bar{u}^{(1)} > \bar{u}^{(2)} > u_\alpha$. Suppose we have had $\bar{u}^{(\tau_0)}$ for some τ_0 , then $\{u_\alpha, \bar{u}^{(\tau_0)}\}$ is a pair of sub-super-solutions of (4.15) for $\tau = \tau_0 + 1$. For the same reason we have $\bar{u}^{(\tau_0+1)}$. Continuing this procedure we obtain a sequence of radial super-solutions of (1.4) $\bar{u}^{(1)} > \bar{u}^{(2)} > \dots > u_\alpha$. Similarly we may get a sequence of sub-solutions of (1.4) $\underline{u}^{(1)} < \underline{u}^{(2)} < \dots < u_\alpha$ by considering equation

$$u'' + \frac{n-1}{r}u' + (K - \frac{H_1}{\tau})u^p = 0 \quad \text{in } \mathbb{R}^n, \quad \tau = 1, 2, \dots \quad (4.16)$$

Since $\{\bar{u}^{(\tau)}\}$ is strictly decreasing in τ , by standard elliptic estimates the limit $\lim_{\tau \rightarrow \infty} \bar{u}^{(\tau)} = \tilde{u}$ is a regular solution of (1.3) and $\bar{u}^{(\tau)} > \tilde{u} \geq u_\alpha$. By Corollary 4.1 we have $\tilde{u} = u_\alpha$. Similarly $\lim_{\tau \rightarrow \infty} \underline{u}^{(\tau)} = u_\alpha$. The proof is completed.

Q.E.D.

Let $\bar{a}_1^{(\tau)}(\underline{a}_1^{(\tau)})$ and $\bar{b}_1^{(\tau)}(\underline{b}_1^{(\tau)})$ are the coefficients of terms $r^{-(m+\lambda_1)}$ and $r^{-(m+\lambda_2)}$ relating to super(sub)-solution $\bar{u}^{(\tau)}(\underline{u}^{(\tau)})$ respectively. By the constructions of $\bar{u}^{(\tau)}(\underline{u}^{(\tau)})$ and Lemma 4.7 we know that

$$\bar{a}_1^{(\tau)} = a_{1,\alpha} = \underline{a}_1^{(\tau)}, \quad \bar{b}_1^{(\tau)} \geq b_{1,\alpha} \geq \underline{b}_1^{(\tau)}$$

for $\tau = 1, 2, \dots$. More precisely, we have

Lemma 4.8 . Suppose (H.1'). Then the sequences $\{\bar{b}_1^{(\tau)}\}$ and $\{\underline{b}_1^{(\tau)}\}$ have the properties that $\bar{b}_1^{(1)} > \bar{b}_1^{(2)} > \dots > b_{1,\alpha} > \dots > \underline{b}_1^{(2)} > \underline{b}_1^{(1)}$, and

$$\lim_{\tau \rightarrow \infty} \bar{b}_1^{(\tau)} = b_{1,\alpha} = \lim_{\tau \rightarrow \infty} \underline{b}_1^{(\tau)}$$

Proof: For any $\tau \geq 1$, if $\bar{b}_1^{(\tau)} = \bar{b}_1^{(\tau+1)}$, then by Theorem 3.1 we know that

$$\bar{u}^{(\tau)} - \bar{u}^{(\tau+1)} = O\left(\frac{1}{r^{n-2+\varepsilon}}\right) \quad \text{at } r = \infty$$

for some $\varepsilon > 0$. On the other hand, since $\Delta(\bar{u}^{(\tau)} - \bar{u}^{(\tau+1)}) < 0$, by the argument of Theorem 3.8 in [N1] there exists a constant $C > 0$ such that

$$\bar{u}^{(\tau)} - \bar{u}^{(\tau+1)} \geq \frac{C}{r^{n-2}} \quad \text{at } r = \infty$$

The contradiction implies $\bar{b}_1^{(\tau)} > \bar{b}_1^{(\tau+1)}$. Similarly one has $\underline{b}_1^{(\tau)} < \underline{b}_1^{(\tau+1)}$.

To show that $\bar{b}_1^{(\tau)}$ decreases to $b_{1,\alpha}$ as τ goes to ∞ , let

$$w_{(\tau)} = r^{m+\lambda_2}(\bar{u}^{(\tau)} - u_\alpha),$$

then $w_{(\tau)}$ satisfies

$$\begin{aligned} w_{(\tau)}'' + \frac{n-1-2(m+\lambda_2)}{r}w_{(\tau)}' \\ &= r^{m+\lambda_2}\Delta w_{(\tau)} + (m+\lambda_2)(n-2-m-\lambda_2)r^{m+\lambda_2-2}(\bar{u}^{(\tau)} - u_\alpha) \\ &= \left(pL^{p-1} - K(r)\frac{(\bar{u}^{(\tau)})^p - u_\alpha^p}{\bar{u}^{(\tau)} - u_\alpha}r^2 \right) r^{m+\lambda_2-2}(\bar{u}^{(\tau)} - u_\alpha) - r^{m+\lambda_2}\frac{H_1}{\tau}\bar{u}^{(\tau)}. \end{aligned}$$

Let ψ be the right hand side. Recall that $\text{supp}(H) \subset B_1$, by Theorem 3.1, (H.1) and (H.4) we have estimate

$$\begin{aligned} 0 < \psi &\leq p(L^{p-1} - k_\infty(r^m u_\alpha)^{p-1})r^{m+\lambda_2-2}(\bar{u}^{(\tau)} - u_\alpha) \\ &= pk_\infty(u_\infty^{p-1} - (r^m u_\alpha)^{p-1})r^{m+\lambda_2-2}(\bar{u}^{(\tau)} - u_\alpha) \\ &= O(r^{-2-\min(\lambda_1, \gamma)}). \end{aligned}$$

While, at $r = 0$, we have

$$|\psi| = O(r^{m+\lambda_2-2}) + O(r^{m+\lambda_2+l}).$$

Since $l > -2$, it follows $s\psi(s) \in L^1(0, \infty)$.

Case 1° : $n-1-2(m+\lambda_2) \geq 0$. In this case, by the same argument as in the proof of Lemma 4.2, we have

$$\lim_{\tau \rightarrow \infty} w^{(\tau)}(r) = 0$$

uniformly in $[0, \infty)$, hence $\lim_{\tau \rightarrow \infty} \bar{b}_1^{(\tau)} = b_{1,\alpha}$.

Case 2° : $n-1-2(m+\lambda_2) < 0$. For $r > 1$, we have

$$(r^{n-1-2(m+\lambda_2)}w_{(\tau)}')' = \psi > 0$$

because $\text{supp}(H_1(x)) \subset B_1(0)$. Since $w_{(\tau)}(r) \rightarrow \bar{b}_1^{(\tau)} - b_{1,\alpha}$ as $r \rightarrow \infty$, there exists a sequence $\{r_N\}$, such that $w'_{(\tau)}(r_N) \rightarrow 0$ as $N \rightarrow \infty$, and

$$r^{n-1-2(m+\lambda_2)}w'_{(\tau)}(r) < r_N^{n-1-2(m+\lambda_2)}w'_{(\tau)}(r_N)$$

for $r < r_N$. Letting $N \rightarrow \infty$, we obtain $w'_{(\tau)}(r) < 0$ for all $r > 0$. Hence

$$\begin{aligned} \bar{b}_1^{(\tau)} - b_{1,\alpha} &= \lim_{r \rightarrow \infty} w_{(\tau)}(r) \\ &< \lim_{r \rightarrow \infty} w_{(\tau)}(1). \end{aligned}$$

Since $\lim_{\tau \rightarrow \infty} w_{(\tau)}(1) = 0$, we have $\bar{b}_1^{(\tau)} - b_{1,\alpha} \rightarrow 0$ as $\tau \rightarrow \infty$.

Similarly we can prove this result for $\underline{b}_1^{(\tau)}$'s.

Q.E.D.

Now we can complete the proof of Theorem 2.

Proof of Theorem 2(ii): For any given $\varepsilon > 0$, by Proposition 4.4 and Lemma 4.8 there exists τ_0 such that $\|\bar{u}^{(\tau_0)} - u_\alpha\|_{m+\lambda_2} < \varepsilon$. On the other hand, since $\underline{u}^{(\tau_0)} < u_\alpha < \bar{u}^{(\tau_0)}$ and $\underline{b}_1^{(\tau_0)} < b_{1,\alpha} < \bar{b}_1^{(\tau_0)}$, there exists $\delta > 0$ such that if $\|\varphi - u_\alpha\|_{m+\lambda_2} < \delta$, then $\underline{u}^{(\tau_0)} < \varphi < \bar{u}^{(\tau_0)}$. It follows from Proposition 4.1 that $\|u(\cdot, t; \varphi) - u_\alpha\|_{m+\lambda_2} < \varepsilon$.

To show that u_α is weak asymptotic stable with respect to the norm $\|\cdot\|_{m+\lambda_2}$, we need to show there exists $\delta > 0$, if $\|\varphi - u_\alpha\|_{m+\lambda_2} < \delta$, it implies

$$\lim_{t \rightarrow \infty} \|u(\cdot, t; \varphi) - u_\alpha\|_{m+\lambda'} \rightarrow 0$$

for every $\lambda' < \lambda_2$. Choosing δ so small that $\underline{u}^{(1)} < \varphi < \bar{u}^{(1)}$ provided $\|\varphi - u_\alpha\|_{m+\lambda_2} < \delta$. Again, by Proposition 4.1 we have

$$\underline{u}^{(1)} < u(\cdot, t; \underline{u}^{(1)}) < u(\cdot, t; \varphi) < u(\cdot, t; \bar{u}^{(1)}) < \bar{u}^{(1)}$$

and both $u(\cdot, t; \bar{u}^{(1)})$ and $u(\cdot, t; \underline{u}^{(1)})$ are monotone in t . Since u_α is the only steady state between $\bar{u}^{(1)}$ and $\underline{u}^{(1)}$, which implies

$$\lim_{t \rightarrow \infty} u(\cdot, t; \bar{u}^{(1)}) = u_\alpha = \lim_{t \rightarrow \infty} u(\cdot, t; \underline{u}^{(1)}) \quad \text{in } \mathbb{R}^n. \quad (4.17)$$

Now for every $\lambda' < m + \lambda_2$ and $R > 0$, it follows from (4.17), Lemma 4.7 and Lemma 4.8 that

$$\begin{aligned}
& | (1 + |x|)^{\lambda'} (u(\cdot, t; \varphi) - u_\alpha) | \\
& \leq \begin{cases} C(1 + |x|)^{\lambda'} |x|^{-(m+\lambda_2)} & \text{if } |x| \geq R \\ (1 + R)^{\lambda'} \| u(\cdot, t; \varphi) - u_\alpha \|_{L^\infty(B_R)} & \text{if } |x| < R \end{cases} \\
& \leq \begin{cases} CR^{\lambda'-(m+\lambda_2)} & \text{if } |x| \geq R \\ (1 + R)^{\lambda'} \| u(\cdot, t; \varphi) - u_\alpha \|_{L^\infty(B_R)} & \text{if } |x| < R. \end{cases}
\end{aligned}$$

Letting $t \rightarrow \infty$ we get

$$\lim_{t \rightarrow \infty} \sup \| u(\cdot, t; \varphi) - u_\alpha \|_{\lambda'} \leq CR^{\lambda'-(m+\lambda_2)}$$

Since R is arbitrary, it follows $\lim_{t \rightarrow \infty} \| u(\cdot, t; \varphi) - u_\alpha \|_{\lambda'} = 0$. Thus we complete the proof.

Q.E.D.

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