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Lin, F., & Li, Y. (1991). Boundary C1, α Regularity for Variational Inequalities. *Communications on Pure and Applied Mathematics, 44* (6), 715-732.
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BOUNDARY $C^{1,\alpha}$ REGULARITY FOR VARIATIONAL INEQUALITIES

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§1. Introduction.

In this paper we will consider the regularity problem for the following obstacle problem.

$$\inf \int_\Omega |\nabla v|^p \, dx \tag{1.1}$$

among the functions in $\overline{W}(\varphi, \psi)$, where Ω is a bounded C^2 domain in $\Re^n$ $(n \geq 2)$ and φ and ψ are C^2-functions defined on $\overline{\Omega}$ with $\psi \leq \varphi$, and $1 < p < \infty$ such that

$$\overline{W}(\varphi, \psi) = \{v \in W^{1,p}(\Omega) : v - \varphi \in W_0^{1,p}(\Omega) \text{ and } v \geq \psi \text{ a.e. in } \Omega\}$$

Remark 1.1. The C^2 assumptions on ψ and φ are purely technical to avoid complications, as the reader may find out later.

Because of the convexity of the integrand, (1.1) has a unique solution u satisfying the variational inequality (see, for example [LQ]).

$$\int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla (v - u) \geq 0, \quad \forall\ v \in \overline{W}(\varphi, \psi) \tag{1.2}$$

[†]The author was supported in part by the Presidential Young Investigator Award and by the National Science Foundation

*The author was supported in part by the National Science Foundation

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When $p = 2$, (1.1) reduces to the Dirichlet integral and it has been studied very extensively — for example, by Brezis, Caffarelli, Friedman, Giaquinta, Jensen, Kinderlehrer, Lewy, Lions, Stampacchia and Ural'tseva ([BK], [BS], [C], [CK], [Ch], [F], [G1,2], [J], [KD1], [KD2], [KDS], [LeS1,2,3], [LiS] and [Mo]). In particular, it is shown that soluiotns u can be as smooth as ψ. On the other hand, even the counterpart of (1.1) without obstacles are shown by DiBenedetto [Di], Evans [E], Lewis [L], Lieberman [LB], Lin [LN], Torksdorff [T1,2], Uhlenbeck [U] and Ural'tzeva [Ur] to have a solution at most $C^{1,\alpha}$ on $\overline{\Omega}$ in general for some $\alpha > 0$. Therefore, the best regularity in general for (1.1) is $C^{1,\alpha}$ for some $\alpha > 0$.

It is the purpose of this paper to show the $C^{1,\alpha}$ regularity of solution u of (1.1) up to the boundary. Our proofs of both interior and boundary regularity are based on some ideas used in [CK] and a Kryslov-type estimate for solutions of uniformly elliptic equations of non-divergent type near the boundary [K], [W]. Here are our main results.

Theorem 1.1. *Let u be the solution of (1.20) in Ω with $u \in \overline{W}(\varphi, \psi)$ and φ and ψ are C^2 functions on $\overline{\Omega}$. Then there exist $\tau = \tau(n,p) > 0$ and $C = C(n,p,\Omega,R)$ with $R \in (0,1)$ such that*

$$\|u\|_{C^{1,\tau}(\Omega_R)} \leq C\left(\|\varphi\|_{C^2(\Omega)} + \|\psi\|_{C^2(\Omega)}\right). \tag{1.3}$$

where $\Omega_R = \{x \in \Omega : \ dist\ (x, \partial\Omega) > R\}$.

Concerning the boundary regularity, we have

Theorem 1.2. *Let u be the solution of (1.2) in Ω, where $u \in \overline{W}(\varphi, \psi)$ and φ, ψ are C^2 functions on Ω. Then there exist $\alpha = \alpha(n,p) > 0$, $\rho = \rho(\Omega) > 0$ and $C = C(n,p,\Omega)$ such that*

$$\|u - \varphi\|_{C^{1,\alpha}(\Gamma_\rho)} \leq C(\|\varphi\|_{C^2(\Omega)} + \|\psi\|_{C^2(\Omega)}) \tag{1.4}$$

where $\Gamma_\rho = \{x \in \Omega : \ dist\ (x, \partial\Omega) \leq \rho\}$.

Remark 1.2. Once we have (1.3) and (1.4), the global $C^{1,\alpha}$ regularity of u becomes a direct consequence.

We would like to mention some earlier results in this direction . In particular, the continuity of u was proved in [MZ], [RT]. In [LQ], the interior $C^{1,4/p^2}$ estimate was obtained when $n = 2$. See also [BI], [H], and [Sa].

§**2. Some Preliminaries.** Let u be the solution of problem (1.1) and define

$$K = \{a \in \Omega : u(a) = \psi(a)\}. \tag{2.1}$$

which is a relatively closed set in Ω by [MZ] and we have

$$\begin{cases} \operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0 & \text{in } \Omega \setminus K \\ \operatorname{div}(|\nabla u|^{p-2}\nabla u) \le 0 & \text{weakly in } \Omega. \end{cases} \tag{2.2}$$

Also it is clear by a simple truncation [GT], [LUr] that

$$\|u\|_{L^\infty(\Omega)} \le M \equiv \max\{\|\varphi\|_{L^\infty(\Omega)}, \|\psi\|_{L^\infty(\Omega)}\} \tag{2.3}$$

For (1.1), let u^ε be the unique minimizer of the associated problem

$$\inf \int_\Omega (\varepsilon + |\nabla v|^2)^{p/2}\, dx \quad \text{in } \overline{W}(\varphi, \psi). \tag{2.4}$$

The advantage of (2.4) is that it is non-degenerate and furthermore

$$u^\varepsilon(x) \to u(x) \text{ in } W^{1,p}(\Omega), \quad \nabla u^\varepsilon(x) \to \nabla u(x) \text{ a.e. in } \Omega$$

as $\varepsilon \to 0$ (see, for example [Di] and [RT]). Therefore, to obtain $C^{1,\alpha}$ regularity for u, it is sufficient to prove these results for u^ε independently for $\varepsilon > 0$.

Correspondingly, we have

$$\int_\Omega (\varepsilon + |\nabla u^\varepsilon|^2)^{\frac{p-2}{2}} \nabla u^\varepsilon \cdot \nabla(v - u^\varepsilon) \ge 0, \ \forall\ v \in \overline{W}(\varepsilon, \psi)$$

or

$$\begin{cases} \operatorname{div}\left((\varepsilon + |\nabla u^\varepsilon|^2)^{\frac{p-2}{2}} \nabla u^\varepsilon\right) \le 0 & \text{weakly in } \Omega \\ \operatorname{div}\left((\varepsilon + |\nabla u^\varepsilon|^2)^{\frac{p-2}{2}} \nabla u^\varepsilon\right) & \text{in } \Omega \setminus K^\varepsilon \end{cases} \tag{2.5}$$

where $K^\varepsilon = \{a \in \Omega : u^\varepsilon(a) = \psi(s)\}$.

Because of the non-degenernate property of (2.4), we have that u^ε are smooth for $\varepsilon > 0$ ([ ]) and hence u^ε satisfies

$$\begin{cases} a^\varepsilon_{ij} D_{ij} u^\varepsilon \le 0 & \text{in } \Omega \\ a^\varepsilon_{ij} D_{ij} u^\varepsilon = 0 & \text{in } \Omega \setminus K^\varepsilon \end{cases} \tag{2.6}$$

with $a^{\varepsilon}_{ij}(x) = \delta_{ij} + (p-2)D_i u^{\varepsilon}(x) D_j u^{\varepsilon}(x)(\varepsilon + |\nabla u^{\varepsilon}|^2)^{-1}$.

It is clear that $(a^{\varepsilon}_{ij}(x))$ is uniformly elliptic in Ω with $\lambda = \min\{1, p-1\}$ and $\Lambda = \max\{1, p-1\}$ independently of $\varepsilon > 0$.

Unless otherwise indicated, C and various Hölder exponents α, β, γ. will denote constants depending only on n, p, δ and $\partial\Omega$ Let

$$M \equiv \max\{\|\varphi\|_{L^{\infty}(\Omega)}, \|\psi\|_{L^{\infty}(\Omega)}\}$$
$$M_1 \equiv \max\{\|\nabla\varphi\|_{L^{\infty}(\Omega)} \|\nabla\psi\|_{L^{\infty}(\Omega)}\}$$

and

$$M_2 \equiv \max\{|\nabla\varphi|_{c^{\delta}(\Omega)}, |\nabla\psi|_{c^{\delta}(\Omega)}\}$$

To make the presentation clean, we will omit ε in the future because all the estimates obtained will be independent of $\varepsilon > 0$.

Let $a \in \Omega$ such that $B_{4R}(a) \in \Omega$ for some $R > 0$. Then we have

Lemma 2.1. *Let v satisfy*

$$\begin{cases} a_{ij}(x) D_{ij} v \le 0 & \text{in } B_{4R} \\ v \ge 0 \end{cases} \tag{2.7}$$

with $v(a) \le \lambda$ and $\sup_G v \le \lambda$ where $G =$ supp $(a_{ij}(x)D_{ij}v)$. Then for each $\alpha \in (0,3)$ there exists $C_\alpha > 0$ such that

$$0 \le v(x) \le C_\alpha \lambda, \quad \forall\ x \in B_{\alpha r}(a).$$

Proof. Let v_0 be the solution of (2.7) with boundary value $v_0 = v$, i.e.,

$$\begin{cases} a_{ij}(x) D_{ij} v_0 = 0 & \text{in } B_{4R} \\ v_0 = v & \text{on } \partial B_{4R} \end{cases} \tag{2.8}$$

Then we have $v - v_0 \ge 0$ in B_{4R} and v_0 is non-negative in B_{4R} since v is on ∂B_{4R}. This implies that

$$0 \le v_0(a) \le v(a) \le \lambda$$

and hence the Harnack inequality ([Mr], [Sf]) implies tht

$$0 \le v_0(x) \le C'_\alpha \lambda \quad \text{in } B_{\alpha R}(a), \text{ for } \alpha \in (0,3). \tag{2.9}$$

On the other hand, $v - v_0$ is a non-negative solution of (2.7) in $B_{4R} \setminus G$. Therefore, the maximum principle shows that

$$0 \leq v(x) - v_0(x) \leq \sup_{\partial(B_{4R}\setminus G)} (v - v_0) \leq \sup_G v \leq \lambda \quad \forall\, x \in B_{4R}.$$

Therefore we have by (2.9) that

$$0 \leq v(x) \leq v_0(x) + \lambda \leq (C'_\alpha + 1)\lambda. \qquad \text{Q.E.D.}$$

Lemma 2.2. *Let u be the super-solution of (2.6) and $a \in K \cap \Omega$ such that $B_{4R}(a) \subset \Omega$, and assume that there exists a solution h of (2.8) in $B_{4R}(a)$ such that $\sup_{B_{4R}(a)} |h - \psi| \leq \lambda$. Then*

$$\sup_{B_{4R}(a)} |u - \psi| \leq C\lambda. \tag{2.10}$$

Proof. Let $w(x) = u(x) - h(x) + \lambda$ in $B_{4R}(a)$. Then w is a supersolution of (2.6). And since $u(x) \geq \psi(x) \geq h(x) - \lambda$ in B_{4R} we have $w \geq 0$ in $B_{4R}(a)$ with $w(a) = \psi(a) - h(a) + \lambda \leq 2\lambda$ and

$$w(x) = \psi(x) - h(x) + \lambda \leq 2\lambda \quad \text{in } K \cap B_{4R}.$$

Now applying Lemma 2.1, we get that

$$0 \leq w(x) \leq c\lambda \quad \text{in } B_{2R}(a),$$

which implies tht

$$-\lambda \leq u(x) - h(x) \leq (c-1)\lambda.$$

Thus

$$\sup_{B_{2R}} |u - \psi| \leq (C+1)\lambda. \qquad \text{Q.E.D.}$$

Lemma 2.3. *Let $u, a \in K$ and $R > 0$ be the same as in Lemma 2.2.*

(i) If ω is the modulus of continuity of ψ at a, then

$$\sup_{B_r(a)} (u - \psi) \leq C_1 \omega(2r), \quad \forall\, r \leq R. \tag{2.11}$$

(ii) If ω_1 is the modulus of continuity of gradient of ψ at a, then

$$\sup_{B_r(a)} (u-\psi) \le C_2 r\omega_1(2r), \quad \forall\, r \le R. \tag{2.12}$$

Proof. By using Lemma 2.2, both (2.11) and (2.12) become direct consequences if we let $h(x) = \psi(a)$ and $h(x) = \psi(a) + \nabla\psi(a)\cdot(x-a)$ in $B_{4R}(a)$ respectively for case (i) and (ii). Q.E.D.

Remark 2.1. If (2.12) holds for u in B_R, and if $\omega_1 \to 0$ as $r \to 0$, then we have

$$\begin{aligned} &|u(x)-\psi(a)-\nabla\psi(a)\cdot(x-a)| \\ &\quad \le |u(x)-\psi(x)| + |\psi(x)-\psi(a)-\nabla\psi(a)\cdot(x-a)| \\ &\quad \le (c+1)r\omega_1(2r), \quad \text{for } x \in B_R(a) \end{aligned}$$

with $r = |x-a|$, which implies that

$$\nabla u(a) = \nabla\psi(a)$$

and hence

$$|u(x)-u(a)-\nabla u(a)\cdot(x-a)| \le C_3|x-a|\omega_1(2|x-a|), \quad \forall\, x \in B_R(a). \tag{2.13}$$

Lemma 2.4. *Let u be the solution of (1.1) (or (2.4)) and $y \in \Omega$ such that $B_{4R}(y) \subset \Omega \setminus K$. Then*

$$\|\nabla u\|_{L^\infty(B_{3R}(y))} \le C\|u\|_{L^p(B_{4R})}\frac{1}{R^{1+n/p}} \le \frac{C_4 M}{R} \tag{2.14}$$

and

$$|\nabla u(x)-\nabla u(z)| \le C|x-z|^\beta R^{-\frac{n}{p}-\beta}\|\nabla u\|_{L^p(B_{3R})} \le \frac{C_5 M}{R^\beta}|x-z|^\beta, \tag{2.15}$$

for all $x, z \in B_{2r}(y)$, and for some $\beta = \beta(n,p) > 0$

Proof. See [L], [S].

§3. Interior Regularity.

Now in this section, we will prove Theorem 1.1. But first the following lemma is needed.

Lemma 3.1. *Given $R_0 \in (0,1)$ and let $\Omega_{R_0} = \{x \in \Omega : d(x) > R_0\}$, where $d(x) \equiv$ dist $(x, \partial\Omega)$. If u satisfies (2.4), then*

$$|\nabla u|_{L^\infty(\Omega_{R_0})} \leq C_6(\frac{M}{R_0} + M_1) \tag{3.1}$$

and

$$|\nabla u(x) - \nabla u(a)| \leq \frac{C_7 M_{R_0}, \gamma}{R_0^{1+\gamma}}|x-a|^\gamma, \quad \forall\, x \in \Omega_{R_0} \text{ and } a \in \Omega_{R_0} \cap K, \tag{3.2}$$

where $\gamma = \delta\frac{\beta}{1+\beta}$, $M_{R_0}, \gamma = M + R_0 M_1 + R_0^{1+\gamma} M_2$.

Proof. By Lemma 2.3 and Remark 2.1, we have that if $a \in K \cap \Omega$, then $\nabla u(a) = \nabla\psi(a)$ and therefore

$$\|\nabla u\|_{L^\infty(K\cap\Omega)} \leq M_1 \tag{3.3}$$

and furthermore, we have that for all $a \in K \cap \Omega_{\frac{3}{4}R_0}$

$$\begin{cases} |u(x) - u(a)| \leq 2C_1 M_1 |x-a| & \text{by } (2.11) \\ |u(x) - u(a) - \nabla u(a)\cdot(x-a)| \leq 2C_3 M_2 |x-a|^{1+\delta} & \text{by } (2.13) \end{cases} \tag{3.4}$$

for $x \in B_{R_0/4}(a)$. Let $x \in \Omega_{R_0}$ be fixed. Then we have *either* $B_{R_0/4}(x) \cap K = \emptyset$ in which case we have

$$|\nabla u(x)| \leq C_4 M/R_0 \tag{3.5}$$

by Lemma 2.4, *or* $0 <$ dist $(x, K) < R_0/4$. Let $d_x =$ dist (x, K), then there exists some $a \in K \cap \Omega_{\frac{3}{4}R_0}$ such that $R_0/4 > d_x = |x-a|$. Now let

$$w(y) = \frac{u(d_x\, y + a) - u(a)}{d_x} \quad \text{for } y \in B_1(\frac{x-a}{|x-a|}).$$

Then w satisfies equation (2.5) (with a different $\varepsilon > 0$ after this scaling). Hence by the estimate (3.4) we have

$$\|w\|_{L^\infty(B_1)} \leq 4C_1 M_1$$

which implies by Lemma 2.4 that

$$\|\nabla w\|_{L^\infty(B_{1/2})} \leq 4C_1 C_4 M_1.$$

In particular, by letting $y = \frac{x-a}{|x-a|}$, we have

$$|\nabla u(x)| \le 4C_1 C_4 M_1 \tag{3.6}$$

Therefore, we have by (3.3), (3.5) and (3.6) that (3.1) is valid.

To prove (3.2), let $x \in \Omega_{R_0}$ and $a \in K \cap \Omega_{R_0}$ with $0 < |x-a| < R_0/8$. For otherwise, (3.2) is implied by (3.1) already.

First assume that $|x-a| \le 4d_x \equiv 4$ dist (x, K). As before, let

$$v(y) = w(y) - \nabla u(a) y, \quad y \in B_1(\frac{x-a}{|x-a|}).$$

If we have

$$|\nabla u(a)| \le 4C_3 M_2\, d_x^\delta \tag{3.7}$$

then

$$|w(z)| \le 12 C_3 M_2\, d_x^\delta \tag{3.8}$$

in $B_1(\frac{x-a}{|x-a|})$ because $\|v\|_{L^\infty(B_1)} \le 4C_3 M_2\, d_x^\delta$ by (3.4).

But once we have (3.8), we conclude that

$$|\nabla u(x)| = |\nabla w(\frac{x-a}{|x-a|})| \le 12 C_3 C_4 M_2\, d_x^\delta \tag{3.9}$$

by the same reason as we derived (3.6). Therefore, we have

$$|\nabla u(x) - \nabla u(a)| \le (1 + 12C_4) C_3 M_2\, d_x^\delta \le (1+12C_4) C_3 M_2 |x-a|^\delta.$$

Now suppose (3.7) is false. Then we have that

$$\left\| \frac{v}{|\nabla u(a)|} \right\|_{L^\infty(B_1)} \le \frac{4C_3 M_2 d_x^\delta}{|\nabla u(a)|} = \sigma \le 1. \tag{3.10}$$

In particular, we have

$$\left| \frac{1}{|\nabla u(a)|} \left[w(\frac{x-a}{|x-a|}) - \nabla u(a) \frac{x-a}{|x-a|} \right] \right| \le \sigma$$

Therefore, we have

$$\left| \frac{v(y)}{|\nabla u(a)|} - \frac{1}{|\nabla u(a)|} v(\frac{x-a}{|x-a|}) \right| \le 2\sigma \quad \forall\ y \in B_1\left(\frac{x-a}{|x-a|}\right) \tag{3.11}$$

But

$$
(3.12)\qquad \begin{cases} \left|\dfrac{v(y)}{|\nabla u(a)|}-\dfrac{1}{|\nabla u(a)|}v(\dfrac{x-a}{|x-a|})\right| \\ = \left|\dfrac{1}{|\nabla u(a)|}[w(y)-w(\dfrac{x-a}{|x-a|})]-\dfrac{\nabla u(a)}{|\nabla u(a)|}(y\dfrac{x-a}{|x-a|})\right| \\ \geq \left|\dfrac{1}{|\nabla u(a)|}\nabla w(\dfrac{x-a}{|x-a|})\cdot(y-\dfrac{x-a}{|x-a|})-\dfrac{\nabla u(a)}{|\nabla u(a)|}(y-\dfrac{x-a}{|x-a|})\right| \\ \qquad -C_4C_5(2+\sigma)|y-\dfrac{x-a}{|x-a|}|^{1+\beta} \end{cases}
$$

by Lemma 2.4 (2.15) because from (3.10) we have that $\frac{w}{|\nabla u(a)|}$ is bounded by $(2+\sigma)$ and satisfies equation (2.5) (with a different $\varepsilon>0$ after scaling) in $B_1(\frac{x-a}{|x-a|})$.

Therefore (3.11) and (3.12) imply that

$$
\begin{aligned}
&\left|\frac{1}{|\nabla u(a)|}\nabla w(\frac{x-a}{|x-a|})(y-\frac{x-a}{|x-a|})-\frac{\nabla u(a)}{|\nabla u(a)|}(y-\frac{x-a}{|x-a|})\right| \\
&\qquad \leq 2\sigma+C_4C_5(2+\sigma)|y-\frac{x-a}{|x-a|}|^{1+\beta}.
\end{aligned}
$$

Since $\sigma\leq 1$, we can choose an appropriate $y\in B_1$ such that $2\sigma=C_4C_5(2+\sigma)|y-\frac{x-a}{|x-a|}|^{1+\beta}$ and furthermore $\nabla w(\frac{x-a}{|x-a|})-\nabla u(a)$ is parallel to $y-\frac{x-a}{|x-a|}$. Now for such a choice, we have

$$
\begin{aligned}
\left|\frac{1}{|\nabla u(a)|}(\nabla w(\frac{x-a}{|x-a|})-\nabla u(a))\right| &\leq 4C_8|y-\frac{x-a}{|x-a|}|^{\beta} \\
&= 4C_8\sigma^{\frac{\beta}{1+\beta}}=\widetilde{C}_8M_2^{\frac{\beta}{1+\beta}}d_x^{\delta\frac{\beta}{1+\beta}}|\nabla u(a)|^{-\frac{\beta}{1+\beta}}
\end{aligned}
$$

Noticing that $\nabla w(\frac{x-a}{|x-a|})=\nabla u(x)$, we obtain by (3.3) that

$$
\begin{aligned}
(3.13)\qquad |\nabla u(x)-\nabla u(a)| &\leq \widetilde{C}_8M_2^{\frac{\beta}{1+\beta}}d_x^{\delta\frac{\beta}{1+\beta}}|\nabla u(a)|^{\frac{\beta}{1+\beta}} \\
&\leq C_9M_1^{\frac{\beta}{1+\beta}}M_2^{\frac{\beta}{1+\beta}}d_x^{\delta\frac{\beta}{1+\beta}} \\
&\leq C_9M_1^{\frac{1}{1+\beta}}M_2^{\frac{\beta}{1+\beta}}|x-a|^{\delta\frac{\beta}{1+\beta}} \\
&\leq C_{10}(M_1+M_2)|x-y|^{\delta\frac{\beta}{1+\beta}}
\end{aligned}
$$

The case when $|x-a|>4d_x$ can be handled similarly, because when $R_0/8>|x-a|\geq 4\,d_x$, we have that $d_x<R_0/32$. And hence there exists some $b\in B_{R_0/32}(x)\cap K$ with $|x-b|\leq d_x<R_0/32$. Therefore (3.13) implies that

$$
\begin{aligned}
|\nabla u(x)-\nabla u(a)| &\leq |\nabla u(x)-\nabla u(b)|+|\nabla u(b)-\nabla u(a)| \\
&\leq C_9M_1^{\frac{1}{1+\beta}}M_2^{\frac{\beta}{1+\beta}}|x-b|^{\delta\frac{\beta}{1+\beta}}+|\nabla\psi(b)-\nabla\psi(a)| \\
&\leq C_9M_1^{\frac{1}{1+\beta}}M_2^{\frac{\beta}{1+\beta}}|x-b|^{\delta\frac{\beta}{1+\beta}}+M_2|b-a|^{\delta}.
\end{aligned}
$$

This completes the proof. Q.E.D.

Now we are ready to prove Theorem 1.1. Let $\tau = \delta\frac{\beta}{1+\beta+\delta}$ as in Lemma 3.1. We will prove that

$$|\nabla u(x) - \nabla u(y)| \leq \frac{C_{11}M_{R_0},\gamma}{R_0^{1+\gamma}}|x-y|^{\tau}, \quad \text{for } x, y \in \Omega_{R_0} \tag{3.14}$$

Assume that d_x and d_y are strictly positive and $0 < |x-y| < R_0/(12 + 2^{1+\frac{1+\beta}{\delta}})$, for otherwise, (3.1) and (3.2) would imply (3.14).

Case 1. $\max\{d_x, d_y\} \geq R_0/(6 + 2^{(1+\beta)/\delta})$, say without loss of generalilty that $d_x \geq R_0/(6 + 2^{\frac{1+\beta}{\delta}})$. Then u becomes a solution of (2.5) in $B_{R_0/(6+2^{\frac{1+\beta}{\delta}})}(x)$ and $y \in B_{R_0/(12+2^{1+\frac{1+\beta}{\delta}})}(x)$ because $|x-y| < R_0/(12 + 2^{1+\frac{1+\beta}{\delta}})$. Therefore, (2.15) implies that

$$|\nabla u(x) - \nabla u(y)| \leq \frac{C_{12}M}{R_0^{\beta}}|x-y|^{\beta} \leq \frac{C_{13}M}{R_0^{\tau}}|x-y|^{\tau}.$$

Case 2. Assume that $\max\{d_x, d_y\} = d_x < R_0/(6 + 2^{\frac{1+\beta}{\delta}})$. First, let us consider the possibility

$$0 < |x-y| \leq \max\{d_x^{\frac{\gamma}{\tau}}, d_y^{\frac{\gamma}{\tau}}\} = d_x^{\frac{\gamma}{\tau}} \tag{3.15}$$

Then again u is a solution of (2.50) in $B_{d_x}(x)$ such that $y \in B_{d_x/2}(x)$ because of (3.15). Then (2.15) implies that

$$|\nabla u(x) - \nabla u(y)| \leq \frac{C_{14}M}{d_x^{\beta}}|x-y|^{\beta} \leq C_{14}M|x-y|^{\tau}.$$

Second, if $|x-y| > \max\{d_x^{\frac{\gamma}{\tau}}, d_y^{\frac{\gamma}{\tau}}\}$, then we have $d_x = |x-a|$, $d_y = |y-b|$ for some $a, b \in K \cap \Omega_{\frac{5}{6}R_0}$.

Thus we have by (3.2)

$$\begin{aligned}
|\nabla u(x) - \nabla u(y)| &\leq |\nabla u(x) - \nabla u(a)| + |\nabla u(a) - \nabla u(b)| + \nabla u(b) - \nabla u(y)| \\
&\leq \frac{C_7 M_{R_0,\gamma}}{R_0^{1+\gamma}}|x-a|^{\gamma} + |\nabla\psi(a) - \nabla\psi(b)| + \frac{C_7 M_{R_0,\gamma}}{R_0^{1+\gamma}}|b-y|^{\gamma} \\
&\leq \frac{C_7 M_{R_0,\gamma}}{R_0^{1+\gamma}}\, d_x^{\gamma} + M_2|a-b|^{\delta} + \frac{C_7 M_{R_0,\gamma}}{R_0^{1+\gamma}}\, d_y^{\gamma} \\
&\leq \frac{2C_7 M_{R_0,\gamma}}{R_0^{1+\gamma}}|x-y|^{\tau} + 3^{\delta}M_2|x-y|^{\frac{\delta\tau}{\gamma}}
\end{aligned}$$

because $|a-b| \leq |a-x| + |x-y| + |y-b| \leq 3|x-y|^{\frac{\tau}{\gamma}}$.

Thus the proof of Theorem 1.1 is completed with $\tau = \frac{\delta\beta}{1+\beta+\delta}$. Q.E.D.

§4. Boundary Regularity.

In this section, we establish a boundary estimate analogous to Lemma 2.3. And for that purpose we need the following version of a Krylov-type boundary estimate for solutions of boundary value problems for elliptic equations of non-divergent form. (See e.g. [K] and [W] for a proof of it.)

Let $p \in \partial\Omega \in C^2$, $\Omega_{p,r} = \{x \in \Omega : |x-p| < r\}$ and $\Delta_{p,r} = \partial\Omega \cap B_r(p)$. Recall that if $\partial\Omega \in C^2$, then there exists $\rho \in (0,1)$ such that $d(x) \in C^2(\Gamma_\rho)$ where $\Gamma_\rho \equiv \{x \in \overline{\Omega} :\ \text{dist}\ (x, \partial\Omega) \le \rho\}$

Lemma 4.1. *Let w be a solution of the following equation.*

$$\begin{cases} a_{ij}D_{ij}w = f \quad \text{in } \Omega_{p,\rho'} \\ \lambda I \le (a_{ij}(x)) \le \lambda^{-1}I,\ \ \lambda > 0 \text{ fixed, for a.e } x \in \Omega_{p,\rho'} \\ w = \varphi \quad \text{on } \Delta_{p,\rho'} \end{cases} \tag{4.1}$$

where $\varphi \in C^{1,\delta}$ and $f \in L^t(\Omega_{p,\rho'})$ for some $t > n$ and $\rho' \in (0,\rho)$. Then there exist $\mu = \mu(n,\lambda,\delta,t,\rho')$, $C_{15} = C(n,\lambda,\delta,t,\rho')$ and $A(q)$ defined on $\Delta_{p,\frac{2}{3}\rho'}$ such that

$$|w(x) - \varphi(x) - A(q)\nu(q)(x-q)| \le C_{15}(M_2 + \|f\|_{L^q(\Omega_{p,\rho'})})|x-q|^{1+\mu} \tag{4.2}$$

for any pair $(x,q) \in \Omega_{p,\frac{2}{3}\rho'} \times \Delta_{p,\frac{2}{3}\rho'}$, where $\nu(q)$ is the unit inner normal of $\partial\Omega$ at q, and

$$|A(q)| \le C_{15}[\|w-\varphi\|_{L^\infty(\Omega_{p,\rho'})} + M_2] \quad \text{for } q \in \Delta_{p,\frac{2}{3}\rho'}. \tag{4.3}$$

.

Remark 4.1. If w satisfies (4.2), then it is easy to see that

$$\nabla w(q) = \nabla\varphi(q) + A(q)\nu(q), \quad \forall q \in \Delta_{p,\frac{2}{3}\rho'} \tag{4.4}$$

and furthermore, if $q, s \in \Delta_{p,\frac{2}{3}\rho'}$ and $x \in \Omega_{p,\frac{2}{3}\rho'}$ we have by applying (4.2) for x at points q and then subtracting that

$$|A(q)\nu(q)(x-q) - A(s)\nu(s)(x-s)| \le C_{15}(M_2 + \|f\|_{L^t})(|x-q|^{1+\mu} + |x-s|^{1+\mu})$$

Now define $\Pi_c = \{x \in \Omega :\ \text{dist}\ (x,\partial\Omega) = c\}$ for $c \in (0,\rho)$. Then there exists $x \in \Pi_{|\frac{q-s}{2}|} \cap \Delta_{p,\frac{2}{3}\rho'}$ such that

$$\nu(q)(x-q) = \nu(t)(x-t) \approx |q-t|$$

with $|x-q|$ and $|x-s| \approx |q-t|$. Therefore, for such choice of x, we have from the above inequality that

$$|A(q) - A(s)| \leq C_{16}(M_2 + \|f\|_{L^t(\Omega_{p,\rho'})})|q-s|^{\mu}. \tag{4.5}$$

¿From now on we will let r_0 be fixed in $(0, \rho]$ and $\theta \in (0, 2/3]$. Assume $0 \in \partial\Omega$, $r_k = \theta^k r_0$ and $\Omega_k = \Omega_{0,r_k}$ and $\Delta_k = \Delta_{0,r_k}$. Define h_k to be the solution of the following boundary value problem

$$\begin{cases} a_{ij}D_{ij}h_k = 0 & \text{in } \Omega_k \\ h_k = u & \text{on } \partial\Omega_k \end{cases} \tag{4.6}$$

First, by the maximum principle we have

$$u(x) \geq h_k(x) \quad \text{for } x \in \Omega_k \tag{4.7}$$

But then $u - h_k$ is a positive super solution vanishing on $\partial\Omega_k$ and becoming a solution in $\Omega_k \setminus K$. Therefore, we have

$$\begin{aligned} \sup_{\Omega_k}(u - h_k) &= \sup_{\partial(\Omega_k \setminus K)}(u - h_k) \\ &\leq \sup_{x \in K \cap \Omega_k}(u - h_k) \\ &= \sup_{x \in K \cap \Omega_k}(\psi - h_k)^+ \end{aligned} \tag{4.8}$$

Combining (4.6) and (4.7) together, we have

$$0 \leq u(x) - h_k(x) \leq \sup_{x \in K \cap \Omega_k}(\psi - h_k) \quad \text{in } \Omega_k. \tag{4.9}$$

On the other hand, we have

$$|\psi(x) - \psi(0) - \nabla\psi(0) \cdot x| \leq M_2|x|^{1+\delta} \quad \text{for } x \in \Omega_k$$

Therefore, it is clear that

$$h_k(x) \geq \psi(0) + \nabla\psi(0) \cdot x - M_2 r_k^{1+\delta} \quad \text{in } \Omega_k \tag{4.10}$$

because the latter is itself a solution of (4.6) and is less than or equal to h_k on $\partial\Omega_k$.

Hence, we obtain by (4.10) that

$$\sup_{\Omega_k}(\psi - h_k)_+ \leq 2M_2 r_k^{1+\delta}$$

and combining it with (4.9), we obtain

$$0 \le u(x) - h_k(x) \le 2M_2 r_k^{1+\delta} \text{ in } \Omega_k. \tag{4.11}$$

However, we have h_k satisfies equation (4.6) with smooth boundary value φ. Hence we apply Lemma 4.1 and Remark 4.1 to obtain the following

$$\text{(4.12)} \qquad \begin{cases} |h_k(x) - \varphi(x) - A_k(q)\nu(q)(x-q)| \le C_{15} M_2 |x-q|^{1+M} \\ \qquad \text{for any pair } (x,q) \in \Omega_{k+1} \times \Delta_{k+1}, \\ |A_k|_{L^\infty(\Delta_{k+1})} \le C_{16}[\dfrac{1}{r_k}\|h_k - \varphi(x)\|_{L^\infty(\Omega_k)} + M_2 r_k^\delta] \\ |A_k(q) - A_k(p)| \le C_{16}\mu_2 |q-p|^\mu \quad \text{in } \Delta_{k+1} \times \Delta_{k+1} \end{cases}$$

Now we need to estimate the difference $h_{k+1} - h_k$. As in (4.7), we first conclude that

$$h_{k+1}(x) - h_k(x) \ge 0 \quad \text{in } \Omega_{k+1}. \tag{4.12}$$

On the other hand since $u(x) \ge h_{k+1}(x)$ in Ω_{k+1} and (4.11) holds, we have

$$0 \le h_k(x) - h_{k+1}(x) \le 2M_2 r_k^{1+\delta} \quad \text{in } \Omega_{k+1}. \tag{4.13}$$

Also, we have for h_{k+1} the following

$$\text{(4.14)} \qquad \begin{cases} |h_{k+1}(x) - \varphi(x) - A_{k+1}(q)\nu(q)(x-q)| \le C_{15} M_2 |x-q|^{1+\mu} \\ \qquad \text{for any pair } (x,q) \in \Omega_{k+2} \times \Delta_{k+2}, \\ |A_{k+1}|_{L^\infty(\Delta_{k+2})} \le C_{16}[\dfrac{1}{r_{k+1}}\|h_{k+1} - \varphi(x)\|_{L^\infty(\Omega_{k+1})} + M_2 r_{k+1}^\delta] \\ |A_{k+1}(q) - A_{k+1}(p)| \le C_{16} M_2 |q-p|^\mu \quad \text{in } \Delta_{k+2} \times \Delta_{k+2} \end{cases}$$

By (4.13) and (4.14), we have

(4.15)

$$|(A_{k+1}(q) - A_k(q))\nu(q)(x-q)| \le 2C_{15} M_2 |x-q|^{1+\mu} + 2M_2 r_k^{1+\delta}$$
$$\text{for any } (x,q) \in \Omega_{k+2} \times \Delta_{k+2}.$$

¿From (4.15) we achieve the following by choosing $x = q + \rho r_{k+2}\nu(q)$

$$|A_{k+1}(q) - A_k(q)| \le C_{17} M_2 \max\{r_{k+2}^\mu, r_k^{1+\delta}/r_{k+2}\}$$

i.e.,

$$|A_{k+1}(q) - A_k(q)| \le C_{18} M_2 \theta^{k\mu} r_0^{\mu}, \quad q \in \Delta_{k+2} \tag{4.16}$$

And hence in particular, we have

$$A_{k+1}(0) - A_k(0)| \le C_{17} M_2 \theta^{k\mu} r_0^{\mu}, \quad k = 0, 1, 2, \ldots \tag{4.17}$$

which implies from (4.17) that

$$\begin{cases} \lim\limits_{k\to\infty} A_k(0) = A(0) \quad \text{exists and} \\ |A(0)| \le C_{18}(M_2 r_0^{\mu} \dfrac{1}{1-\theta^{\mu}} + \dfrac{M}{r_0}) \quad \text{and} \\ |A_k(0) - A(0)| \le \dfrac{C_{18} M_2}{1-\theta^{\mu}} \theta^{k\mu} r_0^{\mu} = \dfrac{C_{18} M_2}{1-\theta^{\mu}} r_k^{\mu} \end{cases} \tag{4.18}$$

Now (4.11), (4.12) and (4.18) imply that

$$|u(x) - \varphi(x) - A(0)\nu(0) \cdot x| \le C_{19} M_2 (|x|^{1+\mu} + r_k^{1+\mu})$$

or simply

$$|u(x) - \varphi(x) - A(0)\nu(0) \cdot x| \le C_{20} M_2 r_{k+1}^{1+\mu} \quad \text{in } \Omega_{k+1}. \tag{4.19}$$

Now it is standard to derive from (4.19) the following

$$|u(x) - \varphi(x) - A(0)\nu(0) \cdot x| \le C_{21} M_2 |x|^{1+\mu} \quad \text{in } \Omega_{k_0} \tag{4.20}$$

for some $k_0 = k_0(\rho)$. We have from (4.20) that

$$\nabla u(0) = \nabla \varphi(0) + A(0)\nu(0) \tag{4.21}$$

and hence

$$|u(x) - u(0) - \nabla u(0) \cdot x| \le C_2 1 M_2 |x|^{1+\mu}$$

with

$$|\nabla u(0)| \le M_1 + C_{18}(\frac{M}{r_0} + \frac{1}{1-\theta^{\mu}} r_0^{\mu} M_2)$$

by (4.18). This gives the following theorem

Theorem 4.2. *Let u be the solution of (2.4). Then there exist $0 < \mu = \mu(n, p, \rho)$, $C_{22} = C(n, p, \rho)$ and $r_0 = r_0(\rho) > 0$ such that*

$$\begin{cases} |u(x) - \varphi(x) - A(p)\nu(p)(x-p)| \leq C_{22} M_2 |x-p|^{1+\mu} \\ \qquad \text{if } x \in \Omega_{p,r_0} \text{ and } p \in \partial\Omega \end{cases} \tag{4.22}$$

and

$$\nabla u(p) = \nabla\varphi(p) + A(p)\nu(p)$$

which satisfies

$$|\nabla u(p)| \leq \frac{C_{22}}{r_0}(M + r_0 M_1 + r_0^{1+\mu} M_2), \quad p \in \partial\Omega.$$

Remark 4.2. From (4.22) and the fact that $\partial\Omega \in C^2$, we obtain again the following (see Remark 4.1).

$$|A(q) - A(p)| \leq C_{23} M_2 |q-p|^{\mu}. \tag{4.23}$$

Noticing that $Dd(p) = \nu(p)$, we have

Corollary 4.3. *Let u be as in Theorem 4.2. Then we have*

$$u(x) = \varphi(x) + d(x)A(x) \quad \text{in } \Gamma_{r_0}$$

such that

$$\begin{cases} \|A\|_{L^\infty(\Gamma_{r_0})} \leq C_{24}(M + r_0 M_1 + r_0^{1+\mu} M_2)/r_0 \\ [A]_{\mu,\Gamma_{r_0}} \leq C_{24} M_2. \end{cases} \tag{4.24}$$

Once we have obtained the crucial estimates (4.24) we can prove the following theorem on the boundary regularity of solutions of (2.4), using methods of proof similar to that for Theorem 1 in §3. We therefore omit the proof here.

Theorem 4.4. *Let u be the solution of (2.4). Then $u \in C^{1,\alpha}(\Gamma_{r_0})$ such that*

$$\begin{cases} |\nabla u|_{L^\infty(\Gamma_{r_0})} \leq C_{25}(M + r_0 M_1 + r_0^{1+\alpha} M_2)/r_0 \\ [\nabla u]_{\alpha,\Gamma_0} \leq C_{25} M_2 \end{cases} \tag{4.25}$$

where $\alpha = \frac{\mu\beta}{(1+\beta+\mu)}$ and $C = C(n, p, \rho)$.

Proof of Theorem 1.2. Since the estimate (4.25) is independent of $\varepsilon > 0$ and since $\nabla u^\varepsilon(x) \to \nabla u(x)$ a.e. x in Ω, the result follows immediately from Theorem 1.1 and (4.25). Hence, the proof of Theorem 1.2 is complete. Q.E.D.

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