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Remarks on a Semilinear Elliptic Equation on Rn

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1. INTRODUCTION

Existence and symmetry properties of solutions are among the major questions in the study of partial differential equations. In this paper we consider the following semilinear elliptic equation

$$\begin{aligned} &\Delta u - u + Q\,|u|^{p-1}u = 0 \qquad \text{in } \mathbb{R}^n \\ &u \geqslant 0, u \not\equiv 0 \qquad \text{in } \mathbb{R}^n, \qquad \text{and} \qquad u \to 0 \qquad \text{at } \infty, \end{aligned} \tag{1.1}$$

where p is a constant satisfying

$$1 < p < \frac{n+2}{n-2}. \tag{1.2}$$

(For $n = 2$, the number $(n+2)/(n-2)$ is considered to be ∞.)

We study the existence of solutions of Eq. (1.1) with some restrictions on Q in Section 2 below. In Sections 3 and 4 below, the symmetry properties and nonexistence of solutions are investigated. We find a class of radially symmetric potentials Q for which positive nonradial solutions of (1.1) exist, and a nonexistence theorem is proved for certain radial potentials.

Equation (1.1) is referred to as a (nonautonomous) scalar field equation. It arises in various branches of applied mathematics, for example, in the study of standing wave solutions of nonlinear Klein-Gordon equations and of nonlinear Schrödinger equations. For existence theory of Eq. (1.1), a major difficulty is that in $\mathbb{R}^n$ we no longer have Sobolev Compact Embedding Theorems. Nevertheless, the important special case $Q \equiv 1$ and its generalizations have been investigated extensively by various authors, which in particular include Nehari [10]. Synge [13], Berger [2]. Coffman [3], Strauss [12], Berestycki and Lions [1], and McLeod and Serrin [9]. In 1963 Nehari [10] showed that in $\mathbb{R}^3$ Eq. (1.1) with $Q \equiv 1$ has a *positive radial* solution provided that $1 < p \leqslant 4$, and that in case $p = 5$, such a solution does not exist. Nehari's results may be extended to general p and n with $1<p<(n+2)/(n-2)$ for existence and $p \geqslant (n+2)/(n-2)$ for nonexistence (see, e.g., [2, 11]). This nonexistence result together with the fact that any solution of (1.1) with $Q \equiv 1$ must be radial (see [5, 6]) gives us a nonexistence result of positive solution for (1.1) in case $p \geqslant (n+2)/(n-2)$. For general potentials Q, existence theorems have been established recently under various kinds of hypotheses on Q by Lions [8] and by Ding and Ni [4]. However, very simple examples (see Ni [11] for more details) show that in general one cannot hope to solve (1.1) even for bounded Q's. On the other hand, a result recently given by Ding and Ni [4] shows that the solvability is guaranteed for any nonnegative bounded Q provided that it is radial. The results presented in Section 2 below improve some of Lions in [8]. Symmetry properties of solutions have been studied by Gidas, Ni, and Nirenberg in a series of elegant papers [5, 6]. Our results here (Section 3 below) indicate that the radial symmetry of solutions of (1.1) is, in general, very sensitive to perturbations of the potential Q. In [4], the existence of positive radial solutions of (1.1) has been studied by Ding and Ni for a radial potential Q. It seems that our nonexistence result in Section 4 below shows that Corollary 4.8 in Ding and Ni [4] is optimal and thus completes the theory in some sense for radial cases.

We would like to point out that the results in Sections 2 and 3 can be extended to more general second order elliptic operators than Δ.

2. Existence Results

2.1. *Preliminaries.* In this section we study the existence of solutions of Eq. (1.1). We shall use a variational approach, namely the so-called "Concentration–Compactness Principle" developed by Lions (see [8]) in solving Eq. (1.1).

First, for convenience, some notations need to be introduced. Let

$$J(Q)[u] = \int_{\mathbb{R}^n} Q(x)\, |u|^{p+1}(x)\, dx, \tag{2.1}$$

where $Q(x)$ is continuous and bounded in $\mathbb{R}^n$ with $Q^+(x) \not\equiv$ in $\mathbb{R}^n$ and $u \in L^{p+1}(\mathbb{R}^n)$, where $Q^+(x) = \max\{Q(x), 0\}$. Next for $\lambda > 0$, we define

$$I_\lambda(Q) = \inf\{\, \|u\|^2 \colon u \in H^1(\mathbb{R}^n) \quad \text{and} \quad J(Q)[u] = \lambda\}. \tag{2.2}$$

Recall that $H^1(\mathbb{R}^n)$ is the space of the closure of $C_0^\infty(\mathbb{R}^n)$ under the following norm

$$\|u\|^2 = \|u\|^2_{H^1} = \int_{\mathbb{R}^n} (|\nabla u|^2 + u^2)\, dx.$$

Finally, let us denote Q^* as $\limsup_{x \to \infty} Q(x)$ and Q_* as $\inf_{x \in \mathbb{R}^n} Q(x)$.

Remarks 2.1. By (1.2), it is clear that $H^1(\mathbb{R}^n) \subset L^{p+1}(\mathbb{R}^n)$, so that $J(Q)[\cdot]$ is well defined in $H^1(\mathbb{R}^n)$.

2.2. Since it is assumed that $Q^+ \not\equiv 0$ in $\mathbb{R}^n$, we have that $\{u \in H^1(\mathbb{R}^n) : J(Q)[u] = \lambda\}$ is not empty for every $\lambda > 0$.

Now, one may ask the following

QUESTION. *Is there a* $u \in H^1(\mathbb{R}^n)$ *such that* $J(Q)[u] = \lambda$ *and* $I_\lambda(Q) = \|u\|^2$?

This is a problem of existence of minimizers, and *we will denote it by* $(P_\lambda(Q))$ *or simply by* (P). It is clear that to establish the existence of solutions of Eq. (1.1), one can instead prove the existence for $(P_\lambda(Q))$ for some $\lambda > 0$ because such a minimizer will be a solution of Eq. (1.1) after a scaling.

It is known (see [4, 8]) that

(A) (P) *always possesses a positive minimizer if* $Q^* = Q_*$, *or* $Q^* \leqslant 0$.

(B) (P) *has no minimizers if* $\lim_{x \to \infty} Q(x) = \sup_{x \in \mathbb{R}^n} Q(x)$ *and* Q *is not a constant.*

In view of (A), we shall assume that $Q^* > 0$ for the rest of this paper.

For $(P_\lambda(Q))$ we can always choose minimizing sequences and it is known (see [8]) that if $\{u_m\}$ is a minimizing sequence of $(P_\lambda(Q))$, there exists a subsequence, say, without loss of generality $\{u_m\}$ itself, and a sequence $\{y_m\}$ in $\mathbb{R}^n$, such that for any $\varepsilon > 0$, there is a $R_\varepsilon < +\infty$ so that

$$\int_{B_{R_\varepsilon}(y_m)} |u_m|^{p+1}(x)\, dx > \lambda'_m - \varepsilon \qquad \text{for all} \quad m \geqslant 1, \tag{2.3}$$

where

$$\lambda'_m = \int_{\mathbb{R}^n} |u_m|^{p+1}(x)\, dx \qquad \text{and} \qquad \lambda'_m \to \lambda' > 0 \qquad \text{as} \quad m \to \infty \tag{2.4}$$

and $\{u_m\}$ converges to some function u in $H^1(\mathbb{R}^n)$ weakly. By the lower semicontinuity of norm, we have $\|u\|^2 \leqslant I_\lambda(Q)$. Therefore, to show that u is indeed a minimizer of $(P_\lambda(Q))$, what we have to show is that $J(Q)[u] \geqslant \lambda$!

Remark 2.3. It is clear that we may assume that $\{u_m\}$ and u are non-negative in this section, because if we replace u by $|u|$, $J(Q)[\cdot]$ remains the same and $\|\cdot\|$ can be only reduced (see [7, pp. 152]).

For the sake of convenience we state the following fact as a lemma and give a simple proof.

LEMMA 2.1. *Suppose that* $\{u_m\}$ *is a minimizing sequence of* $(P_\lambda(Q))$ *which converges to some function* u *in* $H^1(\mathbb{R}^n)$ *weakly. If there exists a bounded sequence* $\{y_m\}$ *in* $\mathbb{R}^n$ *such that the inequality* (2.3) *holds for all* $\varepsilon > 0$, *then* $J(Q)[u] = \lambda$.

Proof. Since $\{y_m\}$ is bounded, (2.3) becomes $\forall \varepsilon > 0$, $\exists R_\varepsilon < +\infty$, so that

$$\int_{B_{R_\varepsilon}(0)} u_m^{p+1}(x)\,dx > \lambda'_m - \varepsilon \tag{2.3$'$}$$

and correspondingly, since

$$\lambda = J(Q)[u_m] = \int_{B_{R_\varepsilon}(0)} Q(x)\,u_m^{p+1}(x)\,dx + \int_{\mathbb{R}^n \setminus B_{R_\varepsilon}(0)} Q(x)\,u_m^{p+1}(x)\,dx,$$

i.e.,

$$\int_{B_{R_\varepsilon}(0)} Q(x)\,u_m^{p+1}(x)\,dx = \lambda - \int_{\mathbb{R}^n \setminus B_{R_\varepsilon}(0)} Q(x)\,u_m^{p+1}(x)\,dx$$

by (2.3)′ and (2.4), we have then

$$\lambda - \|Q\|_{L^\infty}\varepsilon \leqslant \int_{B_{R_\varepsilon}(0)} Q(x)\,u_m^{p+1}(x)\,dx \leqslant \lambda + \|Q\|_{L^\infty}\,\varepsilon. \tag{2.5}$$

But on $B_{R_\varepsilon}(0)$, we have the Sobolev Compactness Theorem. Hence let $m \to \infty$ in (2.5), it becomes

$$\lambda - \|Q\|_{L^\infty}\varepsilon \leqslant \int_{B_{R_\varepsilon}(0)} Q(x)\,u^{p+1}(x)\,dx \leqslant \lambda + \|Q\|_{L^\infty}\varepsilon$$

which of course implies that

$$\int_{\mathbb{R}^n} Q(x)\,u^{p+1}(x)\,dx = \lambda. \qquad \text{Q.E.D.}$$

2.2. THEOREM 2.1. *If* $\forall \varepsilon > 0$, *we have*

$$|\{x \in \mathbb{R}^n,\ Q(x) \geqslant Q_* + \varepsilon\}| < \infty.$$

Then $(P_\lambda(Q))$ *has at least one positive minimizer for every* $\lambda > 0$.

Proof. Let $\{u_m\}$, $\{y_m\}$, and u be as in (2.3) and (2.4). There are two possible cases:

First, there exists a bounded subsequence of $\{y_m\}$, then the conclusion follows from the argument in Lemma 2.1.

Second, if $y_m \to \infty$ as $m \to \infty$. Let

$$\tilde{u}_m(x) = u_m(x + y_m).$$

Hence for $\tilde{u}_m(x)$, we have

$$\|\tilde{u}_m\|^2 = \|u_m\|^2 \to I_\lambda(Q) \qquad \text{as} \qquad m \to \infty,$$

and $\forall \varepsilon > 0$, $\exists R_\varepsilon < +\infty$, so that

$$\int_{B_{R_\varepsilon}(0)} \tilde{u}_m^{p+1}(x)\, dx > \lambda'_m - \varepsilon,$$

where again we have

$$\lambda'_m = \int_{\mathbb{R}^n} \tilde{u}_m^{p+1}(x)\, dx = \int_{\mathbb{R}^n} u_m^{p+1}(x)\, dx.$$

Now, since $\{\tilde{u}_m\}$ is bounded in $H^1(\mathbb{R}^n)$, there is a subsequence of $\{\tilde{u}_m\}$, say $\{\tilde{u}_m\}$ itself again converges to a function $\tilde{u}$ in $H^1(\mathbb{R}^n)$ weakly, with

$$\|\tilde{u}\|^2 \leqslant I_\lambda(Q).$$

Next we want to show that

$$J(Q)[\tilde{u}] \geqslant \lambda$$

which will end the proof.

From our assumption on Q, it is clear that for any $\varepsilon > 0$, there is a $\tilde{R}_\varepsilon < +\infty$, such that

$$|\{Q > Q_* + \varepsilon\} \backslash B_{\tilde{R}_\varepsilon}(0)| < \varepsilon.$$

Since $y_m \to^m \infty$, there exists N_ε, such that $|y_m| > R_\varepsilon + \tilde{R}_\varepsilon$ for $m \geqslant N_\varepsilon$ (R_ε as in (2.3)). Therefore for $m \geqslant N_\varepsilon$

$$\begin{aligned}
&\int_{B_{R_\varepsilon}(0)} Q(x)\, \tilde{u}_m^{p+1}(x)\, dx - \int_{B_{R_\varepsilon}(y_m)} Q(x)\, u_m^{p+1}(x)\, dx \\
&\quad = \int_{B_{R_\varepsilon}(y_m)} [Q(x - y_m) - Q(x)]\, u_m^{p+1}(x)\, dx \\
&\quad = \int_{B_{R_\varepsilon}(y_m) \cap \{Q \geqslant Q_* + \varepsilon\}} [Q(x - y_m) - Q(x)]\, u_m^{p+1}(x)\, dx \\
&\qquad + \int_{B_{R_1}(y_m) \cap \{Q < Q_* + \varepsilon\}} [Q(x - y_m) - Q(x)]\, u_m^{p+1}(x)\, dx
\end{aligned}$$

$$\geqslant -(\|Q\|_{L^\infty} - Q_*) \int_{\{Q \geqslant Q_* + \varepsilon\} \setminus B_{R_\varepsilon}(0)} u_m^{p+1}(x)\, dx$$

$$-\varepsilon \int_{B_{R_\varepsilon}(y_m) \cap \{Q < Q_* + \varepsilon\}} u_m^{p+1}(x)\, dx$$

$$\geqslant -(\|Q\|_{L^\infty} - Q_*) \left(\int_{\{Q \geqslant Q_* + \varepsilon\} \setminus B_{R_\varepsilon}(0)} dx \right)^{1/q'} \left(\int_{\mathbb{R}^n} u_m^{2n/(n-2)}(x)\, dx \right)^{1/p'}$$

$$-\varepsilon \lambda'_m \geqslant -C(n, Q, p, \lambda)\, \varepsilon^{1/q'} - \varepsilon \lambda'_m,$$

where $p'(p+1) = 2n/(n-2)$, and $1/p' + 1/q' = 1$ and C is a constant depending only on n, Q, p, and λ. And because $\int_{B_{R_\varepsilon}(y_m)} Q(x)\, u_m^{p+1}(x)\, dx > \lambda - \|Q\|_{L^\infty} \varepsilon$ by (2.3) and (2.4), we have

$$\int_{B_{R_\varepsilon}(0)} Q(x)\, \tilde{u}_m^{p+1}(x)\, dx > \lambda - \|Q\|_{L^\infty} \varepsilon - C\varepsilon^{1/q'} - \varepsilon \lambda'_m,$$

and applying the Compactness Theorem on $B_{R_\varepsilon}(0)$ for $\{\tilde{u}_m\}$, we have

$$\int_{B_{R_\varepsilon}(0)} Q(x)\, \tilde{u}^{p+1}(x)\, dx \geqslant \lambda - \|Q\|_{L^\infty} \varepsilon - C\varepsilon^{1/q'} - \varepsilon \lambda'$$

for any $\varepsilon > 0$. Therefore, we finally get

$$\int_{\mathbb{R}^n} Q(x)\, \tilde{u}^{p+1}(x)\, dx \geqslant \lambda.$$

This completes the proof. Q.E.D.

THEOREM 2.2. *Let $D(r) = \int_{|x| \leqslant r} [Q(x) - Q^*]\, dx$, for $r \geqslant 0$. If $D(r) \geqslant 0$ in $\mathbb{R}^+$ and is not identically 0 then $(P_\lambda(Q))$ has at least one minimizer in $H^1(\mathbb{R}^n)$ for every $\lambda > 0$.*

Proof. First we want to show that under the above assumption on Q, we have for every $\lambda > 0$ the following:

$$I_\lambda(Q) < I_\lambda(Q^*). \tag{2.6}$$

Let u_0 be a positive minimizer of $(P_1(1))$ in $\mathbb{R}^n$. Then u_0 must be radially symmetric, $u'_0(r) < 0$ for $r > 0$ and more

$$\lim_{r \to \infty} r^{(n-1)/2} e^r u_0(r) = \mu > 0 \qquad \text{(see [6])}. \tag{2.7}$$

And u_0 is also a minimizer of $(P_{Q^*}(Q^*))$, but since

$$I_\lambda = \lambda^{2/(p+1)}\, I_1 = \left(\frac{\lambda}{Q^*} \right)^{1/(p+1)} I_{Q^*}$$

for any Q with $Q^*>0$, we get that

$$I_\lambda(Q)<I_\lambda(Q^*),\ \forall\lambda>0 \qquad \text{iff } I_{Q^*}(Q)<I_{Q^*}(Q^*).$$

Now, by the hypotheses on Q, we have

$$\begin{aligned}
&\int_{\mathbb{R}^n} Q(x)\,u_0^{p+1}(x)\,dx-\int_{\mathbb{R}^n} Q^*u_0^{p+1}(x)\,dx\\
&\quad=\int_{\mathbb{R}^n}[Q(x)-Q^*]\,u_0^{p+1}(x)\,dx=\lim_{r\to+\infty}\int_{|x|\leqslant r}[Q(x)-Q^*]\,u_0^{p+1}(x)\,dx\\
&\quad=\lim_{r\to+\infty}\int_0^r dt\int_{|x|=t}[Q(x)-Q^*]\,u_0^{p+1}(t)\,dS_x\\
&\quad=\lim_{r\to+\infty}\int_0^r u_0^{p+1}(t)\,D'(t)\,dt\\
&\quad=\lim_{r\to+\infty}\left[u_0^{p+1}(r)\,D(r)-(p+1)\int_0^r D(t)\,u_0^p(t)\,u_0'(t)\,dt\right]\\
&\quad=-(p+1)\int_0^r D(t)\,u_0^p(t)\,u_0'(t)\,dt>0
\end{aligned}$$

by (2.7) which implies that $I_{Q^*}(Q)<I_{Q^*}(Q^*)$.

Second, we want to show that $\{y_m\}$ has a bounded subsequence. If not, suppose $y_m\to^m\infty$. But we have from (2.3) and (2.4)

$$\int_{B_{R\varepsilon}(y_m)} Q(x)\,u_m^{p+1}(x)\,dx>\lambda-\|Q\|_{L^\infty}\varepsilon,\qquad \forall\varepsilon>0.$$

Since

$$\int_{\mathbb{R}^n} Q^*u_m^{p+1}(x)\,dx\geqslant\int_{B_{R\varepsilon}(y_m)} Q^*u_m^{p+1}(x)\,dx$$
$$=\int_{B_{R_\varepsilon}(y_m)}[\limsup_{y\to\infty} Q(y)]\,u_m^{p+1}(x)\,dx \tag{2.8}$$

and $y_m\to\infty$ as $m\to\infty$, we will have the following: let $\lambda_m^*=\int_{\mathbb{R}^n} Q^*u_m^{p+1}(x)\,dx$, which converges to $Q^*\lambda'>0$ by (2.4), and with (2.8), we have that $Q^*\lambda'\geqslant\lambda$. But

$$I_{Q^*\lambda'}(Q^*)\leqslant\liminf_{m\to\infty}\|u_m\|^2,$$

giving us that $I_{Q^*\lambda'}(Q^*)\leqslant I_\lambda(Q)$, in particular

$$I_\lambda(Q^*)\leqslant I_\lambda(Q)\qquad\text{since}\qquad\lambda\leqslant Q^*\lambda',$$

contradicting the above result. Therefore, $\{y_m\}$ does have a bounded subsequence. This ends the proof by Lemma 2.1. Q.E.D.

Remark 2.4. *From the argument of Theorem* 2.2 *it follows that Eq.* (1.1) *possesses a solution if*

$$\int_{\mathbb{R}^n} Q(x)\, u_0^{p+1}(x)\, dx > \limsup_{x \to \infty} Q(x).$$

2.3. EXAMPLES. (1) For any $f(x) \in C^0(\mathbb{R}^n)$, $f(x) \geqslant 0$, and $\operatorname{supp} f \subset B_1(0)$, let $Q_f(x) = 1 + \sum_{k=1}^{\infty} f(k^2(x - kv))$ for some $v \in S^{n-1}$. Then $Q_* = 1$, $Q^* = 1 + \|f\|_{L^\infty} = \sup_{x \in \mathbb{R}^n} Q(x)$, but the hypothesis of Theorem 2.1 is satisfied. Note that $D(r) < 0$ for r large enough if $f \not\equiv 0$ in $\mathbb{R}^n$.

(2) Letting $Q_0(x) = 1 + e^{-|x|^n} \cos(|x|^n)$, we see that $Q_0^* = 1 > Q_{0*}$. Theorem 2.2 can be applied for Q_0:

$$\begin{aligned} D_0(r) &= \omega_n \int_0^r t^{n-1} e^{-t^n} \cos(t^n)\, dt \\ &= \frac{\omega_n}{2n} \left[1 + \sqrt{2}\, e^{-r^n} \sin\left(r^n - \frac{\pi}{4} \right) \right] > 0 \end{aligned}$$

for all $r > 0$.

3. AN EXAMPLE OF POSITIVE NONRADIAL SOLUTIONS

3.1. *Preliminaries.* We shall consider only radial potentials in this section. It is well known that every positive solution of Eq. (1.1) must be radially symmetric if Q is a positive constant, and it is also the case for the Dirichlet Problem with boundary value zero on $\partial B_R(0)$ (see [5, 6]). But nevertheless, for certain type $Q(r)$, the existence of positive nonradial solutions in $B_R(0)$ has been proved in [4] for large R. Here we will give an example showing the existence of positive nonradial solutions in $\mathbb{R}^n$ with $n \geqslant 3$.

Let $H_r^1(\mathbb{R}^n)$ be the closure of compactly supported smooth radial functions on $\mathbb{R}^n$ in $H^1(\mathbb{R}^n)$. For a given $Q(r) \in C^0(\overline{\mathbb{R}^+})$, which is bounded and $Q^+(r) \not\equiv 0$ in $\overline{\mathbb{R}^+}$, define

$$M_r(Q) = \sup\{J(Q)[u] \colon u \in H_r^1(\mathbb{R}^n) \quad \text{and} \quad \|u\| = 1\} \tag{3.1}$$

and

$$M(Q) = \sup\{J(Q)[u] \colon : u \in H^1(\mathbb{R}^n) \quad \text{and} \quad \|u\| = 1\}. \tag{3.2}$$

Remarks. 3.1. It is obvious that $M_r(Q) \leqslant M(Q)$ and we may assume that the maximizers (if they exist) are nonnegative. And for bounded Q in $\mathbb{R}^n$ we always have $M(Q) < \infty$:

By the Sobolev inequality we know that for all u on the unit sphere in $H^1(\mathbb{R}^n)$ we have

$$\int_{\mathbb{R}^n} |u|^{2n/(n-2)}(x)\, dx \leqslant c(n),$$

where $C(n)$ is some positive constant depending only on n.

Now because of condition (1.2), we have for some positive constant $\tilde{C}(p)$ that

$$t^{p+1} \leqslant \tilde{C}(p)(t^2 + t^{2n/(n-2)}) \text{ in } \overline{\mathbb{R}^+}.$$

Hence for any u in $H^1(\mathbb{R}^n)$ with $\|u\| = 1$, we have the following

$$\begin{aligned} J(Q)[u] &\leqslant \sup_{\mathbb{R}^n} Q \cdot \int_{\mathbb{R}^n} |u|^{p+1}(x)\, dx \\ &\leqslant \tilde{C}(p) \sup_{\mathbb{R}^n} Q \cdot \int_{\mathbb{R}^n} (|u|^2 + |u|^{2n/(n-2)})(x)\, dx \\ &\leqslant \tilde{C}(p)(1 + C(n)) \cdot \sup_{\mathbb{R}^n} Q \\ &< \infty. \end{aligned}$$

Q.E.D.

3.2. It is well known (see [12]) that every $u \in H_r^1(\mathbb{R}^n)$ is almost everywhere equal to a function $\tilde{u}(x)$, which is continuous for $x \neq 0$ and such that

$$|\tilde{u}(x)| \leqslant C_n\, |x|^{(1-n)/2}\, \|u\|, \qquad \text{for} \quad \|x\| \geqslant \alpha_n, \tag{3.3}$$

where C_n and α_n are positive constants depending only on n.

3.3. By a compactness result (see [1,12]) for $H_r^1(\mathbb{R}^n)$, $M_r(Q)$ is realized for every such Q.

3.2. *Example.* We are now ready to construct an example for which a positive nonradial solution of (1.1) exists. Let

$$\varphi_0(t) = 1 + \cos\left(t - \frac{\pi}{2}\right) \chi_{[0,\pi]}(t), \qquad t \in \mathbb{R},$$

where χ_E denotes the characteristic function of set $E \subset \mathbb{R}$. Set

$$\varphi_\alpha(t) = \varphi_0(t - \alpha); \tag{3.4}$$

we know that $M_r(\varphi_\alpha)$ (and $M(\varphi_\alpha)$, resp.) is achieved by some positive function in $H_r^1(\mathbb{R}^n)$ (in $H^1(\mathbb{R}^n)$, resp.) [4, 8]. Denote u_α as a maximizer which assumes $M_r(\varphi_\alpha)$ in $H_r^1(\mathbb{R}^n)$. But by (3.3), if $\alpha \geqslant \alpha_n + \pi/2$, we have

$$\begin{aligned}
&J(1)[u_\alpha] - J(\varphi_\alpha)[u_\alpha) \\
&\quad = \int_{\alpha \leqslant |x| \leqslant \alpha+\pi} \left[1 - \left(1 + \cos\left(|x| - \frac{\pi}{2} - \alpha\right)\right)\right] u_\alpha^{p+1}(|x|)\, dx \\
&\quad \geqslant - \int_{\alpha \leqslant |x| \leqslant \alpha+\pi} u_\alpha^{p+1}(|x|)\, dx \\
&\quad \geqslant - \left(\int_{\alpha \leqslant |x| \leqslant \alpha+\pi} u_\alpha^{2}(|x|)\, dx\right)^{((p+1)/2)\lambda} \\
&\qquad \left(\int_{\alpha \leqslant |x| \leqslant \alpha+\pi} u_\alpha^{q}(|x|)\, dx\right)^{((p+1)/q)(1-\lambda)},
\end{aligned}$$

where $1 = ((p+1)/2)\,\lambda + ((p+1)/q)(1-\lambda)$, $2 < p+1 < q$ by the Holder Inequality. Since $\|u_\alpha\| = 1$, we then have by choosing $q = 2n/(n-2) > p+1$

$$\begin{aligned}
&J(1)[u_\alpha] - J(\varphi_\alpha)[u_\alpha] \\
&\quad \geqslant - \left(\int_{\alpha \leqslant |x| \leqslant \alpha+\pi} u_\alpha^{q}(|x|)\, dx\right)^{((p+1)/q)(1-\lambda)} \\
&\quad \geqslant -C(p, n)\, \alpha^{-(p-1)(n-1)/2},
\end{aligned}$$

where $C(p, n)$ is a positive constant depending only on p and n.

Combining with the fact that $M_r(\varphi_\alpha) \geqslant M_r(1)$, we finally obtain that

$$M_r(\varphi_\alpha) \to M_r(1) \qquad \text{as} \qquad \alpha \to +\infty.$$

Now observing that $M_r(1) = M(1)$ (see [6]), and that $M(\varphi_\alpha) \geqslant M(\varphi_{-\pi/2}) > M(1)$ for all $\alpha \geqslant -\pi/2$, because by the maximum principle each maximizer must be strictly positive, we conclude that $M_r(\varphi_\alpha)$ is strictly less than $M(\varphi_\alpha)$ for α sufficiently large, which in turn implies that *maximizers attaining $M(\varphi_\alpha)$ must be nonradial for large* α. Because otherwise, if one of the positive maximizers, say u_α, is radially symmetric about some point x_0 in $\mathbb{R}^n$, then we know that u_α is a solution of Eq. (1.1) with the potential φ_α and more we have that $(u_\alpha - \Delta u_\alpha)/u_\alpha^p = \varphi_\alpha$ is also radially symmetric about x_0, which implies that x_0 must be the origin because φ_α is radially symmetric only about the origin.

Therefore u_α is in $H_r^1(\mathbb{R}^n)$ and $M(\varphi_\alpha)$ must thus be equal to $M_r(\varphi_\alpha)$ which contradicts our argument for large α.

Remark 3.4. It is clear from this proof that nonradial solutions exist for various positive perturbations of constant $Q \equiv 1$.

4. A Nonexistence Result

While one could expect the existence of solutions of Eq. (1.1) for bounded Q under mild restrictions, it was not known in general, for the situations where Q is unbounded near infinity. Recently, in a paper by Ding and Ni [4], it was proved that Eq. (1.1) always possesses a positive radial solution in $\mathbb{R}^n$ provided that $Q(x)$ is radially symmetric and

$$0 \leqslant Q(x) \leqslant (\text{positive constant})(1+|x|)^l,$$

where $0 \leqslant l < (n-1)(p-1)/2$.

Our concern here is the nonexistence of solutions of (1.1) for radial potentials; for this we have obtained

THEOREM 4.1. *There is no positive radial solution of* (1.1), *if* $Q(r) \geqslant 0$ *and* $Q(r)\, r^{-(n-1)(p-1)/2}$ *is nondecreasing, where* $Q(r) \in C^{0,1}(\overline{\mathbb{R}^+})$ *and* $n \geqslant 3$.

4.1. *Preliminaries.* Suppose that $u(x) \in C^2(\mathbb{R}^n)$ is a solution of (1.1). Set $V(x) = K(|x|)\, u(x)$ where $0 < K \in C^2(\mathbb{R}^+)$. Then

$$\Delta V(x) - \frac{2K'(r)}{rK(r)}\, x \cdot \nabla V(x) - \left\{1 + \frac{K''(r)}{K(r)} + \frac{(n-1)\, K'(r)}{rK(r)} - \frac{2[K'(r)]^2}{K^2(r)}\right\} V(x) + Q(r)\, K^{1-p}(r)\, V^p(x) = 0 \tag{4.1}$$

in $\mathbb{R}^n \backslash \{0\}$, where $r = |x|$.

Now, for a special $K(r) = r^{(n-1)/2}$, we have

$$\Delta V(x) - \frac{n-1}{r^2}\, x \cdot \nabla V(x) - \left\{1 + \frac{(n-1)(n-3)}{4r^2}\right\} V(x) + Q(r)\, r^{-(n-1)(p-1)/2}\, V^p(x) = 0 \tag{4.2}$$

in $\mathbb{R}^n \backslash \{0\}$.

If we assume further that u is radial (and so is V), we then have

$$V''(r) - \left\{1 + \frac{(n-1)(n-3)}{4r^2}\right\} V(r) + Q(r)\, r^{-(n-1)(p-1)/2}\, V^p(r) = 0 \tag{4.3}$$

in $\mathbb{R}^+$.

4.2. *Some lemmas.* Here we will first prove some lemmas for Theorem 4.1. The case $n = 3$ appears to be easier in (4.3) and we have a better understanding about it.

LEMMA 4.1. *The initial value problem*

$$\begin{aligned} V''(t) - V(t) + V^p(t) = 0, \qquad p > 1, \\ V(0) = 0, \qquad V'(0) = a > 0 \end{aligned} \tag{4.4}$$

does not possess any positive solution in $(0, \infty)$.

Proof. Suppose V is a positive solution of Eq. (4.4). We then have

$$V''(t) = V(t)[1 - V^{p-1}(t)] = \begin{cases} >0 & \text{if } \ 0 < V < 1 \\ <0 & \text{if } \ V > 1. \end{cases} \tag{4.5}$$

Claim. $\exists t_0 > 0$, so that $V'(t_0) = 0$.

Suppose not; i.e., we suppose that $V'(t) > 0$ for $t > 0$.

(i) If $V < 1$ for all $t > 0$, then $V'' > 0$ by (4.5) together with $V'(t) > 0$ for $t > 0$ will imply that $V(t) \to +\infty$ as $t \to +\infty$, a contradiction;

(ii) If for some t, $V(t) \geqslant 1$. Now, let t_1 be the point where $V = 1$. Therefore $V(t) \geqslant V(t_1 + 1) > 1$ for $t \geqslant t_1 + 1$. Since $V' > 0$ for $t > 0$, then, V'' is decreasing in $(t_1 + 1, \infty)$ since $V^{(3)} < 0$ there, which in turn implies that

$$\begin{aligned} V'(t) &= V'(t_1 + 1) + \int_{t_1 + 1}^{t} V''(s)\, ds \\ &\leqslant V'(t_1 + 1) + V''(t_1 + 1)(t - t_1 - 1) \to -\infty \end{aligned}$$

as $t \to +\infty$, a contradiction.

Therefore, let t_0 be the first point where $V'(t_0) = 0$. By the uniqueness and time-invertible property of Eq. (4.4), we conclude that $V(2t_0) = 0$, a contradiction again. This completes the proof. Q.E.D.

LEMMA 4.2. *Every positive solution for the following initial value problem*

$$\begin{aligned} V''(t) - \left(1 + \frac{c}{t^2}\right) V(t) + W(t)\, V^p(t) = 0 \qquad \textit{for} \quad t > 0 \\ V(0) = 0, \qquad V'(0) = a \geqslant 0, \end{aligned} \tag{4.6}$$

where $c > 0$, $p > 1$, *W is a nondecreasing function in* $C^{0,1}(\overline{\mathbb{R}^+})$ *with* $W(0) \geqslant 0$, $W \not\equiv 0$, *and* $(1 - c)\, a^2 \geqslant 0$, *satisfies* $V(t) = 0(e^{-\alpha t^2})$ *at* ∞ *for some* $\alpha > 0$.

Proof. Let V be such a solution. We first claim that $\exists\ t_0 > 0$, so that $V'(t_0) = 0$. For otherwise, since $V(0) = 0$, $V'(0) \geqslant 0$, and $V(t) > 0$ for $t > 0$, we must have that $V'(t) > 0$ for $t > 0$. Now let $\beta = \lim_{t \to +\infty} W(t)$ (β could be ∞). We discuss three possible cases.

(i) $\beta<\infty$ and $\beta V^{p-1}(t)<1$ for all $t>0$. Then $V''(t)>0$ for all $t>0$. It gives a contradiction as in Lemma 4.1 (i);

(ii) $\beta<\infty$ and there exists a point t_1 at which $\beta V^{p-1}(t_1)=1$. Then $V(t)\geqslant V(t_1+1)>(1/\beta)^{1/(p-1)}$, for all $t\geqslant t_1+1$ since $V'>0$ and

$$V''(t)=V(t)\left[\left(1+\frac{c}{t^2}\right)-W(t)\,V^{p-1}(t)\right]\leqslant -d(c,\beta)<0$$

for all t large enough, where d is a constant depending only on c and β for all large t. But as in Lemma 4.1 (ii), this again gives a contradiction;

(iii) $\beta=\infty$. This case can be handled as in case (ii) above. Therefore, our assertion follows.

Multiplying Eq. (4.6) by $V'(t)$ and integrating over $[\varepsilon, t]$ for any $\varepsilon>0$, we have

$$\frac{1}{2}V'^2(t)-\frac{1}{2}V'^2(\varepsilon)-\frac{1}{2}\int_\varepsilon^t\left(1+\frac{c}{s^2}\right)(V^2)'(s)\,ds$$
$$+\frac{1}{p+1}\int_\varepsilon^t W(s)(V^{p+1})'(s)\,ds=0.$$

Integration by parts gives

$$\frac{1}{2}V'^2(t)-\frac{1}{2}V'^2(\varepsilon)-\frac{1}{2}\left(1+\frac{c}{t^2}\right)V^2(t)+\frac{1}{2}\left(1+\frac{c}{\varepsilon^2}\right)V^2(\varepsilon)$$
$$-\int_\varepsilon^t\frac{c}{s^3}V^2(s)\,ds+\frac{1}{p+1}W(t)\,V^{p+1}(t)-\frac{1}{p+1}W(\varepsilon)\,V^{p+1}(\varepsilon)$$
$$-\frac{1}{p+1}\int_\varepsilon^t W'(s)\,V^{p+1}(s)\,ds=0.$$

Now, letting $\varepsilon\to 0$ and noting that $V(\varepsilon)/\varepsilon\to V'(0)$, we obtain

$$V'^2(t)=\left(1+\frac{c}{t^2}\right)V^2(t)+2c\int_0^t\frac{V^2(s)}{s^3}\,ds-\frac{2}{p+1}W(t)\,V^{p+1}(t)$$
$$+\frac{2}{p+1}\int_0^t W'(s)\,V^{p+1}(s)\,ds+(1-c)\,a^2. \tag{4.7}$$

Next, we multiply (4.6) by $V(t)$ and subtract (4.7) from

$$V(t)\,V''(t)-V'^2(t)=-2c\int_0^t\frac{V^2(s)}{s^3}\,ds-\frac{p-1}{p+1}W(t)\,V^{p+1}(t)$$
$$-\frac{2}{p+1}\int_0^t W'(s)\,V^{p+1}(s)\,ds-(1-c)\,a^2,$$

therefore

$$\left[\frac{V'(t)}{V(t)}\right]' = -\frac{2c}{V^2(t)}\int_0^t \frac{V^2(s)}{s^3}\,ds - \frac{p-1}{p+1}\,W(t)\,V^{p-1}(t)$$
$$-\frac{2}{(p+1)\,V^2(t)}\int_0^t W'(s)\,V^{p+1}(s)\,ds - \frac{(1-c)\,a^2}{V^2(t)}.$$

Integrating over $[t_0, t]$ for $r > t_0$, we have

$$\begin{aligned}\frac{V'(t)}{V(t)} = &-2c\int_{t_0}^t \frac{dr}{V^2(r)}\int_0^r \frac{V^2(s)}{s^3}\,ds - \frac{p-1}{p+1}\int_{t_0}^t W(r)\,V^{p-1}(r)\,dr\\ &-\frac{2}{p+1}\int_{t_0}^t \frac{dr}{V^2(r)}\int_0^r W'(s)\,V^{p+1}(s)\,ds - (1-c)\,a^2\int_{t_0}^t \frac{dr}{V^2(r)}.\end{aligned} \tag{4.8}$$

Thus $V'(t) < 0$ for all $t > t_0$; i.e., $V(t)$ decreases in (t_0, ∞) and

$$\begin{aligned}\frac{V'(t)}{V(t)} &\leqslant -2c\int_{t_0}^t \frac{dr}{V^2(r)}\int_0^r \frac{V^2(s)}{s^3}\,ds\\ &\leqslant -2c\int_{t_0}^t \frac{dr}{V^2(t_0)}\int_0^{t_0} \frac{V^2(s)}{s^3}\,ds\\ &= -2\alpha(t-t_0),\end{aligned}$$

where $\alpha = c[V(t_0)]^{-2}\int_0^{t_0} V^2(s)\,s^{-3}\,ds$, and which gives us the desired result. Q.E.D.

LEMMA 4.3. *Suppose V is a positive solution for the following initial value problem*

$$\begin{aligned} V''(t) - V(t) + W(t)\,V^p(t) = 0 \qquad &\text{for} \quad t > 0\\ V(0) = 0, \qquad V'(0) = a \geqslant 0,&\end{aligned} \tag{4.9}$$

where $p > 1$ and W is as in Lemma 4.2. Then $V(t)\cdot e^{\alpha t^2}$ is bounded in $\mathbb{R}^+$ for some $\alpha > 0$, provided that W is not a constant.

The proof follows from Eq. (4.8), where we have

$$\begin{aligned}\frac{V'(t)}{V(t)} = -\frac{p-1}{p+1}\int_{r_0}^t W(r)\,V^{p-1}(r)\,dr - \frac{2}{p+1}\int_{t_0}^t \frac{dr}{V^2(r)}\int_0^r W'(s)\,V^{p+1}(s)\,ds\\ -a^2\int_{t_0}^t \frac{dr}{V^2(r)}\end{aligned} \tag{4.10}$$

and we make use of the following

$$\frac{V'(t)}{V(t)} \leqslant -\frac{2}{p+1} \int_{t_0}^{t} \frac{dr}{V^2(r)} \int_0^r W'(s)\, V^{p+1}(s)\, ds.$$

Then the very same arguments above work provided that W is not a constant, i.e., $\exists \tilde{t} > 0$, so that

$$\int_0^{\tilde{t}} W'(s)\, V^{p+1}(s)\, ds > 0.$$

4.3. *Proof of Theorem* 4.1. Suppose u is a positive radial solution of Eq. (1.1) with Q satisfying the hypotheses of Theorem 4.1. Since $Q(r) \geqslant 0$, we have

$$\Delta u - u \leqslant 0$$

which implies that (see [6]) for some $\mu > 0$:

$$u(x) \geqslant \mu e^{-|x|}/|x|^{(n-1)/2} \qquad \text{near} \quad \infty.$$

Hence, if we set $V(r) = r^{(n-1)/2}\, u(r)$, we get

$$V(r) \geqslant \mu e^{-r} \qquad \text{for large} \quad r \tag{4.11}$$

and by (4.3) we see that V satisfies Eq. (4.6). Now (4.11) contradicts the conclusions of Lemmas 4.1, 4.2, and 4.3. Q.E.D.

Remark 4.1. Theorem 4.1 shows that the existence result obtained by Ding and Ni in [4] (namely, Corollary 4.8 in [4]) is optimal in a certain sense.

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