Estimates in the Generalized Morrey Space for Linear Parabolic Systems

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Estimates in the Generalized Morrey Spaces for Linear Parabolic Systems

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I HEREBY RECOMMEND THAT THE THESIS PREPARED UNDER MY SUPERVISION BY Matthew Scott McBride ENTITLED Estimates in the Generalized Morrey Spaces for Linear Parabolic Systems BE ACCEPTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF Master of Science.

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Abstract

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The purpose of this paper is to study the parabolic system $u_t^i - D_\alpha(a_{ij}^{\alpha\beta} D_\beta u^j) = -div\, f^i$ in the generalized Morrey Space $L_\varphi^{2,\lambda}$. We would like to understand the regularity of the solutions of this system. It will be shown that 1: if $a_{ij}^{\alpha\beta} \in C(\overline{Q_T})$ then $Du \in L_\varphi^{2,\lambda}$, and 2: if $a_{ij}^{\alpha\beta} \in VMO(Q_T)$ then $Du \in L_\varphi^{2,\lambda}$. Moreover we will be able to obtain estimates on the gradient of the solutions to the system, which will tell us about the regularity of the solutions.

Contents

1 Introduction

In this paper we will be investing the following linear parabolic systems of the form

(1-1) $u_t^i - D_\alpha \left(a_{ij}^{\alpha\beta}(x,t) D_\beta u^j \right) = -div\, f^i(x,t) \qquad i = 1, \ldots, N$

where $i, j = 1, \ldots, N$; $\alpha, \beta = 1, \ldots, n$ and the repeated indices denote summation such as $\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} \xi^i \xi^j = a_{ij} \xi^i \xi^j$

Throughout the paper we assume an uniform ellipticity condition, namely:

(1-2) $\Lambda^{-1} |\xi|^2 \le a_{ij}^{\alpha\beta}(x,t) \xi_\alpha^i \xi_\beta^j \le \Lambda |\xi|^2$ where $\Lambda > 0$, $\xi \in \mathbb{R}^{(n+1)N}, (x,t) \in Q_T, Q_T = \Omega \times [0,T],\ \Omega \subset \mathbb{R}^n$

The main purpose of this paper is to demostrate that one can obtain the gradient estimates in generalized Morrey spaces $L_\varphi^{2,\lambda}$ for weak solutions of (1-1).

2 Preliminaries

Notations:

$B_R(x_0) = \{x \in \mathbb{R}^n |\ |x - x_0| < R\}$-ball in $\mathbb{R}^n$ centered at x_0 with radius R

$z_0 = (x_0, t_0) \in \mathbb{R}^{n+1}$ and $z = (x, t) \in \mathbb{R}^{n+1}$ for $x \in \mathbb{R}^n$ and $t \in (0, T]$

$Q_R(z_0) = B_R(x_0) \times (t_0 - R^2, t_0]$ - parabolic cylinder in $\mathbb{R}^{n+1}$ vertexed at $\underline{z_0}$

Boundary Terms of the Parabolic Cylinder:

The boundary of the parabolic cylinder consists of the lateral walls, the lower boundary, and the lower corners, however we will use $\partial_p Q_R$ to denote the parabolic boundary of the parabolic cylinder

Morrey Space for Parabolic Setting:

(2-1) $L^{p,\lambda}_{\varphi}(Q_T) = \left\{ f \in L^p(Q_T) \,|\, \underset{z_0 \in Q_T, 0 \le \rho \le d}{sup} \frac{1}{\varphi(\rho)} \left(\rho^{-\lambda} \int\int_{Q_T \cap Q_\rho(z_0)} |f|^p \, dz \right)^{\frac{1}{p}} < \infty \right\}$ with $1 \le p < \infty, 0 \le \lambda \le n + 2\,\varphi$ is a continuous function on $[0, d]\,,\ \varphi > 0$ on $(0, d], d$ is the diameter of $Q_T = \Omega \times (0, T]\ \Omega \subset \mathbb{R}^n$.

Lemma 2.1:

$L^{p,\lambda}_{\varphi}$ is a Banach Space under the norm

$$||f||_{L^{p,\lambda}_{\varphi}} = \underset{z_o \in Q_T, 0 \le \rho \le d}{sup} \frac{1}{\varphi(\rho)} \left(\rho^{-\lambda} \int\int_{Q_T \cap Q_\rho(z_0)} |f|^p \right)^{\frac{1}{p}}$$

Proof:

So the "norm" must satisfy the three properties to classify as a norm, then it must be shown that the space is complete. $||f||_{L^{p,\lambda}_{\varphi}} \ge 0$ is trival similarly $||\alpha f||_{L^{p,\lambda}_{\varphi}} = |\alpha|\, ||f||_{L^{p,\lambda}_{\varphi}}$ is quite obvious, though the triangle inequality must be shown since it is not very simple.

Consider $\frac{1}{\varphi(\rho)} \left(\rho^{-\lambda} \int\int_{Q_T \cap Q_\rho(z_0)} |f + g|^p \, dz \right)^{\frac{1}{p}}$

$\le \frac{1}{\varphi(\rho)} \left(\left(\rho^{-\lambda} \int\int_{Q_T \cap Q_\rho(z_0)} |f|^p \, dz \right)^{\frac{1}{p}} + \left(\rho^{-\lambda} \int\int_{Q_T \cap Q_\rho(z_0)} |g|^p \, dz \right)^{\frac{1}{p}} \right)$ via Minkowski's inequality. The one applys the *sup* function to both sides which yields,

$$sup \frac{1}{\varphi(\rho)} \left(\rho^{-\lambda} \int\int_{Q_T \cap Q_\rho(z_0)} |f + g|^p \, dz \right)^{\frac{1}{p}} \le$$

$$sup \frac{1}{\varphi(\rho)} \left(\left(\rho^{-\lambda} \int\int_{Q_T \cap Q_\rho(z_0)} |f|^p \, dz \right)^{\frac{1}{p}} + \left(\rho^{-\lambda} \int\int_{Q_T \cap Q_\rho(z_0)} |g|^p \, dz \right)^{\frac{1}{p}} \right)$$

$\leq sup\dfrac{1}{\varphi(\rho)}\left(\rho^{-\lambda}\displaystyle\int\int_{Q_T\cap Q_\rho(z_0)}|f|^p\,dz\right)^{\frac{1}{p}}+sup\dfrac{1}{\varphi(\rho)}\left(\rho^{-\lambda}\displaystyle\int\int_{Q_T\cap Q_\rho(z_0)}|g|^p\,dz\right)^{\frac{1}{p}}$. This implies that $||f+g||_{L^{p,\lambda}_\varphi}\leq||f||_{L^{p,\lambda}_\varphi}+||g||_{L^{p,\lambda}_\varphi}$ Therefore the triangle inequality is satisfied, and $||\cdot||_{L^{p,\lambda}_\varphi}$ is a norm on $L^{p,\lambda}_\varphi$.

Next it must be shown that $L^{p,\lambda}_\varphi$ is complete under the norm $||\cdot||_{L^{p,\lambda}_\varphi}$. Let $(f_k)_{k=1}^\infty$ be a Cauchy sequence in $L^{p,\lambda}_\varphi$. Tschebyshev's inequality implies that $|\{z\in Q_T\,|\,|f_k(z)-f_m(z)|>\varepsilon\}|\leq\varepsilon^{-p}\displaystyle\int\int_{Q_T\cap Q_\rho(z_0)}|f_k-f_m|^p\,dz$. Therefore, there exists a subsequence $\left(f_{k_j}\right)$ and f such that $f_{k_j}\to f$ *a.e. in* Q_T. For every $\varepsilon>0$ there exists K such that $\left|\left|f_{k_j}-f_k\right|\right|_{L^{p,\lambda}_\varphi}<\varepsilon$ if $k_j,k>K$. Let $k_j\to\infty$ then by Fatou's lemma, one obtains, $||f-f_k||_{L^{p,\lambda}_\varphi}<\varepsilon$ for $k>K$. Thus $f\in L^{p,\lambda}_\varphi$ by $||f||_{L^{p,\lambda}_\varphi}\leq||f-f_k||_{L^{p,\lambda}_\varphi}+||f_k||_{L^{p,\lambda}_\varphi}<\infty$ and $||f-f_k||_{L^{p,\lambda}_\varphi}\to 0\;as\;k\to\infty$. Therefore $L^{p,\lambda}_\varphi$ is complete and hence it is a Banach space. $\square$

Morrey Space for p=2:

We consider the case of $L^{p,\lambda}_\varphi(Q)$ for $p=2$. Define the Morrey space for $p=2$ by:

(2-2) $L^{2,\lambda}_\varphi(Q_T)=\left\{f\in L^2(Q_T)\,|\,\sup_{z_0\in Q,0\leq\rho\leq d}\dfrac{1}{\varphi(\rho)}\left(\rho^{-\lambda}\displaystyle\int\int_{Q_T\cap Q_\rho(z_0)}|f|^2\,dz\right)^{\frac{1}{2}}<\infty\right\}$

Defintion 2.1:

A function $h:[0,d]\to[0,\infty)$ is said to be almost increasing if there exists $K_h\geq 1$ such that $h(s)\leq K_h h(t)$ for $0\leq s\leq t\leq d$.

The next propostion is due to [Hu] which will be useful for the main results of this paper.

Proposition 2.1:

Let H be a non-negative almost increasing function in $[0,R_0]$ and $F(f)>0$ on $(0,R_0]$. Suppose that

(a) There exists $A,B,\varepsilon,\beta>0$ such that $H(\rho)\leq\left(A\left(\frac{\rho}{R}\right)^\beta+\varepsilon\right)H(R)+BF(R)$ for $0\leq\rho\leq R\leq R_0$

(b) There exists $\gamma\in(0,\beta)$ such that $\frac{\rho^\gamma}{F(\rho)}$ is almost increasing in $(0,R_0]$

Then there exists $\varepsilon_0=\varepsilon_0(A,\beta,\gamma)$ and $C=C\left(A,\beta,\gamma,K_H,K\right)$ such that if $\varepsilon<\varepsilon_0$ then $H(\rho)\leq C\frac{F(\rho)}{F(R)}H(R)+CBF(\rho)$.

BMO and VMO Spaces:

Defintion 2.2:

Let $\psi \in C[0,d]$ and $\psi > 0$ on $[0,d]$ then $BMO_\psi(Q)$ is defined by:

(2-4) $BMO_\psi(Q_T) =$

$$\left\{ f \in L^2(Q_T) \,|\, \sup_{z_0 \in Q, 0 \le \rho \le d} \frac{1}{\psi(\rho)} \left(⨌_{Q_T \cap Q_\rho(z_0)} \left| f(z) - f_{Q_T \cap Q_\rho(z_0)}(z_0) \right|^2 dz \right)^{\frac{1}{2}} < \infty \right\}$$

where $f_A = ⨌_A f(z)dz$ and $A \subset \mathbb{R}^{n+1}$

Defintion 2.3:

Letting $\psi = 1$ one defines $VMO(Q_T)$ by:

(2-5) $VMO(Q_T) =$

$\{ f \in BMO(Q_T) \,|\, [f]_{BMO(Q_T:\sigma)} = \}$

$$\left\{ \sup_{z_0 \in Q, 0 \le \rho \le \sigma} \left(⨌_{Q_T \cap Q_\rho(z_0)} \left| f(z) - f_{Q_T \cap Q_\rho(z_0)}(z_0) \right|^2 dz \right)^{\frac{1}{2}} \to 0 \; as \, \sigma \to 0 \right\}$$

Weak Solutions:

We would like to discuss the energy estimates for the system (2-3) $u^i_t - D_\alpha \left(a^{\alpha\beta}_{ij} D_\beta u^j \right) = 0$ in Q_T and $a^{\alpha\beta}_{ij}$ is constant. For $Q_R(z_0) \subset Q_T$, let $u^i \xi^2(x)\eta(t)$ be a test function with $\xi \in C^\infty_0(B_R(x_0))$, $0 \le \xi \le 1$ and $|D\xi| \le \frac{C}{R-\rho}$ with $B_\rho(x_0) \subset B_R(x_0) \subset \Omega$ and $\eta(t) = \begin{cases} \frac{t-(t_0-R^2)}{R^2-\rho^2} & t \in (t_0 - R^2, t_0 - \rho^2) \\ 1 & t \in [t_0 - \rho^2, t_0) \end{cases}$.

Next we multiply the test function by (2-3) and use integration by parts.

$0 = \int \int_{B_R(x_0) \times (t_0 - R^2, t]} \left(u^i_t - D_\alpha \left(a^{\alpha\beta}_{ij} D_\beta u^j \right) \right) u^i \xi^2 \eta$

$= \int \int_{B_R(x_0) \times (t_0 - R^2, t]} u^i_t u^i \xi^2 \eta + a^{\alpha\beta}_{ij} D_\beta u^j D_\alpha \left(u^i \xi^2 \eta \right)$ The boundary term is zero by defition of η and ξ

$= \int \int_{B_R(x_0) \times (t_0 - R^2, t]} \left(\frac{1}{2} |u|^2 \right)_t \xi^2 \eta + \int \int_{B_R(x_0) \times (t_0 - R^2, t]} a^{\alpha\beta}_{ij} D_\beta u^j \left(\xi^2 D_\alpha u^i + 2\xi u^i D_\alpha \xi \right) \eta.$

$= \int \int_{B_R(x_0) \times (t_0 - R^2, t]} \left(\frac{1}{2} |u|^2 \eta \right)_t \xi^2 - \frac{1}{2} |u|^2 \xi^2 \eta_t + \int \int_{B_R(x_0) \times (t_0 - R^2, t]} a^{\alpha\beta}_{ij} D_\beta u^j \left(\xi^2 u^i + 2\xi u^i D_\alpha \xi \right) \eta$

Therefore by uniform ellipticity and Cauchy-Schwartz inquality:

$\int_{B_R(x_0)} \frac{1}{2} |u(x,t)|^2 \xi^2(x) + C \int_{t_0-R^2}^t \int_{B_R(x_0)} \xi^2(x) |Du|^2$

$\le \frac{1}{2} \int_{t_0-R^2}^t \int_{B_R(x_0)} |u|^2 \xi^2 \eta_t + C \int_{t_0-R^2}^t |D\xi|^2 |u|^2 \eta \le C \int_{t_0-R^2}^t \int_{B_R(x_0)} |u|^2 \left(|D\xi|^2 \eta + \frac{1}{2} \xi^2 \eta_t \right)$ Then since $|D\xi| \le \frac{C}{R-\rho}$ and $\eta_t \le \frac{C}{R^2-\rho^2}$ This implies

$\int_{B_R(x_0)} \frac{1}{2} |u(x,t)|^2 \xi^2 + \int_{t_0-R^2}^t \int_{B_R(x_0)} |Du|^2 \xi^2 \eta \le C \int_{t_0-R^2}^t \int_{B_R(x_0)} |u|^2 \left(\frac{1}{(R-\rho)^2} + \frac{1}{R^2-\rho^2} \right)$

Then this implies

$\sup_{t_0-\rho^2 \le t \le t_0} \int_{B_\rho(x_o)} |u(t)|^2 + \int \int_{Q_\rho(z_0)} |Du|^2 \le \frac{C}{(R-\rho)^2} \int \int_{Q_R(z_0)} |u|^2$ We call this last line the energy estimate for (2-3). Then define $V_2(Q_T) = \left\{ u \,|\, u \in L^\infty \left(0, T; L^2(Q_T), Du \in L^2(Q_T) \right) \right\}$ This space is the Sobolov space counterpart for parabolic equations.

Using energy estimates and Sobolev embedding thereom, one can get the Morrey estimate for (2-3) with constant coefficients.

<u>Lemma 2.2:</u>

Let $u \in V_2(Q_T)$ solve the following,

(2-2) $u_t^i - D_\alpha \left(a_{ij}^{\alpha\beta} D_\beta u^j \right) = 0 \ \ in \ Q_T = \Omega \times (0,T]$ Then for $Q_R(z_0) \subset Q_T$ and $0 \leq \rho \leq R$

$\int\int_{Q_\rho(z_0)} |Du|^2 \leq C \left(\frac{\rho}{R}\right)^{n+2} \int\int_{Q_R(z_0)} |Du|^2$

<u>Proof:</u>

By [Sc] one has $⨏⨏_{Q_\rho(z_0)} |u|^2 \leq C \left(\frac{\rho}{R}\right)^2 ⨏⨏_{Q_R(z_0)} |u|^2$. Since $a_{ij}^{\alpha\beta}$ is constant, by differentiating (2-3) we obtain that $D_\alpha u$ is still a solution. Hence $\int\int_{Q_\rho(z_0)} |Du|^2 \leq C \left(\frac{\rho}{R}\right)^{n+2} \int\int_{Q_R(z_0)} |Du|^2$ □

We will now turn our attention to the main results of this paper.

3 Main Results

In the preliminaries section the Morrey estimate we are interested in was shown for $a_{ij}^{\alpha\beta} = constant$ by [Sc]. In this section we will extend this result to system (1-1). We establish the Morrey estimate first for the case $a_{ij}^{\alpha\beta} \in C(\overline{Q_T})$ and second for $a_{ij}^{\alpha\beta} \in L^\infty(Q_T) \cap VMO(Q_T)$.

Theorem 3.1:

Let $u \in V_2(Q_T)$ be a weak solution to $u_t^i - D_\alpha\left(a_{ij}^{\alpha\beta}(z)D_\beta u^j\right) = -div\, f^i\; i = 1, \ldots N\, in\, Q_T$. Let $a_{ij}^{\alpha\beta} \in C(\overline{Q_T})$ and assume the uniform ellipticity condition and $f^i \in L_\varphi^{2,\lambda}(Q_T)$ also assume there exists λ, γ such that $\lambda < \gamma < n+2$ and $\frac{r^{\gamma-\lambda}}{\varphi^2(r)}$ is almost increasing, then $Du \in L_\varphi^{2,\lambda}(Q_T)$ for any $Q\prime \subset\subset Q_T$, for $Q_R(z_0) \subset Q_T$ and $\rho \leq R$. Moreover

$$\int\int_{Q_\rho(z_0)} |Du|^2\, dz \leq C\frac{\rho^\lambda\varphi^2(\rho)}{R^\lambda\varphi^2(R)}\int\int_{Q_R(z_0)} |Du|^2\, dz + C\varphi^2(\rho)\rho^\lambda\, ||f||^2_{L_\varphi^{2,\lambda}}$$

Proof:

let w satisfy (3-1) $\begin{cases} w_t^i - D_\alpha\left(a_{ij}^{\alpha\beta}(z_0)D_\beta w^j\right) = 0 & in\, Q_R(z_0) \\ w = u & on\, \partial_p Q_R(z_0) \end{cases}$ with z_0 a fixed point. Then $v = u - w$ will satisfy

$$(3\text{-}2)\;\begin{cases} v_t^i - D_\alpha\left(a_{ij}^{\alpha\beta}(z_0)D_\beta(v^j)\right) = D_\alpha\left(\left(a_{ij}^{\alpha\beta}(z) - a_{ij}^{\alpha\beta}(z_0)\right)D_\beta u^j\right) - div\, f^i & in\, Q_R(z_0) \\ v = 0 & on\, \partial_p Q_R(z_0) \end{cases}$$

Obviously by lemma 2.2 one obtains $\int\int_{Q_\rho(z_0)} |Du|^2 \leq 2\int\int_{Q_\rho(z_0)}\left(|Dw|^2 + |Dv|^2\right)$
$\leq C\left(\frac{\rho}{R}\right)^{n+2}\int\int_{Q_R(z_0)} |Dw|^2 + \int\int_{Q_\rho(z_0)} |Dv|^2$
$\leq C\left(\frac{\rho}{R}\right)^{n+2}\int\int_{Q_R(z_0)} |Du|^2 + C\int\int_{Q_R(z_0)} |Dv|^2$. Multiplying (3-2) by v, integrating and performing integration by parts, one obtains the following:

$$\int\int_{Q_R(z_0)} v_t^i v^i + \int\int_{Q_R(z_0)} a_{ij}^{\alpha\beta}(z_0)D_\beta v^j D_\alpha v^i \leq \int\int_{Q_R(z_0)}\left|a_{ij}^{\alpha\beta}(z) - a_{ij}^{\alpha\beta}(z_0)\right| |Du|\,|Dv| + |f|\,|Dv|$$

. Since $a_{ij}^{\alpha\beta} \in C(\overline{Q})$,if R is small enough, then $\left|a_{ij}^{\alpha\beta}(z) - a_{ij}^{\alpha\beta}(z_0)\right| < \varepsilon$. Therefore one has
$\leq \varepsilon\int\int_{Q_R(z_0)} |Du|\,|Dv| + \int\int_{Q_R(z_0)} |f|\,|Dv| \leq \varepsilon\int\int_{Q_R(z_0)}\left(|Du|^2 + |Dv|^2\right) + \int\int_{Q_R(z_0)} |f|^2$ via Schwartz inequality. This yields
$\int\int_{Q_R(z_0)} |Dv|^2 \leq \varepsilon\int\int_{Q_R(z_0)} |Du|^2 + C\int\int_{Q_R(z_0)} |f|^2$ Therefore one obtains
$\int\int_{Q_\rho(z_0)} |Du|^2 \leq C\left(\frac{\rho}{R}\right)^{n+2}\int\int_{Q_R(z_0)} |Du|^2 + \varepsilon\int\int_{Q_R(z_0)} |Du|^2 + C\int\int_{Q_R(z_0)} |f|^2$
$\leq \left(C\left(\frac{\rho}{R}\right)^{n+2} + \varepsilon\right)\int\int_{Q_R(z_0)} |Du|^2 + C\varphi^2(R)R^\lambda\, ||f||^2_{L_\varphi^{2,\lambda}}$. Then the desired result follows immediately from proposition 2.1□

The following Reverse Holder Inequality for parabolic equations can be found in [St-Gi] and is used in the proof of lemma 3.1.

Proposition3.1:

Let $u \in V_2(Q_T)$ be a weak solution to $u_t^i - D_\alpha\left(a_{ij}^{\alpha\beta}(z)D_\beta u^j\right) = 0\, in\, Q_T\ i = 1, \dots N$ and $a_{ij}^{\alpha\beta}$ satisfying the uniform ellipticity condition (1-2) , then there exists some $s > 2$ such that $Du \in L_{loc}^s(Q_T)$ and for every $Q_R \subset Q_{4R} \subset Q_T$ one has

$$\left(\fint\!\!\fint_{Q_R} |Du|^s\, dz\right)^{\frac{1}{s}} \leq C\left(\fint\!\!\fint_{Q_{4R}} |Du|^2\, dz\right)^{\frac{1}{2}}$$

Lemma 3.1:

Let $u \in V_2(Q_T)$ be a weak solution to $u_t^i - D_\alpha\left(a_{ij}^{\alpha\beta}(z)D_\beta u^j\right) = 0\, in\, Q_T\ i = 1, \dots N$. Assume that $a_{ij}^{\alpha\beta} \in L^\infty(Q_T) \cap VMO(Q_T)$ and the uniform ellipticity condition (1-2) holds. Then for any $0 < \mu < n+2$ there exist R_0 and C depending only on $n+2, N, \mu, \Lambda, and \left[a_{ij}^{\alpha\beta}\right]_{BMO(Q_T;\sigma)}$ such that for $\rho \leq R \leq \frac{min(R_0, dist(z_0, \partial_p Q_T))}{2}$

$$\int\int_{Q_\rho(z_0)} |Du|^2 \leq C\left(\frac{\rho}{R}\right)^\mu \int\int_{Q_R(z_0)} |Du|^2\, dz.$$

Proof:

Let $\left(a_{ij}^{\alpha\beta}\right)_{z_0 R} = \fint\!\!\fint_{Q_R(z_0)} a_{ij}^{\alpha\beta} dxdt$. As in Thereom 3.1 let w and $v = u - w$ satisfy respectively

$$\text{(3-3)} \begin{cases} w_t^i - D_\alpha\left(a_{ij}^{\alpha\beta}(z_0)D_\beta w^j\right) = 0 & in\ Q_R(z_0) \\ w = u & on\ \partial_p Q_R(z_0) \end{cases} \quad \text{and}$$

$$\text{(3-4)} \begin{cases} v_t^i - D_\alpha\left(\left(a_{ij}^{\alpha\beta}\right)_{z_0 R} D_\beta(v^j)\right) = D_\alpha\left(\left(a_{ij}^{\alpha\beta}(z) - \left(a_{ij}^{\alpha\beta}\right)_{z_0 R}\right) D_\beta u^j\right) & in\ Q_R(z_0) \\ v = 0 & on\ \partial_p Q_R(z_0) \end{cases}$$

Similar to the proof of Theorem 3.1 one obtains

$\int\int_{Q_\rho(z_0)} |Du|^2 \leq C\left(\frac{\rho}{R}\right)^{n+2} \int\int_{Q_R(z_0)} |Du|^2 + C\int\int_{Q_R(z_0)} |Dv|^2$. We multiply (3-4) by v and perform an integration by parts to obtain

$\int\int_{Q_R(z_0)} |Dv|^2 \leq C\int\int_{Q_R(z_0)} \left|a_{ij}^{\alpha\beta}(z) - \left(a_{ij}^{\alpha\beta}\right)_{z_0 R}\right|^2 |Du|^2$ Then by Holder's inequality, along with $a_{ij}^{\alpha\beta} \in VMO$ and by proposition 3.1 one has

$$\leq C\left(\int\int_{Q_R(z_0)} \left|a_{ij}^{\alpha\beta}(z) - \left(a_{ij}^{\alpha\beta}\right)_{z_0 R}\right|^{2p}\right)^{\frac{1}{p}} \left(\int\int_{Q_R(z_0)} |Du|^{2q}\right)^{\frac{1}{q}}$$

$$= C|Q_R(z_0)|\left(\fint\!\!\fint_{Q_R(z_0)} \left|a_{ij}^{\alpha\beta}(z) - \left(a_{ij}^{\alpha\beta}\right)_{z_0 R}\right|^{\frac{2s}{s-2}}\right)^{\frac{s-2}{s}} \left(\fint\!\!\fint_{Q_R(z_0)} |Du|^s\right)^{\frac{2}{s}}$$

$\leq C\,|Q_R(z_0)|\,\varepsilon\left(\iint_{Q_R(z_0)}|Du|^s\right)^{\frac{2}{s}} \leq C\varepsilon \int\int_{Q_{4R}}|Du|^2$ and therefore one gets
$\int\int_{Q_\rho(z_0)}|Du|^2 \leq \left(C\left(\frac{\rho}{R}\right)^{n+2}+\varepsilon\right)\int\int_{Q_{4R}(Z_0)}|Du|^2$. It follows that
$\int\int_{Q_\rho(z_0)}|Du|^2 \leq \left(C\left(\frac{\rho}{R}\right)^{n+2}+\varepsilon\right)\int\int_{Q_R(z_0)}|Du|^2$. Then by using proposition 2.1 one can achieve the desired result.

Our final theorem will establish the generalized Morrey estimate for system (1-1).

Theorem 3.2:

Let $u \in V_2(Q_T)$ be a weak solution to $u_t^i - D_\alpha\left(a_{ij}^{\alpha\beta}(z)D_\beta u^j\right) = -div\, f^i\; i=1,\ldots N\, in\, Q_T$ with the uniform ellipticity condition. Suppose there exist λ,γ such that $\lambda<\gamma<n+2$ and $\frac{r^{\gamma-\lambda}}{\varphi^2(r)}$ is almost increasing. Assume $a_{ij}^{\alpha\beta} \in L^\infty(Q_T)\cap VMO(Q_T)$ and $f^i \in L_\varphi^{2,\lambda}(Q_T)$, then $Du \in L_\varphi^{2,\lambda}(Q)$ for any $Q\prime \subset\subset Q_T$ and for $Q_R \subset Q_T$ and $\rho \leq R$. Moreover one obtains the following interior estimate
$\int\int_{Q_\rho}|Du|^2\,dz \leq C\frac{\rho^\lambda\varphi^2(\rho)}{R^\lambda\varphi^2(R)}\int\int_{Q_R}|Du|^2\,dz + C\varphi^2(\rho)\rho^\lambda\,||f||^2_{L_\varphi^{2,\lambda}}$.

Proof:

Again like in theorem 3.1, let w and $v=u-w$ satisfy respectively

(3-5) $\begin{cases} w_t^i - D_\alpha\left(a_{ij}^{\alpha\beta}(z)D_\beta w^j\right)=0 & in\; Q_R(z_0) \\ w=u & on\; \partial_p Q_R(z_0)\end{cases}$ and

(3-6) $\begin{cases} v_t^i - D_\alpha\left(a_{ij}^{\alpha\beta}(z)D_\beta (v^j)\right)=-div\, f^i & in\; Q_R(z_0) \\ v=0 & on\; \partial_p Q_R(z_0)\end{cases}$ Apply Lemma 3.1 to w one obtains the following

(3-7)$\int\int_{Q_\rho(z_0)}|Du|^2 \leq C\left(\frac{\rho}{R}\right)^\mu\int\int_{Q_R(z_0)}|Du|^2 + C\int\int_{Q_R(z_0)}|Dv|^2$. Also multiplying (3-6) to v and performing integration by parts, just as in the proof of Thereom 3.1 one has
$\int\int_{Q_R(z_0)}|Dv|^2 \leq C\int\int_{Q_R(z_0)}|f|\,|Dv|$. Applying Cauchy-Schwartz's inequality one has
(3-8)$\int\int_{Q_R(z_0)}|Dv|^2 \leq C\int\int_{Q_R(z_0)}|f|^2$. By combinding (3-7), (3-8) and the fact that $f^i \in L_\varphi^{2,\lambda}(Q_T)$ one obtains
$\int\int_{Q_\rho(z_0)}|Du|^2 \leq C\left(\frac{\rho}{R}\right)^\mu\int\int_{Q_R(z_0)}|Du|^2 + C\rho^\lambda\varphi^2(\rho)\,||f||^2_{L_\varphi^{2,\lambda}}$. Therefore the result will follow applying Proposition 2.1 □

4 References

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