7-1994

Evolving Plane Curves by Curvature in Relative Geometries II

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PRELIMINARY REPORT
April 20, 2001 771

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0. Introduction

In this paper we prove the existence of self-similar solutions to the anisotropic curve shortening equation.

Theorem 0.1. *Given any positive C^2 function γ on S^1 there exists a solution to the equation*

$$\frac{\partial X}{\partial t} = \gamma(\theta) k N \tag{0.1}$$

which is self-similar. This means that the evolution shrinks the initial curve without changing its shape.

In (0.1) $X\colon S^1 \times [0,\omega) \to \mathbb{R}^2$ is the position vector of a family of closed convex plane curves, kN is the curvature vector, with k being the curvature and N the inward pointing normal given by $N = -(\cos\theta, \sin\theta)$. The weight function $\gamma(\theta) = \gamma(N)$ is a function of the normal vector to the curve at each point but does not depend on position in the plane.

Equation (0.1) has two significant interpretations. It can be seen as the generalization of the "curve shortening" problem ([Ga8]) to Minkowski geometry or as a simplified model of the motion of the interface of a metal crystal as it melts ([AnGu],[Ta1] and [Ga8]).

The proof illustrates most of the techniques that have been used recently in understanding geometric evolution equations as described in [Ha3].

It is not hard to show that the self-similar solutions correspond to positive, 2π periodic solutions of the equation

$$\frac{d^2h}{d\theta^2} + h = \frac{\gamma(\theta)}{h} \tag{0.2}$$

and that $h(\theta)$ is the support function of the suitably normalized self-similar solution. (This follows from equation (2.1) or (5.2).

This allows us to re-interpret theorem (0.1) as

Corollary 0.2. *Every positive, smooth function on the circle $\gamma(\theta)$ can be written as*

$$\gamma(\theta) = h\left(\frac{d^2h}{d\theta^2} + h\right) = \frac{h}{k} \tag{0.3}$$

where h is the support function of the self-similar solution and $k = (\frac{d^2h}{d\theta^2} + h)^{-1}$ is its curvature.

In effect we solve the ODE (0.2) by reducing it to a related parabolic PDE! Rather than start with a solution of (0.2) and perturb it to a periodic solution, the strategy is to start with a periodic and positive function and evolve it to a solution of (0.2).

These theorems are corollaries of a stronger result.

Theorem7.8. *Every convex curve which evolves by equation (0.1) evolves to a point in finite time and if the family of curves is renormalized to enclose constant area then each infinite sequence of curves has a convergent subsequence which converges to the shape of a self-similar solution.*

This complements the uniqueness results obtained by the first author in [Ga3, Theorem 4.4, and Corollaries A and B]:

Theorem 0.4. *If γ is smooth, strictly positive and symmetric (i.e. $\gamma(\mathbf{v}) = \gamma(-\mathbf{v})$ or $\gamma(\theta) = \gamma(\theta + \pi)$) then there is a unique self-similar solution to (0.1)and every renormalized family of curves converges to this shape.*

Several related problems remain unsolved:

(1) If γ is not symmetric is there a unique self-similar solution. Is there a generalization of the Minkowski isoperimetric quotient as described in [Ga8] which serves as a Liapunov function?
(2) Can one can show that every embedded curve (not just convex curves) flows to a "round point" under the flow (0.1)? (see [Gr1] for the case when γ is identically 1).
(3) What is the appropriate generalization of these results to non-smooth functions? This is a particularly intriguing question since it would connect our result with a series of results by J. Taylor [Ta1] (see also [An-Gu, p377] which involve flows in which γ is crystalline, that is its Wulff shape is a polygon.

Here is a synopsis of the paper:

In section 1 we derive evolution equations for the support function h, the curvature k and other geometric quantities of the curves. In section 2 we review the comparison principle for parabolic evolution equations.

The short time existence result of Angenent, reviewed in section 3 guarantees a smooth solution for a short time for any initial curve with Hölder continuous curvature. The solution lasts as long as the curvature of the evolving curve remains Hölder continuous.

In section 4 we derive a gradient bound for the curvature in terms of its average value. This means that the maximal solution must last until the curvature becomes infinite. The computation should be compared to the computation in [Ga2, Lemma 2.1] and the computation of the Harnack inequality for curve shortening [Ha3].

In section 5 we show that the curvature can become infinite only if the enclosed area decreases to zero (the cusp theorem). Since the area is a linear function of time, this allows us to define a normalized flow for which the area enclosed by the curve remains constant. (See also [Ts1] and [An1].) We also derive the entropy estimate (compare with [Ha3]). We show that the entropy remains bounded for the normalized flow.

In section 6 we derive several geometric estimates which show that an upper bound on the entropy integral yields lower bounds on the minimum width of the normalized curve, which in turn yields an upper bound on the diameter and the length.

The main theorem is proved in section 7 where we derive as monotonicity theorem for the support function. ¿From this and the gradient estimates we conclude that the supremum norm of $\overline{k}$ remains finite. It follows immediately that a subsequence of the normalized solutions converges to a solution of (0.1) which represents a self-similar solution.

We wish to thank R. Hamilton for several illuminating conversations.

1. Definitions and notation

Let $X(u,t)\colon S^1 \times [0,\omega) \to \mathbb{R}^2$ be the position vector of the evolving convex curve. Then $v = \left|\frac{\partial X}{\partial u}\right|$ is the speed of the parametrization and $ds = v du$ is the unit of arclength. The unit tangent vector and unit normal vector are represented by T and N respectively and used to define the angle θ via the formulae $T = (-\sin\theta, \cos\theta)$ and $N = -(\cos\theta, \sin\theta)$. Note that the normal points towards the interior of a curve traced in the counterclockwise direction. The curvature of the curve is $k = \frac{\partial\theta}{\partial s} = \theta_s$. We will frequently use subscripts to denote partial derivatives.

The support function is the distance from the origin to the tangent line of the curve, in other words $h = - < X, N >$.

Fig 1.

Since the curves are convex the angle θ gives a regular parametrization of the curve. One can define $X(\theta(u,t),t) = X(u,t)$ and the support function h can be considered either as a function of u or of θ. Beginning in section 4 we will use only the latter definition. In particular the notation h_t will denote the partial derivative of h keeping θ fixed (i.e. $\frac{\partial h}{\partial t}\big|_\theta$).

All of the geometric quantities of interest to us can be expressed in terms of the support function:

Lemma 1.1.

$$\frac{1}{k} = h_{\theta\theta} + h \tag{1.1}$$

$$L = \int^{2\pi} h\, d\theta \tag{1.2}$$

and

$$A = \frac{1}{2}\int_0^{2\pi} h^2 - (h_\theta)^2 \, d\theta \tag{1.3}$$

where k is the curvature, L the length of the curve, and A the area enclosed by the curve.

Proof. Equation (1.1) follows from differentiating the definition of h and using the Frenet formulas. Equation (1.2) follows immediately from (1.1). The final equation follows from the first two and the representation of area as $2A = \int xdy - ydx$. See [Ga1] or any book on convex geometry for more details.

2. Evolution equations and the comparison principle

We reformulate the evolution equation (0.1) as an evolution equation for the support function h and derive the evolution equations for the other geometric quantities.

Lemma 2.1. *The evolution equations for the support function, curvature, length and area are*

$$h_t = -\gamma k \tag{2.1}$$

$$k_t = k^2(\gamma k)_{\theta\theta} + \gamma k^3 \tag{2.2}$$

$$L_t = -\int \gamma k \, d\theta \tag{2.3}$$

$$A_t = -\int \gamma \, d\theta \tag{2.4}$$

Proof. Intuitively (2.1) is obtained from (0.1) by observing that the distance to the support line should decrease according to the velocity of the contact point of the support line and the curve, i.e. at the rate γk where k is the curvature at the contact point. A more rigorous derivation is to observe that differentiating $X_u = vT$ and comparing the coefficients of T and N proves that $v_t = -k^2 v$ and $T_t = k_s N$. From the definition of θ in terms of T and N we observe that this last equation implies that $\theta_t = k_s$.

The partial derivative with respect to t holding θ fixed is obtained from the partial derivative holding u fixed via the chain rule

$$\left.\frac{\partial X(\theta(u,t),t)}{\partial t}\right|_u = \left.\frac{\partial X}{\partial t}\right|_\theta + \left.\frac{\partial X}{\partial \theta}\frac{\partial \theta}{\partial t}\right|_u = \left.\frac{\partial X}{\partial t}\right|_\theta + \frac{T}{k}\frac{\partial k}{\partial s}$$

Finally applying this to the definition of h and observing that, by definition $\left.\frac{\partial N}{\partial t}\right|_\theta = 0$ we obtain $\left.\frac{\partial h}{\partial t}\right|_\theta = -< \left.\frac{\partial X}{\partial t}\right|_\theta, N >= -\gamma k$.

This new equation, with the changed parametrization, is equivalent to adding a tangential component to the velocity vector of the original equation (0.1). The equivalence is discussed in more detail in [E-G] and also in [An-Gu].

Equations (2.2)—(2.4) are obtained by differentiating equations (1.1)—(1.3) and using (2.1).

Proposition 2.2 Comparison theorem for ODE. *Let $y(t)$ and $z(t)$ be functions defined on $[0, t_2]$ and satisfying*

1. $y'(t) = g(t, y), \qquad z'(t) = l(t, z),$
2. $g(t, x) > l(t, x)$ *for all* $t \in [0, t_2]$ *and*
3. $y(0) > z(0)$

then $y(t) \geq z(t)$ for all $t \in [0, t_2]$.

The same conclusion holds if $y(t)$ and $z(t)$ are Lipschitz functions and for all $t \geq 0$ the forward derivatives satisfy the inequalities

4. $\liminf_{\epsilon \searrow 0} \frac{y(t+\epsilon) - y(t)}{\epsilon} \geq g(t, y)$ *and*
5. $\limsup_{\epsilon \searrow 0} \frac{z(t+\epsilon) - z(t)}{\epsilon} \leq l(t, y).$

Proof.

Let $t_1 = \sup\{t \mid t \in [0, t_2] \text{ and for all } s \leq t, \quad y(s) \geq z(s)\}$. The continuity of y and z guarantee that $y(t_1) = z(t_1)$. Using the hypothesis on the forward derivatives and choosing δ small there exists an $\epsilon > 0$ such that for all $t \in [t_1, t_1 + \epsilon)$

$$\frac{y(t) - y(t_1)}{t - t_1} \geq g(t_1, y(t_1)) - \delta \geq l(t_1, y(t_1)) + \delta \geq \frac{z(t) - z(t_1)}{t - t_1}$$

from which it follows that $y(t) \geq z(t)$ for $t \in [t_1, t_1 + \epsilon)$ contradicting the definition of t_1.

Corollary 2.3. *If $y'(t) = g(t, y(t))$ and $g(t, y)$ is continuous and Lipshitz in y then the strict inequalities can be replaced by non-strict inequalities. That is if $g(t, x) \geq l(t, x) \geq \limsup_{\epsilon \searrow 0} \frac{z(t+\epsilon) - z(t)}{\epsilon}$ and $y(0) \geq z(0)$ then $y(t) \geq z(t)$ for all positive t.*

A similar result holds if z satisfies a well posed differential equation.

Proof. The essential observation is that the solutions to the differential equation are continuous with respect to the variation of parameters and initial conditions. In fact let $y_\epsilon(t)$ satisfy $y_\epsilon(t) = g(t, y_\epsilon) + \epsilon$ and $y_\epsilon(0) = y(0) + \epsilon$ then $y_\epsilon(t) \geq z(t)$ and $y_\epsilon(t)$ converges uniformly to $y(t)$ on compact subsets, hence $y(t) \geq z(t)$, for positive t.

The infimum of a one parameter family of smooth functions is continuous, but not usually differentiable. It is however Lipshitz and its forward derivative is given by equation (2.5) below:

Lemma 2.4 (see [Ha1]**).** *If $y(t) = \sup_{\theta \in S^1}\{h(\theta, t)\}$ where $h(\theta, t)$ is a smooth function then $y(t)$ is Lipshitz and*

$$\limsup_{\epsilon \searrow 0} \frac{y(t+\epsilon) - y(t)}{\epsilon} = \sup\{h_t(\theta, t) \mid \theta \textit{ such that } h(\theta, t) = y(t)\}. \tag{2.5}$$

Proof. For any $\delta > 0$ there is some θ such that for all small ϵ

$$\frac{y(t+\epsilon) - y(t)}{\epsilon} \geq \frac{h(\theta, t+\epsilon) - h(\theta, t)}{\epsilon} \geq h_t(\theta, t) - \delta.$$

This proves the lower bound on the forward difference of y.

Let ϵ_i be positive sequence converging to zero, then for some choice of θ_i, $y(t+\epsilon_i) = h(\theta_i, t+\epsilon_i)$ and

$$\frac{y(t+\epsilon_i) - y(t)}{\epsilon_i} \leq \frac{h(\theta_i, t+\epsilon_i) - h(\theta_i, t)}{\epsilon_i} = h_t(\theta_i, c_i).$$

The compactness of S^1 and the smoothness of h now imply the upper bound for the forward derivative of y.

A symmetric result holds for the infimum of a one parameter family of functions.

An easy corollary of these results, typical of the way in which we will use them, is that

Corollary 2.5. *The function $k_{\max}(t) = \sup_{\theta \in S^1} k(\theta, t)$ satisfies*

$$k_{\max}(t) \leq \frac{1}{\left(\frac{1}{k_{\max}(0)} - 2t\right)^{1/2}} \tag{2.6}$$

and

$$k_{\max}(t) \geq \frac{1}{(2\omega - 2t)^{1/2}} \tag{2.7}$$

where ω is the time when $k_{\max}$ blows up.

Proof. ¿From equation (2.2) and lemma 2.4 it follows that the forward difference of $k_{\max}(t)$ is less than or equal to ${k_{\max}}^3$. From corollary 2.3 $k_{\max}(t)$ must be less than the solution of $y' = y^3$ with initial value $y(0) = k_{\max}(0)$. This proves inequality (2.6).

For the inequality (2.7) we observe, from corollary 2.3 that once $k_{\max}(t)$ is smaller than a solution of $y' = y^3$ it must stay smaller. If $k_{\max}(t) \leq (2\omega + \delta - 2t)^{-1/2}$ for any t then we obtain a contradiction since in this case $k_{\max}$ remains finite at $t = \omega$ and the blowup must occur at a later time. We conclude that $k_{\max}(t) \geq (2\omega + \delta - 2t)^{-1/2}$ for all $t < \omega$ and for any δ which proves the second inequality.

Remark 2.6. We will write $z_t(t)$ even for a Lipshitz function. This should be interpreted as either the limsup or liminf of the forward differences. For example $z_t(t) \geq z(t)^2$ means that the liminf of the forward differences of z is always greater than the square of the value of z.

3. Review of short time existence.

The existence of a solution to (0.1) for a short time and the conclusion that the solution remains smooth (if $\gamma(\theta)$ is smooth) as long as its $C^{2+\alpha}$ norm remains bounded is proved in S. Angenent's paper [An1]. It is also a special case of Tso's paper on the Gauss curvature flow [Ts1].

Angenent outlines the proof of the following short time existence theorem for initial curves which have Hölder continuous curvature and which lie on an arbitrary two manifold. $\Omega(M)$ is the space of constant speed parameterized curves on M and $S^1(M)$ is the unit tangent bundle.

Theorem 3.1, (Angenent)[An1]. *Assume that*

$$X_t = V(T,k)N \tag{3.1}$$

where $V\colon S^1(M)\times\mathbb{R}\to\mathbb{R}$ *is* $C^{1,1}$ *and satisfies* $\frac{\partial V}{\partial k}>0$ *for all* $(T,k)\in S^1(M)\times\mathbb{R}$.

Let α_0 *be a regular curve with Hölder continuous curvature. Then there exists a unique maximal solution* $\alpha\colon[0,t_{\max})\to\Omega(M)$ *with initial value* $\alpha(0)=\alpha_0$.

Furthermore $t_{\max}>0$ *and* $\alpha(t)\in C([0,t_{\max});\Omega(M))$ *and for each* $t\in[0,t_{\max})$, $\alpha(t)$ *has continuous curvature and normal velocity and of course satisfies the equation (3.1).*

If V *is a* C^{m+1} *function, for some* $m\geq 1$, *then the solution* $\alpha(t)$ *is a* $C^{m+2+\beta}$ *curve for any* $t\in(0,t_{\max})$, *and any* $0<\beta<1$.

In the simple case we are concerned with M is $\mathbb{R}^2$ and $V(T,k)=\gamma(\theta)k$, hence $\frac{\partial V}{\partial k}=\gamma>0$. Furthermore V is C^{m+1} provided γ is C^{m+1}. Observe that if $t_{\max}$ is finite then the limit of $\alpha(t)$ as t approaches $t_{\max}$ must be a curve whose curvature is not Hölder continuous since otherwise one could extend the solution further.

The next point to recall from Angenent's discussion in [An1] is that the higher derivatives of the solution exist and are bounded on the compact subsets of $(0,t_{\max})$. In fact one obtains

$$\left\|\frac{\partial^{k+l}\alpha}{\partial t^k\partial u^l}\right\|_\infty \leq c(t_1,M,k,l,\gamma)t^{-(k+l)}$$

where α is a solution to (3.1) whose $C^{2+\alpha}$ norm at $t=0$ is bounded by M. The function c is an increasing function of M and of t_1, $0<t_1<t_{\max}$. This means that even if the initial curve is not C^∞ it becomes infinitely smooth instantly provided that $\gamma(\theta)$ is infinitely smooth, and that if the curvature of each curve $\alpha(t)$ has a Hölder constant which is uniformly bounded by M for each t in an interval $[0,t_1)$ then all of the higher time and space derivatives of the evolving curve are uniformly bounded in compact subsets of $(0,t_1)$ in terms of M.

In the following sections we show that a Hölder bound on the curvature can be obtained from more geometric quantities, such as the enclosed area.

4. Gradient estimates

By considering the evolution equation for $(\log\gamma k)_t$ we are able to derive an estimate on the second derivative of k in terms of the curvature. This in turn allows us to estimate the derivative k_θ in terms of the average curvature and *a fortiori* in terms of the maximum curvature

Lemma 4.1. *Let* $u=(\log\gamma k)_t=\dfrac{(\gamma k)_t}{\gamma k}=k(\gamma k)_{\theta\theta}+\gamma k^2$ *then* u *satisfies:*

$$u_t=\gamma k^2u_{\theta\theta}+2k(\gamma k)_\theta u_\theta+2u^2 \tag{4.1}$$

Proof. This is a straight forward calculation. First

$$u_t=k_t(\gamma k)_{\theta\theta}+k(\gamma k_t)_{\theta\theta}+2\gamma k k_t$$

Since $u=\dfrac{k_t}{k}$ this yields

$$\begin{aligned}u_t&=k\left((\gamma k)_{\theta\theta}+\gamma k\right)\frac{k_t}{k}+\gamma k k_t+k(\gamma k u)_{\theta\theta}\\&=u^2+\gamma k^2u+k(\gamma k)_{\theta\theta}u+2k(\gamma k)_\theta u_\theta+\gamma k^2u_{\theta\theta}\end{aligned}$$

which simplifies to (4.1)

Lemma 4.2. *It follows that* $z(t) = \min_\theta u(\theta, t)$ *is a Lipschitz function and satisfies*

$$z_t \geq 2z^2 \geq 0 \tag{4.2}$$

and therefore

$$k(\gamma k)_{\theta\theta} + \gamma k^2 \geq z(t) \geq z(0) = \frac{-1}{C_1 + t} \geq -(C_1)^{-1} \quad \textit{for } t > 0 \tag{4.3}$$

for some non-negative constant C_1 *which depends on the initial conditions on* k *and its derivatives.*

Remark 4.3. (If we choose $C_1 = 0$ then for positive time the left hand side of equation (4.3) is bounded independent of the bounds on the initial derivatives of k.)

Proof. This is a direct application of the comparison and maximum principles described in lemma 2.2 together with the fact that the solution to equations of the form $y' = y^2$ are of the form $y(t) = (C - t)^{-1}$.

Lemma 4.4. *There is a positive constant* ϵ_k *depending on the curvature of the initial curve such that* $k(\theta, t) \geq \epsilon_k$.

Proof.

We have that $(k_{\min})_t > (k_{\min})^3$ hence in view of the comparison principle and the fact that the solutions to $y' = y^3$ are of the form $y(t) = \pm(C - 2t)^{-1/2}$ or $y(t) \equiv 0$ we see that $k(\theta, t) \geq \left(\frac{1}{(k_{\min}(0))^2} - t\right)^{-1/2}$ which implies the inequality in the statement with $\epsilon_k = k_{\min}(0)$.

Corollary 4.5 The Gradient Estimate. *We have the following estimate on the gradient. (See Angenent* [An3] *for a similar estimate obtained using a different technique.)*

$$\|(\gamma k)_\theta\|_\infty \leq \int \gamma k \, d\theta + \frac{2\pi}{(C_1 + t)\epsilon_k} \leq 2\pi \|\gamma\|_\infty k_{\max} + \frac{2\pi}{(C_1 + t)\epsilon_k} \tag{4.4}$$

Proof. Let p be a maximum point of γk and proceeding counter clockwise (the direction of increasing θ) from p to any x in S^1 yields

$$(\gamma k)_\theta(x) - (\gamma k)_\theta(p) \geq -\int_p^x \gamma k + \frac{1}{(C_1 + t)\epsilon_k} \, d\theta$$

hence

$$-(\gamma k)_\theta(x) \leq \int_{S^1} \gamma k \, d\theta + \frac{2\pi}{(C_1 + t)\epsilon_k}.$$

Similarly, again proceeding counter clockwise,

$$(\gamma k)_\theta(p) - (\gamma k)_\theta(x) \geq -\int_x^p \gamma k + \frac{2\pi}{(C_1 + t)\epsilon_k} \, d\theta$$

and (4.4) follows.

The gradient bound allows us to estimate the maximum curvature in terms of integrals of the curvature. In particular the following is true:

Corollary 4.6. *There exist constants δ, C_1 and C_2 which depend only on the curvature of the initial curve such that*

$$k_{\max}(t) \leq C_1 \int_{S^1} \gamma k \, d\theta + C_2 \tag{4.5}$$

and if $k(\theta_0, t) = k_{\max}(t)$ then

$$\frac{k_{\max}}{2} \leq k(\theta, t) \qquad \text{when } |\theta - \theta_0| \leq \delta \tag{4.6}$$

Remark 4.7. We can conclude that the time ω at which the supremum norm of curvature approaches infinity is the same as $t_{\max}$ (and not some later time) and that the average curvature of the maximal solution to (0.1) must go to infinity as t approaches $t_{\max}$. In the next section we show that ω also coincides with the time when the area enclosed by the curve reaches zero.

5. The entropy estimate and the area estimate

One could also use equation (4.1) to obtain an upper bound on the value of $u_{\min}$ relative to the time to blowup, however, by integrating u first over S^1 one obtains an upper bound on the average of u relative to the time to blowup which is more powerful. This is the entropy bound discussed by R. Hamilton in [Ha3]. Let $\mathcal{E}(t) = \int_{S^1} \gamma \log(\gamma k) \, d\theta$ be the "entropy".

Proposition 5.1.

$$\frac{\partial \mathcal{E}(t)}{\partial t} = \int \gamma (\log \gamma k)_t \, d\theta \leq \frac{\int \gamma \, d\theta}{2} \frac{1}{\omega - t} \tag{5.1}$$

where ω is the time at which the curvature becomes infinite.

Proof. We begin with the following calculation, using the notation of lemma 4.1.

$$\begin{aligned}
\mathcal{E}_{tt} = \left(\int \gamma u \right)_t &= \int (\gamma k)^2 u_{\theta\theta} + 2(\gamma k)^2 u_\theta^2 + 2\gamma u^2 \, d\theta \\
&= \int -2(\gamma k)(\gamma k)_{\theta_\theta} + 2(\gamma k)^2 u_\theta^2 + 2\gamma u^2 \, d\theta \\
&= \int -2(\gamma k)^2 u_\theta^2 + 2(\gamma k)^2 u_\theta^2 + 2\gamma u^2 \, d\theta \\
&= \int 2\gamma u^2 \, d\theta \\
&\geq 2 \frac{\left(\int \gamma u \, d\theta \right)^2}{\int \gamma \, d\theta} = 2 \frac{(\mathcal{E}_t)^2}{\int \gamma \, d\theta}
\end{aligned}$$

We used equation (4.1) and for the last line the Schwartz inequality. Setting $y = \int \gamma u \, d\theta$ yields

$$y_t \geq \frac{2}{\int \gamma \, d\theta} y^2.$$

Equation (5.1) now follows from the comparison lemma 2.2 and the fact that the entropy must be infinite at time $t = \omega$.

The next step is to show that the blow-up time ($t = \omega$) for the curvature is also when the area enclosed by the curve goes to zero.

A region of zero area cannot be bounded by a simple curve with finite curvature, so ω is less than or equal to the time at which the area is zero.

That the two are equal is the main result of Tso's paper [Ts1] (which also proves an analogous result for Gauss curvature flow in all dimensions). In the case where $\gamma(\theta) = \gamma(\theta+\pi)$ the equality follows from theorem 8.1 in Angenent's paper [An2]. We will illustrate here that this fact is also the essential content of the "cusp theorem" (see [Ga-Ha] or [Gr1].)

Consider all numbers β such that the open set $\{\theta \mid \gamma k > \beta\}$ is the union of disjoint intervals I_i whose length is at most π.

Definition 5.2. Let $\beta^* = \sup\{\beta \mid \{\theta \mid \gamma k > \beta\} = \bigcup I_i \text{ and } |I_i| \leq \pi\}$.

Proposition 5.3 (Cusp theorem). *If $\beta^* \leq M$ on $[0, t_1)$ with $t_1 < \infty$ then $k_{\max}(t)$ is uniformly bounded on $[0, t_1)$.*

Proof outline. (See [Ga-Ha] for an analogous calculation.) Let $\beta < \beta^* \leq M$. Then

$$\left(\int_{S^1} \gamma \log(\gamma k)\right)_t = \int -(\gamma k)_\theta^2 + (\gamma k)^2 = \int_{S^1} -(\gamma k)_\theta^2 + (\gamma k - \beta)^2 + 2\beta\gamma k - \beta^2$$

Evaluating the integral on the right on each interval $|I_i|$ we find that $\int_{I_i} -(\gamma k)_\theta^2 + (\gamma k - \beta)^2 \leq 0$. This follows from Wirtinger's inequality for intervals because $(\gamma k - \beta)$ is zero at the endpoints and each interval has length no more than π. On the complement of the union of the intervals, $\bigcup I_i$, we have $|\gamma k| \leq \beta$. Combining these estimates and using (2.3) proves

$$\left(\int_{S^1} \gamma \log(\gamma k)\right)_t \leq 2\beta \int_{S^1} \gamma k = -2\beta L_t \leq -2ML_t.$$

Integrating this inequality proves that the entropy $\mathcal{E}(t) = \int \gamma \log(\gamma k)$ remains finite on the interval $[0, t_1)$.

Using corollary 4.6 and the lower bound on k given in lemma 4.4 we conclude that $k_{\max}(t)$ remains finite as long as $\int \gamma \log(\gamma k)$ is finite. This proves the desired result.

Proposition 5.4. *We have $\beta^* < \|\gamma\|_\infty L(0)/A$, hence from the previous lemma, as long as the area is strictly greater than zero on a time interval the curvature remains uniformly bounded.*

Proof outline. This is a strictly geometric estimate. (See [Ga-Ha] for a detailed proof.) There is some interval of length π on which $\gamma(\theta)k(\theta) \geq \beta^*$. This implies that the convex curve $\alpha(t)$ is enclosed in a cigar shaped figure whose area is less than $L(0)\|\gamma\|_\infty/\beta^*$.

Fig 2.

Remark 5.5. The above two propositions prove that $A(\omega) = 0$. Using equation (2.4) we determine that $\omega = \frac{A(0)}{\int \gamma \, d\theta}$. Notice that the information given by propositions 5.1 and 5.4 is complementary. The first tells how fast the entropy blows up, while the second identifies the time of blow-up but doesn't give a sharp bound on the rate of blow-up of the entropy.

This suggests that we consider the flow in which the figure is homothetically expanded so as to keep the enclosed area constant.

Proposition 5.6. *Let $\mu = (2\omega - 2t)^{-1/2}$. Rescaling the geometric quantities by magnifying $\mathbb{R}^2$ by μ and replacing time t by $\tau = -\frac{1}{2}\log(2\omega - 2t)$ transforms the equations (2.1)—(2.4) into the rescaled equations:*

$$\overline{h}_\tau = (\mu h)_\tau = -\gamma \overline{k} + \overline{h} \tag{5.2}$$

$$\overline{k}_\tau = \left(\frac{k}{\mu}\right)_\tau = \overline{k}^2 (\gamma \overline{k})_{\theta\theta} + \gamma \overline{k}^3 - \overline{k} \tag{5.3}$$

$$\overline{L}_\tau = -\int \gamma \overline{k} \, d\theta + \overline{L} \tag{5.4}$$

$$\overline{A}_\tau = 0 \tag{5.5}$$

and in fact

$$\overline{A} \equiv \frac{1}{2} \int \gamma \, d\theta \tag{5.6}$$

The entropy of the normalized flow decreases:

Corollary 5.7 (The normalized entropy estimate).

$$\overline{\mathcal{E}}_\tau(\tau) = \left(\int \gamma \log \gamma \overline{k} \, d\theta \right)_\tau \leq 0 \tag{5.7}$$

and

$$\overline{\mathcal{E}}(\tau) = \int \gamma \log \gamma \overline{k} \, d\theta \leq \overline{C}_E \tag{5.8}$$

where $\overline{C}_E$ depends only on the initial curve.

Proof. This calculation follows from (5.1) and the definition of the normalized curvature in equation (5.3).

The gradient estimate for $\overline{k}$ is:

Corollary 5.8.

$$\|(\gamma \overline{k})_\theta\|_\infty(t) \leq \int \gamma \overline{k} \, d\theta + \frac{2\pi(\omega - t)^{1/2}}{\epsilon_k (C_1 + t)} \leq 2\pi \|\gamma\|_\infty \overline{k}_{\max} + \frac{2\pi(\omega - t)^{1/2}}{\epsilon_k (C_1 + t)}$$

Remark 5.9. We have determined the time of blow-up of the curvature, but we have not yet determined its rate. This means that the normalized curvature may be unbounded as τ goes to infinity. Similarly we have no lower bound on the normalized curvature, no lower bound on the size of $\overline{h}$ and no upper bound on $\overline{h}$ or the normalized length. In sum we have no control on the shape of the normalized curve. Once one of these bounds is established (the purpose of the next two sections) the other bounds follow quickly and the existence of subsequences converging to a self similar solution is established.

6. Geometry estimates

These elementary geometric inequalities are analogous to the geometric inequalities obtained by R. Hamilton for the Gauss curvature evolution equation and similar to the estimates obtained by W. Firey [Fi] for the same flow. These estimates depend only on the geometry of the curve. We will apply them to the curves obtained from the normalized flow.

We first obtain a lower bound on the width of the curve in terms of the upper bound $\overline{C}_E$ on the entropy. This also implies a lower bound on the inradius of the curve. From there we obtain upper bounds on the diameter of the normalized curve and on its length in terms of the area (which remains fixed) and the bound $\overline{C}_E$ on entropy.

Lemma 6.1. *Let $w(\theta_0)$ be the width in the direction parallel to $-N = (\cos\theta, \sin\theta)$.*

$$\log w(\theta_0) = \log(h(\theta_0) + h(\theta_0 + \pi)) \geq C_w(\gamma) - \frac{\overline{\mathcal{E}}}{\gamma_{\min}}$$

where C_w depends only upon the anisotropic factor γ.

Remark 6.2. I am indebted to R. Hamilton for showing me the identical estimate which he had obtained for the Gauss curvature flow.

Proof. First we see that

$$2w(\theta_0) = \int \frac{|\sin(\theta - \theta_0)|}{\overline{k}}\, d\theta$$

since $d\theta = \overline{k} ds$. Dividing by 2π, taking logarithm and using the Jensen inequality for concave functions we obtain

$$\log\left(\frac{w(\theta_0)}{\pi}\right) = \log\left(\int \frac{|\sin(\theta - \theta_0)|}{\overline{k}} \frac{d\theta}{2\pi}\right) \geq \int \log\left(\frac{|\sin(\theta - \theta_0)|}{\overline{k}}\right) \frac{d\theta}{2\pi}$$

hence

$$2\pi \log(w(\theta_0)) \geq 2\pi \log(\pi) + \int_0^{2\pi} \log(|\sin(\theta - \theta_0)|)\, d\theta - \int_0^{2\pi} \log \overline{k}\, d\theta$$

$\log(|\sin(\theta)|)$ is integrable (in fact its integral is negative and greater than $4\pi \log \frac{2}{\pi} - 2\pi$) and the result follows from estimating the integral of $\log \overline{k}$ in terms of the normalized entropy.

Lemma 6.3. *The inradius of a convex curve (that is the radius of the largest circle inscribed inside the curve) satisfies*

$$r_{\text{in}} \geq \frac{1}{3} \min_\theta w(\theta)$$

Proof. Let K be a convex curve with inradius 1. If there are only two points of contact between the inradius circle and the convex boundary then these points must be antipodal and the minimum width is equal to the inradius.

Otherwise one can find 3 points of contact between the inradius circle and the convex boundary and the intervals between each pair of contact points are less than π. Taking tangent lines at each of the points of contact one obtains a triangle with the same inradius which encloses the original convex body and therefore has greater minimal width.

This reduces the problem to finding the triangle circumscribing the unit circle whose minimum width is the largest. The minimum width is clearly the shortest altitude. Suppose one of the three altitudes of a triangle is strictly smaller than the other two. Orient the triangle so that this altitude is vertical, then rotate the points of contact of the other two sides downward on the inradius circle. This clearly increases the size of the vertical altitude. If the motion is small enough the vertical altitude is still the shortest, hence the minimum altitude has been increased. This argument shows that only a triangle with at least two altitudes equal could be a critical point.

If the triangle has two shortest altitudes equal, place the base connecting the two vertices of the short altitudes on the bottom as in Figure 3. We see that rotating the points of contact of the triangle with the circle symmetrically upward we move the points A and B outward and that the points a and b move monotonically in the vertical direction. As long as the a and b lie below the points of contact with the circle this action increases the length of both of the short altitudes while decreasing the vertical altitude only slightly. This hypothesis is satisfied since the points of a and b coincide with the points of contact only for the equilateral triangle and since they move monotonically a and b must lie below the points of contact when the side altitudes are shorter than the vertical altitude. This proves that the equilateral triangle is the only candidate for the maximum minimum width among triangles.

The family of triangles is completely described by the three contact points lying on the circle, and the altitudes and minimum width vary continuously hence there exists a minimum, which from the above argument must be the equilateral triangle. Calculating the minimum width of the equilateral triangle gives the desired estimate.

Fig 3.

Lemma 6.4. *The diameter of the curve is bounded by its enclosed area divided by* *its minimum width. In addition the diameter bounds the length of the curve.*

$$D \leq \frac{\overline{A}}{\min_\theta w(\theta)} \qquad \overline{L} \leq \pi D \tag{6.1}$$

Proof. Choose a line segment which realizes the diameter. The projection onto the line perpendicular to the line segment is not less than the minimum width. By connecting the ends of the diameter line segment with points which project into the boundary of the "shadow" of the convex curve on the perpendicular line we obtain a quadrilateral enclosed in the convex figure, a quadrilateral with the same diameter and the same width, whose area is exactly the diameter times the width. (see Figure 4.)

The second inequality, the isodiametric inequality is standard. (See Santalo [Sa1].) This completes the proof of the lemma.

Fig 4.

7. The monotonicity theorem and the curvature estimate.

We can now obtain an upper bound for the curvature $\overline{k}$ for all time. This is equivalent to showing that the unnormalized curvature blows up at a rate proportional to $(\omega - t)^{-1/2}$.

Let

$$H(\tau, a, b) = \int_{S^1} \gamma \log \overline{h}\, d\theta \tag{7.1}$$

where $\overline{h} = \overline{h}(\tau, \theta, a, b)$ is the support function, relative to the point (a, b), of a curve $\alpha(\tau)$ which is evolving under the normalized equation.

Lemma 7.1(see [Fi]**) Monotonicity Lemma.** *H evolves according to the equation:*

$$H(\tau, a, b)_\tau = \int_{S^1} \gamma - \frac{\gamma^2 \overline{k}}{\overline{h}}\, d\theta \leq 0$$

Proof. This is a straightforward calculation using (5.2). The inequality follows from the Schwartz inequality:

$$\int \frac{\gamma^2 \overline{k}}{\overline{h}} \int \frac{\overline{h}}{\overline{k}} \geq \left(\int \gamma \right)^2$$

and the identities $\int \gamma = 2\overline{A} = \int \frac{\overline{h}}{\overline{k}}$ which are easily derived from (1.1), (1.3) and (5.5).

Proposition 7.2. *Let ϵ_{in} be the minimum value of the inradius of the family of evolving normalized curves. From the geometric estimates of lemma 6.1, lemma 6.3 and the entropy estimate 5.7 we know that ϵ_{in} is positive.*

For any initial curve α_0 there is a choice of origin so that $H(\tau)$ is uniformly bounded below by $\log(\epsilon_{\text{in}}) \int \gamma \, d\theta$ for all $\tau > 0$. In addition the evolving normalized curve remains within some fixed ball.

Proof. Let α_0 be our initial curve. Using the continuous dependence of the solutions $\overline{h}(\tau, \theta, a, b)$ on initial conditions we show that it is possible to choose a point (a, b) within the curve α_0 (and hence an initial support function $\overline{h}(0, \theta, a, b)$) so that the quantity $\int \gamma \log \overline{h} \, d\theta$ remains greater than $\log(\epsilon_{\text{in}}) \int \gamma \, d\theta$.

¿From lemma 6.1 and corollary 6.3 the inradius of the normalized curve $\alpha(t)$ is bounded below for all time by a positive constant ϵ_{in}. Let τ_n be a fixed sequence diverging to infinity. For each τ_n there is a choice of (a_n, b_n) which makes $\overline{h}(\tau_n, \theta, a_n, b_n) \geq \epsilon_{\text{in}}$ (take (a_n, b_n) to be the coordinates of the center of the inradius circle). Since $H(\tau, a_n, b_n)$ is decreasing we have $H(\tau, a_n, b_n) \geq \log(\epsilon_{\text{in}}) \int \gamma \, d\theta$ for all $\tau < \tau_n$. The points (a_n, b_n) lie in a compact set and therefore have a subsequence which converges to (a_∞, b_∞). Then for any fixed τ and any δ there exists N such that for $n > N$ we have $\tau_n > \tau$ and

$$H(\tau, a_\infty, b_\infty) \geq H(\tau_n, a_\infty, b_\infty) \geq H(\tau_n, a_n, b_n) - \delta \geq \left(\log(\epsilon_{\text{in}}) \int \gamma \, d\theta \right) - \delta$$

Since the choice of δ is arbitrary this completes the proof.

Remark 7.3. Hereafter we assume that the choice of origin is one for which the integral H remains finite for all τ.

Corollary 7.4. *For this choice of the origin there are constants $\overline{h}_{\min} > 0$ and $\overline{h}_{\max}$ such that $\overline{h}_{\min} \leq \overline{h}(\theta, t) \leq \overline{h}_{\max}$.*

Proof. Since H is bounded below the point (a, b) is enclosed by each convex curve $\alpha(\tau)$ and therefore the support function cannot become larger than the diameter which is bounded by lemma 6.4.

For the lower bound we first show that the derivative h_θ is bounded. Since $\overline{h}_{\theta\theta} + \overline{h} = 1/\overline{k}$ we have

$$|\overline{h}_\theta| \leq \int \left| \frac{1}{\overline{k}} \right| + |h| \, d\theta \leq \overline{L} + 2\pi D \leq 3\pi D$$

The derivative bound allows us to conclude that $\overline{h}(\theta, \tau) \leq 2\overline{h}_{\min}(\tau)$ for all θ in an interval whose size is independent of time. Writing

$$H(\tau) = \int_{\{\theta | \overline{h}(\theta,\tau) < 2\overline{h}_{\min}\}} \gamma \log \overline{h} \, d\theta + \int_{\{\theta | \overline{h}(\theta,\tau) \geq 2\overline{h}_{\min}\}} \gamma \log \overline{h} \, d\theta$$

one sees that the second integral is bounded above in terms of the diameter and therefore $\overline{h}_{\min}(\tau)$ cannot approach zero without forcing the value of $H(\tau)$ to negative infinity. This completes the proof of the lower bound.

Theorem 7.5. *The normalized curvature $\overline{k}$ is uniformly bounded for all time.*

Proof. There are two components to this proof. The first is the observation that the growth of $\overline{k}_{\max}(\tau)$ does not depend explicitly on τ. In fact if $\overline{k}_{\max} < b$ at time τ then there is a constant 2δ depending only on b but not on τ such that $\overline{k}_{\max}(t) < 2b$ for all $t \in [\tau, \tau+\delta]$.

This follows from the fact that $\overline{k}$ satisfies an autonomous equation (5.3) and from the comparison argument 2.2.

The second ingredient is that the average curvature is frequently small. Indeed the following monotone sum converges

$$\sum_{n=0}^{\infty} \int_{n\delta}^{(n+1)\delta} \int_{S^1} \gamma - \frac{\gamma^2 \overline{k}}{\overline{h}} \, d\theta \, dt = \lim_{\tau\to\infty} H(\tau) - H(0) > -\infty$$

and therefore

$$\lim_{n\to\infty} \int_{n\delta}^{(n+1)\delta} \int_{S^1} \gamma - \frac{\gamma^2 \overline{k}}{\overline{h}} \, d\theta \, dt = 0. \tag{7.2}$$

Further, from the mean value theorem, there exists $\xi \in [n\delta, (n+1)\delta]$ such that

$$\int_{n\delta}^{(n+1)\delta} \int_{S^1} \gamma - \frac{\gamma^2 \overline{k}}{\overline{h}} \, d\theta \, dt = \int_{S^1} \gamma - \frac{\gamma^2(\theta)\overline{k}(\theta,\xi)}{\overline{h}(\theta,\xi)} \, d\theta. \tag{7.3}$$

¿From (7.2) and (7.3) it follows that for any $\epsilon > 0$ and all sufficiently large n there exists exists $\xi_n \in [n\delta, (n+1)\delta]$ such that

$$\frac{1}{\overline{h}_{\max}} \int \gamma^2(\theta)\overline{k}(\theta,\xi_n) \, d\theta \le \int \gamma \, d\theta + \epsilon$$

Using corollary 5.8 it follows that $\overline{k}_{\max}(\xi_n) < b$ and that b depends on the initial curve, but not on τ_n. The bound on the growth of $\overline{k}_{\max}$ now implies that $\overline{k}_{\max}(\tau) < 2b$ on $[\tau_n, \tau_n + 2\delta] \supseteq [(n+1)\delta, (n+2)\delta]$.

This completes the proof.

Now we modify the monotonicity formula above to show convergence. Let

$$J(t,a,b) = \int_{S^1} \overline{h}_\theta^2 - \overline{h}^2 + 2\gamma \log \overline{h} \, d\theta \tag{7.4}$$

where $\overline{h} = \overline{h}(\tau,\theta,a,b)$ is the support function, relative to the point (a,b), of a curve $\alpha(t)$ which is evolving under the normalized equation.

Lemma 7.6. *$J(\tau,a,b)$ evolves under the normalized flow according to*

$$J_\tau = -2\int \frac{(\overline{h}-\gamma\overline{k})^2}{\overline{h}\overline{k}} \, d\theta \le \frac{-2}{\overline{k}_{\max}\overline{h}_{\max}} \int (\overline{h}-\gamma\overline{k})^2 \, d\theta \le -C \int (\overline{h}-\gamma\overline{k})^2 \, d\theta. \tag{7.5}$$

where C depends only upon the initial curve. Further $J(\tau) \ge -2\overline{A}(0) + 2\log \overline{h}_{\min} \int \gamma \, d\theta$. Together these imply that $\lim_{\tau\to\infty} J_\tau = 0$.

Proof. This is an easy calculation using (5.3). The second inequality follows from the bounds in lemma 7.4 and theorem 7.5. The lower bound on J follows from proposition 7.2 and from equations (1.3) and (5.6).

Remark 7.7. In fact the quantities J and $2H$ have the same derivatives because the first two terms of equation (7.4) are twice the area enclosed by the curve which remains constant. The derivative formulas of lemma 7.6 and lemma 7.1 are algebraically equivalent up to a factor of 2.

Theorem 7.8. *There exists a self-similar solution to equation (0.1)*

Proof.

Let τ_n be any sequence of times diverging to infinity. $\overline{k}(\cdot, \tau_n)$ and $\overline{h}(\cdot, \tau_n)$ are equicontinuous functions (by proposition 5.8 and the argument in corollary 7.4) and must therefore contain a converging subsequence. Each converging subsequence, in view of lemma 7.6, must converge to a solution of the equation $\overline{h} - \gamma\overline{k} = 0$, that is the equation defining a self-similar solution.

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