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Some minor-closed classes of signed graphs

Daniel Slilaty* and Xiangqian Zhou*

September 18, 2012

Abstract

We define four minor-closed classes of signed graphs in terms of embeddability in the annulus, projective plane, torus, and Klein bottle. We give the full list of 20 excluded minors for the smallest class and make a conjecture about the largest class.

1 Introduction

A *signed graph* is a pair $\Sigma = (G, \sigma)$ in which G is a graph and $\sigma : E(G) \to \{+1, -1\}$. A walk or cycle with edges $e_1, e_2, \ldots, e_m$ in Σ is called *positive* (respectively *negative*) when $\sigma(e_1)\sigma(e_2)\ldots\sigma(e_m) = +1$ (respectively $\sigma(e_1)\sigma(e_2)\ldots\sigma(e_m) = -1$). An uncommon feature in graph theory is a *half edge* which has one end attached to a vertex and one end unattached. (We do not allow half edges to be included in walks.) A *joint* of Σ is a half edge or negative loop and J_Σ is the set of joints of Σ.

The notion of how a signed graph Σ embeds in the annulus was introduced in [7]. Our notion here is slightly more general in its use of joints. A connected signed graph Σ without half edges embeds in the annulus when the underlying graph may be embedded in the annulus so that the positive cycles are exactly the contractible ones. Such an embedding in the annulus corresponds to an embedding in the plane (by capping the holes of the annulus with disks) in which exactly zero or two facial boundary walks are negative. (The total number of faces with negative boundary walks must be even.) In [7], a connected signed graph Σ with half edges H_Σ was said to embed in the annulus when $\Sigma \backslash H_\Sigma$ embeds in the annulus and then the elements of H_Σ may be drawn in without crossings as curves from their endpoints to the boundary rings of the annulus. In this paper, we will say that a connected signed graph Σ embeds in the annulus when there is a bipartition (H, L) of J_Σ where the elements of H that are not half edges are made into half edges and the elements of L that are not negative loops are made into negative loops and then Σ embeds in the annulus as defined in [7]. When Σ may be embedded in the annulus, we call Σ *annular*.

Let $\mathcal{A}$ be the collection of signed graphs for which each connected component is annular. Minors of signed graphs will be discussed later and we will show that $\mathcal{A}$ is a minor-closed class of signed graphs. Our main result is Theorem 1.1. When drawing signed graphs, positive edges are drawn as solid curves and negative edges are drawn as dashed curves.

Consider the 20 signed graphs in Figure 1. Let $\mathcal{E}$ be the collection of signed graphs consisting of the 20 signed graphs in Figure 1 along with the signed graphs obtainable from those in Figure 1 by replacing some subset of the negative loops with half edges.

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Theorem 1.1. *If Σ is a signed graph, then $\Sigma \in \mathcal{A}$ iff Σ does not contain an $\mathcal{E}$-minor. In other words, $\mathcal{E}$ is the collection of excluded minors for $\mathcal{A}$.*

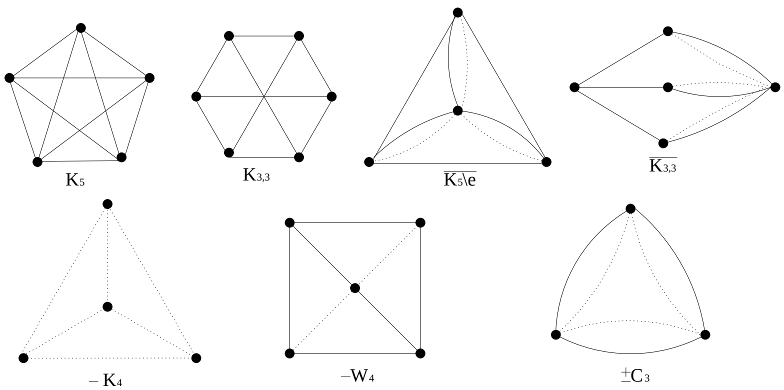
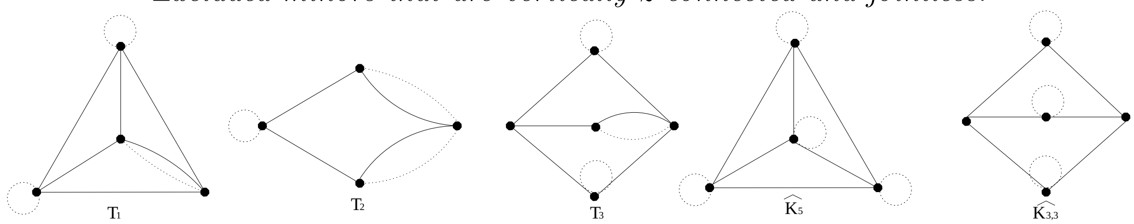
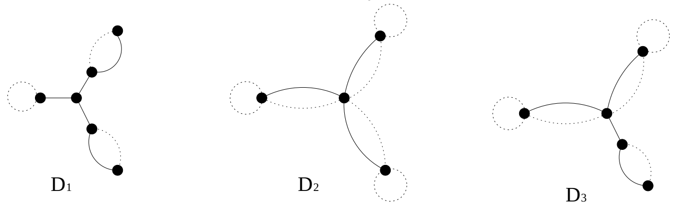


Figure 1.

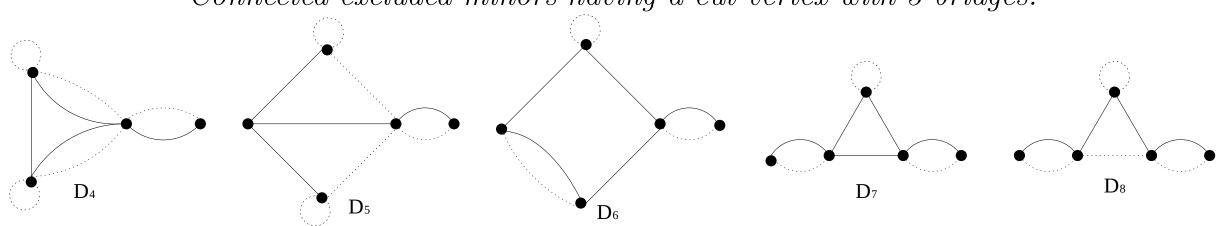
Excluded minors that are vertically 2-connected and jointless.



Excluded minors that are vertically 2-connected and have joints.



Connected excluded minors having a cut vertex with 3 bridges.



Connected excluded minors having cut vertices with at most two bridges each.

This definition of embedding a signed graph $\Sigma = (G, \sigma)$ in the annulus is natural from the standpoint of homology and minors as the reader will see in Section 3. There is also another motivation from matroid theory that we will spend the rest of this introduction discussing. Here the reader should be familiar with matroid theory as in [3]. We will not use matroid theory anywhere else in the paper.

Two classic theorems of Whitney [11] tell us that the intersection of the classes of graphic matroids and cographic matroids is exactly the class of matroids of planar graphs. The first theorem says if G is planar with topological dual graph G^*, then $M^*(G) = M(G^*)$. The second says if $M^*(G) = M(H)$ for some graph H, then G is planar. The consequence of this is that $\{M^*(K_5), M^*(K_{3,3})\}$ is the set of excluded minors for graphic matroids within the class of cographic matroids because $\{K_5, K_{3,3}\}$ is the set of excluded minors for planarity within the class of graphs.

A signed graph Σ embeds in the projective plane when its underlying graph embeds so that every facial boundary walk is positive. In [10] and also in [4] it is shown that the intersection of the classes of connected cographic matroids and connected frame matroids of signed graphs† is exactly the class of connected cographic matroids of projective-planar graphs. Analogously to the results of Whitney we get that $\{M^*(G_1), \ldots, M^*(G_{29})\}$‡ is the complete set of excluded minors for frame matroids of signed graphs within the class of cographic matroids.

Along with the notion of embedding and duality of a connected signed graph Σ in the annulus, there are notions of embedding and duality of Σ in the torus and Klein bottle (which we will define later). In [7], [8], [9] it is shown that if Σ is connected and embeds in one of these three surfaces with topological dual signed graph Σ^*, then $M^*(\Sigma) = M(\Sigma^*)$.

Now let $\mathcal{T}$ be the collection of signed graphs that embed in the torus, let $\mathcal{K}$ be the collection of signed graphs that embed in the Klein bottle, and let $\mathcal{P}$ be the collection of ordinary graphs that embed in the projective plane. A signed graph is called *balanced* when it has no half edges and every closed walk is positive. If Σ is unbalanced but is balanced after removing its joints, then Σ is said to be *joint unbalanced*. Given a joint unbalanced signed graph Σ, we define the ordinary graph G_Σ as follows. The vertices of G_Σ are the vertices of Σ along with a new vertex, call it v. For each edge that is a link or positive loop of Σ, place this edge in G_Σ (without a sign, of course) on the same endpoint(s). For each joint of Σ place a link in G_Σ from the endpoint of the joint to the new vertex v. It is easy to show that $M(\Sigma) = M(G_\Sigma)$. We call this transformation from Σ to G_Σ the *joint-unbalanced transformation*. Given Proposition 1.2 (which is proven by Propositions 3.1–3.3), the previous paragraph, and some other evidences we make Conjecture 1.3. If true, the conjecture will make the identification between the excluded minors for $\mathcal{T}\cup\mathcal{K}\cup\mathcal{A}\cup\mathcal{P}$ and the excluded minors for the class of frame matroids of signed graphs. One way to settle Conjecture 1.3 would simply be to find the excluded minors for $\mathcal{T}\cup\mathcal{K}\cup\mathcal{A}\cup\mathcal{P}$ and check if their dual matroids are signed-graphic or not. A possible first step in finding the excluded minors for $\mathcal{T}\cup\mathcal{K}\cup\mathcal{A}\cup\mathcal{P}$ would be to find the excluded minors for $\mathcal{A}$.

Proposition 1.2. *Each of $\mathcal{A}$, $\mathcal{K}\cup\mathcal{A}\cup\mathcal{P}$, $\mathcal{T}\cup\mathcal{A}$, and $\mathcal{T}\cup\mathcal{K}\cup\mathcal{A}\cup\mathcal{P}$ is closed under taking minors of signed graphs up to the joint-unbalanced transformation.*

Conjecture 1.3. *If a connected matroid M is in the intersection of the class of frame matroids of signed graphs and duals of frame matroids of signed graphs, then there is a signed graph $\Sigma \in \mathcal{T}\cup\mathcal{K}\cup\mathcal{A}\cup\mathcal{P}$ such that $M = M(\Sigma)$.*

2 Preliminaries

Each edge of a graph has two ends. When the ends of an edge are attached to different vertices, the edge is called a *link* and when the ends are attached to the same vertex, the edge is called a *loop*. In this paper we also use the less common ideas of *half edges* and *loose edges*: a half edge has one end attached to a vertex and the other unattached and a loose edge has both ends unattached.

Given $X \subseteq E(G)$, we write $V(X)$ to denote the set of vertices that are used as endpoints by edges in X. By $G{:}X$ we mean the subgraph of G with edges X and vertices $V(X)$. For $k \geq 1$ a

†See [6] or [12] for the definition of the frame matroid of a signed graph. We denote the frame matroid of Σ by $M(\Sigma)$.

‡Here $G_1, \ldots, G_{29}$ are the 2-connected excluded minors for projective planarity within the class of graphs (see [2, pp.247–251].

k-separation of G is a partition (A, B) of $E(G)$ such that $|A|, |B| \geq k$ and $|V(X) \cap V(Y)| = k$. We say that G is k-connected when it has no r-separation for $r < k$. A *vertical k-separation* of G is a k-separation (A, B) of G such that $V(A) \setminus V(B) \neq \emptyset$ and $V(B) \setminus V(A) \neq \emptyset$. We say that G is *vertically k-connected* when it has at least $k+1$ vertices and no vertical r-separation for $r < k$.

A *signed graph* Σ is a pair (G, σ) where G is a graph and σ is a labeling of the links and loops of G by elements of the multiplicative group $\{+1, -1\}$. A positive edge or unsigned edge is drawn as a solid curve and a negative edge is drawn as a dashed curve. A *joint* is an edge that is either a negative loop or half edge.

Given a signed graph $\Sigma = (G, \sigma)$ and an edge cut η in Σ we define $\Sigma^{\eta} = (G, \sigma^{\eta})$ where $\sigma^{\eta}(e) = \sigma(e)$ for $e \notin \eta$ and $\sigma^{\eta}(e) = -\sigma(e)$ for $e \in \eta$. This operation of reversing signs on an edge cut is called *switching*. Since the symmetric difference of two edge cuts is again an edge cut, equality by switching is an equivalence relation on signings of G.

Proposition 2.1 (Zaslavsky [12]). *If $\Sigma = (G, \sigma)$ and $\Upsilon = (G, \upsilon)$ are signed graphs, then Σ and Υ have the same collection of positive and negative cycles iff Σ and Υ are switching equivalent.*

A minor of a signed graph is obtained by a sequence of edge deletions, edge contractions, deletions of isolated vertices, and switchings. Deleting edges is done in the obvious way. Contracting a positive loop or loose edge is the same as deleting it. Contracting a positive link is done in the usual way. Contracting a negative link is done by first switching on an edge cut containing that edge and then contracting as usual (this makes contraction well defined up to switching). Contraction of a joint incident to vertex v is done by un-attaching the ends of the edges incident to v and then deleting e and v. Thus any link incident to v becomes a half edge incident to its other endpoint and any edge with all attachments on v becomes a loose edge. This notion of contraction on signed graphs was formulated in [12] so as to make it correspond to contraction in frame matroids of signed graphs. A minor of Σ obtained without contracting any joints is called a *link minor.*

Proposition 2.2 ([6]). *If Σ and Υ are signed graphs and Υ is jointless, then Σ has Υ as a minor iff $\Sigma \setminus J_{\Sigma}$ has Υ as a link minor.*

A walk $e_1, e_2, \dots, e_m$ in Σ is called *positive* (respectively *negative*) when $\sigma(e_1)\sigma(e_2)\dots\sigma(e_m) = +1$ (respectively $\sigma(e_1)\sigma(e_2)\dots\sigma(e_m) = -1$). A cycle in a signed graph is called positive when all closed walks on that cycle are positive; otherwise the cycle is called negative. A signed graph is called *balanced* when it has no joints and no negative cycles. A *balancing* vertex is a vertex whose removal leaves a balanced subgraph. When a signed graph has two distinct balancing vertices, then it takes on a very restricted structure.

Theorem 2.3 ([13]). *If Σ is a connected signed graph with two distinct balancing vertices x and y, then there is a bipartition (A, B) of $E(\Sigma)$ in which $\Sigma{:}A$ and $\Sigma{:}B$ are both balanced and $V(A) \cap V(B) = \{x, y\}$.*

Consider the two signed graphs in Figure 1 named $-K_4$ and $\pm C_3$. When a signed graph has neither of these as a minor, then it takes on a special structure as given in Theorem 2.4.

Theorem 2.4 (Gerards [1, Thm. 3.2.3]). *If Σ is a signed graph without $-K_4$- or $\pm C_3$-minor, then either $\Sigma \setminus J_{\Sigma}$*

(1) is balanced,

(2) has a balancing vertex,

(3) is annular,

(4) is isomorphic to the signed graph of Figure 2, or

(5) has a 1-, 2-, or 3-split into two signed graphs without $-K_4$- or $\pm C_3$-minor.

In Part (5) *k-splits* are defined for signed graphs without a balancing vertex. If (A_1, A_2) is a 1-separation of Σ, then $\Sigma{:}A_1$ and $\Sigma{:}A_2$ is called a 1-*split* of Σ. If (A_1, A_2) is a 2-separation of Σ with $|A_i| \geq 3$ when $\Sigma{:}A_i$ is unbalanced, then because Σ doesn't have a balancing vertex we can assume that $\Sigma{:}A_1$ is unbalanced. We call $\Sigma{:}A_1 \cup P$ and $\Sigma{:}A_2 \cup P$ a 2-*split* of Σ where P is a single or double link on $V(A_1) \cup V(A_2)$ defined as follows: a double edge of different signs when $\Sigma{:}A_2$ is unbalanced and a single link of the unique sign that makes $\Sigma{:}A_2 \cup P$ balanced otherwise. If (A_1, A_2) is a 3-separation of Σ with $\Sigma{:}A_1$ unbalanced and $\Sigma{:}A_2$ balanced with $|A_2| \geq 4$, then we call $\Sigma{:}A_1 \cup P$ and $\Sigma{:}A_2 \cup P$ a 3-*split* of Σ where P is a triad attached to the three vertices in $V(A_1) \cap V(A_2)$ and signed so that $\Sigma{:}A_2 \cup P$ is balanced.

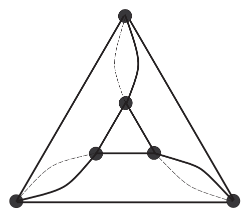


Figure 2.

A signed graph Σ is called *tangled* when it does not have a balancing vertex and yet does not have two vertex-disjoint negative cycles. Proposition 2.5 is an important structural fact about tangled signed graphs. A full structural characterization of tangled signed graphs is given in [10] and [5]

Proposition 2.5 (Qin, Slilaty [6]). *If Σ is a tangled signed graph, then Σ has $-K_4$ or $\pm C_3$ as a link minor.*

3 Minor-closed classes of signed graphs

Given a compact surface S with holes, we denote the closed surface obtained by capping the holes of S by $\mathsf{S}^\bullet$. We make the convention that a graph G embedded in S touches the boundaries of the holes at only the unattached ends of half edges and loose edges and that each unattached end of a half or loose edge is at the boundary of a hole. We say that G is *cellularly embedded* in S when the induced embedding of G minus its half and loose edges in $\mathsf{S}^\bullet$ is cellular. The *faces* of a cellular embedding of G in S are the 2-cells of $\mathsf{S} \setminus G$. In Figure 3 we have a graph cellularly embedded in the annulus A with five faces. Note that the region having a hole with no half edges on it does not count as a face. Of course any connected graph that is embedded in a surface but not cellularly embedded may be extended to a cellular embedding.

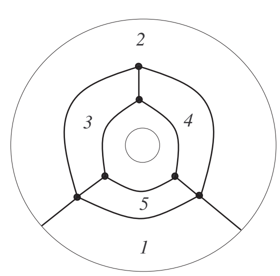


Figure 3.

Given a graph G, let $Z(G)$ be the integer cycle space of G. Loose and half edges do not contribute to the cycle space of a graph, that is, if G' is G with its half and loose edges removed, then $Z(G') = Z(G)$. If G is connected and cellularly embedded in a surface S, then let $B(G)$ be the subspace of $Z(G)$ generated by the facial boundary walks of G in S. (As defined this does not include faces whose boundaries have half edges on them, e.g., faces 1 and 2 in Figure 3.) Invariance of homology gives us that for any two connected graphs G and H cellularly embedded in S, $Z(G)/B(G) \cong Z(H)/B(H)$. This quotient group, denote it by $H(\mathsf{S})$, is well defined for S up to isomorphism and is called the first homology group of S calculated with integer coefficients. So now given any graph G embedded in S, there is up to isomorphism a canonical map $\natural\colon Z(G) \to H(\mathsf{S})$.

Now if the underlying graph of $\Sigma = (G, \sigma)$ is embedded in S such that all facial boundary walks are positive (faces without half edges on them), then there is a unique map $\mu\colon H(\mathsf{S}) \to \mathbb{Z}_2$ such that $\widehat{\sigma} = \mu\natural$. Here $\widehat{\sigma}\colon Z(G) \to \mathbb{Z}_2$ is the homomorphism induced by σ. Note that two signings σ_1 and σ_2 on G satisfy $\widehat{\sigma}_1 = \widehat{\sigma}_2$ iff they are switching equivalent.

Signed graphs in the annulus For the annulus A one can calculate that $H(\mathsf{A}) \cong \mathbb{Z}$. The only nonzero $\mu\colon \mathbb{Z} \to \mathbb{Z}_2$ is the usual quotient map defined by $\mu(1) = 1$. We say that a signed graph $\Sigma = (G, \sigma)$ embeds in A when its underlying graph with joints removed embeds such that $\widehat{\sigma} = \mu\natural$ and then joints may be added back in by individually choosing to embed each one as a negative loop or half edge. Loose edges may always be added in onto the boundaries of holes.

Proposition 3.1. *If a signed graph Σ is embedded in the annulus, then for any edge e there is an induced embedding of $\Sigma \backslash e$ in A and an induced embedding of Σ/e in either A or in two distinct copies of A.*

Proof. That the deletion has an induced embedding is evident. For Σ/e, the induced embedding is evident when e is a link, positive loop, or loose edge. In the case that e is a half edge, we can cut the embedding of Σ in A as indicated on the left in Figure 4 to obtain an embedding of Σ/e in A. In the case that e is a negative loop, we can cut the embedding of Σ in A as indicated on the right in Figure 4 to obtain an embedding of Σ/e in one or two copies of A. □

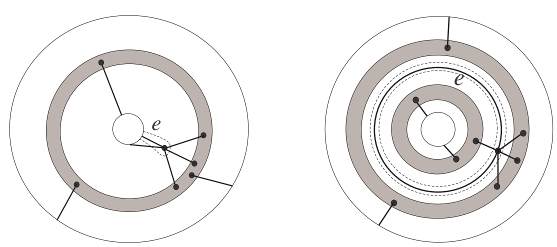


Figure 4.

Signed graphs in the torus For the torus T, one can calculate that $H(\mathsf{T}) \cong \mathbb{Z} \times \mathbb{Z}$. Up to homeomorphism of T, there is only one possible nonzero $\mu\colon H(\mathsf{T}) \to \mathbb{Z}_2$ and it is defined by $\mu(1, 0) = \mu(0, 1) = 1$. We say that a signed graph $\Sigma = (G, \sigma)$ embeds in T when its underlying graph embeds such that $\widehat{\sigma} = \mu\natural$.

Proposition 3.2. *If Σ is a signed graph embedded in T, then for any edge e there is an induced embedding of $\Sigma \backslash e$ in T and an induced embedding of Σ/e in either T or A.*

Proof. That the deletion has an induced embedding is evident. For Σ/e, the induced embedding is evident when e is a link or positive loop. There can be no half an loose edges in an embedding in T.

Finally, assume that e is a negative loop. Up to homeomorphism e is embedded as shown in Figure 5 and then an induced embedding of Σ/e in A is obtained by cutting T as indicated. □

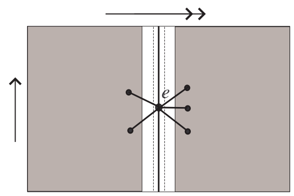


Figure 5.

Signed graphs in the Klein bottle For the Klein bottle K, one can calculate that $H(\mathsf{K}) \cong \mathbb{Z}\times\mathbb{Z}_2$. Up to homeomorphism of K, there are two possible nonzero possibilities for $\mu\colon H(\mathsf{T}) \to \mathbb{Z}_2$: the first is defined by $\mu(1,0) = \mu(1,1) = 1$ and the second by $\mu(1,0) = \mu(0,1) = 1$. We say that a signed graph $\Sigma = (G, \sigma)$ embeds in K when its underlying graph embeds such that $\widehat{\sigma} = \mu\natural$ where μ is the latter map.

Proposition 3.3. *If Σ is a signed graph embedded in K, then for any edge e there is an induced embedding of $\Sigma\backslash e$ in K and either*

- *there is an induced embedding of Σ/e in K or A or*
- *Σ/e is joint unbalanced and there is an induced embedding of Σ/e in the projective plane after applying the joint-unbalanced transformation.*

Proof. That the deletion has an induced embedding is evident. For Σ/e, the induced embedding is evident when e is a link or positive loop. There can be no half an loose edges in an embedding in K. Finally, assume that e is a negative loop. The negative loop is embedded in either the $(0,1)$- or $(1,0)$-homology class. For e in the $(0,1)$-homology class, Σ is as shown (up to homeomorphism) on the left in Figure 6. For e in the $(1,0)$-homology class, Σ is as shown (up to homeomorphism) on the right in Figure 6. In the former case, if we cut K as indicated in the figure then we get an embedding of Σ/e in the A. In the latter case, the endpoint of e must be a balancing vertex for Σ because any other cycles in the $(1,0)$- and $(0,1)$-homology classes must pass through e and these are the only homology classes that can contain negative cycles. Thus Σ/e is joint unbalanced and if we cut K as indicated in the figure, then we get an embedding of Σ/e in the Möbius band where the unattached ends of the half edges are on the boundary. Thus we can transform Σ/e by the joint-unbalanced transformation to an ordinary graph G embedded in the projective plane. □

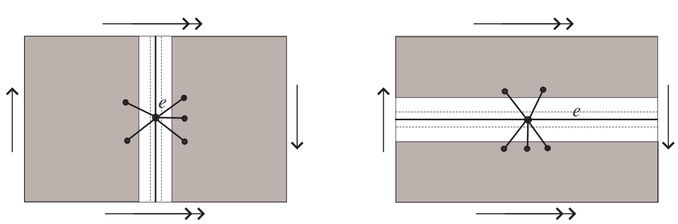


Figure 6.

4 Lemmas and Proofs

Proposition 4.1. *Each of the signed graphs in Figure 1 is minor-minimally not in $\mathcal{A}$.*

Sketch of proof. Checking that the proposition holds for each signed graph in Figure 1 is routine; we will provide proofs for $-W_4$, T_2, and D_2 only and leave the rest to the reader. We refer to the labelings of these three signed graphs in Figure 7.

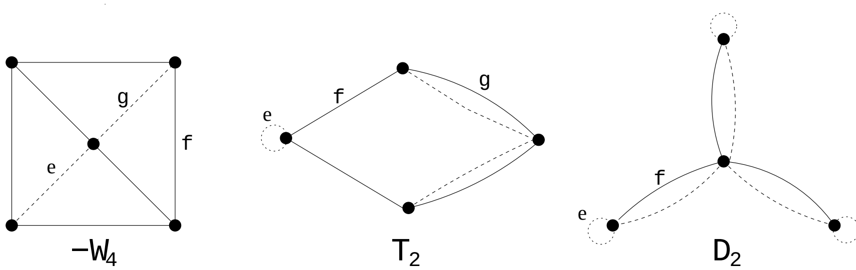


Figure 7. *The signed graphs $-W_4$, T_2, and D_2*

For the signed graph $-W_4$, there is only one way to imbed $-W_4\backslash e$ in the annulus as shown in the first figure in Figure 8. It is clear that we can not add the edge e to the embedding, and hence, $-W_4$ is not embeddable in the annulus. Next we show that every proper minor of $-W_4$ embeds in the annulus. By symmetry, it suffices to show that $-W_4\backslash e$, $-W_4/e$, $-W_4\backslash f$, and $-W_4/f$ are embeddable. Figure 8 shows an embedding for each of the four signed graphs.

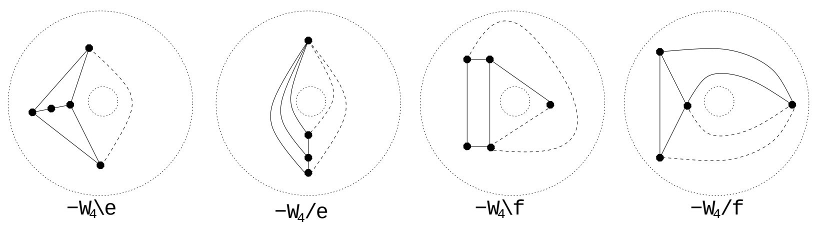


Figure 8.

For the signed graph T_2, it is easy to see that $T_2\backslash e$ must embed in the annulus as shown in Figure 9. The negative loop e can not be added to the embedding either as a half edge or as a negative loop. So T_2 is not embeddable in the annulus. For the minor-minimality, by symmetry, we only need to show the deletion and the contraction of the three edges e, f, and g are embeddable. Figure 9 shows an embedding for each of the six signed graphs.

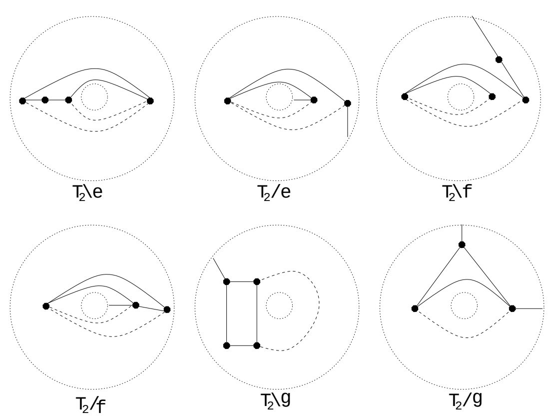


Figure 9.

For the signed graph D_2, we first remove all joints, the resulting signed graph must embed in the annulus with one inner-most negative digon, one outer-most negative digon, and the other negative digon lying in between. It is clear that we can add the joints to the inner-most and the outer-most digon, but we can not add the joints to the third digon. Therefore, D_2 is not embeddable. Next we show all proper minors of D_2 are embeddable. By symmetry, it suffices to show that the deletion and contraction of the edges e and f are embeddable. Figure 10 shows an emdedding for each of these signed graphs. □

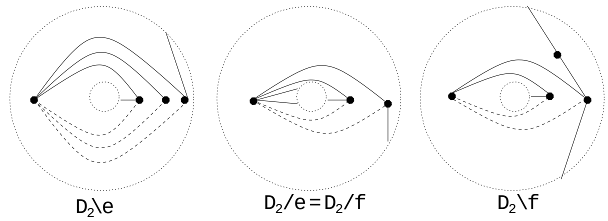


Figure 10. *Embeddings of minors of D_2*

Proposition 4.2. *If Σ is connected and embeds in the annulus and contains a joint e at vertex v, then e may be drawn as a half edge except when v is a cut vertex of Σ and Σ contains the rooted minor at v and e in Figure 11.*

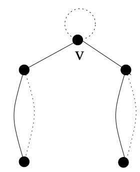


Figure 11.

Proof. If Σ has an embedding in the annulus with v on the inner or outer face, then e may be drawn as a half edge. So assume that v is not on one of these faces in any embedding of Σ in the annulus and so e is drawn as a loop in any embedding in Σ. Thus there is a vertical 1-separation (O, I) of $\Sigma \setminus J_\Sigma$ at v where $\Sigma{:}O$ and $\Sigma{:}I$ are the outer and inner components of some embedding of $\Sigma \setminus J_\Sigma$. Now we must have inner and outer facial cycles in this embedding that block e from being drawn as a half edge and so since Σ is connected we get the desired rooted minor. □

Proof of Theorem 1.1. That each of the signed graphs in Figure 1 is minor-minimally not in $\mathcal{A}$ is given by Proposition 4.1. So now say that Σ does not embed in the annulus. If the underlying graph of Σ is nonplanar, then Σ has a link minor Υ whose underlying graph is K_5 or $K_{3,3}$. If Υ is balanced, then $\Upsilon \cong K_5$ or $K_{3,3}$, as required. If Υ is not balanced, then either Υ has a balancing vertex or is tangled because K_5 and $K_{3,3}$ do not have any vertex-disjoint pairs of cycles. If Υ is tangled, then Υ has a $-K_4$- or $\pm C_3$-link minor by Proposition 2.5, as required. If Υ has a balancing vertex, then Υ switches to a signed graph that is $K_{3,3}$ with one edge negated, K_5 with one edge negated, or K_5 with two edges incident to the same vertex negated. In the first case contracting the negative edge of Υ yields $-W_4$, in the second case contracting the negative edge of Υ yields $\overline{K_5 \backslash e}$, and in the last case Υ has a 2-edge deletion isomorphic to $-W_4$.

So for the remainder of the proof say that Σ is an excluded minor for the annulus whose underlying graph is planar. Now if Σ is joint unbalanced (i.e., $\Sigma \setminus J_\Sigma$ is balanced) then Σ is annular iff G_Σ is planar. That is Σ is annular iff $\Sigma \setminus J_\Sigma$ has a planar embedding in which all of the all joint vertices are on a single face. (They must all be on a single face rather than on two faces because a signed graph embedded in the annulus has all of its positive cycles embedded contractibly and all of its negative cycles embedded non-contractibly. Having the boundary of the annulus accessible from two different faces requires the signed graph to have negative cycles.) So since Σ is minor-minimally non-annular, G_Σ is minor-minimally non-planar. Thus $G_\Sigma \cong K_5$ or $K_{3,3}$ and so $\Sigma \cong \widehat{K}_5$ or $\widehat{K}_{3,3}$.

So for the remainder of the proof we may assume that Σ is an excluded minor for the annulus that is not joint unbalanced and whose underlying graph is planar. We may also assume that Σ does not have $-K_4$ or $\pm C_3$ as a link minor and so we can apply Theorem 2.4 to Σ. We split the proof into three cases based on the connectivity of Σ. In Case 1 Σ is vertically 2-connected and is jointless, in Case 2 Σ is vertically 2-connected and has a joint, and in Case 3 Σ is connected but not vertically 2-connected.

Case 1: We split this case into five subcases given by Theorem 2.4. In Case 1.1 Σ is balanced, in Case 1.2 Σ has a balancing vertex and has no 2-split, in Case 1.3 Σ is isomorphic to the signed graph in Figure 2, and in Case 1.4 Σ has a 3-split and no 2-split, and in Case 1.5 Σ has a 2-split. Note that Σ cannot have parallel edges of the same sign.

Case 1.1: Since Σ is balanced and planar, Σ embeds in the annulus, a contradiction.

Case 1.2: Since Σ must be vertically 3-connected, the planar embedding of Σ is unique up to exchanging parallel edges and each facial boundary walk is a cycle in Σ. Assume that the parallel edges are embedded such that the number of negative facial cycles is a minimum. Because Σ is not annular, this number of negative facial cycles is at least 3 but since the symmetric difference of all facial cycles is empty, the number of negative ones must be even. Thus the number of negative facial cycles is at least 4 and all of these facial cycles contain the balancing vertex, call it v, of Σ. (Assume Σ is switched so that all negative edges are incident to v.) Let Υ be the subgraph of Σ obtained by taking the union of all of the facial boundary cycles containing v. By vertical 3-connectivity Υ consists of a positive cycle R not containing v and the edges of Σ incident to v which then all connect v to R; furthermore, v must have at least 3 adjacent vertices on R. Also by vertical 3-connectivity Υ has a unique embedding in the plane up to exchanging parallel edges. Say again that Υ is embedded so as to minimize the number of negative faces. Now, if there is no negative face in this embedding of length at least 3, then v must be adjacent to at least four vertices on R by double edges. This is because faces of length 2 would be the only negative faces and we must have an even number of these that is greater than 2. Thus Υ contains a $\overline{K_5\backslash e}$-subdivision. So let T be a negative face of Υ of length at least 3. Say that e_1 and e_2 are positive and negative links (respectively) of T incident to v and let r_1 and r_2 be their endpoints on R. Let $r_3, \ldots, r_n$ be the remaining vertices of R adjacent to v. In Case 1.2.1 say that r_2 and v have a single edge between them and in Case 1.2.2 say that r_2 and v have a double edge between them.

Case 1.2.1: It cannot be that the edges between v and $r_3, \ldots, r_n$ are all negative because then we may embed Υ (and so Σ) with two negative faces, a contradiction. So let r_i be the first vertex in $r_3, \ldots r_n$ with a positive link connected to v. So now if there is some $r_j \in \{r_{i+1}, \ldots, r_n\}$ with a negative link to v, then Υ has a $-W_4$-subdivision. So suppose that all edges connecting $r_{i+1}, \ldots, r_n$ are positive. So now in order to have at least four negative faces in the embedding of Υ it must be that both r_1 and r_i have double edges connecting to v. But then Υ (and so Σ) embeds with two negative faces, a contradiction.

Case 1.2.2: Between v and r_3, there must be a positive link or else we can reverse the embedding of the r_2v-edges and reduce the number of negative faces in the embedding of Υ, a contradiction. So now if there is any negative link from v to one of $r_4, \ldots, r_n$ then Υ contains a $-W_4$-subdivision. So assume that all links from v to $r_4, \ldots, r_n$ are positive. So in order for the embedding of Υ to have at least 4 negative faces, either r_1 and v or r_3 and v have a double edge between them. If both have a double edge, then Υ has a $\overline{K_5\backslash e}$-subdivision. If only one has a double edge, then we may re-embed Υ (and so Σ) with at most two negative faces, a contradiction.

Case 1.3: If we contract the edges of a positive triangle in the signed graph of Figure 2, then we obtain $\overline{K_5\backslash e}$.

Case 1.4: Let Σ_1 and Σ_2 be the terms of the 3-split in which Σ_2 is balanced. Since Σ_1 is a proper minor of Σ, Σ_1 embeds in the annulus. Because Σ is planar and is vertically 3-connected (if Σ was not vertically 3-connected then Σ would have a 2-split) and Σ_2 is balanced, we may identify the planar embedding of Σ_2 along the annular embedding of Σ_1 to obtain an embedding of Σ in the annulus, a contradiction.

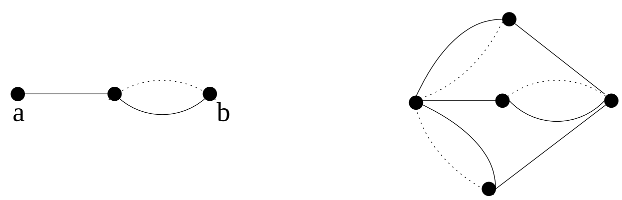
Case 1.5: Say that there is a 2-split of Σ at vertices x and y and let $B_1, \ldots, B_n$ $(n \geq 2)$ be the $\{x, y\}$-bridges of Σ that each contain a vertex other than x and y. The only other possibilities for $\{x, y\}$-bridges of Σ are single xy-links. Let Σ_i and $\overline{\Sigma}_i$ be the terms in the 2-split of Σ where B_i is contained in Σ_i and the remaining $\{x, y\}$-bridges of Σ are all contained in $\overline{\Sigma}_i$.

First we claim that each B_i is unbalanced. If B_i is balanced, then $\overline{\Sigma}_i$ is a proper minor of Σ and so embeds in the annulus. Since Σ is itself a planar graph and Σ_i is balanced, we may obtain an embedding of Σ in the annulus attaching the planar embedding of Σ_i to the embedding of $\overline{\Sigma}_i$, a contradiction.

Second we claim that $n \geq 3$. If $n = 2$, then it must be that Σ_1 and $\overline{\Sigma}_1$ do not have vertical 2-separations at x and y. By minimality both Σ_1 and $\overline{\Sigma}_1$ embed in the annulus and since there are no vertical 2-separation of Σ_1 and $\overline{\Sigma}_1$ at x and y, the negative digons of Σ_1 and $\overline{\Sigma}_1$ are both along the outer rim of the annulus in each embedding. Thus we may then embed all of Σ in the annulus by identifying the embeddings of Σ_1 and $\overline{\Sigma}_1$ along the negative digons at x and y.

Third we claim that for each B_i, that at least one of x and y is not a balancing vertex of B_i. If x and y are both balancing vertices of B_i, then by Theorem 2.3 there are $\{x, y\}$-bridges C_1 and C_2 of B_i where each C_i is balanced. Furthermore, since B_i contains no xy-links, each C_i has at least three vertices. Thus B_i is not an $\{x, y\}$-bridge of Σ, a contradiction.

So by the previous three paragraphs and the fact that each Σ_i must be vertically 2-connected, each B_i contains a rooted link minor as shown on the left in Figure 4 where $\{a, b\} = \{x, y\}$.



Thus Σ contains either $\overline{K}_{3,3}$ or the right-hand signed graph of Figure 4 as a link minor. In the latter case $\Sigma \cong \overline{K}_{3,3}$ and in the former case the right-hand signed graph contains a T_2-minor.

Case 2: Because Σ is not joint unbalanced, we can split this case into the following three subcases. In Case 2.1 $\Sigma \setminus J_\Sigma$ has a unique balancing vertex, in Case 2.2 $\Sigma \setminus J_\Sigma$ has two distinct balancing vertices, and in Case 2.3 $\Sigma \setminus J_\Sigma$ does not have a balancing vertex.

Case 2.1: Since $J_\Sigma \neq \emptyset$, pick some $l \in J_\Sigma$ (say with endpoint v) and so $\Sigma \backslash l$ embeds in the annulus. Let C_1 and C_2 be the innermost and outermost negative cycles of an embedding of $\Sigma \backslash l$ on the the annulus. Since $\Sigma \setminus J_\Sigma$ has a unique balancing vertex, call it b, C_1 intersects C_2 at b only. Now the vertex v must be embedded between C_1 and C_2 in this embedding of $\Sigma \backslash l$. If there is a path γ_1 from v to C_1 that avoids C_2 and a path γ_2 from v to C_2 that avoids C_1, then $C_1 \cup C_2 \cup \gamma_1 \cup \gamma_2 \cup l$ contains a T_2-minor. So without loss of generality, every path from v to C_2 must first intersect C_1. Let Γ be the union of all paths in Σ from v to in C_1. By assumption, no $\gamma \subseteq \Gamma$ intersects C_2. Now let $v_1, \ldots v_k$ be all of the endpoints in C_1 (in some cyclic ordering around C_1) of paths in Γ. By vertical connectivity $k \geq 2$. However, it cannot be that $v_1, \ldots, v_k$ are all of the vertices of C_1, because then there can be no path from C_1 to C_2 that avoids b which makes b a cut vertex of Σ, a contradiction of vertical 2-connectivity. Let δ be the $v_1 v_k$-path in C_1 that contains $v_2, \ldots, v_{k-1}$ (when $k \geq 3$) and does not contain b in its interior. Now the subgraph $H = \Gamma \cup l$ may be reembedded as shown in Figure 12 which yields an embedding of Σ (a contradiction) unless there is a joint l' of Σ on an interior vertex w of δ. By vertical 2-connectivity there exists a path α between C_1 and C_2 that avoids b as shown in Figure 12 and this contains a T_3-minor.

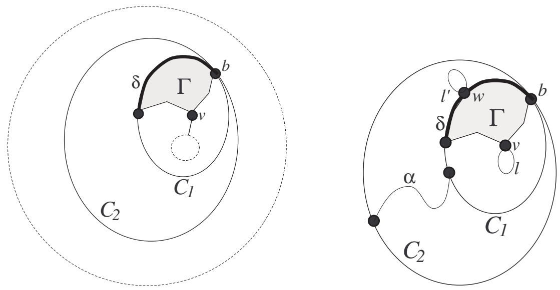


Figure 12.

Case 2.2: Let x and y be two distinct balancing vertices of $\Sigma \setminus J_\Sigma$ and so we have a bipartition (A, B) of $E(\Sigma) \setminus J_\Sigma$ as given in Theorem 2.3. Switch Σ so that its negative links are exactly the edges of B incident to x. In the next paragraph we show that either $|A| = 1$ or $|B| = 1$ (assume the latter after the next paragraph).

Assume that $|A|, |B| \geq 2$. Let Σ_A be the signed graph obtained from $\Sigma{:}A$ by attaching all negative loops of Σ with endpoints in $V(A)$ and also a negative xy-link. Let $\Sigma{:}B$ be the signed graph obtained from Σ by attaching all negative loops of Σ with endpoints in $V(B)$ and a positive xy-link. Since $|A|, |B| \geq 2$, each of Σ_A and Σ_B is a proper minor of Σ and so each embeds in the annulus. The embeddings must look as shown in Figure 13 and so we can remove the new xy-links of Σ_A and Σ_B and paste the two embeddings together to obtain an embedding for Σ in the annulus, a contradiction.

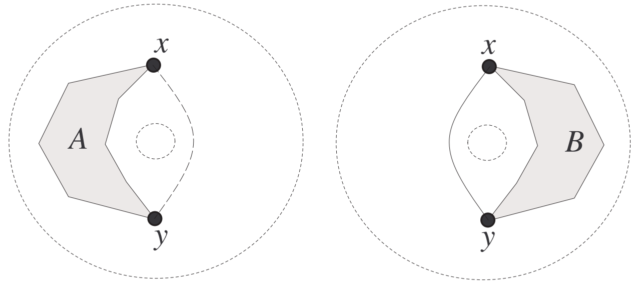


Figure 13.

So now after switching Σ has one negative link, call it e, and the underlying graph of Σ is planar. Let $j_1, \ldots, j_m$ be the joints of Σ with endpoints $v_1, \ldots, v_m$. Now by Proposition 4.2 and the fact that Σ is vertically 2-connected, Σ will embed in the annulus iff there is a planar embedding of $\Sigma \setminus \{e, j_1, \ldots, j_m\}$ with $x, y, v_1, \ldots, v_m$ all on a single facial walk. So now let G_Σ be the ordinary graph obtained from $\Sigma \setminus \{e, j_1, \ldots, j_m\}$ by adding a new vertex v adjacent to $x, y, v_1, \ldots, v_m$ where e_i is the $v_i v$-link and e_x and e_y are the xv-link and yv-link. Note that G_Σ will be planar iff Σ is annular. Thus G_Σ is nonplanar and by the minimality of Σ, each $G_\Sigma \backslash e_i$ is planar and $G_\Sigma \setminus \{e_x, e_y\}$ is planar. In Case 2.2.1 say that both $G_\Sigma \backslash e_x$ and $G_\Sigma \backslash e_y$ are planar and in Case 2.2.2 say without loss of generality that $G_\Sigma \backslash e_y$ is not planar.

Case 2.2.1: Here there must be a K_5 or $K_{3,3}$ subdivision K in G_Σ that uses all of the edges incident to v. If K is a subdivision of K_5, then Σ contains a minor isomorphic to T_1. If K is a subdivision of $K_{3,3}$, then Σ contains a minor isomorphic to the signed graph of Figure 14. If we contract the negative link of the signed graph in Figure 14, then we obtain T_2.

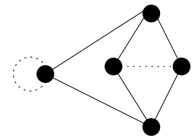


Figure 14.

Case 2.2.2: Here there must be a K_5 or $K_{3,3}$ subdivision K in G_Σ that uses all of the edges incident to v except for e_y. If K is a subdivision of K_5, then Σ contains a subgraph S which consists of a subdivision of K_4 (with x as one branch vertex) along with three joints at its other three branch vertices. Now since Σ is vertically 2-connected, there is a path γ from y to S in $\Sigma \backslash x$. So now $S \cup \gamma \cup e$ contains a T_1-minor. If K is a subdivision of $K_{3,3}$, then Σ contains a subgraph S which consists of a subdivision of $K_{2,3}$ in which the three vertices of the one partite set are x, $a \neq y$, and $b \neq y$ with joints attached to a and b. Because Σ is vertically 2-connected, there is a path γ from y to S in $\Sigma \backslash x$. If the endpoint of γ in S is either a or b, then $S \cup \gamma \cup e$ contains as a minor the signed graph of Figure 14 which has a T_2-minor. If the endpoint of γ in S is not a or b, then $S \cup \gamma \cup e$ contains T_3 as a minor.

Case 2.3: Let l be a joint of Σ with endpoint v. By minimality $\Sigma \backslash l$ embeds in the annulus. Let C_1 and C_2 be the innermost and outermost negative cycles of an embedding of $\Sigma \backslash l$ on the the annulus. Since $\Sigma \setminus J_\Sigma$ does not have a balancing vertex, C_1 and C_2 are vertex disjoint. The remainder of this case is similar to Case 2.1.

Case 3: Of course if Σ has a D_1-minor, then $\Sigma \cong D_1$. So in the remainder of Case 3 assume that Σ has no D_1-minor. Since Σ is connected but not vertically 2-connected, then there is a cut vertex v of Σ. Denote the v-bridges of Σ by $B_1, \ldots, B_k$ (note that $k \geq 2$).

First we claim that each B_i is not balanced. If it were, then $\Sigma \setminus E(B_i)$ is a proper minor of Σ and so embeds in the annulus. Since the underlying graph of Σ is planar, B_i is planar and balanced and so we can embed Σ in the annulus, a contradiction.

Second we claim that each $|E(B_i)| \neq 1$. By way of contradiction assume that $|E(B_1)| = 1$; that is, $E(B_1)$ is a single joint, call it e. Thus $\Sigma \backslash e$ embeds in the annulus and since v is a cut vertex of Σ we can add e to the embedding of $\Sigma \backslash e$ as either a loop or half edge, a contradiction.

Third we claim that each B_i is not joint unbalanced. By way of contradiction, assume without loss of generality that B_1 is joint unbalanced. Let Σ_1 be B_1 along with a new joint added at v and let $\overline{\Sigma}_1$ be $\Sigma \setminus E(B_1)$ along with a new joint at v and all isolated vertices removed. (This new joint is an edge not already present in Σ.) Since $|E(B_i)| \geq 2$ and $k \geq 2$, Σ_1 and $\overline{\Sigma}_1$ are both proper minors of Σ and so embed in the annulus. By Proposition 4.2, the joint of $\overline{\Sigma}_1$ at v may be embedded as a half edge unless we have the rooted minor of $\overline{\Sigma}_1$ at v shown in Figure 11. The latter case cannot hold, however, because then Σ will have a D_1-minor. So if the joint of $\overline{\Sigma}_1$ at v is embedded as a half edge, then because Σ_1 is joint unbalanced and embeds in the annulus we can combine the two embeddings to get an embedding of Σ, a contradiction.

In Case 3.1 assume that Σ has at least three v-bridges and in Case 3.2 assume Σ has exactly two v-bridges. In each case let Σ_i be B_i with a joint attached to v and let $\overline{\Sigma}_i$ be $\Sigma \setminus E(B_i)$ with all isolated vertices removed and a joint attached to v. Since each $|E(B_i)| \geq 2$, both Σ_i and $\overline{\Sigma}_i$ are proper minors of Σ and so both embed in the annulus.

Case 3.1: We claim that v is not a balancing vertex for Σ_i. Assume without loss of generality that v is a balancing vertex for Σ_1. Thus Σ_1 has only the one joint at v and no others. Now $\overline{\Sigma}_1$ embeds in the annulus and has at least two v-bridges and each of thee v-bridges is not balanced and not joint unbalanced. Thus we can embed the joint of $\overline{\Sigma}_1$ at v as a negative loop in between two concentric v-bridges. We may now combine the embeddings of Σ_1 and $\overline{\Sigma}_1$ by placing Σ_1 along the negative loop at v in the embedding of $\overline{\Sigma}_1$ to obtain an embedding of Σ, a contradiction.

So now since v is not a balancing vertex of Σ_i and each Σ_i is not joint unbalanced, each Σ_i contains one of the signed graphs from Figure 15 rooted at v. So since $k \geq 3$ and Σ does not have a D_1-minor, we get that Σ has a D_2- or D_3-minor.

Figure 15.

Case 3.2: We claim that either B_1 or B_2 must consist only of a negative digon. If not, then let Σ_i' be B_i along with a negative digon attached to v. Since each B_i has at least two edges and is not joint unbalanced, our assumption gives us that each Σ_i' is a proper minor of Σ and so embeds in the annulus. Since v is not a cut vertex of B_i, the negative digon of Σ_i' is on an outer face of the embedding of Σ_i'. Thus we may combine the embeddings of Σ_1' and Σ_2' to obtained an embedding of Σ, a contradiction. So assume without loss of generality that B_1 is a negative digon.

In Case 3.2.1 say that B_2 has a cut vertex, call it u, and in Case 3.2.2 say that B_2 is vertically 2-connected.

Case 3.2.1: If Σ has three or more u-bridges, then it has a D_2- or D_3-minor as in Case 3.1. So we may assume that Σ has exactly two u-bridges. One of these u-bridges is contained in B_2 and does not contain the vertex v, call it B_1'. The other u-bridge, call it R, contains B_1 and so also contains the vertex v; let $B_2' = R \cap B_2$. By the same argument as at the beginning of Case 3.2 we get without loss of generality that either B_1' or B_2' consists only of a negative digon. Thus we specifically get that B_1' is a negative digon. Now either B_2' is balanced or not.

If B_2' is balanced, then because B_2' is planar, it must be that there is not an embedding of B_2' with u and v both on the same face. If there were then we could embed Σ in the annulus, a contradiction. Thus the graph G obtained from B' by adding a uv-link is non-planar. Thus there is a K_5- or $K_{3,3}$-subdivision in G using e. Thus Σ has one of the signed graphs of Figure 16 as a minor. In the left-hand case, Σ contains a $\widehat{K}_5$-minor and in the right-hand case Σ contains a $\widehat{K}_{3,3}$-minor.

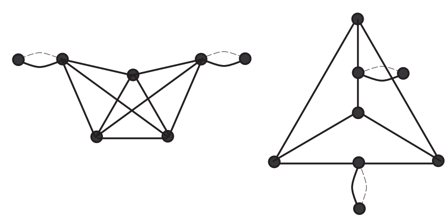


Figure 16.

If B_2' is not balanced, then we claim that B_2' has a joint not incident to u or v. If not, then let Σ' be the signed graph obtained by contracting one of the links in B_1 and so Σ' is a proper minor of Σ and has a joint, call it f, at v. Thus Σ' embeds in the annulus and since v is not a cut vertex of Σ', f may be embedded as a half edge (say drawn to the outer ring of the annulus). Furthermore, since Σ' has only two u-bridges, the digon B_1' is a facial cycle on the inner ring of the annulus. So now because there are no joints in B_2' not incident to either u or v, we can extend this embedding of Σ' to an embedding of Σ in the annulus, a contradiction. So let $w \notin \{u, v\}$ be the endpoint of a joint l in B_2'. It cannot be that w is a cut vertex of Σ because (as in the third claim at the beginning of Case 3) we could embed $\Sigma \backslash l$ in the annulus with B_1 and B_1' on the inner and outer rings of the annulus and then extend the embedding of $\Sigma \backslash l$ to all of Σ by adding l as a loop, a contradiction. Thus there is a uv-path γ in $B_2' \backslash w$. Now there is also a wv-path α_v and a wu-path α_u in B_2' as well. It cannot be that either one of α_v and α_u contains an internal vertex of γ because Σ is D_1-free. Thus Σ contains a D_7- or D_8-minor.

Case 3.2.2: Let Σ' be obtained from Σ by contracting one of the links in B_1. Thus Σ' is obtained from Σ by replacing B_1 with a joint, call it e. Since Σ' is a proper minor of Σ it embeds in the annulus. Since Σ' is vertically 2-connected we may assume that all joints in Σ' are embedded as half edges by Proposition 4.2. Given that B_2 is unbalanced and assuming that e is drawn to the inner ring of the annulus, let C_1 be the innermost negative cycle of B_2. Thus v is a vertex of C_1. Now it must be that we cannot reembed e as a negative loop because otherwise we could simply extend the embedding of Σ' to an embedding of Σ, a contradiction. Thus there is some other joint e_2 on vertex w of C_1 that is embedded as a half edge to the inner ring of the annulus. Let C_2 be the outermost negative cycle of the embedding of Σ'. In Case 3.2.2.1 say that C_2 is vertex-disjoint from C_1, in Case 3.2.2.2 say that C_2 intersects C_1 in a single vertex, in Case 3.2.2.3 say that C_2 intersects C_1 in a single path of positive length, and in Case 3.2.2.4 say that C_2 intersects C_1 in several paths.

Case 3.2.2.1: Since Σ' is vertically 2-connected, there are vertex-disjoint paths γ_1 and γ_2 that connect C_1 to C_2. One can now check that Σ contains a D_6-minor.

Case 3.2.2.2: Either $C_1 \cap C_2 = v$, $C_1 \cap C_2 = w$, or $C_1 \cap C_2 \notin \{v, w\}$. If $C_1 \cap C_2 = v$, then say that γ is a path not containing v that connects C_1 to C_2. We can now reembed Σ by drawing e as a half edge to the outer ring of the annulus. Since we cannot then reembed e as negative loop (or else we can extend to an embedding of Σ) there is another joint e_3 in Σ' with endpoint u on C_2. One can now check that there is a D_4-minor in Σ.

If $C_1 \cap C_2 = w$, then we can reembed Σ' with e_2 drawn as a half edge to the outer ring of the annulus. Thus we can extend to an embedding of all of Σ (a contradiction) unless there is another joint e_3 on a vertex $u \notin \{v, w\}$ of C_1. One can check that there is now a D_8-minor in Σ.

If $C_1 \cap C_2 \notin \{v, w\}$, then one can check that there is a D_8-minor in Σ.

Case 3.2.2.3: Let $\gamma = C_1 \cap C_2$. Now either $v, w \notin \gamma$, $v \in \gamma$ and $w \notin \gamma$, $v \notin \gamma$ and $w \in \gamma$, or $v, w \in \gamma$. If $v, w \notin \gamma$, then one can check that there is a D_6-minor in Σ.

If $v \in \gamma$ and $w \notin \gamma$, then we may reembed Σ' with e drawn as a half edge to the outer boundary of the annulus. We would be able to extend this embedding of Σ' to an embedding of Σ (a contradiction) unless there is a joint e_3 in Σ' on a vertex $u \in C_2 \setminus \gamma$. One can now check that there is a D_5-minor in Σ.

If $v \notin \gamma$ and $w \in \gamma$, then we can reembed Σ' with w drawn as a half edge to the outer boundary of the annulus. We could now extend this embedding of Σ' to an embedding of Σ (a contradiction) unless there is a joint e_3 of Σ on a vertex $u \in C_1 \setminus \gamma$. One can now check that there is a D_6-minor in Σ.

If $v, w \in \gamma$, then we could reembed Σ' with one of e and e_2 as a half edge to the outer boundary of the annulus and the other as a half edge to the inner boundary. Some such embedding would extend to an embedding of Σ (a contradiction) unless there are joints f_1 and f_2 on vertices $u_1 \in C_1 \setminus \gamma$ and $u_2 \in C_2 \setminus \gamma$. One can now check that there is a D_5-minor in Σ.

Case 3.2.2.4: The embedding of $\Sigma' \setminus J_{\Sigma'}$ is as shown on the left of Figure 17. Note that any of the grey lobes may be twisted to reembed $\Sigma' \setminus J_{\Sigma'}$. We can then reembed so that e is the only half edge drawn to the inner rim of the annulus and so replace e with a negative loop and then extend to an embedding of Σ (a contradiction) unless there are joints e_3 e_4 with endpoints $v_3 \neq v$ and $v_4 \neq v$ in the interiors of the lobes as shown on the right of Figure 17. No matter where v is on C_1 we can now find a D_5-minor in Σ.

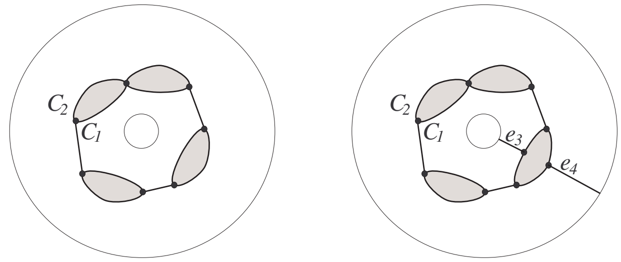


Figure 17.

□

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