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Bounding and Stabilizing Realizations of Biased Graphs With a Fixed Group

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Bounding and stabilizing realizations of biased graphs with a fixed group

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Dedicated to the memory of Dr. Edward Neudauer.

Abstract

Given a group Γ and a biased graph $(G, \mathcal{B})$, we define a what is meant by a Γ-realization of $(G, \mathcal{B})$ and a notion of equivalence of Γ-realizations. We prove that for a finite group Γ and $t \geq 3$, that there are numbers $n(\Gamma)$ and $n(\Gamma, t)$ such that the number of Γ-realizations of a vertically 3-connected biased graph is at most $n(\Gamma)$ and that the number of Γ-realizations of a nonseparable biased graph without a $(2C_t, \emptyset)$-minor is at most $n(\Gamma, t)$. Other results pertaining to contrabalanced biased graphs are presented as well as an analogue to Whittle's Stabilizer Theorem for Γ-realizations of biased graphs.

1 Introduction

A biased graph is a pair $(G, \mathcal{B})$ in which G is a graph and $\mathcal{B}$ is a collection of cycles in G for which every theta subgraph of G contains either zero, one, or three cycles from $\mathcal{B}$ (i.e., not exactly two cycles from $\mathcal{B}$). The canonical example of biased graphs comes from gain graphs. Let Γ be a group, G an ordinary graph, and φ a labeling of the oriented edges of G with elements of Γ such that the label on the reverse orientation of an edge is the inverse of the original label. The pair (G, φ) is called a Γ-gain graph and a cycle C in G is in the set $\mathcal{B}_\varphi$ when the product of group labels around C is the identity element of Γ. The pair $(G, \mathcal{B}_\varphi)$ is a biased graph.

Biased graphs and gain graphs were first used within matroid theory by Zaslavsky [18, 19, 20] as a tool for representing frame matroids, in particular, Dowling Geometries and their minors. The centrality of the matroid variety of Dowling Geometries and their minors within general matroid theory was first displayed by Kahn and Kung [9] and more recently by Geelen, Gerards, and Whittle [5, 7]. Also, gain graphs coming from 1-skeletons of 2-dimensional cellular complexes have been used to interesting effect in matroid theory by Funk [4] and Slilaty [14, 15]. The simple fact that biased graphs and gain graphs provide a link between matroid theory and the topology of cellular complexes is interesting in itself.

Given a biased graph $(G, \mathcal{B})$ and a group Γ, we define a Γ-*realization* of $(G, \mathcal{B})$ to be a Γ-gain graph (G, φ) such that $\mathcal{B}_\varphi = \mathcal{B}$. This concept of realizability of biased graphs over a group Γ is analogous

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to the concept of representability of general matroids over a field $\mathbb{F}$. Inequivalent representations of matroids over a fixed field $\mathbb{F}$ has been a topic of much research and conjecture; in particular, the following question originally inspired by a conjecture of Kahn [10]. Does there exist a fixed $t \geq 3$ such that for any finite field $\mathbb{F}$ there is there a number $n(\mathbb{F})$ such that every t-connected (or vertically t-connected) matroid has at most $n(\mathbb{F})$ inequivalent $\mathbb{F}$-representations. Geelen and Whittle [8] proved in very impressive fashion that for each prime-order field $GF(p)$, there is $n(p)$ such every 4-connected matroid has at most $n(p)$ inequivalent $GF(p)$-representations. For non-prime fields, the results are weaker. In [6] it is shown that any fixed level of vertical t-connectivity is not sufficient for finding such a universal bound for any arbitrary finite field of non-prime order. Recently, Geelen, Gerards, Huynh, and Van Zwam have announced a proof of the following result. For any given finite field $\mathbb{F}$, there are numbers $t(\mathbb{F})$ and $n(\mathbb{F})$ such that any $t(\mathbb{F})$-connected matroid has at most $n(\mathbb{F})$ inequivalent representations. Another approach to this topic is to exclude sequences of matroids with unbounded numbers of representations and then try to find universal bounds. Again in [8, Corollary 12.8], Geelen and Whittle showed that, for any finite field $\mathbb{F}$ and $k \geq 3$, there exists $n(\mathbb{F}, k)$ such that any 3-connected frame matroid with no free k-spike minor, no free k-swirl minor, no $U_{2,k}$-minor, and no $U_{k-2,k}$-minor has at most $n(\mathbb{F}, k)$ inequivalent $\mathbb{F}$-representations.

In this paper we define a notion of equivalence of Γ-realizations of a biased graph $(G, \mathcal{B})$ that is analogous to the notion of equivalence of $\mathbb{F}$-representations of a matroid M and prove the following results. First, given a finite group Γ, there is $n(\Gamma)$ such that every vertically 3-connected biased graph has at most $n(\Gamma)$ inequivalent Γ-realizations (Theorem 4.3). Second, given a finite group Γ and $k \geq 3$, there exists $n(\Gamma, k)$ such that any nonseparable biased graph without a $(2C_k, \emptyset)$-minor has at most $n(\Gamma, k)$ inequivalent Γ-realizations (Theorem 4.4). Excluding $(2C_k, \emptyset)$-minors in this second result is necessary by Proposition 4.2. For a prime number p and $\Gamma \in \{\mathbb{Z}_{p^m-1}, (\mathbb{Z}_p)^m\}$, this second result is essentially implied by Geelen and Whittle's result [8, Corollary 12.8]. Third, for each prime p there is $n(p)$ such that every nonseparable biased graph has at most $n(p)$ inequivalent $\mathbb{Z}_p$-realizations (Corollary 4.5). Given the results for general matroids, these tighter results for biased graphs are not so surprising. We do feel, however, that it is somewhat surprising that the proofs of our results are so short given the difficulty of proving Geelen and Whittle's results in [8].

Central to the proofs of these three theorems about the number of inequivalent Γ-realizations of arbitrary biased graphs is knowing something about the number of Γ-realizations of contrabalanced biased graphs (i.e., biased graphs with no balanced cycles). Our theorems on contrabalanced biased graphs (which are perhaps of independent interest) are Theorems 3.1, 3.2, 4.9, and 4.10.

Another important tool that has been of use in the study of inequivalent representations of matroids is Whittle's Stabilizer Theorem from [17]. We present Theorem 5.1 as an analogue of Whittle's Theorem for Γ-realizations of biased graphs.

2 Preliminaries

Graphs A graph G consists of a collection of vertices (i.e., topological 0-cells), denoted by $V(G)$, and a set of edges (i.e., topological 1-cells), denoted by $E(G)$, where an edge has two ends each of which is attached to a vertex. A *link* is an edge that has its ends incident to distinct vertices and a *loop* is an edge that has both of its ends incident to the same vertex. The degree of a vertex in G is the number of ends of edges attached to that vertex and a graph is said to be *k-regular* when all of its vertices have degree k. A *path* is either a single vertex or a connected graph with two vertices of degree 1 and the remaining vertices of degree 2 each. The *length* of a path is the number of edges in it. A *cycle* is a connected 2-regular graph and the *length* of a cycle is the number of edges in it. The cycle of length n is denoted C_n. The *wheel* with rim cycle of length n is denoted by W_n. Let $\mathcal{C}(G)$ denote the set of all cycles in G. If G is a simple graph and $n \geq 2$, then by nG we mean the graph

obtained from G by replacing each link by n parallel links on the same two vertices. We refer to the complete graph on t vertices by K_t and the graph nK_2 is called an *n-multilink*.

If $X \subseteq E(G)$, then we denote the subgraph of G consisting of the edges in X and all vertices incident to an edge in X by $G{:}X$. The collection of vertices in $G{:}X$ is denoted by $V(X)$. For $k \geq 1$, a *k-separation* of a graph is a bipartition (A, B) of the edges of G such that $|A| \geq k$, $|B| \geq k$, and $|V(A) \cap V(B)| = k$. A *vertical k-separation* (A, B) of G is a k-separation where $V(A) \setminus V(B) \neq \emptyset$ and $V(B) \setminus V(A) \neq \emptyset$. A graph on at least $k + 2$ vertices is said to be *vertically k-connected* when it is connected and there is no vertical r-separation for $r < k$. A graph on $k + 1$ vertices is said to be vertically k-connected when it has a spanning complete subgraph. A graph G that is connected and does not have a 1-separation is said to be *nonseparable*. Nonseparable graphs are always loopless and a graph with at least three vertices is nonseparable iff it is loopless and vertically 2-connected.

Given a graph G, an *oriented edge* e is an element of the edge set $E(G)$ together with a direction along it. An oriented edge e has a *head* $\mathbf{h}(e)$ and a *tail* $\mathbf{t}(e)$. The reverse orientation is denoted e^{-1}. The collection of oriented edges of G is denoted by $\vec{E}(G)$ and for $X \subseteq E(G)$, we let $\vec{X}$ be the corresponding subset of $\vec{E}(G)$ with $|\vec{X}| = 2|X|$. When considering an oriented edge e, the underlying unoriented edge is also referred to as e when it causes no confusion. A *walk* w in G is a product of oriented edges $e_1 e_2 \cdots e_n$ for which $\mathbf{h}(e_i) = \mathbf{t}(e_{i+1})$ for each $i \in \{1, \ldots, n-1\}$. The walk w is sometimes called a *uv-walk* where u is the tail of e_1 and v is the head of e_n. The *uv*-walk w is *closed* when $u = v$. The *reverse walk* of w is $w^{-1} = e_n^{-1} \cdots e_1^{-1}$.

For two graphs G and H an *isomorphism* $\iota \colon G \to H$ is a bijection $\iota \colon (V(G) \sqcup \vec{E}(G)) \to (V(H) \sqcup \vec{E}(H))$ where $\iota(V(G)) = V(H)$, $\iota(\vec{E}(G)) = \vec{E}(H)$, $\iota \mathbf{h} = \mathbf{h}\iota|_{\vec{E}(G)}$, and $\iota \mathbf{t} = \mathbf{t}\iota|_{\vec{E}(G)}$. We will reserve the letter ι for graph isomorphisms.

Given disjoint subsets $K, D \subseteq E(G)$, by $G/K \backslash D$ we mean the *minor* obtained from G by deleting the edges in D and contracting the edges in K. Given graphs G and H, we say that G has an H-minor, when there is $G/K \backslash D$ that is isomorphic to H up to deletion of isolated vertices from $G/K \backslash D$. Given a minor $G/K \backslash D$ of a graph G, one can always choose $K' \subseteq K$ such that $G{:}K'$ is a maximal forest of $G{:}K$. Hence if $D' = D \cup (K \backslash K')$, then $G/K' \backslash D' = G/K \backslash D$. We say that the minor $G/K' \backslash D'$ is obtained with contraction on an *acyclic set*.

Gain Functions   Given a group Γ and a graph G, a Γ-*gain function* on G is a function $\varphi \colon \vec{E}(G) \to \Gamma$ satisfying $\varphi(e^{-1}) = \varphi(e)^{-1}$. A Γ-*gain graph* is a pair (G, φ) where G is a graph and φ a Γ-gain function. Gain graphs are called "voltage graphs" within the field of topological graph theory and are sometimes called "group-labeled graphs" as well. A $\mathbb{Z}_2$-gain graph is most often called a *signed graph*. Given any walk $e_1 \cdots e_n$ we define $\varphi(e_1 \cdots e_n) = \varphi(e_1) \cdots \varphi(e_n)$. This also yields the relations $\varphi(w^{-1}) = \varphi(w)^{-1}$ for any walk w and $\varphi(w_1 w_2) = \varphi(w_1)\varphi(w_2)$ for any uv-walk w_1 and vz-walk w_2.

If C is a cycle in G, then let w_C be a closed Eulerian walk along C. (Of course, w_C is only well defined up to a choice of starting vertex and direction around C; however, for any two possible choices w_1 and w_2 for w_C, there is a walk w on C such that either $w_1 = w w_2 w^{-1}$ or $w_1 = w w_2^{-1} w^{-1}$. Hence a cycle C satisfies $\varphi(w_C) = 1$ for all possible choices of w_C or $\varphi(w_C) \neq 1$ for all possible choices of w_C. Now define a cycle C in G to be *balanced* with respect to φ when $\varphi(w_C) = 1$ and let $\mathcal{B}_\varphi$ be the collection of cycles in G that are balanced with respect to φ.

Given a Γ-gain function φ on G and a function $\eta \colon V(G) \to \Gamma$, define the gain function φ^η by $\varphi^\eta(e) = \eta(\mathbf{t}(e))^{-1}\varphi(e)\eta(\mathbf{h}(e))$. We call η a *switching function*. Note that if w is a uv-walk in G, then $\varphi^\eta(w) = \eta(u)^{-1}\varphi(w)\eta(v)$. Therefore a cycle C is balanced with respect to φ iff C is balanced with respect to φ^η, i.e., $\mathcal{B}_\varphi = \mathcal{B}_{\varphi^\eta}$. Proposition 2.1 follows immediately from the definitions.

Proposition 2.1. *If φ is a Γ-gain function on a graph G and η_1 and η_2 are switching functions, then $(\varphi^{\eta_1})^{\eta_2} = \varphi^{\eta_1 \eta_2}$.*

When two Γ-gain functions φ and ψ satisfy $\varphi^\eta = \psi$ for some η, we say that φ and ψ are *switching*

equivalent. Two Γ-gain functions φ and ψ are defined to be *equivalent* when $\alpha\varphi^\eta = \psi$ for some switching function η and some automorphism α of Γ.

Proposition 2.2. *Let F be a maximal forest of a graph G and φ a Γ-gain function on G.*

(1) *There is a switching function η such that $\varphi^\eta(e) = 1$ for all oriented edges e in F.*

(2) *If $G_1, \ldots, G_n$ are the connected components of G, v_i is a vertex of G_i, and $g_1, \ldots, g_n \in \Gamma$, then η in Part (1) may be chosen so that each $\eta(v_i) = g_i$. Furthermore, η is unique with respect to $(v_1, g_1), \ldots, (v_n, g_n)$.*

Proof. Construct η on each connected component of G_i of G as follows. Let $T_i \subseteq F$ be the spanning tree of G_i. Inductively we construct η on G_i by first picking $\eta(v_i) = g_i$ and then orient each edge of T_i in the direction towards v_i. Inductively assume that η is defined for all vertices at a distance up to $t \geq 0$ from v_i in the tree T_i. Given u at a distance $t + 1$ from from v_i, let e be the edge of T_i connecting u to its parent vertex u_p in T_i. Now set $\eta(u) = \varphi(e)\eta(u_p)$. Notice that this choice for $\eta(u)$ is the unique choice that makes $1 = \eta(u)^{-1}\varphi(e)\eta(u_p) = \varphi^\eta(e)$. $\square$

Given a maximal forest F of G, a Γ-gain function φ is said to be *F-normalized* when $\varphi(e) = 1$ for all oriented edges e in F.

Proposition 2.3. *Let G be a connected graph, T a spanning tree of G, and φ and ψ two T-normalized Γ-gain functions on G.*

(1) *The following are equivalent.*

    i. There is a switching function η such that $\varphi^\eta = \psi$.

    ii. $\varphi = g^{-1}\psi g$ for some $g \in \Gamma$.

    iii. There is a constant switching function $\eta \equiv g$ for some $g \in \Gamma$ such that $\varphi = \psi^\eta$.

    iv. There is an inner automorphism α of Γ such that $\varphi = \alpha\psi$.

(2) *If Γ is abelian, then $\varphi^\eta = \psi$ for some η iff $\varphi = \psi$.*

Proof. For Part(1), the definition of switching functions and inner automorphisms implies that *ii.*, *iii.*, and *iv.* are equivalent. Certainly *iii.* $\rightarrow$ *i.* so it is left to show that *i.* $\rightarrow$ *iii.*. Since φ is T-normalized, φ^η is T-normalized iff η is constant. So since $\varphi^\eta = \psi$ is T-normalized, we have our result. For Part (2) if $\varphi^\eta = \psi$, then by Part (1) η is constant; however, since Γ is abelian $\varphi = \varphi^\eta = \psi$. $\square$

Given a Γ-gain function φ on a graph G and a minor $G' = G/K \backslash D$ of G, we wish to give an *induced* Γ-gain function $\varphi|_{G'}$. If e is an edge of G and $G' = G\backslash e$, then $\varphi|_{G'}$ is defined on $G\backslash e$ by restriction. If e is a link and $G' = G/e$, then $\varphi|_{G'}$ is defined up to switching as follows. Since e is a link and not a loop, there is switching function η, such that $\varphi^\eta(e) = 1$. Now $\varphi|_{G'}$ is defined up to switching by restriction of φ^η to $E(G)\backslash e$. If e is a loop, then $G/e = G\backslash e$ and so we define $\varphi|_{G/e} = \varphi|_{G\backslash e}$. In other words, we always insist that loops are deleted rather than contracted. It is not usually the case in graph theory that one needs to distinguish between contracting a loop and deleting a loop, but in the context of biased graphs the distinction is necessary. So now for $G' = G/K\backslash D$, we can define $\varphi|_{G'}$ (up to switching) iteratively, taking care to delete loops as they occur rather than contracting them. One can define $\varphi|_{G'}$ globally (again, up to switching) as follows. Let $G{:}K'$ be a maximal forest in $G{:}K$ and let $D' = D \cup (K\backslash K')$ and we have $G/K\backslash D = G/K'\backslash D'$. Let F be a maximal forest of G whose edges contain K'. Let φ^η be the F-normalization of φ and so we define $\varphi|_{G'}$ by restricting φ to $E(G') = E(G)\backslash(K \cup D)$.

Biased Graphs Recall that a *biased graph* is a pair $(G, \mathcal{B})$ where G is a graph and $\mathcal{B}$ is a collection of cycles in G (called *balanced*) for which any theta subgraph contains either 0, 1, or 3 cycles from $\mathcal{B}$. That is, no theta subgraph contains exactly two cycles from $\mathcal{B}$. In the language of matroids, $\mathcal{B}$ is *linear class* of circuits of the cycle matroid $M(G)$. The biased graph $(G, \mathcal{B})$ is called *balanced* when $\mathcal{B} = \mathcal{C}(G)$ (i.e., when $\mathcal{B}$ is the collection of all cycles in G) and is called *contrabalanced* when $\mathcal{B} = \emptyset$. Any loop ℓ whose corresponding cycle is unbalanced is often called a *joint*. A set of edges X in $(G, \mathcal{B})$ (or a subgraph H of G) is said to be *balanced* when every cycle in $G{:}X$ (or in H) is balanced. For two biased graphs $(G, \mathcal{B})$ and $(H, \mathcal{S})$ an *isomorphism* $\iota\colon (G, \mathcal{B}) \to (H, \mathcal{S})$ consists of an underlying graph isomorphism $\iota\colon G \to H$ that takes $\mathcal{B}$ to $\mathcal{S}$. A biased graph $(G, \mathcal{B})$ is said to be vertically k-connected (or nonseparable) when its underlying graph is vertically k-connected (or nonseparable). A *simple biased graph* is a biased graph without balanced cycles of length 1 or 2 and without two joints at the same vertex. Note that a simple biased graph need not have an underlying graph that is a simple graph. The canonical example of a biased graph is given in Proposition 2.4.

Proposition 2.4 (Zaslavsky [18])**.** *If φ is a Γ-gain function on a graph G, then $(G, \mathcal{B}_\varphi)$ is a biased graph.*

Given a biased graph $(G, \mathcal{B})$ and a group Γ, a Γ-*realization* of $(G, \mathcal{B})$ is a Γ-gain function φ for which $\mathcal{B}_\varphi = \mathcal{B}$. A Γ-realization φ of a contrabalanced biased graph $(G, \emptyset)$ is called a Γ-*antivoltage*. Antivoltages were first formally studied by Zaslavsky [21]. When we refer to the number of Γ-realizations of $(G, \mathcal{B})$, we mean the number of Γ-realizations up to equivalence. We denote this number of Γ-realizations by $\mathsf{N}_\Gamma(G, \mathcal{B})$.

Proposition 2.5. *If Γ is a finite group, then $(nK_2, \emptyset)$ is Γ-realizable iff $n \le |\Gamma|$. Furthermore, if $n = |\Gamma|$, then $\mathsf{N}_\Gamma(nK_2, \emptyset) = \frac{(|\Gamma|-1)!}{|\mathrm{Aut}(\Gamma)|}$ where $\mathrm{Aut}(\Gamma)$ is the automorphism group of Γ.*

Proof. Let u and v be the vertices of nK_2 and $e_1, \ldots, e_n$ be the edges of nK_2 all oriented from u to v. Certainly a Γ-gain function φ is a realization of $(nK_2, \emptyset)$ iff $\varphi(e_i) \ne \varphi(e_j)$ for each $i \ne j$. This requires that $n \le |\Gamma|$. Now if $n = |\Gamma|$ and φ is a Γ-realization of $(nK_2, \emptyset)$, then we may assume that φ is T-normalized on the spanning tree containing the single edge e_1. So now there are $(|\Gamma| - 1)!$ ways to choose $\varphi(e_2), \ldots, \varphi(e_n)$. Therefore two T-normalized Γ-realizations are equivalent iff the choices for $e_2, \ldots, e_n$ are the same up to some automorphism of Γ (see Proposition 2.3). $\qquad\square$

Let $(G, \mathcal{B})$ be a biased graph and e an edge in G. Define $(G, \mathcal{B})\backslash e = (G\backslash e, \mathcal{B}|_{G\backslash e})$ where $\mathcal{B}|_{G\backslash e} = \mathcal{B} \cap \mathcal{C}(G\backslash e)$. If e is a link, then define $(G, \mathcal{B})/e = (G/e, \mathcal{B}|_{G/e})$ where $\mathcal{B}|_{G/e}$ is the collection of minimal elements of $\{C \in \mathcal{C}(G/e) : C \in \mathcal{B} \text{ or } C \cup e \in \mathcal{B}\}$. If e is a balanced loop, then $(G, \mathcal{B})/e = (G, \mathcal{B})\backslash e$. (Again we insist on balanced loops being deleted rather than contracted.) If e is an unbalanced loop (i.e., a joint), then the contraction $(G, \mathcal{B})/e$ is defined in different ways depending on the various matroids of the biased graph under consideration [18, 19]: either the frame matroid or the lift matroid. We will not consider contractions of unbalanced loops in biased graphs (as we do not consider contraction of loops in ordinary graphs either). This restriction actually has no effect on the topic of realizability as explained in the paragraph after Proposition 2.6.

A *link minor* of $(G, \mathcal{B})$ is a minor that is obtained without contracting any unbalanced loops. Thus a link minor $(G, \mathcal{B})/K\backslash D$ must always satisfy that K is a balanced edge set and so $(G, \mathcal{B})/K\backslash D = (G', \mathcal{B}|_{G'})$ for $G' = G/K\backslash D$. Also we get that $(G, \mathcal{B})/K\backslash D = (G, \mathcal{B})/K'\backslash D'$ for some $G{:}K'$ that is a maximal forest in $G{:}K$ and $D' = D \cup (K\backslash K')$, that is, the link minor $(G', \mathcal{B}|_{G'}) = (G, \mathcal{B})/K'\backslash D'$ can always be obtained with contraction of an acyclic set. We say that biased graph $(G, \mathcal{B})$ has an $(H, \mathcal{S})$-link minor when there is link minor $(G', \mathcal{B}|_{G'})$ of $(G, \mathcal{B})$ that is isomorphic to $(H, \mathcal{S})$ up to deletion of isolated vertices in $(G', \mathcal{B}|_{G'})$.

Given a Γ-realization φ of biased graph $(G, \mathcal{B})$ and a link minor $(G', \mathcal{B}|_{G'})$ of $(G, \mathcal{B})$, we call the Γ-realization $\varphi|_{G'}$ of $(G', \mathcal{B}|_{G'})$ of Proposition 2.6 the *induced* Γ-realization of $(G', \mathcal{B}|_{G'})$. The proof of Proposition 2.6 is routine.

Proposition 2.6. *If φ is a Γ-realization of $(G, \mathcal{B})$ and $(G', \mathcal{B}|_{G'})$ is a link minor of $(G, \mathcal{B})$, then the induced gain function $\varphi|_{G'}$ is a Γ-realization of $(G', \mathcal{B}|_{G'})$.*

Proposition 2.6 tells us that Γ-realizability of biased graphs is a link-minor-closed property. Those familiar with contractions of unbalanced loops in biased graphs know that $(G, \mathcal{B})/e = (G \backslash e, \mathcal{C}(G))$ for an unbalanced loop e when considering the lift matroid of $(G, \mathcal{B})$ and when considering the frame matroid of $(G, \mathcal{B})$, the contraction $(G, \mathcal{B})/e$ is obtained from a subgraph of $(G, \mathcal{B})$ by the addition of unbalanced loops. Furthermore, for any unbalanced loop f, $(H, \mathcal{S}) \backslash f$ is Γ-realizable iff $(H, \mathcal{S})$ is Γ-realizable. Hence, for either type of contraction, Proposition 2.6 can actually be extended to general minors of biased graphs rather than just link minors. For these reasons we do not consider contractions of loops in biased graphs.

Proposition 2.7. *If φ is a Γ-realization of $(G, \mathcal{B})$ and $\iota \colon (H, \mathcal{S}) \to (G, \mathcal{B})$ is an isomorphism, then $\varphi\iota$ is Γ-realization of $(H, \mathcal{S})$.*

Now if $(G, \mathcal{B})$ contains an $(H, \mathcal{S})$-link minor, then there is an isomorphism $\iota \colon (H, \mathcal{S}) \to (G', \mathcal{B}_{G'})$ for some link minor $(G', \mathcal{B}_{G'})$ of $(G, \mathcal{B})$. So if φ is a Γ-realization of $(G, \mathcal{B})$, then $(\varphi|_{G'})\iota$ is a Γ-realization of $(H, \mathcal{S})$.

3 Contrabalanced biased graphs and antivoltages

Theorem 3.1 is a simple modification of a theorem in [2].

Theorem 3.1. *If Γ is a finite group, then there are finitely many vertically 3-connected and simple biased graphs $(G, \emptyset)$ that are Γ-realizable.*

Proof. Let $n = |\Gamma| + 1$ so the biased graph $(nK_2, \emptyset)$ is not Γ-realizable by Proposition 2.5. First, there are at most $(n-1)\binom{v}{2}$ edges in a simple and loopless biased graphs $(G, \emptyset)$ on v vertices that does not contain a $(nK_2, \emptyset)$-subgraph. Second, Oporowski, Oxley, and Thomas showed [11] that there is N such that if $v \geq N$ then any vertically 3-connected graph G on v vertices contains either a W_n- or $K_{3,n}$-minor.

So now by way of contradiction, if there are infinitely many vertically 3-connected and simple $(G, \emptyset)$ that are Γ-realizable, then the first fact in the previous paragraph implies that there is some such contrabalanced and simple biased graph $(G, \emptyset)$ on $v \geq N$ vertices. The second fact in the previous paragraph implies that $(G, \emptyset)$ contains either a $(W_n, \emptyset)$- or $(K_{3,n}, \emptyset)$-link minor. However, $(W_n, \emptyset)$ and $(K_{3,n}, \emptyset)$ both contain an $(nK_2, \emptyset)$-link minor, a contradiction of the assumption that $(G, \emptyset)$ is Γ-realizable. $\qquad\square$

Theorem 3.2. *Let Γ be a finite group and $k \geq 3$. Let $\mathcal{G}_{2,k}$ be the class of graphs that are nonseparable, have minimum degree at least three, and have no $2C_k$-link minor. There are finitely many $G \in \mathcal{G}_{2,k}$ such that $(G, \emptyset)$ is Γ-realizable.*

Proof. The reader of this proof should be familiar with the canonical tree decomposition of a nonseparable graph G. This tree decomposition and its properties are described in [3] and also [16]. (For a more recent and succinct presentation see [12, pp.308–315].) Suppose that $G \in \mathcal{G}_{2,k}$ and $(G, \emptyset)$ is Γ-realizable. Let T be the canonical tree decomposition of G with $V(T) = \{G_1, \ldots, G_n\}$ and $E(T) = \{e_1, \ldots, e_{n-1}\}$. For each $G_i \in V(T)$ there is $t_i \geq 3$ such that G_i is one of the following: a t_i-cycle, a t_i-multilink, or a vertically 3-connected simple graph. Furthermore, there are no two adjacent vertices in T that are both cycles or both multilinks. For each j, the edge $e_j \in E(T)$ connecting $G_{j_1}, G_{j_2} \in V(T)$ is labeled with two edges, one from each of G_{j_1} and G_{j_2}, over which the 2-sum is taken.

First we claim that T does not contain a path of length $2|\Gamma| + 2$. Let P be a path in T with $n \geq 2$ vertices. At most every other vertex in P is a cycle-labeled vertex and so at least $\lfloor n/2 \rfloor$ of the vertices in P are labeled by multilinks or vertically 3-connected graphs. From [13, (3.1)] it follows that for any two edges e and f in a vertically 3-connected simple graph G_i, there is a K_4-minor in G_i that contains e and f. Thus there is a 3-multilink minor in G_i using e and f. So if a vertex of P is labeled by a 3-connected graph G_i, then we have a 3-multilink minor of G_i containing the two edges of G_i used in the 2-sums indicated by the two edges of P incident to G_i. Therefore, there is a minor of G isomorphic to $B_1 \oplus_2 B_2 \oplus_2 \ldots \oplus_2 B_{\lfloor n/2 \rfloor}$, where each B_i is a multilink of size at least three. Thus G has a $\lfloor n/2 \rfloor$-multilink link minor. Since $(G, \emptyset)$ is Γ-realizable, G cannot have a $(|\Gamma| + 1)$-multilink link minor and so we must have that $\lfloor n/2 \rfloor \leq |\Gamma|$ and so $n \leq 2|\Gamma| + 1$.

Second we claim that the maximum degree of a vertex in T is bounded. By Theorem 3.1, there is a maximum number of edges, call it d_3, of a vertically 3-connected and simple graph G such that $(G, \emptyset)$ is Γ-realizable. Thus any vertex of T labeled by a vertically 3-connected graph has maximum degree d_3. A vertex labeled by a t-multilink has $t \leq |\Gamma|$ and so such a vertex has degree at most $|\Gamma|$ in T. If a vertex G_i of T is labeled by a t-cycle, then because G does not contain a $2C_k$-minor, the vertex G_i has degree at most $\min\{t, k\}$ in T. Thus the maximum degree of a vertex in T is at most $\max\{d_3, |\Gamma|, k\}$.

Third we claim that the labels on the vertices of T are chosen from a finite set. This comes from the following three facts. (I) Theorem 3.1 says there are only finitely many vertically 3-connected labels possible. (II) The t-multilink labels have $3 \leq t \leq |\Gamma|$. (III) The t-cycle labels must have $3 \leq t < 2k$. Fact (III) is true by the following argument. Since G has minimum degree 3 and is loopless, at least every other edge of the t-cycle must be indicated in a 2-sum by T. This now implies that G contains a $2C_{\lceil t/2 \rceil}$-link minor. Since G does not contain a $2C_k$-minor, we get that $t < 2k$.

Therefore, since T has bounded degree, T has bounded diameter, and T has finitely many possible labels on its vertices, there are only finitely many possibilities for T. Thus there are only finitely many possibilities for G. $\qquad \square$

Proposition 3.3. *If p is a prime and $n \geq (p-1)^2 + 1$, then the biased graph $(2C_n, \emptyset)$ is not $\mathbb{Z}_p$-realizable.*

Proof. Label the edges of $(2C_n, \emptyset)$ by $e_1, \ldots, e_n, f_1, \ldots, f_n$ where e_i, f_i are a parallel pair. Assume that the edges are all oriented in the same direction around the cycle. By way of contradiction, assume that $n \geq (p-1)^2 + 1$ and φ is a $\mathbb{Z}_p$-realization of $(2C_n, \emptyset)$. (Note that φ is an additive gain function rather than a multiplicative one.) We may assume that φ is T-normalized for the spanning tree T on edges $e_1, \ldots, e_{n-1}$ and so $\varphi(f_i) \neq 0$ for each $i \in \{1, \ldots, n\}$, $\varphi(e_n) \neq 0$, and $\varphi(f_n) \neq 0$. Since $\frac{n-1}{p-1} \geq p - 1$, there is a nonzero element $a \in \mathbb{Z}_p$ and $F \subseteq \{f_1, \ldots, f_{n-1}\}$ of order at least $p - 1$ with $\varphi(f) = a$ for each $f \in F$. Now let $m \in \{1, \ldots, p-1\}$ be such that $ma = -\varphi(e_n)$ in $\mathbb{Z}_p$. So now there is a length-n cycle C in $(2C_n, \emptyset)$ consisting of m edges from F, the $n - 1 - m$ edges from $\{e_1, \ldots, e_{n-1}\}$ that are not parallel to these edges chosen from F, and the edge e_n. Notice that $\varphi(w_C) = ma + \varphi(e_n) = -\varphi(e_n) + \varphi(e_n) = 0$, a contradiction of the fact that φ is a $\mathbb{Z}_p$-realization of $(2C_n, \emptyset)$. $\qquad \square$

Theorem 3.2 and Proposition 3.3 yield Corollary 3.4.

Corollary 3.4. *Let $\mathcal{G}_2$ be the class of graphs that are nonseparable and have minimum degree at least three. For each prime p, there are finitely many $G \in \mathcal{G}_2$ such that $(G, \emptyset)$ is $\mathbb{Z}_p$-realizable.*

4   Bounding the number of realizations

Given an arbitrary group Γ, we are interested in knowing if there is some number $n(\Gamma)$ such that $\mathsf{N}_\Gamma(G, \mathcal{B}) \leq n(\Gamma)$ for all biased graphs $(G, \mathcal{B})$. Of course such an upper bound would require that Γ is

finite. It also requires that $(G, \mathcal{B})$ is nonseparable by Proposition 4.1. For biased graphs $(G_1, \mathcal{B}_1)$ and $(G_2, \mathcal{B}_2)$ that intersect in at most single vertex, $(G_1, \mathcal{B}_1) \cup (G_2, \mathcal{B}_2) = (G_1 \cup G_2, \mathcal{B}_1 \cup \mathcal{B}_2)$ is clearly a biased graph.

Proposition 4.1. *If $(G, \mathcal{B}) = (G_1, \mathcal{B}_1) \cup (G_2, \mathcal{B}_2)$ where $G_1 \cap G_2$ is at most a single vertex, then for any finite group Γ, $\mathsf{N}_\Gamma(G, \mathcal{B}) = \mathsf{N}_\Gamma(G_1, \mathcal{B}_1)\mathsf{N}_\Gamma(G_2, \mathcal{B}_2)$.*

Proposition 4.2 shows that there is still no such upper bound even if we insist on G being nonseparable; a class of nonseparable biased graphs with unbounded numbers of realizations being $(2C_n, \emptyset)$.

Proposition 4.2. *If Γ is a finite group such that $|\Gamma| \neq 4$ and $|\Gamma|$ is not prime, then $\mathsf{N}_\Gamma(2C_n, \emptyset) \geq \frac{1}{|\mathrm{Aut}(\Gamma)|}2^{n-1}$.*

Proof. By Sylow's Theorems Γ has a proper subgroup Λ of order at least 3. Denote the edges of $(2C_n, \emptyset)$ by $e_1, \ldots, e_n, f_1, \ldots, f_n$ where e_i, f_i are a parallel pair. Assume that the edges are all oriented in the same direction around the cycle. Given spanning tree T on edges $e_1, \ldots, e_{n-1}$ define a T-normalized Γ-realization φ of $(2C_n, \emptyset)$ by arbitrarily picking $\varphi(f_i) \in \Lambda - \{1\}$ for each $i \in \{1, \ldots, n-1\}$ and arbitrarily picking $\varphi(e_n)$ and $\varphi(f_n)$ to be two distinct elements in $\Gamma - \Lambda$. Since φ is T-normalized, Proposition 2.3 implies that these choices for $\varphi(f_1), \ldots, \varphi(f_{n-1})$ among nonidentity elements of Λ gives us at least $\frac{1}{|\mathrm{Aut}(\Gamma)|}(|\Lambda| - 1)^{n-1} \geq \frac{1}{|\mathrm{Aut}(\Gamma)|}2^{n-1}$ distinct Γ-realizations of $(2C_n, \emptyset)$. $\square$

Whereas being nonseparable does not suffice to place a general bound on the number of Γ-realizations of biased graphs, insisting on G being vertically 3-connected and loopless does yield such a bound (Theorem 4.3). Furthermore, the class of biased graphs $(2C_n, \emptyset)$ presented in Proposition 4.2 is in essence the only obstacle preventing such an upper bound for nonseparable biased graphs (Theorem 4.4 and Corollary 4.5). Hence Theorems 4.3 and 4.4 and Corollary 4.5 are the best possible results for existence of such bounds.

Theorem 4.3 (Main Result I). *If Γ is a finite group, then there is a number $n(\Gamma)$ such that if $(G, \mathcal{B})$ is vertically 3-connected and loopless, then $\mathsf{N}_\Gamma(G, \mathcal{B}) \leq n(\Gamma)$.*

Theorem 4.4 (Main Result II). *If Γ is a finite group and $k \geq 3$, then there is a number $n(\Gamma, k)$ such that if $(G, \mathcal{B})$ is nonseparable and contains no $(2C_k, \emptyset)$-link minor, then $\mathsf{N}_\Gamma(G, \mathcal{B}) \leq n(\Gamma, k)$.*

Corollary 4.5 (Main Result III). *If p is prime, then there is a number $n(p)$ such that if $(G, \mathcal{B})$ is nonseparable, then $\mathsf{N}_{\mathbb{Z}_p}(G, \mathcal{B}) \leq n(p)$.*

Corollary 4.5 follows immediately from Theorem 4.4 and Proposition 3.3. The rest of this section is dedicated to the proof of Theorems 4.3 and 4.4.

Proposition 4.6. *Let e be a link in a biased graph $(G, \mathcal{B})$. If φ and ψ are Γ-realizations of $(G, \mathcal{B})$ whose induced realizations $\varphi|_{G/e}$ and $\psi|_{G/e}$ of $(G, \mathcal{B})/e = (G/e, \mathcal{B}|_{G/e})$ are switching equivalent, then φ and ψ are switching equivalent.*

Proof. Without loss of generality, we may assume that G is connected. Pick a spanning tree T of G that contains the link e. By Proposition 2.2 assume that φ and ψ are both T-normalized. Now T/e is a spanning tree of G/e and the induced realizations $\varphi|_{G/e}$ and $\psi|_{G/e}$ are both (T/e)-normalized. By Proposition 2.3 there is a constant switching function η (say $\eta(v) \equiv g \in \Gamma$ for all $v \in V(G)$) such that $g^{-1}(\varphi|_{G/e}(f))g = (\varphi|_{G/e})^\eta(f) = \psi|_{G/e}(f)$ for all edges f in G/e. So now since $\varphi|_{G/e}$ and $\psi|_{G/e}$ are (T/e)-normalized and φ and ψ are T-normalized, we get that $g^{-1}(\varphi(f))g = \psi$ for all edges f in G and so $\varphi^\mu = \psi$ where $\mu \equiv g$ on all of G, as required. $\square$

Proposition 4.7. *Let e be a link of biased graph $(G, \mathcal{B})$ and let Γ be a group.*

(1) $\mathsf{N}_\Gamma(G, \mathcal{B}) \leq \mathsf{N}_\Gamma((G, \mathcal{B})/e)$.

(2) *If e is in some balanced cycle of $(G, \mathcal{B})$, then* $\mathsf{N}_\Gamma(G, \mathcal{B}) \leq \mathsf{N}_\Gamma((G, \mathcal{B})\backslash e)$.

Proof. Part (1) follows from Proposition 4.6. For Part (2), consider a Γ-realization φ of $(G, \mathcal{B})$. The induced Γ-realization $\varphi|_{G\backslash e}$ of $(G\backslash e, \mathcal{B}|_{G\backslash e})$ extends uniquely to φ because e is in a balanced cycle of $(G, \mathcal{B})$. $\qquad\square$

Theorem 4.9 is necessary for the proof of Theorem 4.3 and it is derived from a result of Bixby. Given a graph G, let $\mathsf{si}(G)$ denote the underlying simple graph of G and $\mathsf{co}(G)$ the cosimplification of G, that is, repeatedly contract one edge in a coparallel pair until no coparallel pairs remain. A pair of edges e, f in a graph G are *coparallel* when they form a bond.

Theorem 4.8 (Bixby [1]). *If G is a vertically 3-connected and simple graph on at least five vertices, then for each edge e of G, either $\mathsf{co}(G\backslash e)$ or $\mathsf{si}(G/e)$ is vertically 3-connected and simple.*

Theorem 4.9. *If $(G, \mathcal{B})$ is a vertically 3-connected and loopless biased graph and Γ is a group, then there is a link minor $(G', \mathcal{B}|_{G'})$ of $(G, \mathcal{B})$ such that the following hold:*

- *$(G', \mathcal{B}|_{G'})$ is vertically 3-connected and loopless,*

- *$(G', \mathcal{B}|_{G'})$ is either contrabalanced or has four vertices, and*

- *$\mathsf{N}_\Gamma(G, \mathcal{B}) \leq \mathsf{N}_\Gamma(G', \mathcal{B}|_{G'})$.*

Proof. Given a vertically 3-connected and loopless biased graph $(G, \mathcal{B})$, we will find sequence of biased graphs $(G_1, \mathcal{B}_1), \ldots, (G_n, \mathcal{B}_n)$ which satisfies the following properties. Proposition 4.7 will then imply that $\mathsf{N}_\Gamma(G, \mathcal{B}) \leq \mathsf{N}_\Gamma(G_n, \mathcal{B}_n)$.

- $(G, \mathcal{B}) = (G_1, \mathcal{B}_1)$.

- Each $(G_i, \mathcal{B}_i)$ is vertically 3-connected and loopless.

- $(G_{i+1}, \mathcal{B}_{i+1})$ is obtained from $(G_i, \mathcal{B}_i)$ by either deletion of a link in a balanced cycle, contraction of a link that is not parallel to any other link in G_i, or deletion of a link in a balanced cycle followed by contractions of links that are not parallel to any other link in G_i.

- $(G_n, \mathcal{B}_n)$ either is contrabalanced or has four vertices.

If $(G_i, \mathcal{B}_i)$ is contrabalanced or has four vertices, then let $n = i$ and we are done. So assume that $\mathcal{B}_i \neq \emptyset$ and G_i has at least five vertices. If $(G_i, \mathcal{B}_i)$ has a pair of parallel edges e_1, e_2 such that e_1 is in a balanced cycle, then let $(G_{i+1}, \mathcal{B}_{i+1}) = (G_i, \mathcal{B}_i)\backslash e_1$. This has no effect on vertical 3-connectivity or looplessness. So assume that every balanced cycle of $(G_i, \mathcal{B}_i)$ consists of edges that are not parallel to other edges in G_i and let e be an edge in a balanced cycle of $(G_i, \mathcal{B}_i)$. Since $\mathsf{si}(G_i)$ is vertically 3-connected and simple, Theorem 4.8 implies that either G_i/e is vertically 3-connected (and also loopless because e is not parallel to any other edge in G_i) or $\mathsf{co}(\mathsf{si}(G_i\backslash e))$ is vertically 3-connected and simple. In the former case let $(G_{i+1}, \mathcal{B}_{i+1}) = (G_i, \mathcal{B}_i)/e$ and we are done. In the latter case $\mathsf{si}(G_i)\backslash e = \mathsf{si}(G_i\backslash e)$ is vertically 3-connected after cosimplification. Consider a pair of coparallel edges e_1, e_2 in $\mathsf{si}(G_i\backslash e)$. For each $i \in \{1, 2\}$, let E_i be the collection of edges in G_i parallel to e_i. Since $\mathsf{si}(G_i)$ is vertically 3-connected, e_1, e_2, and e form an edge cut of size three in $\mathsf{si}(G_i)$; furthermore, since e was in a balanced cycle C, one of e_1 and e_2 (say e_1) is in C with e. Since no edge of C was parallel to any other edge of G_i we must have that $E_1 = \emptyset$. So now $G_i\backslash e/e_1$ is loopless, $\mathsf{si}(G_i\backslash e/e_1) = \mathsf{si}(G_i)\backslash e/e_1$, and the number of coparallel pairs of edges in $\mathsf{si}(G_i\backslash e/e_1)$ is strictly less than the number of coparallel pairs of edges in $\mathsf{si}(G_i\backslash e)$. We may now repeat this process of cosimplification until we have obtained $G_i\backslash e/e_1/\ldots/e_m$ that is vertically 3-connected and loopless and we let $(G_{i+1}, \mathcal{B}_{i+1}) = (G_i, \mathcal{B}_i)\backslash e/e_1/\ldots/e_m$.

Now since G and $\mathcal{B}$ are both finite, this process will eventually halt with our desired outcome. $\quad\square$

Proof of Theorem 4.3. There are only finitely many vertically 3-connected and loopless contrabalanced biased graphs that are Γ-realizable by Theorem 3.1. Thus there is r_c such that for any vertically 3-connected and loopless $(G, \emptyset)$, we have that $\mathsf{N}_\Gamma(G, \emptyset) \leq r_c$. Also, since Γ is finite, there are only finitely many vertically 3-connected and loopless simple biased graphs on four vertices that are Γ-realizable by Proposition 2.5. Thus there there is r_4 such that $\mathsf{N}_\Gamma(G, \mathcal{B}) \leq r_4$ for any $(G, \mathcal{B})$ that is a vertically 3-connected and loopless simple biased graph on four vertices. Let $n(\Gamma) = \max\{r_c, r_4\}$ and so if $(G_0, \mathcal{B}_0)$ is vertically 3-connected and loopless and either contrabalanced or having at most four vertices, then $\mathsf{N}_\Gamma(G_0, \mathcal{B}_0) \leq n(\Gamma)$. Theorem 4.9 now implies our result. $\qquad\square$

Theorem 4.10. *If $(G, \mathcal{B})$ is a nonseparable biased graph and Γ is a group, then there is a link minor $(G', \mathcal{B}|_{G'})$ of $(G, \mathcal{B})$ such that the following hold:*

- *$(G', \mathcal{B}|_{G'})$ is nonseparable and contrabalanced,*
- *$(G', \mathcal{B}|_{G'})$ either has minimum degree at least three or is a loop, and*
- *$\mathsf{N}_\Gamma(G, \mathcal{B}) \leq \mathsf{N}_\Gamma(G', \mathcal{B}|_{G'})$.*

Proof. Given a nonseparable biased graph $(G, \mathcal{B})$, we will find a sequence of biased graphs $(G_1, \mathcal{B}_1), \ldots, (G_n, \mathcal{B}_n)$ which satisfies the following. Proposition 4.7 will then imply our result.

- $(G, \mathcal{B}) = (G_1, \mathcal{B}_1)$.
- Each $(G_i, \mathcal{B}_i)$ is nonseparable.
- $(G_{i+1}, \mathcal{B}_{i+1})$ is obtained from $(G_i, \mathcal{B}_i)$ by either deletion of a link in a balanced cycle or contraction of a link that is not parallel to any other link in G_i.
- $(G_n, \mathcal{B}_n)$ is contrabalanced and either has minimum degree at least three or is a loop.

If $(G_i, \mathcal{B}_i)$ is contrabalanced and either has minimum degree at least 3 or is a loop, then let $n = i$ and we are done. So assume that either $B_i \neq \emptyset$ or G_i has a vertex of degree 2 and at least two vertices. If $(G_i, \mathcal{B}_i)$ has a vertex of degree 2, then contract one of its incident links to get $(G_{i+1}, \mathcal{B}_{i+1})$. Since G_i is nonseparable so is G_{i+1}. So now assume that the minimum degree of $(G_i, \mathcal{B}_i)$ is three and $\mathcal{B}_i \neq \emptyset$. If there is a pair of parallel edges e_1, e_2 such that e_1 is in a balanced cycle, then let $(G_{i+1}, \mathcal{B}_{i+1}) = (G_i, \mathcal{B}_i)\backslash e_1$ (this, again, has no effect on separability). So assume that every balanced cycle of $(G_i, \mathcal{B}_i)$ consists of edges that are not parallel to other edges in G_i. Let e be an edge in a balanced cycle of $(G_i, \mathcal{B}_i)$. It is well known that either $G_i\backslash e$ or G_i/e is nonseparable. Perform this deletion or contraction to obtain $(G_{i+1}, \mathcal{B}_{i+1})$. Since G and $\mathcal{B}$ are both finite, this process will eventually halt with our desired outcome. $\qquad\square$

Proof of Theorem 4.4. There are only finitely many contrabalanced biased graphs that are Γ-realizable, are nonseparable, have minimum degree three, and do not contain a $(2C_k, \emptyset)$-minor by Theorem 3.2. Thus there is r_c such that the number of Γ-realizations of such a contrabalanced biased graph is at most r_c. Now let $n(\Gamma, k) = \max\{r_c, |\Gamma|\}$ and our result follows from Theorem 4.10. $\qquad\square$

5 A stabilizer theorem

Given a finite group Γ and $k \geq 3$, finding the finite collection of contrabalanced and Γ-realizable biased graphs in Theorems 3.1 and 3.2 would allow one to calculate the bounds in Theorems 4.3 and 4.4. Of course, the finite collections could be large enough so as to make this approach impractical. Another approach would be to find an appropriate class of stabilizers for Γ-realizations. This requires a stabilizer theorem analogous to Whittle's Stabilizer Theorem [17]. We present Theorem 5.1 as this analogue.

Let $\mathfrak{M}$ be a link-minor-closed class of Γ-realizable biased graphs. Consider a biased graph $(G, \mathcal{B}) \in \mathfrak{M}$ containing a link minor $(G', \mathcal{B}|_{G'})$. We say that $(G, \mathcal{B})$ is Γ-*stabilized* by $(G', \mathcal{B}|_{G'})$ when any two Γ-realizations φ and ψ of $(G, \mathcal{B})$ are switching equivalent iff the induced realizations $\varphi|_{G'}$ and $\psi|_{G'}$ are switching equivalent. Of course, switching equivalence of φ and ψ implies switching equivalence of $\varphi|_{G'}$ and $\psi|_{G'}$; however, when φ and ψ are not switching equivalent it may be that $\varphi|_{G'}$ and $\psi|_{G'}$ are switching equivalent. In more intuitive terms, $(G, \mathcal{B})$ is Γ-*stabilized* by $(G', \mathcal{B}|_{G'})$ when any Γ-realization of $(G', \mathcal{B}|_{G'})$ that does extend to a Γ-realization of $(G, \mathcal{B})$ does so uniquely up to switching. If $(H, \mathcal{S})$ is isomorphic to a link minor of $(G, \mathcal{B})$ up to deletion of isolated vertices, then we say that $(G, \mathcal{B})$ is Γ-stabilized by $(H, \mathcal{S})$ when it is Γ-stabilized by any link minor of $(G', \mathcal{B}|_{G'}) \cong (H, \mathcal{S})$ up to deletion of isolated vertices. Finally, a biased graph $(H, \mathcal{S}) \in \mathfrak{M}$ is called an $\mathfrak{M}$-*stabilizer* (or $(H, \mathcal{S})$ *stabilizes* $\mathfrak{M}$) when any nonseparable $(G, \mathcal{B}) \in \mathfrak{M}$ that contains an $(H, \mathcal{S})$-link minor is Γ-stabilized by $(H, \mathcal{S})$. The inclusion of nonseparability in this definition is necessary because, of course, it would not be possible to stabilize separable biased graphs in $\mathfrak{M}$ (that can have arbitrarily many 1-separations) by a given link minor with a fixed number of 1-separations (see Proposition 4.1).

Given a biased graph $(H, \mathcal{S})$, a *digon split* of $(H, \mathcal{S})$ is a biased graph $(G, \mathcal{B})$ such that $(G, \mathcal{B})/e \backslash f \cong (H, \mathcal{S})$ where $G{:}\{e, f\}$ is a cycle of length 2.

Theorem 5.1 (Main Result IV, The Stabilizer Theorem). *Let $\mathfrak{M}$ be a link-minor-closed class of Γ-realizable biased graphs and let $(H, \mathcal{S}) \in \mathfrak{M}$ be nonseparable. If $(H, \mathcal{S})$ satisfies one of the following, then $(H, \mathcal{S})$ is an $\mathfrak{M}$-stabilizer.*

(1) Γ is abelian and $(H, \mathcal{S})$ Γ-stabilizes every nonseparable single-edge extension of $(H, \mathcal{S})$ in $\mathfrak{M}$ and every nonseparable digon split of $(H, \mathcal{S})$ in $\mathfrak{M}$.

(2) $(H, \mathcal{S})$ is not realizable over any proper subgroup of Γ and $(H, \mathcal{S})$ Γ-stabilizes every nonseparable single-edge extension of $(H, \mathcal{S})$ in $\mathfrak{M}$ and every nonseparable digon split of $(H, \mathcal{S})$ in $\mathfrak{M}$.

If Γ is nonabelian, then the assumption in Part (2) of Theorem 5.1 that $(H, \mathcal{S})$ is not realizable over any proper subgroup of Γ is necessary by the following example using $\Gamma = \langle a, b : a^3 = 1, ab = ba^2 \rangle$ (i.e., the nonabelian group of order six). There are six automorphisms of Γ defined by $a \mapsto a^i$ for $i \in \{1, 2\}$ and $b \mapsto ba^j$ for $j \in \{0, 1, 2\}$; furthermore, all automorphisms of Γ are inner automorphisms (i.e., defined by conjugation).

Let $H = 3K_3$ with vertices v_0, v_1, v_2 and edges $e_{i,j}$ for $i, j \in \{0, 1, 2\}$ where $\mathbf{t}(e_{i,j}) = v_i$ and $\mathbf{h}(e_{i,j}) = v_{i+1}$ where addition is modulo 3. Define $\varphi(e_{i,j}) = a^j$ and let $\mathcal{S} = \mathcal{B}_\varphi$. One can show that the only Γ-realizations of $(H, \mathcal{S})$ up to switching equivalence are φ and $\alpha\varphi$ where $\alpha(a) = a^2$. Any Γ-realizable and nonseparable single-edge extension $(H \cup e, \mathcal{S}')$ that can be realized using $\varphi(e) \in \{b, ba, ba^2\}$ may be realized up to inner automorphism (i.e., up to switching by Proposition 2.3) by $\varphi(e) = b$. Furthermore, the edge e will not be in any balanced cycles of $(H \cup e, \mathcal{S}')$ and so it is not possible to realize $(H \cup e, \mathcal{S}')$ by using $\varphi(e) \in \{1, a, a^2\}$. Any Γ-realizable and nonseparable single-edge extension $(H \cup e, \mathcal{S}')$ that can be realized using $\varphi(e) \in \{1, a, a^2\}$ has e in some balanced cycle of $\mathcal{S}'$ and so this would be the only possibility for $\varphi(e) \in \Gamma$. Similar reasoning will apply for a nonseparable digon split of $(H, \mathcal{S})$. Now, however, a 2-edge extension $(H \cup \{e, f\}, \mathcal{S}'')$ in which e and f have the same head and tail endpoints that is realizable by $\varphi(e) = b$ and $\varphi(f) = ba$ is also realizable using $\varphi(e) = b$ and $\varphi(f) = ba^2$ and these two realizations are not switching equivalent.

For Part (2) of Theorem 5.1, Proposition 5.2 is a mechanism that overcomes situations like the one presented the previous example.

Proposition 5.2. *Let $(G, \mathcal{B})$ be a connected biased graph that is Γ-realizable but not realizable over any proper subgroup of Γ. Let T be a spanning tree of G and φ a T-normalized Γ-realization of $(G, \mathcal{B})$. If η is a switching function such that $\varphi^\eta = \varphi$, then $\eta \equiv g$ and g in the center of Γ.*

Proof. Evidently, if $\eta \equiv g$ and g is in the center of Γ, then $\varphi^\eta = \varphi$. Conversely, suppose that $\varphi^\eta = \varphi$. Since φ is T-normalized, Proposition 2.3 implies that $\eta \equiv g$ for some $g \in \Gamma$. So now $g^{-1}\varphi g = \varphi^\eta = \varphi$

can only be the case if $g^{-1}\varphi(e)g = \varphi^\eta(e) = \varphi(e)$ for every $e \in \vec{E}(G)$. Note that φ is a Λ-gain function on G where Λ is the subgroup of Γ generated by $\{\varphi(e) : e \in \vec{E}(G)\}$. As such φ is a Λ-realization of $(G, \mathcal{B})$ and so $\Lambda = \Gamma$. Therefore $g^{-1}hg = h$ for all $h \in \Gamma$ which implies that g is in the center of Γ. $\square$

Proof of Theorem 5.1. In the proof we repeatedly use Proposition 2.3 Part (1) rather than Part (2) so as to include the details necessary for the case where Γ is nonabelian. If Γ is abelian, the details are easier. Suppose that $(H, \mathcal{S})$ is nonseparable and stabilizes every nonseparable single-edge extension and digon split of $(H, \mathcal{S})$ in $\mathfrak{M}$. Consider $(G, \mathcal{B}) \in \mathfrak{M}$ that is nonseparable and contains a link minor $(G', \mathcal{B}|_{G'}) = (G, \mathcal{B})/K\backslash D$ that is isomorphic to $(H, \mathcal{S})$ possibly after deletion of isolated vertices. Take Γ-realizations φ and ψ of $(G, \mathcal{B})$ whose induced gain functions $\varphi|_{G'}$ and $\psi|_{G'}$ are switching equivalent. We will show that φ and ψ are switching equivalent as well.

Among all choices for $K' \subseteq K$ and $D' = D \cup (K\backslash K')$ such that $(G, \mathcal{B})\backslash D'/K' = (G, \mathcal{B})\backslash D/K$ up to deletion of isolated vertices, choose a minimal K'. Since G' is nonseparable up to deletion of isolated vertices, the minimality of K' implies that $(G, \mathcal{B})\backslash D'$ is also nonseparable up to deletion of isolated vertices and $G{:}K'$ is a forest. Denote $(G, \mathcal{B})\backslash D' = (G\backslash D', \mathcal{B}|_{G\backslash D'})$ minus any isolated vertices that it may have by $(\overline{G}, \mathcal{B}|_{\overline{G}})$. Let T' be the edges of a spanning tree of G' and so $\overline{T} = T' \cup K'$ is the edge set of a spanning tree of $\overline{G}$. Say that $T \supseteq \overline{T}$ is the edge set of a spanning tree of G and now assume that φ and ψ are $(G{:}T)$-normalized. Thus $\varphi|_{\overline{G}}$ and $\psi|_{\overline{G}}$ are $(\overline{G}{:}\overline{T})$-normalized. Since $G' = \overline{G}/K'$ is obtained with contraction on an acyclic set and since $\varphi|_{G'}$ and $\psi|_{G'}$ are switching equivalent, Proposition 4.6 implies that $\varphi|_{\overline{G}}$ and $\psi|_{\overline{G}}$ are switching equivalent. Since $\varphi|_{\overline{G}}$ and $\psi|_{\overline{G}}$ are $(\overline{G}{:}\overline{T})$-normalized, $(\varphi|_{\overline{G}})^\eta = \psi|_{\overline{G}}$ for some constant switching function $\eta \equiv g \in \Gamma$ on $\overline{G}$.

Now since $\overline{G}$ is a nonseparable subgraph of nonseparable graph G, either $\overline{G} = G$ (in which case there is nothing left to prove) or there is $G_0 \subset G_1 \subset \cdots \subset G_n$ such that $\overline{G} = G_0$, $G = G_n$, and G_{i+1} is obtained by appending a path (call it γ_{i+1}) to G_i where γ_{i+1} is internally disjoint from G_i and has both endpoints in G_i. Label the endpoints of γ_{i+1} by u_{i+1} and v_{i+1}.

Consider the subgraph of $\overline{G}$ whose vertex set is $V(\overline{G})$ and whose edge set is K'. The connected components of this subgraph are the vertices of G'. Let π be the vertex projection from $V(\overline{G})$ to $V(G')$. Now either $\pi(u_1) = \pi(v_1)$ or $\pi(u_1) \neq \pi(v_1)$.

In the case that $\pi(u_1) \neq \pi(v_1)$, then let $T_1 \supseteq \overline{T}$ be the edge set of a spanning tree of G_1. Thus K' is contained in $K'' = T_1\backslash T'$ and G_1/K'' is a nonseparable single-edge extension of G'. By assumption $\varphi|_{G_1/K''}$ and $\psi|_{G_1/K''}$ are switching equivalent and since $G_1{:}K''$ is acyclic $\varphi|_{G_1}$ and $\psi|_{G_1}$ are switching equivalent by Proposition 4.6.

In the case that $\pi(u_1) = \pi(v_1)$, let e be an edge of the unique u_1v_1-path in $\overline{G}{:}K'$. Again let $T_1 \supseteq \overline{T}$ be the edge set of a spanning tree of G_1 and now $K'\backslash e$ is contained in $K'' = (T_1\backslash e)\backslash T'$ and G_1/K'' is a nonseparable digon split of G'. By assumption $\varphi|_{G_1/K''}$ and $\psi|_{G_1/K''}$ are switching equivalent and since $G_1{:}K''$ is acyclic $\varphi|_{G_1}$ and $\psi|_{G_1}$ are switching equivalent by Proposition 4.6.

Recall that the *Betti number* (or *cyclomatic number*) of a graph G is $\beta(G) = |E(G)| - |V(G)| + c(G)$ where $c(G)$ is the number of connected components of G. Note that $\beta(G_i) = \beta(\overline{G}) + i$. Take $i \geq 1$ and inductively assume that if N is a nonseparable graph such that $\overline{G} \subseteq N \subseteq G$ and $\beta(N) \leq \beta(\overline{G}) + i$, then $\varphi|_N$ and $\psi|_N$ are switching equivalent. Consider $G_{i+1} = G_{i-1} \cup \gamma_i \cup \gamma_{i+1}$. In Case 1 assume that neither u_{i+1} nor v_{i+1} is an interior vertex of γ_i, in Case 2 assume that exactly one of u_{i+1} and v_{i+1} is an interior vertex of γ_i, and in Case 3 assume that both u_{i+1} and v_{i+1} are interior vertices of γ_i.

Case 1 Let $G'_i = G_{i-1} \cup \gamma_{i+1}$ and note that G'_i is nonseparable and that $\beta(G'_i) = \beta(G_i) = \beta(\overline{G}) + i$. Inductively $\varphi|_{G_i}$ and $\psi|_{G_i}$ are switching equivalent and $\varphi|_{G'_i}$ and $\psi|_{G'_i}$ are switching equivalent. Let T_{i-1} be a spanning tree of G_{i-1} and extend T_{i-1} to spanning trees T_i of G_i and T'_i of G'_i. Note that $T_{i+1} = T_i \cup T'_i$ is a spanning tree of G_{i+1} and assume that $\varphi|_{G_{i+1}}$ and $\psi|_{G_{i+1}}$ are T_{i+1}-normalized. Thus there is a constant switching function $\eta \equiv g \in \Gamma$ such that $(\varphi|_{G_i})^\eta = g^{-1}(\varphi|_{G_i})g = \psi|_{G_i}$ and a constant switching function $\mu \equiv h \in \Gamma$ such that $(\psi|_{G'_i})^\mu = h^{-1}(\psi|_{G'_i})h = \varphi|_{G'_i}$. These together imply that $(\varphi|_{G_{i-1}})^\eta = g^{-1}(\varphi|_{G_{i-1}})g = \psi|_{G_{i-1}}$ and $(\psi|_{G'_{i-1}})^\mu = h^{-1}(\psi|_{G'_{i-1}})h = \varphi|_{G'_{i-1}}$ which now imply

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that

$$(\varphi|_{G_{i-1}})^{\eta\mu} = ((\varphi|_{G_{i-1}})^{\eta})^{\mu} = (\psi|_{G_{i-1}})^{\mu} = \varphi|_{G_{i-1}}.$$

Therefore by Proposition 5.2, gh is an element in the center of group Γ and so for any oriented edge e on the path γ_i and we have that

$$\varphi|_{G_i}(e) = (gh)^{-1}(\varphi|_{G_i}(e))(gh) = h^{-1}(g^{-1}(\varphi|_{G_i}(e))g)h = h^{-1}(\psi|_{G_i}(e))h.$$

Hence we cen extend $\mu \equiv g$ to all of G_{i+1} and then $(\psi|_{G_{i+1}})^{\mu} = \varphi|_{G_{i+1}}$, that is, $\psi|_{G_{i+1}}$ and $\varphi|_{G_{i+1}}$ are switching equivalent.

Case 2 Without loss of generality assume that assume that u_{i+1} is an interior vertex of γ_i and v_{i+1} is not an interior vertex of γ_i. Furthermore, we may assume that $v_{i+1} \neq u_i$. Let γ' be the subpath of γ_i from u_i to u_{i+1} and let $G'_i = G_{i-1} \cup \gamma' \cup \gamma_{i+1}$. Note that G'_i is nonseparable and that $\beta(G'_i) = \beta(G_i) = \beta(\overline{G}) + i$. Now let T_{i-1} be a spanning tree of G_{i-1} and note that $T_{i-1} \cup \gamma'$ is also a tree. Extend $T_{i-1} \cup \gamma'$ to spanning trees T_i of G_i and T'_i of G'_i. Again $T_{i+1} = T_i \cup T'_i$ is a spanning tree of G_{i+1}. Now assume that $\varphi|_{G_{i+1}}$ and $\psi|_{G_{i+1}}$ are T_{i+1}-normalized and the rest of the details of this case are analogous to those of Case 1.

Case 3 Without loss of generality we may assume that $u_i, u_{i+1}, v_{i+1}, v_i$ appear in this order along γ_i. Thus $\gamma_i = \alpha_1 \cup \alpha_2 \cup \alpha_3$ where α_1 is from u_i to u_{i+1}, α_2 is from u_{i+1} to v_{i+1} and α_3 is from v_{i+1} to v_i. Let $G'_i = G_{i-1} \cup \alpha_1 \cup \alpha_3 \cup \gamma_{i+1}$ and note that G'_i is nonseparable and that $\beta(G'_i) = \beta(G_i) = \beta(\overline{G}) + i$. Now let T_{i-1} be a spanning tree of G_{i-1} and note that $T_{i-1} \cup \alpha_1 \cup \alpha_3$ is also a tree. Extend $T_{i-1} \cup \alpha_1 \cup \alpha_3$ to spanning trees T_i of G_i and T'_i of G'_i. Again $T_{i+1} = T_i \cup T'_i$ is a spanning tree of G_{i+1}. Now assume that $\varphi|_{G_{i+1}}$ and $\psi|_{G_{i+1}}$ are T_{i+1}-normalized and the rest of the details of this case are analogous to those of Case 1. $\square$

Acknowledgements

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