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Projective-planar graphs with no K3,4-minor. II.

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Projective-planar graphs with no $K_{3,4}$-minor. II.

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Abstract

The authors previously published an iterative process to generate a class of projective-planar $K_{3,4}$-free graphs called 'patch graphs'. They also showed that any simple, almost 4-connected, nonplanar, and projective-planar graph that is $K_{3,4}$-free is a subgraph of a patch graph. In this paper, we describe a simpler and more natural class of cubic $K_{3,4}$-free projective-planar graphs which we call *Möbius hyperladders*. Furthermore, every simple, almost 4-connected, nonplanar, and projective-planar graph that is $K_{3,4}$-free is a minor of a Möbius hyperladder. As applications of these structures we determine the page number of patch graphs and of Möbius hyperladders.

1 Introduction

The graph $K_{3,4}$ plays a particular role within the class of vertically 3-connected graphs. That is an expression of a vertically 3-connected graph G as a Y-sum $G_1 \oplus_Y G_2$ is guaranteed to have at least one planar term when G has no $K_{3,4}$-minor. Conversely, if G_1 is $K_{3,4}$-free and G_2 is planar, then $G = G_1 \oplus_Y G_2$ is still $K_{3,4}$-free (see [2, Sec.3]). In [2], the authors of this paper gave an iterative description of a class $\mathcal{PG}$ of projective-planar graphs that do not contain a minor isomorphic to $K_{3,4}$. The graphs in $\mathcal{PG}$ are called *patch graphs*. Additionally, it was shown that any simple, almost 4-connected, nonplanar, and projective-planar graph that is $K_{3,4}$-free is either isomorphic to K_6 or is a subgraph of a graph in $\mathcal{PG}$. The vertically 3-connected variety (rather than almost 4-connected variety) is then obtained by Y-summing planar graphs onto one of these almost 4-connected graphs. In this paper, we define a more natural class $\mathcal{MH}$ of cubic projective-planar graphs called *Möbius hyperladders*. We then

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show that every graph in $\mathcal{PG}$ is a minor of a graph in $\mathcal{MH}$ (Theorem 4.1) and every graph in $\mathcal{MH}$ is a subgraph of a graph in $\mathcal{PG}$ (Theorem 4.2). Therefore we get that any simple, almost 4-connected, nonplanar, and projective-planar graph that is $K_{3,4}$-free is either isomorphic to K_6 or is a minor of a graph in $\mathcal{MH}$ and that all graphs in $\mathcal{MH}$ are $K_{3,4}$-free. A motivating reason for exploring this new class is (as the reader will see) that $\mathcal{MH}$ is a more natural and easily described class than $\mathcal{PG}$. This may likely make statements concerning $K_{3,4}$-free projective-planar graphs more approachable. As an example of the simpler nature of $\mathcal{MH}$ over $\mathcal{PG}$, we will show that the page number of any graph in either class is 3 (Theorems 5.1 and 5.2). The page-number-3 result for $\mathcal{MH}$ is, of course, implied by the page-number-3 result for $\mathcal{PG}$ because of Theorem 4.2; however, we include both proofs for comparison's sake.

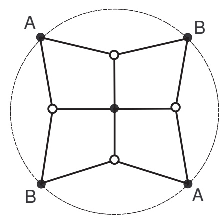


Figure 1.
The topologically unique embedding of $K_{3,4}$ on the projective plane.

2 Preliminaries

Given a simple graph H, a graph G is said to *contain an H-minor* if for each vertex of $v \in V(H)$ there corresponds a connected subgraph $\nu(v) \subseteq G$ such that the subgraphs $\nu(v)$ are pairwise disjoint and for any u and v that are adjacent in H, there exists $u' \in \nu(u)$ and $v' \in \nu(v)$ that are adjacent in G. If G does not contain an H-minor, we will say that G is *H-free*. Given $N = \{v_1, v_2, \ldots, v_k\} \subset V(H)$ and a set $M = \{v'_1, v'_2, \ldots, v'_k\} \subset V(G)$, we will say that G contains an *H-minor with N rooted on M* if G contains an H-minor such that each $v'_i \in \nu(v_i)$.

Given a set of edges X in G, let $V(X)$ denote the vertices of G incident to edges in X. For $k \geq 0$, a *k-separation* of G is a bipartition (A_1, A_2) of the edges of G with nonempty parts such that each $|A_i| \geq k$ and $|V(A_1) \cap V(A_2)| = k$. The k-separation is called *vertical* when $V(A_1) \setminus V(A_2) \neq \emptyset$ and $V(A_2) \setminus V(A_1) \neq \emptyset$. A graph on at least $k+1$ vertices is called *vertically k-connected* when every vertical t-separation has $t \geq k$. We use vertical k-connectivity rather than Tutte k-connectivity to allow for multiple edges. A vertically 3-connected graph G is *almost 4-connected* when any vertical 3-separation (A_1, A_2) has some $|V(A_i)| = 4$. This is similar to but weaker than the usual notion of internal 4-connectivity.

Given two graphs G_1 and G_2, a 1-sum $G_1 \oplus_1 G_2$ is the identification of G_1 and G_2 along some specified vertex and a 2-sum $G_1 \oplus_2 G_2$ is obtained by identifying G_1 and G_2 along some specified link and then deleting that link. If G_1 and G_2 both contain a 3-valent vertex, then a *Y-sum* $G_1 \oplus_Y G_2$ is obtained by identifying the neighbors of these 3-valent vertices in some specified ordering and then removing the 3-valent vertices. The proof of Proposition 2.1 is straightforward, while the proof of Proposition 2.2 can be found in [2, Proposition 3.2].

Proposition 2.1. *For each $k \in \{1,2\}$, G and H are both vertically k-connected and $K_{3,4}$-free if and only if $G \oplus_k H$ is vertically k-connected and $K_{3,4}$-free.*

Proposition 2.2.

(1) If G is simple, vertically 3-connected, and $K_{3,4}$-free, then G is obtained by taking one simple, almost 4-connected, $K_{3,4}$-free graph and then taking Y-sums with planar graphs with possible subdivisions of edges before each sum.

(2) If G is $K_{3,4}$-free and P is planar, then $G \oplus_Y P$ is $K_{3,4}$-free.

The *representativity* of an embedding of a graph in a surface is the minimum number of intersection points of the graph and a noncontractible curve in the surface. An embedding with representativity k is called a *k-representative embedding.* Suppose that a graph G has a 2-representative embedding on the projective plane. That is, some noncontractible curve γ is contained in two faces and intersects the graph in precisely two vertices, say A and B. This curve γ is called a 2-*representative cut* of the embedding.

Following [2], given a graph G embedded in the projective plane, certain faces with boundary cycles of length four will be designated as *patches.* Patches are drawn as shaded regions in the interior of their faces. As defined in [2], a *patch graph* is a pair $(G, \mathcal{P})$ where G is an embedding of a graph in the projective plane with designated vertices A and B on a 2-representative cut and $\mathcal{P}$ is a collection of patches (possibly empty) which together are constructed iteratively as follows. We call the patch graph $(4K_2, \{P_0\})$ shown in Figure 2 the *initial patch graph.*

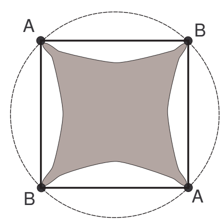
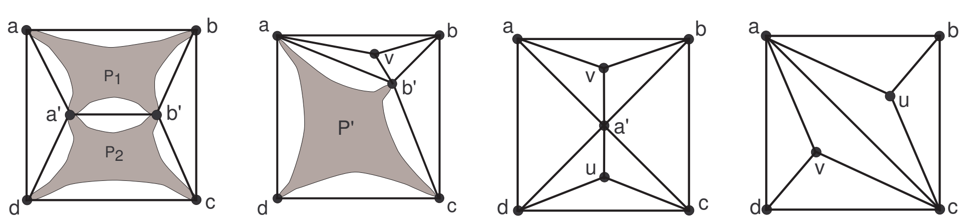


Figure 2.
The initial patch graph embedded on the projective plane.

All patch graphs are constructed from the initial patch graph $(4K_2, \{P_0\})$ by applying sequences of operations H, Y, X, and I. These operations involve replacing a patch with one of the configurations shown in Figure 3 or a similar configuration obtained by rotating or flipping the interior of the patch while leaving the boundary fixed. Since operations X and I remove a patch and do not introduce any new ones we call them *terminal patching operations.* In Figure 4, a typical patch graph rooted at A and B on a 2-representative cut is shown.

Figure 3.



Operations H, Y, X, and I, respectively.

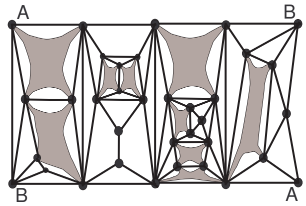


Figure 4.
An example of a subgraph of a patch graph.

The main theorem proven in [2] is the following result.

Theorem 2.3. *If H is a simple, almost 4-connected, nonplanar, and projective-planar graph that is $K_{3,4}$-free, then $H \cong K_6$ or H is a subgraph of a patch graph. Furthermore, all patch graphs are $K_{3,4}$-free.*

3 Möbius hyperladders

We define a class of cubic graphs embedded in the projective plane called *Möbius hyperladders* as follows: Begin with the *initial Möbius hyperladder* shown on the left in Figure 5 (which is an embedding of K_4). Now all Möbius hyperladders are constructed from the initial hyperladder by a sequence of *rung operations*. Given a positive integer n, the *n-rung operation* takes as input a Möbius hyperladder L and a quadrilateral face P. The operation subdivides two antipodal edges on the boundary of P n times each, then adds n disjoint edges called *rungs* as shown on the right in Figure 5. The resulting embedding is again a Möbius hyperladder.

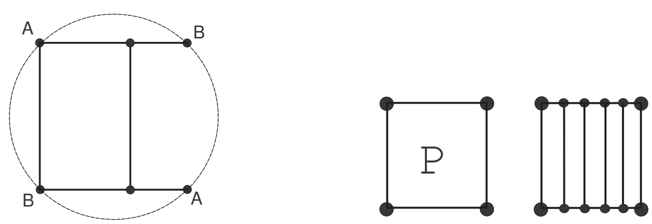


Figure 5.
The initial Möbius hyperladder and a 4-rung operation.

In Figure 6, a typical Möbius hyperladder is shown. Note that the only Möbius hyperladder that is planar is the initial Möbius hyperladder.

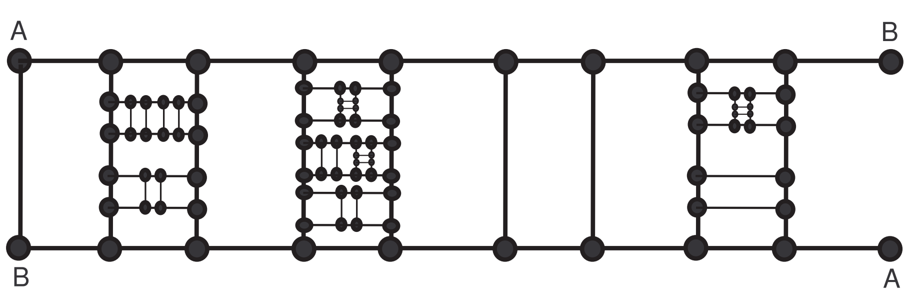


Figure 6.

An example of a Möbius hyperladder.

4 Patch Graphs and Möbius Hyperladders

In this section, we will show that every patch graph is contained as a minor rooted on $\{A, B\}$ in some Möbius hyperladder and, conversely, every Möbius hyperladder is a subgraph of a patch graph.

Theorem 4.1. *If $(G, \mathcal{P})$ is a patch graph, then there exists a Möbius hyperladder L that contains G as a rooted minor on $\{A, B\}$.*

Proof. Consider a patch graph $(G_n, \mathcal{P}_n)$ that is obtained from the initial patch graph by n patching operations. We claim that there is a patch graph $(G', \mathcal{P}')$ that is obtained from the initial patch graph by $n' \geq n$ H-operations such that G_n is a minor of G' and $\mathcal{P}_n \subseteq \mathcal{P}'$. If $n = 0$ this is clearly true. It is also clearly true if the last operation in the construction of $(G_n, \mathcal{P}_n)$ was an H-operation from $(G_{n-1}, \mathcal{P}_{n-1})$. Suppose that the last operation was a Y-operation on a patch $P \in \mathcal{P}_{n-1}$. (The X- and Y-operations are handled similarly.) Notice that the graphical configuration for the Y-operation (see Figure 3) can be obtained as a minor from an H-operation that replaces patch P by two patches P_1 and P_2 followed by another H-operation on P_1. Now the single patch of the Y-operation corresponds to patch P_2 of the first H-operation. Now by induction, there is $(G'', \mathcal{P}'')$ where G_{n-1} is a minor of G'' and $\mathcal{P}_{n-1} \subseteq \mathcal{P}''$. Now perform the two H-operations indicated above on $P \in \mathcal{P}''$ to obtain $(G', \mathcal{P}')$ which will now have G_n a minor of G' and $\mathcal{P}_n \subseteq \mathcal{P}'$.

We now claim that there is a Möbius hyperladder L containing G' as a contraction. If $n' = 0$ then this is clear. Assume $(G', \mathcal{P}')$ is obtained from $(G''', \mathcal{P}''')$ by an H-operation. By induction, there exists a hyperladder L''' that contains G''' as a contraction. Thus there is a bijection between the faces of L''' and faces of G'''. The H-operation on patch P taking G''' to G' can be obtained as a contraction of a 2-rung operation followed by a 1-rung operation on the corresponding face of L'''. □

We also have a converse to Theorem 4.1.

Theorem 4.2. *If L is a Möbius hyperladder, then there exists a patch graph $(G, \mathcal{P})$ such that G contains L as a subgraph.*

Proof. If L is the initial Möbius hyperladder then the result follows; furthermore, every quadrilateral face in L is a patch P in the patch graph $(4K_2, \mathcal{P}_0)$. Now assume that for any Möbius hyperladder L obtained from the initial one by up to k n-rung operations there is a patch graph $(G, \mathcal{P})$ for which L is a subgraph of G and every quadrilateral face of L is a patch in $\mathcal{P}$. Now suppose that L' is obtained from an n-rung operation on a face P of L, where L is obtained from the initial Möbius hyperladder by k such operations. Consider the patch graph $(G, \mathcal{P})$ of the induction hypothesis that has patch $P \in \mathcal{P}$. Now perform n successive H-operations starting on patch P in $(G, \mathcal{P})$ as shown in Figure 7 to obtain $(G', \mathcal{P}')$. Evidently, L' is a subgraph of G' and every quadrilateral face of L' is a patch in $\mathcal{P}'$. □

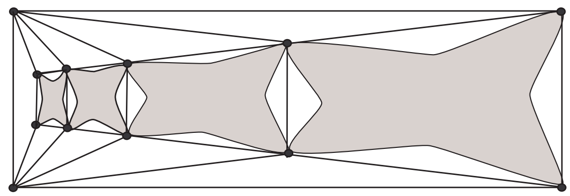


Figure 7.
A 4-rung operation contained within four H-operations.

As a result of Theorems 4.1 and 4.2, we have the following corollary of Theorem 2.3.

Corollary 4.3. *If G is simple, vertically 3-connected, nonplanar, projective-planar and $K_{3,4}$-free, then G is either isomorphic to K_6 or obtained from a simple, almost 4-connected minor of a Möbius hyperladder and then taking Y-sums with planar graphs with possible subdivisions of edges before each sum.*

5 Patch Graphs, Möbius Hyperladders and Book Embeddings

As the reader can see, Möbius hyperladders are described with only one type of iterative operation rather than the four types used for patch graphs. This itself along with their 3-regularity tells us that Möbius hyperladders are perhaps a more accessible family of graphs than patch graphs. One example of this greater accessibility is the simple explanation of the fact that the page number of Möbius hyperladders is three contrasted with the more complicated explanation for the same result with patch graphs.

Yannakakis [3] showed that general planar graphs have page number at most 4 (it is still an open question if this upper bound is sharp) therefore the page number of an apex graph (which includes patch graphs and general projective-planar graphs with 2-representative embeddings) is at most 5. The page number of general projective-planar graphs is not known.

Theorem 5.1. *A non-planar Möbius hyperladder has page number equal to three.*

Theorem 5.1 follows directly from the result of Bekos, Gronemann, and Raftopoulou in [1] but we present a simple inductive proof.

Proof of Theorem 5.1. The first rung operation from the initial hyperladder yields the Möbius hyperladder shown on the left in Figure 8 with an accompanying 3-page book embedding to its right where the third page is indicated by the dashed line. (If n rungs are added vertically rather than horizontally with respect to the rendering of the initial hyperladder in Figure 5, then the resulting hyperladder is isomorphic to the one show in Figure 8.)

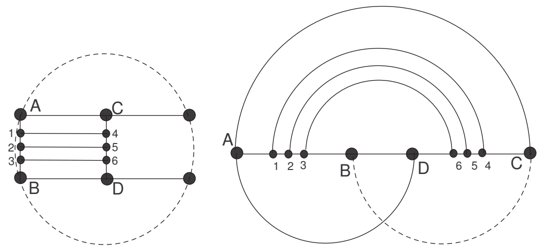


Figure 8.
The Möbius hyperladder coming from one n-rung operation on the initial hyperladder along with a 3-page book embedding.

Furthermore, note that each quadrilateral face of this beginning hyperladder has exactly one pair of opposing sides on the spine of the book embedding. We can now do induction on the number of rung operations to obtain a 3-page book embedding for a general Möbius hyperladder with each quadrilateral face having exactly one pair of opposing sides on the spine. The topological adjustment for the n-rung operation with rungs parallel to the spine is shown in Figure 9. For the n-rung operation with rungs perpendicular to the spine, the endpoints of the rungs are simply added onto the spine itself.

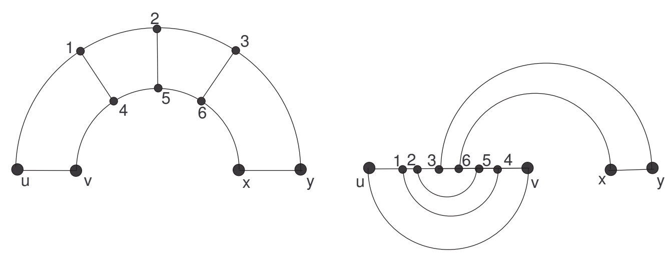


Figure 9.

□

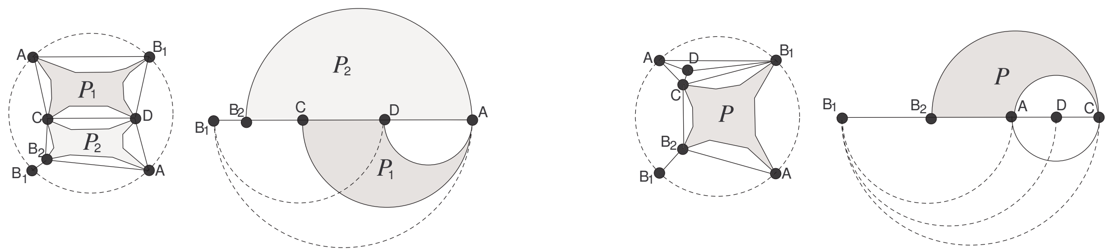
We now show the more general result that the page number of (nonplanar) patch graphs is equal to three.

Theorem 5.2. *A non-planar patch graph has page number equal to three.*

Proof. Consider a patch graph $(G, \mathcal{P})$ that has vertices A and B on the 2-representative cut. If this patch graph is obtained by a single terminal patching operation from the initial patch graph, then G is planar, a contradiction. Thus the first patching operation on the initial patch graph is either an H- or Y-operation. Notice that after an initial H-operation, the four corner vertices on any patch in any intermediate patch graph are distinct. This is not so if the initial patching operation is a Y-operation to obtain $(G_1, \mathcal{P}_1)$. To ensure that all corners of a patch are distinct, we obtain a modified patch graph $(G_1', \mathcal{P}'_1)$ by decontracting an edge at vertex B across the 2-representative cut to obtain vertices B_1 and B_2 as shown in Figure 10. Then utilizing the same patching operations for obtaining $(G, \mathcal{P})$ from $(G_1, \mathcal{P}_1)$, we obtain a modified patch graph $(G', \mathcal{P}')$ from $(G_1', \mathcal{P}'_1)$. We will give a construction of a 3-page embedding of G' with the (B_1, B_2)-edge along the spine. Therefore this edge can be contracted along the spine to obtain a three-page embedding of G.

Three-page embeddings are shown in Figure 10 for the two initial Patch graphs (with parallel edges removed) after an initial H-operation or Y-operation and the decontraction described above. The three pages are called *Up*, *Down*, and *Dashed*. Note that B_1 and B_2 are adjacent along the spine in the both embeddings.

Figure 10.



Now we assume that $(G, \mathcal{P})$ is obtained from $(G', \mathcal{P}')$ by a patching operation on $P \in \mathcal{P}'$. Inductively, we can assume that $(G', \mathcal{P}')$ has an embedding on pages Up, Down, and Dashed with the first two vertices from the left end of the spine being B_1 and B_2. Furthermore, by induction, we have the following facts: All edges on Page Dashed are all incident with B_1. There are two types of patches in $\mathcal{P}'$: *apostrophe patches* and *page patches*. These patches will have edges on the spine designated as "special" and no two patches share a special spine edge.

- Apostrophe patches have B_1 on their boundary, one edge either on Page Up or Down, and a special spine edge. The patch itself is divided between Page Dashed and either Page Up or Down. We do not shade in the part of the apostrophe patch on Page Dashed. Intuitively, Page Dashed resolves the "twisting" of the projective plane.

- Page patches will have two opposing edges that are special spine edges and the remaining two edges either together on page Up or page Down or one on the spine (not special, though) and the other on Page Up or Down. The page patch is entirely on one page (either Up or Down) and is not incident with B_1.

First, assume that $P \in \mathcal{P}'$ is a page patch and the four corners of the patch (which are distinct) are $1, 2, 3, 4$ in order with $(1, 2)$- and $(3, 4)$-edges being the special spine edges. For an H-operation, depending the relationship between the special spine edges of G' and the new edge $(5, 6)$, we modify the 3-page embedding of $(G', \mathcal{P}')$ to obtain a 3-page embedding for $(G, \mathcal{P})$ as shown in Figure 11. In each case, P_1 and P_2 are page patches with special edges chosen appropriately.

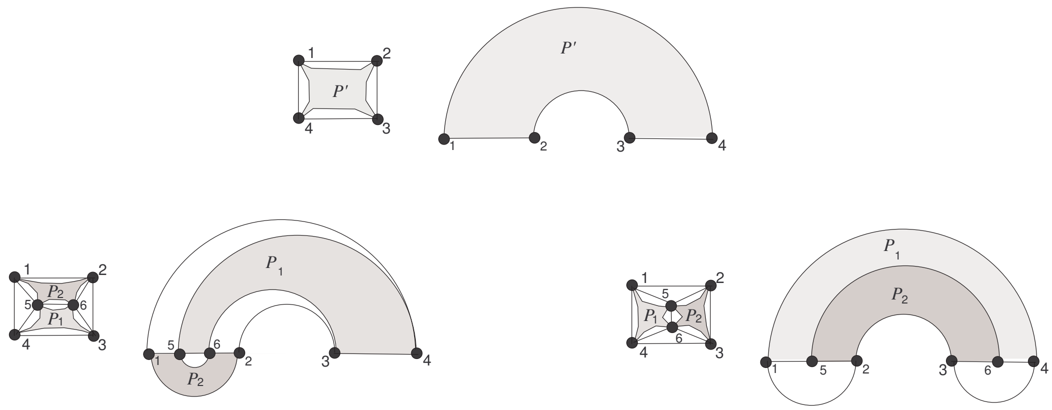
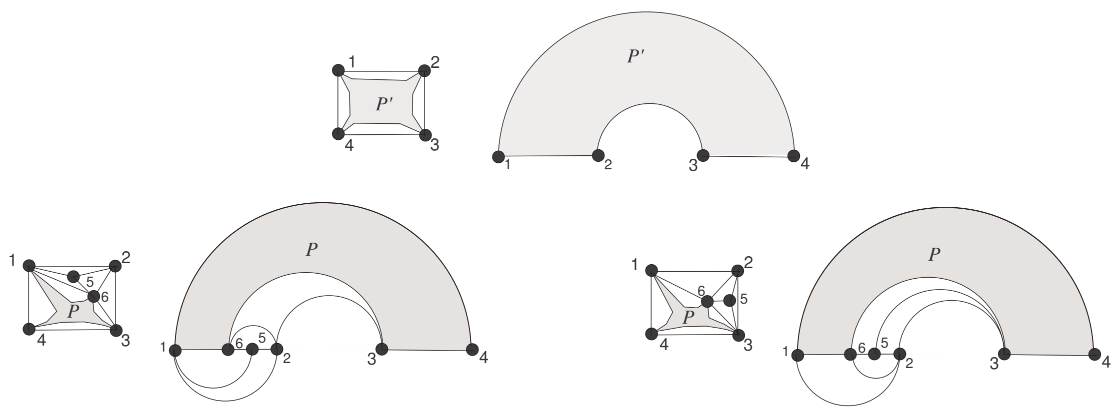


Figure 11.

If the operation on $(G', \mathcal{P}')$ to obtain $(G, \mathcal{P})$ was instead a Y-operation, there are again several cases depending on the orientation of the Y-operation relative to P'. Up to symmetry, we modify the embedding of $(G', \mathcal{P}')$ as shown in Figure 12. In each case, the new patch is a page patch with special edges chosen appropriately.

Figure 12.



If the operation on $(G', \mathcal{P}')$ to obtain $(G, \mathcal{P})$ was instead an X-operation or I-operation, there are again several cases depending on the orientation of the operation relative to P'. Up to symmetry, we modify the embedding of $(G', \mathcal{P}')$ as shown in Figure 13.

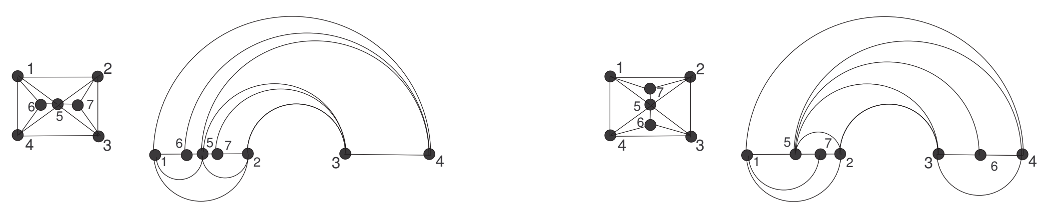
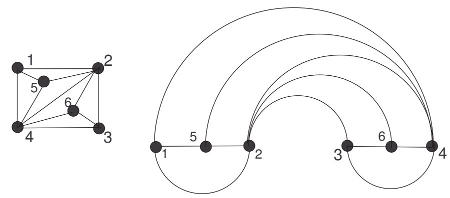


Figure 13.



Now assume that $P \in \mathcal{P}'$ is an apostrophe patch and the four corners of the patch (which are distinct) are $B_1, 2, 3, 4$ in order with $(B_1, 2)$- and $(B_1, 4)$-edges on page Dashed, the $(2, 3)$-edge on Page Up and edge $(3, 4)$ is the special spine edge as in the top of Figure 14. For an H-operation, we modify the 3-page-embedding of $(G', \mathcal{P}')$ to obtain a 3-page embedding for $(G, \mathcal{P})$ as shown in Figure 14 with special edges on the resulting patches chosen appropriately. Note that in the figure on the right, we can add vertex 5 adjacent to vertex 2 along the spine without crossing any other edges on the page and the edge $(2, 5)$ becomes a special edge for patch P_2.

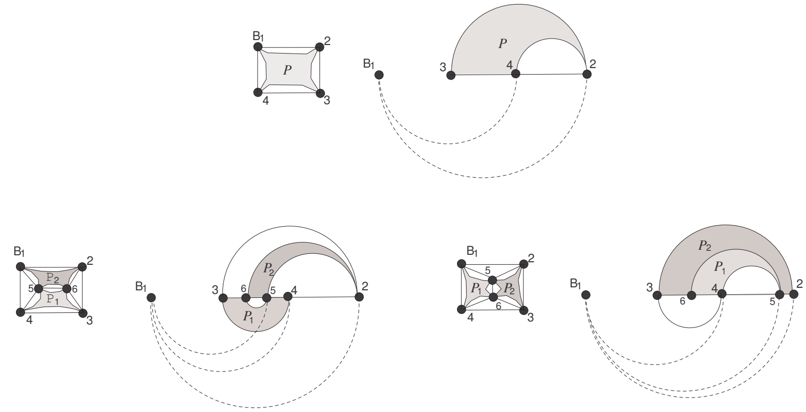
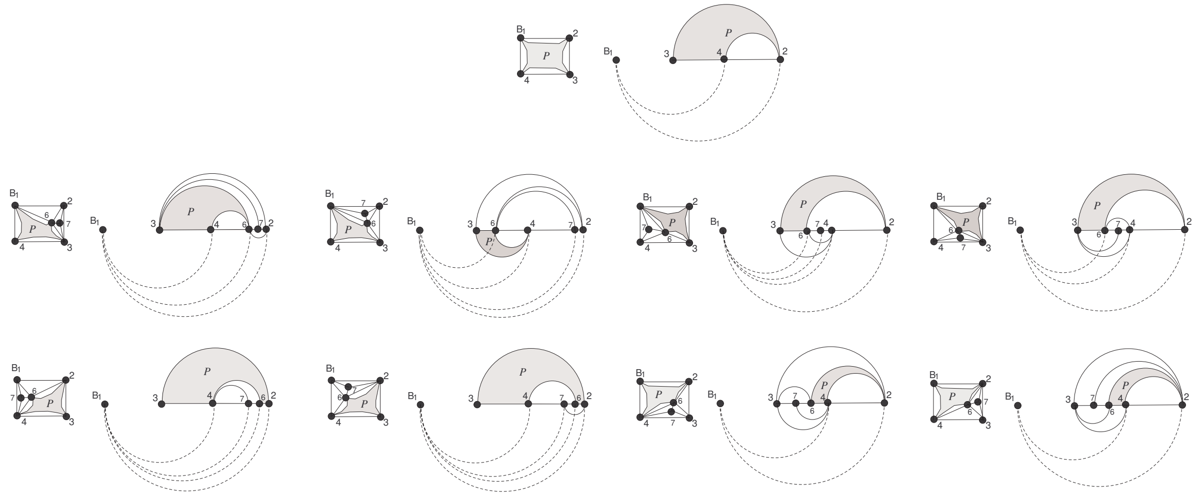


Figure 14.

If the operation on $(G', \mathcal{P}')$ to obtain $(G, \mathcal{P})$ was instead a Y-operation, there is no clear symmetry to help reduce the eight possible cases for the orientation of the Y-operation relative to P. We modify the embedding of $(G', \mathcal{P}')$ as shown in Figure 15 for these eight possibilities. In the cases where vertices are added to the spine in the white area under P, these vertices are in the space adjacent to vertex 2. The resulting patches in each case (apostrophe in six cases, page in two cases) has special edge(s) chosen appropriately.

Figure 15.



If the operation on $(G', \mathcal{P}')$ to obtain $(G, \mathcal{P})$ was instead an X-operation or I-operation, there are again several cases depending on the orientation of the operation relative to P'. Up to symmetry, we modify the embedding of $(G', \mathcal{P}')$ as shown in Figure 16. The crosshatching on the spine represents the part of the spine underneath the white area for P. Again vertices added to this area on the spine are adjacent to vertex 2 on the spine.

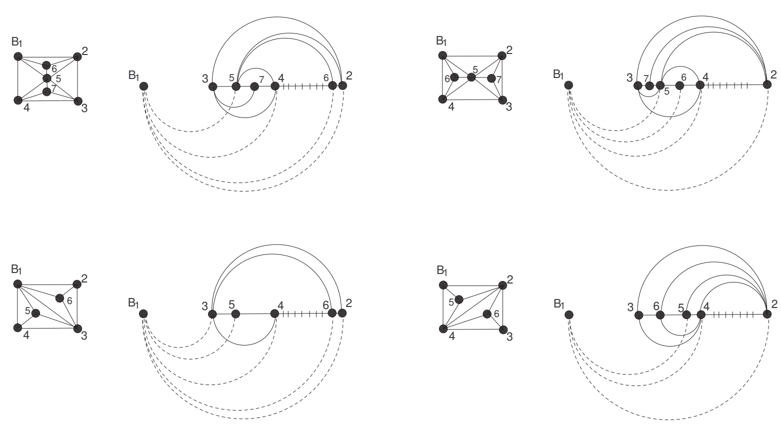


Figure 16.

□

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