11-2019

The Graphs That Have Antivoltages Using Groups of Small Order

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May 17, 2019

Abstract

Given a group Γ of order at most six, we characterize the graphs that have Γ-antivoltages and also determine the list of minor-minimal graphs that have no Γ-antivoltage. Our characterizations yield polynomial-time recognition algorithms for such graphs.

1 Introduction

Given a group Γ and a graph G, a Γ-*antivoltage* on G is a Γ-labeling φ of the oriented edges of G such that:

(A1) $\varphi(e^{-1}) = \varphi(e)^{-1}$ for each oriented edge e in G and

(A2) for any cycle C in G with edges $e_1, \dots, e_n$ in cyclic order along C, $\varphi(e_1)\cdots\varphi(e_n) \neq 1$.

Antivoltages were first formally defined and studied by Zaslavsky [15, 16].

Antivoltages on G are closely related to representations of the bicircular matroid $B(G)$ of the graph G. Let $\mathbb{F}$ be a field. In [15], Zaslavsky shows how $\mathbb{F}^{\times}$-antivoltages on G can be used to create a matrix representation for the bicircular matroid $B(G)$. Zaslavsky conjectured that all $\mathbb{F}$-representations of bicircular matroids are of these antivoltage types. Aside for some degenerate structures, the conjecture is true and was proven by Geelen, Gerards, and Whittle [5] and also by Funk and Slilaty [4]. In [4] it is also shown that the $\mathbb{F}^{\times}$-antivoltage associated to a given $\mathbb{F}$-representation of $B(G)$ is unique up to switching aside for those same degenerate structures.

Neudauer and Slilaty prove the following three results for a fixed finite group Γ. First, there are finitely many 3-connected and loopless graphs which have a Γ-antivoltage. Second, if $|\Gamma|$ is prime, then up to subdivisions, there are finitely many non-separable graphs which have a Γ-antivoltage. Third, let $2C_t$ denote the cycle of length t with each edge doubled. Up to subdivisions, there are finitely many non-separable graphs without a $2C_t$-minor that have a Γ antivoltage.

For some small fields, structural characterizations of the graphs whose bicircular matroids are $\mathbb{F}$-representable are known: $GF(2)$ (Matthews [6]), $GF(3)$ (Sivaraman [11]), and $GF(4)$ and $GF(5)$ (Chun, Moss, Slilaty and Zhou [1]). Since bicircular matroids are also examples of transversal matroids, more general results are known for $GF(2)$ (Sousa and Welsh [3]) and $GF(3)$ (Oxley [8]).

The class of graphs G which have a Γ-antivoltage for $\Gamma \in \{\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4\}$ are essentially known from [1, 11, 14]. In this paper we characterize the class of graphs which have Γ-antivoltages for the remaining groups Γ of order up to six, that is, $\Gamma \in \{\mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_5, \mathbb{Z}_6, S_3\}$. The characterizations are similar to those in [1] and they yield polynomial-time recognition algorithms for such graphs.

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2 Preliminaries

Our graph-theory terminology is mostly standard; we will define some less well-known terms and notations as needed. A graph G is *separable* when there are two subgraphs G_1 and G_2 of G with at least one edge each such that $G_1 \cup G_2 = G$ and $G_1 \cap G_2$ is empty or a single vertex. A graph on at least $k+1$ vertices is k-connected when it is connected and there are no $t < k$ vertices in G whose removal leaves a disconnected subgraph.

If G is a simple graph and $m \geq 2$ an integer, then by mG we mean the graph obtained from G by replacing each edge of G by m parallel edges on the same endpoints. The graph mK_2 is called an *m-multilink*. The graph obtained from a triangle whose edges are replaced by a-, b-, and c-multilinks is denoted by $T_{a,b,c}$. The cycle of length t is denoted by C_t and the wheel with t spokes is denoted by W_t.

In this paper we will make use of the *canonical tree decomposition* of a non-separable graph G. For information on the canonical tree decomposition see one of [2], [9, pp.308–315], or [12] for a full description. In short, if G is non-separable, then there is a unique labeled tree T satisfying the following.

- Each vertex v in T is labeled with either a 3-connected simple graph, a cycle of length at least three, or mK_2 for some $m \geq 3$.
- No two cycle-labeled vertices are adjacent in T and no two multilink-labeled vertices are adjacent in T.
- If e is an edge of T whose endpoints are labeled with graphs G_1 and G_2, then e corresponds to an edge e_i in G_i.
- G is obtained by executing the 2-sums indicated by the vertex labels of T along the edges indicated by the edges of T.

An important consequence of this tree decomposition is that if T_0 is a subtree of T, then the graph G_0 obtained by executing the 2-sums indicated in T_0 is a minor of G.

In a graph G, an *oriented edge* e is an edge together with a specified direction along it. When G is an oriented edge, we denote the reverse oriented edge by e^{-1}. Given a group Γ, a Γ-*antivoltage* is a Γ-labeling φ of the oriented edges of G such that

(A1) $\varphi(e^{-1}) = \varphi(e)^{-1}$ for each oriented edge e in G and

(A2) for any cycle C in G with edges $e_1, \ldots, e_n$ in cyclic order along C, $\varphi(e_1)\cdots\varphi(e_n) \neq 1$.

Of course for the product in (A2) to be well defined, it is necessary to choose a starting vertex in C as well as one of two possible directions along C; however, the requirement that the product does not equal the identity is not affected by any such choice because cyclic shifting conjugates the product and reversing the direction along C inverts the product. A Γ-antivoltage is a special case of a Γ-*gain* function, which is a labeling satisfying Property (A1). We will not, however, discuss gain functions in this paper aside for what is in the next paragraph.

Given a Γ-gain function φ on G, a *switching function* η is a Γ-labeling of the vertices of G. Let φ^η be the Γ-gain function defined by $\varphi^\eta(e) = \eta(u)\varphi(e)\eta(v)^{-1}$ where u is the tail of e and v is the head of e. Evidently the product in Property (A2) is unaffected by switching so φ^η is a Γ-antivoltage if and only if φ is a Γ-antivoltage. If F is a maximal forest in G, then it is easy to show that there is η such that $\varphi^\eta \equiv 1$ on the edges of F (see, for example, [14, Lemma 5.3]). Therefore if G is a connected graph and T a spanning tree of G, then G has a Γ-antivoltage if and only if G has a Γ-antivoltage with each edge in T labeled by the identity.

If G is a subdivision of a graph H, then clearly G has a Γ-antivoltage if and only if H has a Γ-antivoltage. Also, if G is separable, then G has a Γ-antivoltage if and only if each block of G has a

Γ-antivoltage. Given these two properties, one can characterize the graphs having a Γ-antivoltage by characterizing the non-separable ones with minimum degree at least three.

If φ is a Γ-antivoltage on G, then clearly φ restricted to the edges of any subgraph H of G is a Γ-antivoltage for H. If e is a non-loop edge of G, then there is a switching function η such that $\varphi^{\eta}(e) = 1$. Now φ^{η} restricted to the edges of G/e is a Γ-antivoltage on G/e. We have just proven that the class of graphs having a Γ-antivoltage is closed under taking minors. Given a minor-closed class of graphs $\mathcal{M}$, if H is a minor-minimal graph that is not in $\mathcal{M}$, then H is called an *excluded minor* for $\mathcal{M}$.

As just stated, we can characterize those graphs having a Γ-antivoltage by characterizing the non-separable ones with minimum degree at least three. In addition to this we introduce another reduction operation that greatly aids in producing succinct characterizations of graphs with Γ-antivoltages. If G is non-separable and contains an edge e such that $G\backslash e$ is separable, then there is a bijection between the edge sets of cycles in G and the edge sets of cycles in G/e defined by $C \mapsto (C - e)$. We call G/e a *2-bond reduction* of G.

Proposition 2.1. *If G is non-separable and G/e is a 2-bond reduction of G, then G has a Γ-antivoltage if and only if G/e has a Γ-antivoltage.*

Proof. This follows from the fact that there is a spanning tree in G containing e and $C \mapsto (C - e)$ defines a bijection between the edge sets of cycles in G and in G/e. □

If for every edge e in a non-separable graph G the deletion $G\backslash e$ is still non-separable, then we call G *2-bond irreducible.* Since separability is recognizable in polynomial time, so is 2-bond irreducibility. We use the term "2-bond reduction" because if G is non-separable and $G\backslash e$ is separable, then there is a graph G' such that $G'/e' = G$ and in which e and e' form a bond.

Theorem 2.2 is from [7, §3] and is a simple extension of [1, Proposition 2]. For groups of order at most 6 we will find this finite collection of 3-connected graphs.

Theorem 2.2. *If Γ is a finite group, then there are finitely many 3-connected and loopless graphs that have a Γ-antivoltage.*

Theorem 2.3 is also from [7, §3]. If Γ is a non-trivial finite group of non-prime order, then $2C_t$ has a Γ-antivoltage for any t. Theorem 2.3 gives some rough feel for how the class of non-separable graphs having Γ-antivoltages grows.

Theorem 2.3. *Let $\mathcal{G}$ be the class of non-separable graphs with minimum degree 3 and let $\mathcal{G}_t \subset \mathcal{G}$ be those without a $2C_t$-minor.*

(1) If p is a prime, then there are finitely many members of $\mathcal{G}$ that have a $\mathbb{Z}_p$-antivoltage.

(2) If Γ is a finite group, then there are finitely many members of $\mathcal{G}_t$ that have a Γ-antivoltage.

3 The groups $\mathbb{Z}_2$ and $\mathbb{Z}_3$

Theorems 3.1 and 3.2 are proven by Zaslavsky in [13].

Theorem 3.1. *A non-separable graph G has a $\mathbb{Z}_2$-antivoltage if and only if G is a cycle or a single edge.*

Theorem 3.2. *A graph G has a $\mathbb{Z}_2$-antivoltage if and only if G has no $3K_2$-minor.*

The multiplicative group in $GF(4)$ is isomorphic to $\mathbb{Z}_3$ and so Theorem 3.3 essentially follows from [1, Theorem 6].

Theorem 3.3. *A non-separable graph G with minimum degree 3 has a $\mathbb{Z}_3$-antivoltage if and only if $G = 3K_2$.*

Theorem 3.4. *A graph G has a $\mathbb{Z}_3$-antivoltage if and only G has no $4K_2$-minor.*

Proof. Evidently $4K_2$ is a minor-minimal graph without a $\mathbb{Z}_3$-antivoltage. Now let G be a non-separable graph with no $4K_2$-minor and let T be the canonical tree decomposition of G. There can be no 3-connected term in T because K_4 has a $4K_2$-minor. Thus the vertices of T are labeled by cycles and copies of $3K_2$. There cannot be more than one copy of $3K_2$ as this would also create a $4K_2$-minor. Thus G is a subdivision of $3K_2$, as required. □

4 The group $\mathbb{Z}_4$

Theorem 4.1 essentially follows from [1, Theorem 3] and the main result of [4]; however, we furnish a direct proof.

Theorem 4.1. *A non-separable and 2-bond-irreducible graph G with minimum degree 3 has a $\mathbb{Z}_4$-antivoltage if and only if G is K_4 or $2C_n$ for any $n \geq 2$.*

Theorem 4.2. *A graph G has a $\mathbb{Z}_4$-antivoltage if and only if G contains no $5K_2$-minor.*

Proof of Theorems 4.1 and 4.2. It is evident that $5K_2$ is a minor-minimal graph with no $\mathbb{Z}_4$-antivoltage. The reader can confirm that K_4 has a $\mathbb{Z}_4$-antivoltage. A $\mathbb{Z}_4$-antivoltage for $2C_n$ for any $n \geq 2$ is constructed as follows. Let T be a spanning tree of $2C_n$ so T is a path of length $n-1$. Label each edge parallel to an edge in T with a 2 and the remaining two edges of $2C_n$ with 1 and 3. We will now show that any non-separable and 2-bond irreducible graph without a $5K_2$-minor is either K_4 or $2C_n$. This will complete the proof of both theorems.

First assume that G is 3-connected and loopless. Note that W_4 has a $5K_2$-minor and the graph obtained from K_4 by doubling one edge also has a $5K_2$-minor. Thus $G = K_4$.

Now say that G is non-separable and let T be the canonical tree decomposition of G. If G has a 3-connected term, then that term must be K_4. Now the 2-bond irreducibility of G implies that T consists of a single vertex and so $G = K_4$. So we may now assume that T consists of cycle-labeled vertices and 3-multilink and 4-multilink labeled vertices. If there is no cycle-labeled vertex, then G is a subgraph of $4K_2$. Now root T on a vertex labeled by a cycle C_m of length $m \geq 3$. By 2-bond irreducibility, each edge of C_m is indicated in a 2-sum with a multilink. Since G has no $5K_2$-minor, these multilinks are all 3-multilinks. If this is all of T, then $G = 2C_m$, as required. If not, then by 2-bond irreducibility T has a cycle-labeled vertex at distance 2 from C_m and a multilink-labeled vertex at distance 3 from C_m. This, however, would imply that G has a $5K_2$-minor, a contradiction. □

5 The group $\mathbb{Z}_2 \times \mathbb{Z}_2$

The details of the proofs of Theorems 5.1 and 5.2 are nearly identical to those in the proofs of Theorems 4.1 and 4.2.

Theorem 5.1. *A non-separable and 2-bond-irreducible graph G has a $\mathbb{Z}_2 \times \mathbb{Z}_2$-antivoltage if and only if G is $2C_n$ for any $n \geq 2$.*

Theorem 5.2. *A graph G has a $\mathbb{Z}_2 \times \mathbb{Z}_2$-antivoltage if and only if G has no $5K_2$- nor K_4-minor.*

6 The group $\mathbb{Z}_5$

Our main results of this section are Theorems 6.1 and 6.2. Notice that the collection of graphs in Theorem 6.1 is finite; this is the realization of Theorem 2.3 particular to $\mathbb{Z}_5$. If G is a simple graph, then by $\overline{G}$ we mean the complement of G. The "triangular-prism" graph is $\overline{C}_6$.

Theorem 6.1. *If G is non-separable and 2-bond irreducible, then G has a $\mathbb{Z}_5$-antivoltage if and only if G is a minor of one of $\overline{C}_6$, $2C_3$, and $5K_2$.*

Theorem 6.2. *A graph G has a $\mathbb{Z}_5$-antivoltage if and only if G has none of the following graphs as a minor: $6K_2$, $T_{2,2,3}$, $2C_4$, and $K_{3,3}$.*

Proposition 6.3. *Up to switching, group automorphism, and symmetry, the only $\mathbb{Z}_5$ antivoltage on $2C_3$ is shown in Figure 1.*

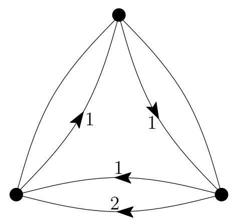

Figure 1: The only $\mathbb{Z}_5$-antivoltage on $2C_3$ up to switching, scaling, and symmetry.

Proof of Proposition 6.3. Up to switching and scalar multiplication, a $\mathbb{Z}_5$-antivoltage of $3K_2$ is as shown in Figure 2 where a, b, and c are all nonzero and $b \neq c$. If we try $a = 1$, then one can check that $\{b, c\} = \{1, 2\}$ is forced. If $a = 4$, then it must be that $\{b, c\} = \{2, 3\}$; however, switching and symmetry take this antivoltage to the first one. If $a = 2$, then it is not possible to choose values for b and c. If $a = 3$, then it is not possible to choose values for b and c. Our result follows. □

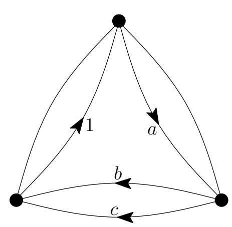

Figure 2: An abstract $\mathbb{Z}_5$-antivoltages up to switching, scaling, and symmetry.

Proposition 6.4. *The graphs $6K_2$, $T_{2,2,3}$, and $2C_4$ are minor-minimal graphs having no $\mathbb{Z}_5$-antivoltage.*

Proof. The proof for $6K_2$ is obvious. Now consider the graph $T_{2,2,3}$. By Proposition 6.3, a $\mathbb{Z}_5$-antivoltage for $T_{2,2,3}$ is as shown in Figure 3 where $x \notin \{0, 1, 2\}$. One can check, however, that neither $x = 3$ nor $x = 4$ works for a $\mathbb{Z}_5$-antivoltage. Thus $T_{2,2,3}$ has no $\mathbb{Z}_5$-antivoltage. Now $T_{2,2,3}$ is minimal because any single-edge deletion or contraction of $T_{2,2,3}$ (aside from $2C_3$) minus its loops has a 2-bond reduction to a subgraph of $5K_2$ and so has a $\mathbb{Z}_5$-antivoltage.

For the graph $2C_4$, any $\mathbb{Z}_5$-antivoltage would have the form as given in Figure 3 because of Proposition 6.3. One can check, however, that no value of $x \in \mathbb{Z}_5$ will yield a $\mathbb{Z}_5$-antivoltage. Now $2C_4$ is minimal because any single-edge deletion or contraction of $2C_4$ reduces to a minor of $2C_3$ and so has a $\mathbb{Z}_5$-antivoltage. □

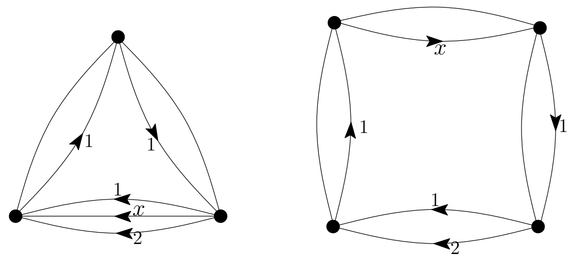

Figure 3: Abstract $\mathbb{Z}_5$-antivoltages for $T_{2,2,3}$ and $2C_4$ up to switching, scaling, and symmetry.

The proof of Proposition 6.5 uses the same idea as Proposition 6.4 of building up antivoltages from $2C_3$ given by Proposition 6.3. Note that there is a unique edge of $K_4 \oplus_2 3K_2$ whose contraction yields $2C_3$.

Proposition 6.5. *Up to switching, scaling, and symmetry, the only $\mathbb{Z}_5$-antivoltages for $K_4 \oplus_2 3K_2$ and W_4 are as shown in Figure 4.*

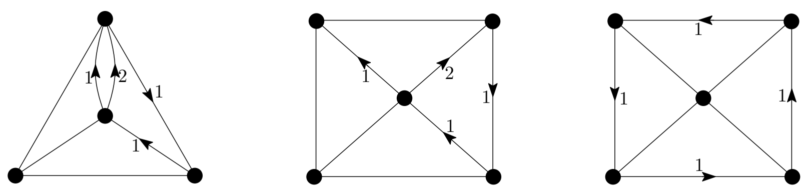

Figure 4: Unique $\mathbb{Z}_5$-antivoltages up to switching, scaling, and symmetry for the graphs shown. The second and third antivoltage are switching equivalent.

Proposition 6.6.

(1) The prism graph $\overline{C}_6$ has a $\mathbb{Z}_5$-antivoltage.

(2) $K_{3,3}$ is a minor-minimal graph without a $\mathbb{Z}_5$-antivoltage.

(3) If G is a 3-connected loopless graph having a $\mathbb{Z}_5$-antivoltage, then G is a minor of the prism graph $\overline{C}_6$; that is, G is one of K_4, $K_4 \oplus_2 3K_2$, W_4, and $\overline{C}_6$.

Proof. Up to isomorphism, there are two single-edge decontractions of W_4: the prism $\overline{C}_6$ and $K_{3,3}$. A $\mathbb{Z}_5$-antivoltage for $\overline{C}_6$ is obtained from the third antivoltage shown in 6.5 by performing a decontraction at the hub vertex. That $K_{3,3}$ has no $\mathbb{Z}_5$-antivoltage again comes from the third antivoltage shown in 6.5. Since every proper minor of $K_{3,3}$ on five edges is either W_4 or a subdivision of K_4, we get that $K_{3,3}$ is a minor-minimal graph without a $\mathbb{Z}_5$-antivoltage. This proves (1) and (2).

For (3) let G be a 3-connected graph having a $\mathbb{Z}_5$-antivoltage. The underlying simple graph $\mathrm{si}(G)$ of G is obtained by a sequence $G_1, \ldots, G_n$ of 3-connected simple graphs such that $G_n = \mathrm{si}(G)$, G_1 is a largest wheel minor in G, and G_i is obtained from G_{i+1} by the deletion or contraction of a single edge.

Since W_5 has a $6K_2$-minor, we get that $G_1 \in \{K_4, W_4\}$. If $G_1 = K_4$, then $G_1 = G_n$. At most one edge of K_4 can be doubled without creating a $6K_2$-minor. Thus G is a subgraph of $K_4 \oplus_2 3K_2$ which is a minor of $\overline{C}_6$.

Now suppose that $G_1 = W_4$. The graph W_4 is K_5 minus a 2-edge matching; however, $K_5 \backslash e$ has a $6K_2$-minor. Decontracting any edge at the hub vertex of W_4 must yield $\overline{C}_6$ rather than $K_{3,3}$ which has no $\mathbb{Z}_5$-antivoltage. Now doubling any edge of W_4 or $\overline{C}_6$ yields a graph with a $6K_2$-minor. Thus G is W_4 or $\overline{C}_6$. $\square$

Proof of Theorems 6.1 and 6.2. We have shown in this section that the graphs listed in Theorem 6.1 all have $\mathbb{Z}_5$-antivoltages. We also know that the graphs listed in Theorem 6.2 are minor-minimal graphs without a $\mathbb{Z}_5$-antivoltage. Now we will show that any non-separable and 2-bond irreducible graph not containing a minor from the list in Theorem 6.2 is a minor of a graph listed in Theorem 6.1. This will complete the proof of both Theorems 6.1 and 6.2.

If G is 3-connected, then the result follows from Proposition 6.6. So suppose that G is non-separable but not 3-connected and consider the tree decomposition T of G. There cannot be more than one 3-connected term in T because if there were then G would have a $K_4 \oplus_2 K_4$-minor, but this graph has a $6K_2$-minor. So in the first case say that T has one 3-connected term and in the second case say that T's terms are all cycles and multi-edges.

Case 1 Let K be the 3-connected term in T. Thus K is simple and a minor of $\overline{C}_6$ by Proposition 6.6. If K is adjacent in T to a cycle-labeled vertex C, then because G is 2-bond irreducible and C has length at least three, C must be adjacent in T to two multilink-labeled vertices. The graph resulting from K, C, and the two multilinks has a $T_{2,2,3}$-minor, a contradiction. Thus K is adjacent to multilink terms only and so G is 3-connected and must be a minor of $\overline{C}_6$.

Case 2 Let v be a nK_2-labeled vertex of T with n maximal. Thus $3 \leq n \leq 5$. If $T = v$, then we have our desired result, otherwise v is adjacent to a cycle-labeled vertex v_m; say the cycle for v_m has length $m \geq 3$. Every edge of the cycle for v_m is indicated in a 2-sum and so $m \leq 3$ in order to avoid a $2C_4$-minor. Furthermore, if $m = 3$, then all of the three multilink neighbors of v_m must be $3K_2$ in order to avoid a $T_{2,2,3}$-minor. Thus $G = 2C_3$. □

7 $\mathbb{Z}_6$-antivoltages

Two infinite families of graphs with $\mathbb{Z}_6$-antivoltages are shown in Figure 5.

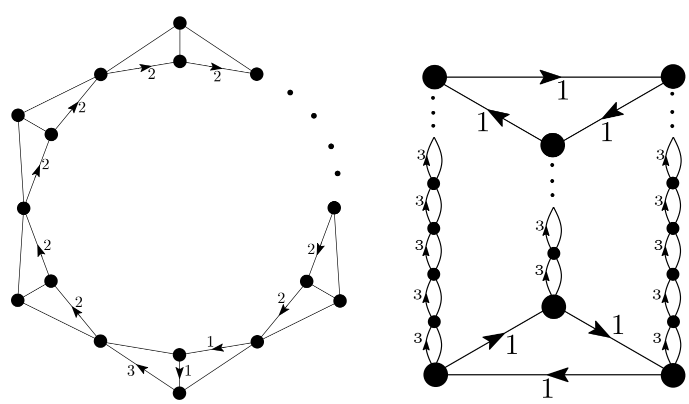

Figure 5: Two infinite classes of graphs having $\mathbb{Z}_6$-antivoltages.

Proposition 7.1. *If G is a connected planar graph with at most n faces, then G has a $\mathbb{Z}_n$-antivoltage.*

Proof. If $f_1, \ldots, f_n$ are the faces of some planar embedding of G, then the clockwise boundary walks of $\partial f_1, \ldots, \partial f_{n-1}$ form a basis for the integer cycle space of G, denote it by $Z_1(G)$. If C is any other cycle of the plane graph G oriented in the clockwise direction, then C is a $\{0, +1\}$-linear combination of $\partial f_1, \ldots, \partial f_{n-1}$. Thus if we let $\varphi(\partial f_i) = 1$ for each $1 \leq i \leq n-1$, then no $\{0, +1\}$-linear combination of $\varphi(\partial f_1), \ldots, \varphi(\partial f_{n-1})$ adds to zero in $\mathbb{Z}_n$. Hence we have a linear transformation from $Z_1(G) \to \mathbb{Z}_n$ that is nonzero on all of the simple cycles of $Z_1(G)$. We now get a $\mathbb{Z}_n$-antivoltage on G by choosing a spanning tree T of G and setting $\varphi(e) = 0$ for each edge in T and for each $e \notin T$, set $\varphi(e)$ equal to the linear transformation value on the single cycle in $T \cup e$ given by $\varphi(\partial f_1) = \cdots = \varphi(\partial f_{n-1}) = 1$. □

Proposition 7.2. *Every planar, 3-connected, simple graph with at most six faces is a minor of one of the graphs of Figure 6. These two graphs both have $\mathbb{Z}_6$-antivoltages.*

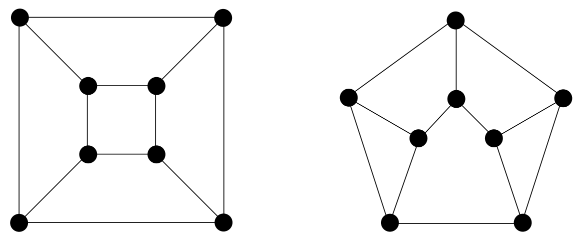

Figure 6: Cubic, 3-connected, planar graphs with 6 faces. These two graph both have $\mathbb{Z}_6$-antivoltages.

Proof of Proposition 7.2. A cubic graph on v vertices has $e = \frac{3}{2}v$ edges. A connected cubic graph embedded in the plane therefore has $f = e - v + 2 = \frac{1}{2}v + 2$ faces. So if $f \leq 6$, then $v \leq 8$. The complete list connected cubic graphs on at most 8 vertices is well known and the first part of our result follows by checking which ones are planar. The second part of the result follows from Proposition 7.1. □

In previous sections we calculated Γ-antivoltages for $|\Gamma| \leq 5$ by hand. We could try doing so again for $\Gamma = \mathbb{Z}_6$, however, the job is more difficult and probably not too enlightening. So we will use the Sage computational software package to compute $\mathbb{Z}_6$-antivoltages on certain small graphs. The computations were performed on an HP EliteBook with an Intel i7 processor running the Windows7 operating system. We include the actual code used for several examples. The interested reader can easily double check the calculations himself. In Proposition 7.3 we give a list of six excluded minors for the class of graphs with $\mathbb{Z}_6$-antivoltages. Later we will see that this is the complete list of excluded minors.

Proposition 7.3. *The following graphs are all minor-minimal graphs without a $\mathbb{Z}_6$-antivoltage: $7K_2$, $K_{3,3}$, K_5, and the graphs shown in Figure 7.*

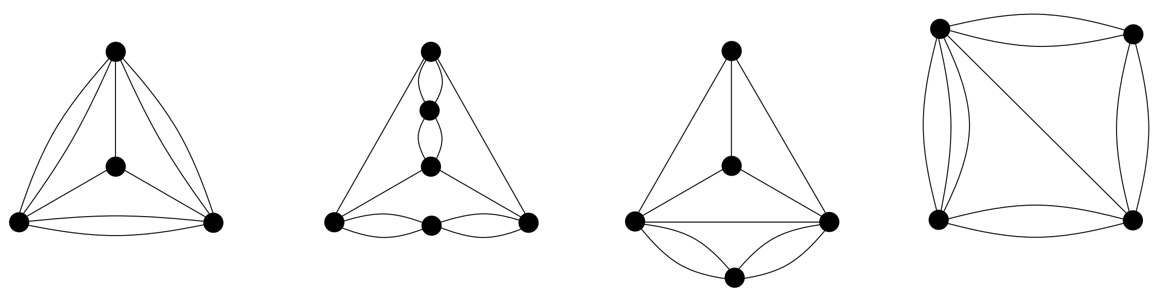

Figure 7: Four excluded minors for the class of graphs with $\mathbb{Z}_6$-antivoltages.

Proof of Proposition 7.3. That $7K_2$ is an excluded minor is immediate. Consider now the first graph of Figure 7, call it G. Up to switching, a $\mathbb{Z}_6$-antivoltage on G has the form shown in Figure 8 where $0 \notin \{a, b, c, d, e, f\}$.

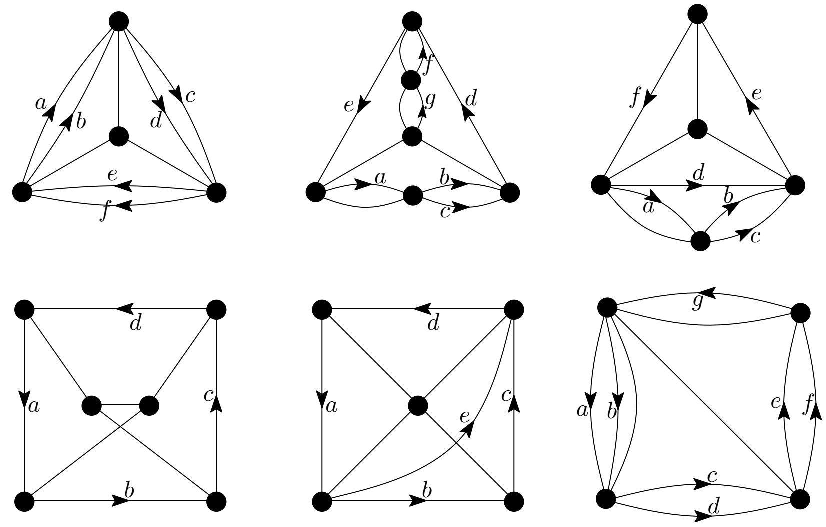

Figure 8: Hypothetical $\mathbb{Z}_6$-antivoltages up to switching.

Now $a, \ldots, f$ yield a $\mathbb{Z}_6$-antivoltage if and only if they satisfy the following relations in $\mathbb{Z}_6$: $a \neq b$, $c \neq d$, $e \neq f$, $a+c \neq 0$, $a+d \neq 0$, $a+e \neq 0$, $a+f \neq 0$, $b+c \neq 0$, $b+d \neq 0$, $b+e \neq 0$, $b+f \neq 0$, $c+e \neq 0$, $c+f \neq 0$, $d+e \neq 0$, $d+f \neq 0$, $a+c+e \neq 0$, $a+d+e \neq 0$, $a+c+f \neq 0$, $a+d+f \neq 0$, $b+c+e \neq 0$, $b+d+e \neq 0$, $b+c+f \neq 0$, $b+d+f \neq 0$. Writing a program in the Sage mathematical software package to perform a brute-force check over all 6-tuples with values from $\mathbb{Z}_6 - \{0\}$ is routine. The actual code we used is as follows. Our computational time was a fraction of a second and yields no solutions.

```
k=0
for a in [1,2,3,4,5]:
    for b in [1,2,3,4,5]:
        for c in [1,2,3,4,5]:
            for d in [1,2,3,4,5]:
                for e in [1,2,3,4,5]:
                    for f in [1,2,3,4,5]:
                        result=(a-b)%6!=0 and (c-d)%6!=0 and (e-f)%6!=0 and
(a+c)%6!=0 and (a+d)%6!=0 and (a+e)%6!=0 and (a+f)%6!=0 and (b+c)%6!=0 and
(b+d)%6!=0 and (b+e)%6!=0 and (b+f)%6!=0 and (c+e)%6!=0 and (c+f)%6!=0 and
(d+e)%6!=0 and (d+f)%6!=0 and (a+c+e)%6!=0 and (a+d+e)%6!=0 and (a+c+f)%6!=0
and (a+d+f)%6!=0 and (b+c+e)%6!=0 and (b+d+e)%6!=0 and (b+c+f)%6!=0 and
(b+d+f)%6!=0
                        if result:
                            k=k+1
                            print(k,[a,b,c,d,e,f])
```

That every proper minor of G has $\mathbb{Z}_6$-antivoltage is as follows. Any single-edge deletion of G has a planar embedding with at most six faces and so has a $\mathbb{Z}_6$-antivoltage by Proposition 7.1. If e is an edge of the central triad of G, then G/e is a minor of the first graph from Figure 5. If e is one of the doubled edges of G, then G/e has a loop e' and so G/e has an antivoltage if and only if $G\backslash e'/e$ has an antivoltage and we already know that any single-edge deletion of G has an antivoltage.

Consider now the second graph of Figure 7, again call it G. Up to switching, a $\mathbb{Z}_6$-antivoltage on G has the form shown in Figure 8 where $0 \notin \{a, b, c, d, e, f, g\}$. Now $a, \ldots, g$ yield a $\mathbb{Z}_6$-antivoltage if and only if they satisfy the following relations in $\mathbb{Z}_6$. The relations are grouped according to the seven cycles in the underlying K_4 structure of G.

- $b \neq c$, $a+b \neq 0$, $a+c \neq 0$,
- $d-f \neq 0$ $d-g \neq 0$, $d-f-g \neq 0$,
- $e+f \neq 0$, $e+g \neq 0$, $e+f+g \neq 0$,
- $a+b+d+e \neq 0$, $a+c+d+e \neq 0$, $b+d+e \neq 0$, $c+d+e \neq 0$,
- $d+e \neq 0$,
- $a+b+d \neq 0$, $a+c+d \neq 0$, $b+d \neq 0$, $c+d \neq 0$, $a+b+d-f \neq 0$, $a+c+d-f \neq 0$, $b+d-f \neq 0$, $c+d-f \neq 0$, $a+b+d-g \neq 0$, $a+c+d-g \neq 0$, $b+d-g \neq 0$, $c+d-g \neq 0$, $a+b+d-f-g \neq 0$, $a+c+d-f-g \neq 0$, $b+d-f-g \neq 0$, $c+d-f-g \neq 0$,
- $a+b+e \neq 0$, $a+c+e \neq 0$, $b+e \neq 0$, $c+e \neq 0$, $a+b+e+f \neq 0$, $a+c+e+f \neq 0$, $b+e+f \neq 0$, $c+e+f \neq 0$, $a+b+e+g \neq 0$, $a+c+e+g \neq 0$, $b+e+g \neq 0$, $c+e+g \neq 0$, $a+b+e+f+g \neq 0$, $a+c+e+f+g \neq 0$, $b+e+f+g \neq 0$, $c+e+f+g \neq 0$.

Writing a Sage program similar to the one above is routine and is left to the reader. The computation takes a fraction of a second on our setup and the result is that there are no possible answers for $a, \dots, g$. That every proper minor of G has an antivoltage is as follows. Deleting an of the edge of G yields a graph having 2-bond reductions that take it to a minor of the second graph class from Figure 5. Contracting an edge that is doubled yields a graph with a loop and so has an antivoltage if and only if the deletion of that edge yields a graph with an antivoltage. Contracting an edge that is not doubled yields a graph having a 2-bond reduction that takes it to a minor of the second graph class from Figure 5.

Consider now the third graph of Figure 7, again call it G. Up to switching, a $\mathbb{Z}_6$-antivoltage on G has the form shown in Figure 8 where $0 \notin \{a, b, c, d, e, f\}$. Now $a, \dots, f$ yield a $\mathbb{Z}_6$-antivoltage if and only if they satisfy the following relations in $\mathbb{Z}_6$. The relations are grouped as follows: the first group are for the K_4 subgraph, the second group are for the K_4-subgraph with d replaced by combinations of a, b, c, and the third group include all possible combinations using a, b, c, d.

- $d+e \neq 0$, $e+f \neq 0$, $f+d \neq 0$, $d+e+f \neq 0$,
- $b+e \neq 0$, $f+b \neq 0$, $b+e+f \neq 0$, $c+e \neq 0$, $f+c \neq 0$, $c+e+f \neq 0$, $a+b+e \neq 0$, $f+a+b \neq 0$, $a+b+e+f \neq 0$, $a+c+e \neq 0$, $f+a+c \neq 0$, $a+c+e+f \neq 0$,
- $b-d \neq 0$, $c-d \neq 0$, $a+b-d \neq 0$, $a+c-d \neq 0$, $b-c \neq 0$, $a+b \neq 0$ and $a+c \neq 0$.

Again a Sage calculation confirms that there are no possibilities for $a, \dots, f$. We get that every single-element deletion and contraction of G has a $\mathbb{Z}_6$-antivoltage as follows. Deleting the d-labeled edge yields a graph that is a minor from the first family of graphs in Figure 5. Contracting the d-labeled edge yields a separable graph in which each block has a $\mathbb{Z}_6$-antivoltage. Consider the edge that is incident with both the e- and f-labeled edges. Deleting this edge, we obtain a graph having 2-bond reductions that take it to a minor from the first class of graphs in Figure 5. Contracting this edge we get a graph that is a minor from the second class of graphs in Figure 5. Deleting or contracting the a-labeled edge takes yields a graph having 2-bond reductions that take it to a graph that is a minor from the first family in Figure 5. Deleting or contracting the f-labeled edge yields a graph with 2-bond reductions taking it to graph that is a minor from the second family of graphs in Figure 5.

Up to switching, a $\mathbb{Z}_6$-antivoltage on $K_{3,3}$ has the form shown in Figure 8. Now $a, \dots, d$ yield a $\mathbb{Z}_6$-antivoltage if and only if they satisfy the following relations in $\mathbb{Z}_6$. The relations are grouped by the number of labels in them. Again a Sage computation yields no solutions for $a, \dots, d$.

- $a+b \neq 0$, $b+c \neq 0$, $c+d \neq 0$, $a+d \neq 0$, $a-c \neq 0$, $b-d \neq 0$,

- $a + b + c \neq 0$, $b + c + d \neq 0$, $a + c + d \neq 0$, $a + b + d \neq 0$,
- $a + b + c + d \neq 0$.

Every single-element contraction of $K_{3,3}$ is W_4, which has a $\mathbb{Z}_6$-antivoltage. Every single-element deletion yields a graph with 2-bond reductions and so is obtained by first contracting an edge from $K_{3,3}$.

Up to switching, a $\mathbb{Z}_6$-antivoltage on $K_5\backslash f$ has the form shown in Figure 8. A Sage computation reveals that, up to automorphism of $\mathbb{Z}_6$, the only $\mathbb{Z}_6$-antivoltage is $a = b = c = d = 1$ and $e = 3$. So a $\mathbb{Z}_6$-antivoltage for K_5 would require the tenth edge f must have $f = 3$, however, this yields $f - d - e + b = 0$, a contradiction. Thus K_5 has no $\mathbb{Z}_6$-antivoltage. Every single-edge deletion of K_5 has a $\mathbb{Z}_6$-antivoltage and every single-edge contraction of K_5 yields a graph that is a minor of the second family from Figure 5.

Consider now the fourth graph of Figure 7, again call it G. Up to switching, a $\mathbb{Z}_6$-antivoltage on G has the form shown in Figure 8 where $0 \notin \{a, b, c, d, e, f, g\}$. Now $a, \ldots, g$ yield a $\mathbb{Z}_6$-antivoltage if and only if they satisfy the following relations in $\mathbb{Z}_6$. The relations are grouped as follows: the first group is for 2-cycles, the second group is for 4-cycles, and the last group is for triangles.

- $a - b \neq 0$, $c - d \neq 0$, $e - f \neq 0$,
- $c+e \neq 0$, $c+f \neq 0$, $d+e \neq 0$, $d+f \neq 0$, $c+e+g \neq 0$, $c+f+g \neq 0$, $d+e+g \neq 0$, $d+f+g \neq 0$, $a + c + e \neq 0$, $a + c + f \neq 0$, $a + d + e \neq 0$, $a + d + f \neq 0$, $a + c + e + g \neq 0$, $a + c + f + g \neq 0$, $a + d + e + g \neq 0$, $a + d + f + g \neq 0$, $b + c + e \neq 0$, $b + c + f \neq 0$, $b + d + e \neq 0$, $b + d + f \neq 0$, $b + c + e + g \neq 0$, $b + c + f + g \neq 0$, $b + d + e + g \neq 0$, $b + d + f + g \neq 0$,
- $a + c \neq 0$, $a + d \neq 0$, $b + c \neq 0$, $b + d \neq 0$, $e + g \neq 0$, $f + g \neq 0$

Again a Sage calculation confirms that there are no possibilities for $a, \ldots, g$. For minimality we proceed as follows. If we delete the diagonal edge we obtain a minor from the first class of graphs in Figure 5. If we contract the diagonal edge we obtain a separable graph in which each block has a $\mathbb{Z}_6$-antivoltage. If we delete the a-labeled edge we obtain a minor from the second family of graphs in Figure 5. If we contract the a-labeled edge we get loops that can be deleted rather than contracted. If we delete the c- or e-labeled edge we obtain graphs having 2-bond reductions to the fat triangles $T_{2,3,3}$ and $T_{2,2,4}$; the first fat triangle is a minor from the first family of Figure 5 and the second is a minor from the second family. Contracting the c- or e-labeled edge yields a loop that may be deleted rather than contracted. □

Our main results in this section are Theorems 7.4 and 7.5. At the end of this section we will present a unified proof of the two results.

Theorem 7.4. *A non-separable 2-bond irreducible graph G has a $\mathbb{Z}_6$-antivoltage if and only if G is a minor of a graph from Figures 5 and 6.*

Theorem 7.5. *A graph G has a $\mathbb{Z}_6$-antivoltage if and only if G contains no minor from among the seven graphs in Proposition 7.3.*

Proposition 7.6. *If G is a 3-connected graph having a $\mathbb{Z}_6$-antivoltage, then G satisfies one of the following.*

(1) G has no W_4-minor and G is a subgraph of one of the graphs in Figure 9

(2) G has a W_4-minor, no W_5-minor and G is a subgraph of one of the graphs in Figure 10.

(3) G has a W_5-minor, no W_6-minor and G is a minor of the second graph of Figure 6.

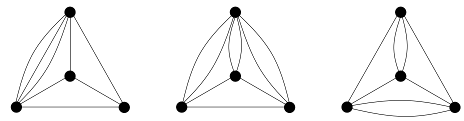

Figure 9: Loopless graphs with $\mathbb{Z}_6$-antivoltages that simplify to K_4.

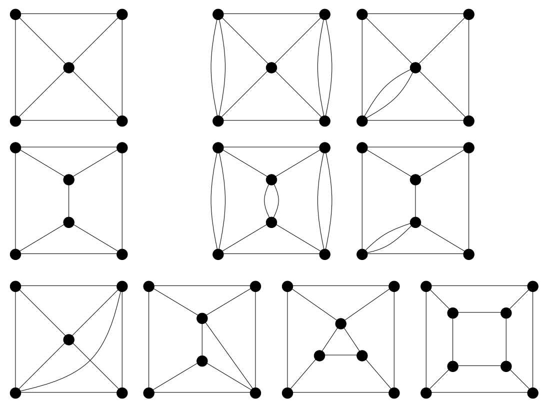

Figure 10: Constructing non-simple 3-connected graphs starting with a 4-Wheel.

Proof. Using Tutte's Wheel Theorem, the underlying simple graph of G may be constructed by single-edge additions and decontractions starting from a largest possible wheel minor and remaining 3-connected and simple at every single-edge operation. Since W_6 has a $7K_2$-minor, the beginning wheel is $K_4 \cong W_3$, W_4, or W_5; however, there is no single-edge operation applicable to K_4 that retains both 3-connectivity and simplicity. After constructing the underlying simple graph, edges may be doubled, tripled, etc. The three graphs shown in Figure 9 are minors of graphs from Figure 5 and so have $\mathbb{Z}_6$-antivoltages. The reader can check that adding any edges to these graphs yields minors from Proposition 7.3. Part (1) now follows.

Up to isomorphism there is only one single-edge addition to W_5 that is 3-connected and simple and this graph has a $7K_2$-minor and so no $\mathbb{Z}_6$-antivoltage. Doubling any edge of W_5 also yields a graph with a $7K_2$-minor and so no $\mathbb{Z}_6$-antivoltage. Up to isomorphism there are two 3-connected, single-edge decontractions of W_5, one that is planar and one non-planar. The non-planar one contains a $K_{3,3}$-minor and so no $\mathbb{Z}_6$-antivoltage. For the planar decontraction, call it P', first consider the graph on the right in Figure 6, call P. There are two edges e and f in P such that $P/\{e, f\} \cong W_5$ and $P/e \cong P/f \cong P'$. There is no single-edge extension of P/f that does not contain a single-edge extension of W_5 save for the extension that doubles edge e. This graph, however, contains the third graph of Figure 7 as a minor and so has no $\mathbb{Z}_6$-antivoltage. So now there is no single-edge extension of P or P/f having a $\mathbb{Z}_6$-antivoltage. The only planar, single-edge decontraction of P' is now P and so Part (3) follows.

For Part (2) we start with W_4. First we construct the planar, 3-connected, simple graphs which contain W_4 and not W_5 as a minor and also have $\mathbb{Z}_6$-antivoltages. (There are six graphs that satisfy these properties and they are labeled (1)–(6) in Figure 11.) The triangular prism $\overline{C}_6$ is the only 3-connected and planar single-edge decontraction of W_4. There is also one 3-connected, single-edge extension ($K_5 \backslash e$), see Figure 11. Both $K_5 \backslash e$ and $\overline{C}_6$ have $\mathbb{Z}_6$-antivoltages as previously shown. Adding an edge to $K_5 \backslash e$ yields K_5 which has no $\mathbb{Z}_6$-antivoltage and any planar decontraction of $K_5 \backslash e$ has a $\overline{C}_6$-minor. Up to isomorphism, there is one single-edge extension of $\overline{C}_6$ (call it Pr^+) and two double-edge extensions that preserve planarity. The single-edge extension has 6 faces and so has a $\mathbb{Z}_6$-antivoltage. The two double-edge extensions (see Figure 11) both have $7K_2$-minors and so have no $\mathbb{Z}_6$-antivoltages. There are two single-edge decontractions of Pr^+ (Figure 11), one has a W_5-minor and the other is a single-edge contraction of the 3-dimensional cube Q_3. The cube Q_3 has a 6 faces and so Q_3 and Q_3/e both have $\mathbb{Z}_6$-antivoltages. Up to isomorphism, there are four planar and simple single-edge extensions of Q_3/e (see the first box in the last row of Figure 11). The first three contain $7K_2$-minors and the fourth contains a W_5-minor. There are two planar and simple single-edge decontractions of Q_3/e (see the second box in the last row of Figure 11). One of these graphs is Q_3 and the other contains a W_5-minor. There are no 3-connected decontractions of Q_3 and up to isomorphism there is one simple single-edge extension of Q_3; however, the latter graph contains a $7K_2$-minor.

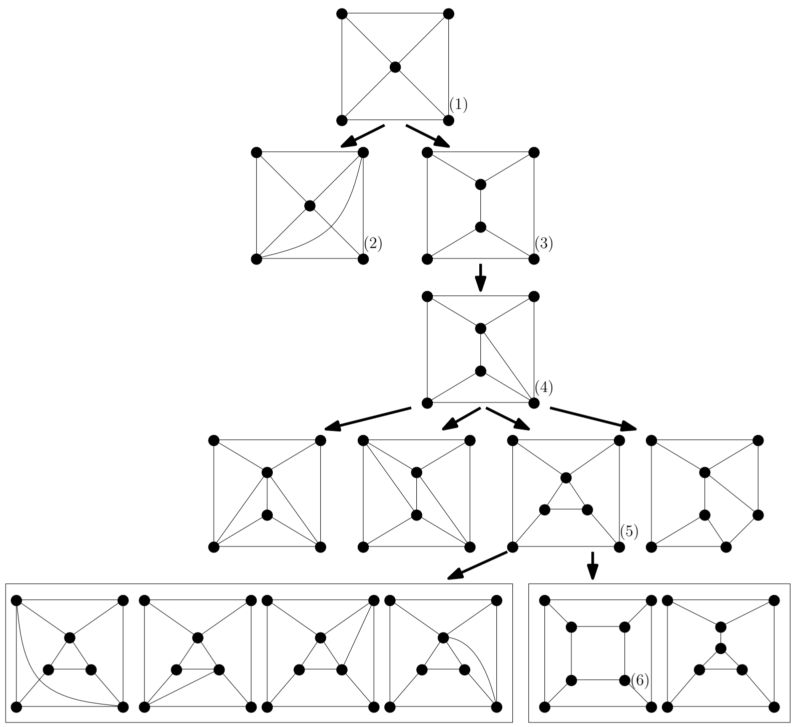

Figure 11: Constructing planar, 3-connected, simple graphs starting with W_4.

Using the 3-connected simple graphs (1)–(6) in Figure 11 we now determine the 3-connected, non-simple graphs with $\mathbb{Z}_6$-antivoltages. The three graphs in the first row of Figure 10 both have $\mathbb{Z}_6$-antivoltages: the first graph is W_4, the second graph is a minor of the second family of graphs from Figure 5 and the third is a minor of W_5. One can check that any other loopless non-simple graph with W_4 as its underlying simple graph has a $7K_2$-minor. In similar fashion the three graphs in the second row of Figure 10 are the only loopless non-simple graphs whose underlying graph is $\overline{C}_6$ that have $\mathbb{Z}_6$-antivoltages. Doubling any edge of one of Graphs (3)–(6) from Figure 11 yields a graph with a $7K_2$-minor. Thus the graphs of Figure 10 are the only 3-connected graphs that have a W_4-minor, no W_5-minor, and a $\mathbb{Z}_6$-antivoltage. $\square$

Proof of Theorems 7.4 and 7.5. If G contains one of the graphs of Proposition 7.3 as a minor, then it has no $\mathbb{Z}_6$-antivoltage. So suppose that G is a non-separable and 2-bond irreducible graphs that does not contain a minor from Proposition 7.3. We will show that G is a minor of a graph from Figures 5 and 6 which will complete our proof.

If G is 3-connected, then we have already determined the possibilities for G in Theorem 7.6. The reader may check that all of the graphs Figures 5 and 6. So say that G is not 3-connected or not simple and let T be the tree decomposition of G. In Case 1, say that T has two or more 3-connected terms. In Case 2, say that T has one 3-connected term. In Case 3, say that T has no 3-connected terms. Throughout the proof recall that any cycle term of T must have every edge summed into some other term because of 2-bond irreducibility.

Proposition 7.7 (Seymour [10, (3.1)]). *If G is a 3-connected and simple graph containing edges e and f, then there is a K_4-minor in G containing both e and f.*

Case 1 For this case, consider the 2-sum $K_4 \oplus_2 K_4^+$ where K_4^+ is K_4 with some edge doubled. Up to isomorphism, there are three possibilities for this 2-sum and all resulting graphs contain a $7K_2$-minor. Hence $K_4 \oplus_2 K_4^+$ has no $\mathbb{Z}_6$-antivoltage.

Let G_1 and G_2 be two 3-connected terms in T of maximum distance in T. If the distance between G_1 and G_2 is at least three, then the two vertices of T between G_1 and G_2 contain either a 3-connected term or an mK_2 term. Using Proposition 7.7 we get that G has $K_4 \oplus_2 3K_2 \oplus K_4$ as a minor which has $K_4 \oplus_2 K_4^+$ as a minor, a contradiction. Thus the distance from G_1 to G_2 is at most two.

If the distance between G_1 and G_2 is exactly two and the vertex between G_1 and G_2 in T is a 3-connected graph or an mK_2, then we again have $K_4 \oplus_2 K_4^+$ as a minor of G, a contradiction. Thus the vertex between G_1 and G_2 is a cycle-labeled vertex, call it C_m of length $m \geq 3$. If there is a 3-connected term summed into C_m that is more than just K_4, we again get a $K_4 \oplus_2 K_4^+$-minor, a contradiction. If a K_4-term summed into C_m is also summed into another graph, then we again get a $K_4 \oplus_2 K_4^+$-minor, a contradiction. Therefore C_m is summed with K_4-labeled terms and multilink terms. In order to avoid a $7K_2$-minor, any such multilink terms must consist of three or four links. We now get that G is a minor of the first graph in Figure 5 unless there is a multilink term that is summed into more terms in addition to C_m. No such term can be 3-connected because such a term would have distance at least three from G_1 in T. Thus this term is a cycle and this cycle is summed into two more multilink terms giving the graph of Figure 12 as a minor in G. This graph, however, contains the third graph of Figure 7 as a minor, a contradiction.

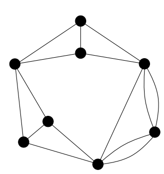

Figure 12: Graph for Case 1

If G_1 and G_2 are adjacent in T, then both G_1 and G_2 are K_4 or we get a $K_4 \oplus_2 K_4^+$-minor, a contradiction. If there are any other terms summed into either G_1 or G_2, then we again get a $K_4 \oplus_2 K_4^+$-minor. Thus G is a minor of the first graph in Figure 5.

Case 2 Let T be rooted at the 3-connected term, call it K. Note than 2-bond irreducibility does not allow a leaf term of T to be a cycle; hence all leaf terms are multilinks. Now the height of T cannot be three or more or we will have the third graph of Figure 7 as a minor, a contradiction. Thus each leaf in T is at level one or two. A leaf at level one acts to multiply a link of K and leaf at level two acts to replace an edge of K with a path of multilinks of length at least two.

If K has no W_4-minor, then $K \cong K_4$ and K has up to three children in the tree T by Proposition 7.6. If K has one child in the tree T, then by Proposition 7.6, G is K with one edge replaced by a path of multilinks of at most three edges each. Thus G is a minor of the first graph in Figure 5. If K has three children in T, then the edges of K indicated in 2-sums by T form a triad by Proposition

7.6. These three edges in the triad are replaced by paths of multilinks with exactly two edges each. Thus G is a minor of the second graph of Figure 5. If K has two children in T, then the two edges of K indicated in 2-sums are either incident or form a matching. In the latter case, we must avoid the second graph of Figure 7 as a minor and so one edge is doubled and the other is replaced with a path of multilinks of any length with exactly two edges in each multilink. This graph is a minor of the second graph of Figure 5. In the former case, we get a minor from a graph in the second graph in Figure 5.

If K has a W_5-minor, then by Theorem 7.6 $G = K$. If K has a W_4-minor but no W_5-minor, then $G = K$ is forced when K is one of the four graphs in the bottom row of Figure 10. So we now assume that K is W_4 or the triangular prism $\overline{C}_6$. If G is W_4, then by Proposition 7.6, K has at most two children in T. If K has two children, then the two edges indicated by 2-sums must be non-incident rim edges of W_4. Thus G is a minor of the second graph of Figure 5. If K has one child in T, then the edge of K indicated in a 2-sum is either a rim edge or a spoke of W_4. If it is a rim edge, then G is again a minor of the second graph of Figure 5. If it is a spoke, then this edge of K may be doubled but cannot be replaced by a path of digons because otherwise we would have the third graph of Figure 7 as a minor, a contradiction. Thus G is from Figure 10. Now if K is $\overline{C}_6$, then K has at most three children in T by Proposition 7.6. If K has two or three children in T, then the edges of K indicated in 2-sums are two or three edges joining the triangles of the prism. Thus G is a minor of the second graph of Figure 5. If K has one child in T, then the one edge of K indicated in a 2-sum is either an edge joining the two triangles (in which case G is again a minor of the second graph of Figure 5) or an edge or one triangle. As in the case where $K \cong W_4$, we get that G is K with this edge doubled which is from Figure 10.

Case 3 Again, the leaves in T must all be multilink-labeled vertices. If T consists of just one vertex, then G is an m-multilink for some $m \leq 6$ which is a minor of the second graph of Figure 5. Root T at a cycle-labeled vertex, say C_m which is a cycle of length $m \geq 3$. First assume that T has height 1. Since each edge of C_m is indicated in a 2-sum (by 2-bond irreducibility) and since G has no $7K_2$-minor, the children of C_m in T are multilinks of at most five edges. If all of the multilinks have at most four edges, then G is a minor of the first graph in Figure 5. If one of the multilinks has five edges, then the remaining multilinks have three edges each so as to avoid a $7K_2$-minor. Thus G is a minor of the second graph of Figure 5. Second assume that T has height at least 2. Since no leaf of T is a cycle, we then get that T has height at least three. If two of the children of C_m themselves have children, then G contains as a minor the graph obtained from the the tree decomposition shown on the left of Figure 13. This graph, however, contains the fourth graph of Figure 7 as a minor, a contradiction.

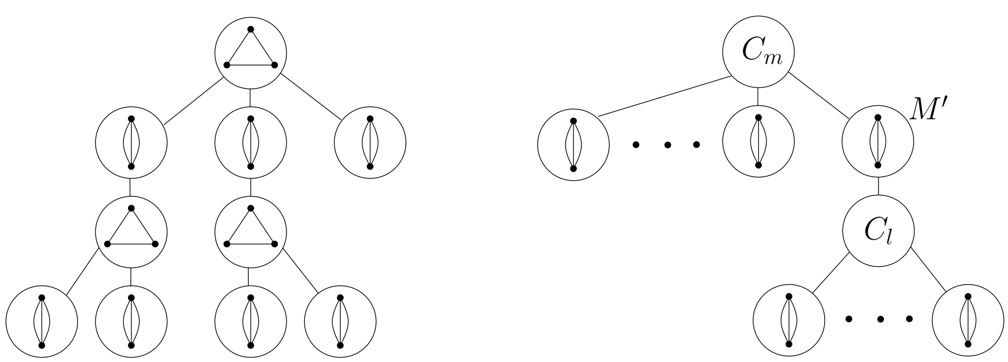

Figure 13: A tree decomposition for Case 3.

Hence at most one child of C_m in T, call it M', has a child itself. Now consider the tree decomposition on the right of Figure 13. The graph obtained from this decomposition is a minor from the second family in Figure 5. If one edge is added to any of the multilinks in this tree aside from M', then the resulting graph from this tree contains the fourth graph of Figure 7 as a minor. On the other hand, one edge may be added to M' and the resulting graph is still a minor from the second family in Figure 5. If two edges are added to M', then the resulting graph has a $7K_2$-minor. So if T has height three, then T contains the right-hand tree decomposition in Figure 13 with maybe a fourth edge added to M'. The only other option would be give M' more than one child which must then look like the child C_l. If M' had two children, then M' can have only 3 edges or else we would obtain a $7K_2$-minor. The resulting graph is again a minor from the second family in Figure 5. So M' has at most two children, which completes our proof. □

8 S_3-antivoltages

Here are two infinite families of graphs having antivoltages over the symmetric group S_3.

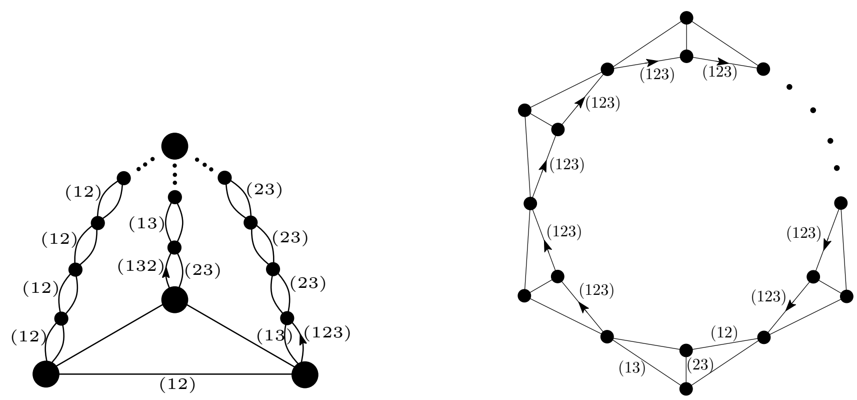

Figure 14: Two infinite families of graphs having S_3-antivoltages.

It is worth noting that S_3 is the first group admitting a nonplanar graph with an antivoltage; however, given Proposition 8.1, K_5 is the only nonplanar graph with an S_3-antivoltage that is also non-separable with minimum-degree-3.

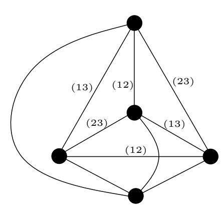

Figure 15: K_5 has an S_3-antivoltages.

In Proposition 8.1 we get eight graphs that are excluded minors for the class of graphs having an S_3-antivoltage. We will see later that this is the complete list of excluded minors.

Proposition 8.1. *The following are minor-minimal graphs with no S_3-antivoltage: $7K_2$, $K_{3,3}$, and the six graphs in Figure 16.*

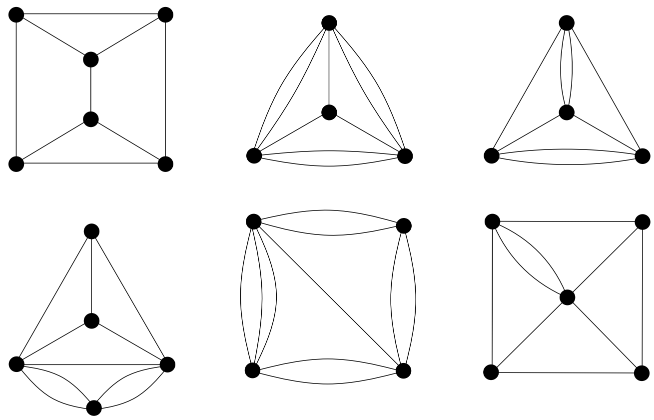

Figure 16: Five excluded minors for the class of graphs with an S_3-antivoltage.

Proof of Proposition 8.1. That $7K_2$ is an excluded minor is clear. Next we calculate all of the possible S_3-antivoltages of the graph W_4 up to switching, symmetry, and group automorphism. Up to switching an S_3-antivoltage on W_4 is as shown on the left in Figure 17.

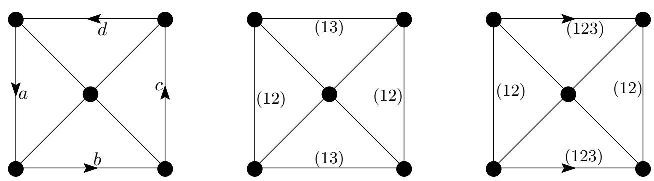

Figure 17: The only S_3-antivoltages for W_4.

Up to automorphism we may assume that $a \in \{(12), (123)\}$. We then run the following code in Sage. Run-time is much slower for permutation groups than for (mod) 6 arithmetic; this calculation took around 30 second on our computational setup.

```
k=0
for a in [G((1,2)),G((1,2,3))]:
    for b in G:
        if b!=G.identity():
            for c in G:
                if c!=G.identity():
                    for d in G:
                        if d!=G.identity():
                            result=(a*b)!=G.identity() and
(b*c)!=G.identity() and (c*d)!=G.identity() and (d*a)!=G.identity() and
(a*b*c)!=G.identity() and (b*c*d)!=G.identity() and (c*d*a)!=G.identity() and
(d*a*b)!=G.identity() and (a*b*c*d)!=G.identity()
                            if result:
                                k=k+1
                                print k,a,b,c,d
```

The computation yields seven results which reduce to the two S_3-antivoltages shown on the right in Figure 17. From this result, one can easily check that $K_{3,3}$, the prism $\overline{C}_6$, and the last graph of Figure

16 do not have S_3-antivoltages. For each edge $e \in K_{3,3}$, $K_{3,3}/e \cong W_4$ and $K_{3,3}\backslash e$ is a subdivision of K_4. So $K_{3,3}$ is minimal with respect to not having an S_3-antivoltage. If e is an edge of the prism $\overline{C}_6$, then $\overline{C}_6/e$ is either W_4 or a minor of the first family in Figure 14 and $\overline{C}_6\backslash e$ is either a subdivision of K_4 or has a 2-bond reduction to $4K_2$. Thus every proper minor of $\overline{C}_6$ has an S_3-antivoltage. If G is the last graph of Figure 16 and e is an edge of G, then G/e minus its loops either has a 2-bond reduction to $5K_2$ or $6K_2$ or is a minor of one of the families from Figure 16 and $G\backslash e$ is either W_4, a subdivision of a minor of the first family of Figure 16, or has a 2-bond reduction to $5K_2$. Thus G is minimal with respect to not having an S_3-antivoltage.

For the second graph in Figure 16, a hypothetical S_3-antivoltage up to switching is as shown in Figure 8 where we, again, up to automorphism assume that $a \in \{(12), (123)\}$. Verifying that there are no antivoltages with a Sage computation similar to the one above shows that there are no S_3-antivoltages. The computation took about twelve minutes in our computational setup. For minimality note the following: if e is one of the non-doubled edges, then G/e and $G\backslash e$ both have a 2-bond reduction that is a minor of the second family Figure 14; and if e is one of the doubled edges, then G/e minus its loop and $G\backslash e$ are both minors of the first family of graphs in Figure 14.

For the third graphs in Figure 16, we perform a similar Sage computations to show that there are no S_3-antivoltages. Minimality is checked by showing that every proper minor minus any loops is a minor of a graph in one of the two families of Figure 14.

For the fourth graph in Figure 16, a hypothetical S_3-antivoltage is shown in Figure 8. We run a similar Sage computation that confirms that there are no S_3-antivoltages. The computation took approximately 12 minutes on our setup. That this graph is minimal may be checked by the reader.

For the fifth graph G in Figure 16, let e be one edge in the parallel triple. A hypothetical antivoltage for $G\backslash e$ is as shown on the left in Figure 18. A Sage computation with $a \in \{(12), (123)\}$ yields only the S_3-antivoltages shown. Note that $a = (12)$ in each of these antivoltages. One can check that none of these antivoltages for $G\backslash e$ extend to G. Again the reader can check for minimality. $\square$

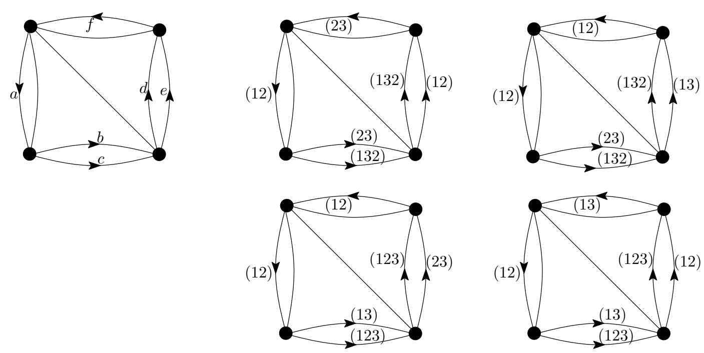

Figure 18: The only S_3-antivoltages for the graph shown.

Theorem 8.2. *If G is a 3-connected graph having an S_3-antivoltage, then G is a minor of K_5 or a minor of a graph from one of the families in Figure 14.*

Proof. Since W_5 contains the last graph of Figure 16 as a minor, we need only check 3-connected graphs coming from K_4 and W_4. Starting with W_4, we cannot decontract the degree-4 vertex because that results in either the triangular prism $\overline{C}_6$ or $K_{3,3}$, both of which are excluded minors. We cannot double a rim edge of W_4 because this graph contains the third graph of Figure 16 as a minor and we cannot double a spoke edge of W_4 because this gives us the last graph of Figure 16. Since K_5 has an S_3-antivoltage, we get that $K_5\backslash e$ has an S_3-antivoltage. In the graph $K_5\backslash e$ no edge may be doubled without creating either the third or last graph of Figure 16 as a minor; additionally, no edge may be decontracted without creating an $\overline{C}_6$- or $K_{3,3}$-minor.

If we start with K_4, then an edge may be tripled but not quadrupled. If one edge is tripled, then the graph is a minor of the second family of graphs in Figure 14. If an edge is tripled, then no other edge may be doubled without creating a $7K_2$-minor. If two edges of K_4 are doubled, then these edges cannot form a matching in K_4 or we get the third graph of Figure 16. Therefore at most three edges are doubled and these edges must be incident to the same vertex in K_4 to avoid the second graph of Figure 16. The resulting graph is K_5/e. □

Theorem 8.3. *A non-separable 2-bond irreducible graph G has an S_3-antivoltage if and only if G is a minor of a graph from Figure 14 or is a minor of K_5.*

Theorem 8.4. *A graph G has a S_3-antivoltage if and only if G contains no minor from among the eight graphs in Proposition 8.1.*

Proof of Theorems 8.3 and 8.4. We already know that the graphs of Proposition 8.1 are minor-minimal graphs having no S_3-antivoltages. Conversely, let G be a non-separable and 2-bond irreducible graph without a minor from Proposition 8.1. We will show that G is a minor of K_5 or a minor of a graph from one of the two families shown in Figure 14. We know that these graphs have S_3-antivoltages and so this will complete the proof of both theorems.

If G is 3-connected and simple, then we have already determined the possibilities for G in Theorem 8.2. So say that G is not 3-connected or not simple and let T be the tree decomposition of G. In Case 1, say that T has two or more 3-connected terms. In Case 2, say that G has one 3-connected term. In Case 3, say that T has no 3-connected terms. Throughout the proof recall that any cycle term of T must have every edge summed into some other term by 2-bond irreducibility.

Case 1 Denote the fourth graph of Figure 16 by F. In Propositions 7.3 and 8.1 we show that F and $7K_2$ are both excluded minors for the class of graphs with Γ-antivoltages for $\Gamma \in \{\mathbb{Z}_6, S_3\}$. In the proof of Case 1 for Theorem 7.4 we show that a non-separable and 2-bond irreducible graph G without a minor from $\{F, 7K_2\}$ is a minor of the second graph from Figure 14. Thus the same proof holds here.

Case 2 Let T be rooted at the 3-connected term, call it K. As in the proof of Case 2 of Theorem 7.4, T cannot have height three or more without creating an F-minor. Also, each leaf term in T is a multilink, and multilinks at level one acts to multiply a link of K and leaf at level two acts to replace an edge of K with a path of multilinks of at least two edges each.

If $K \cong K_5$, $K_5\backslash e$, or W_4, then the rest of T must be empty. So if K does not have a W_4-minor, then $K = K_4$. In order to avoid the second and third graphs of Figure 16, there can be at most three children of K in T and these children must be from among the three edges of K incident to a single vertex. Thus G is a minor of the first graph in Figure 15.

Case 3 The graph $7K_2$ and the fifth graph of Figure 16, call it F_1, are both excluded minors for the classes of graphs with S_3-antivoltages and of graphs with $\mathbb{Z}_6$-antivoltages. In the proof of Case 3 of Theorem 7.4 we show that if a graph G has no minor from $\{7K_2, F_1\}$, then G is a minor of a graph from one of the families in Figure 5. Going over the proof again we actually see that G is a minor of a graph from one of the families in Figure 14. Thus the same proof yields our desired result. □

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