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MODELING AND ANALYSIS OF LASER-BEAM-INDUCED CURRENT IMAGES IN SEMICONDUCTORS*

STAVROS BUSENBERG†, WEIFU FANG‡, AND KAZUFUMI ITO†

Abstract. A mathematical model is developed for the new nondestructive optical technique called laser-beam-induced currents (LBIC), which can be used to detect electrically active regions and defects in semiconductors. The wellposedness of the model equations is shown, and an approximate model that simplifies the numerical implementation of the identification problem is obtained. Some numerical results are presented to show that the approximate model preserves the significant features that the LBIC technique has been experimentally shown to possess.

Key words. semiconductors, laser-beam-induced currents (LBIC), nonlinear elliptic systems, identification problem

AMS(MOS) subject classifications. 35J60, 35Q20

1. Introduction. A nondestructive optical testing technique, called “laser-beam-induced currents (LBIC),” has been recently developed by Bajaj and his colleagues and has been applied to detect electrically active regions and defects in semiconductors (see [1]–[3]). The defects may be due to spatial inhomogeneities or crystal imperfections in the semiconductor material that cannot be directly detected. In the LBIC technique, a focused low-power laser beam is applied to a sample of semiconductor material, and the induced current between two remote contacts on the sample is measured as a function of the laser beam position (the LBIC image of the sample). The electrically active regions of the sample can be easily observed in such an image. In addition to being a nondestructive testing method, this technique also possesses other advantageous features, such as high resolution, and the need for only two remote contacts for measurements of the whole image (see [1]–[3] for details). However, it is not at all clear how the LBIC image is related to quantifiable material properties of the semiconductor, and only one study [7], based on a geophysical analogy, attempts to quantify this relationship.

In this paper, we take a first step in the study of the relationship between defects in a material and its LBIC image by investigating only those defects that cause some variation in the doping profile. The description of other defects may need a more detailed model than we propose in this paper. Nevertheless, our focus on this one kind of defect is justified, since nonuniformity in the doping profile is considered to be a major type of defect in applications (see [1], [2]).

We first derive the model equations to describe the LBIC technique in §2. The model consists of the stationary basic semiconductor equations (see, e.g., [9], [12], [13]) with an extra source term representing the applied laser beam power and a set of appropriately chosen boundary conditions. We assume the defects in the semiconductor that must be identified from the LBIC image are the inhomogeneities in the doping profile. The doping profile determines the built-in potential field in the semiconductor, which, in turn, drives the currents that are measured with the LBIC.
In this model, the LBIC measurements are represented by boundary integrals. Thus, the identification problem in our model involves the relation between boundary integrals and a function that enters as a term in a system of nonlinear partial differential equations (PDEs). Although the wellposedness of the stationary basic semiconductor equations has been analyzed by various authors (see, e.g., [8], [9]), their method cannot be directly applied to our equations. In §3 we extend their method to show the wellposedness of our model equations under appropriate assumptions. Based on the fact that the applied laser beam is of low power, we derive an approximate model in §4 for future study of the inverse problem. The wellposedness of this approximate model is shown, and estimates are given for its range of validity (see §4). Numerical examples (see §5) show that this approximate model preserves the distinguishing features that the LBIC technique has been experimentally shown to possess, such as the bimodal behavior of the induced current when the laser beam crosses a p-n-p junction.

2. Derivation of the model. In this section, we first briefly describe the general drift-diffusion model for semiconductor devices, then present our model for the LBIC problem.

The standard drift-diffusion mathematical model for semiconductor devices that leads to the so-called basic semiconductor equations was first established by Van Roosbroeck in the early 1950s [15]. Since then, it has been extensively studied via various methods, including abstract analysis, asymptotic analysis, and computer simulations (see, e.g., [4], [8], [9], [12], and references therein).

The model describes the flow of electrons and holes in a semiconductor by introducing the three dependent variables: the electrostatic potential $u$, the free conduction electron carrier density $n$, and the free hole carrier density $p$. The model equations are then derived from the Maxwell equations, continuity equations, and the drift-diffusion representation for electron and hole current densities (see, e.g., [9], [12], [13] for details). To present the details of the model equations, let $\Omega$ be a bounded, simply connected domain in $\mathbb{R}^d$ ($d = 1, 2$ or $3$) occupied by a semiconductor and denote the space variable by $x (x \in \Omega)$ and the time variable by $t \geq 0$. Then $u = u(t, x), n = n(t, x), p = p(t, x)$ satisfy the following equations:

$$\nabla \cdot (\epsilon \nabla u) = q(n - p - N),$$

$$q \frac{\partial n}{\partial t} = q(G - R) + \text{div } \vec{J}_n, \quad \text{with } \vec{J}_n = q(D_n \nabla n - \mu_n n \nabla u),$$

$$q \frac{\partial p}{\partial t} = q(G - R) - \text{div } \vec{J}_p, \quad \text{with } \vec{J}_p = q(-D_p \nabla p - \mu_p p \nabla u),$$

where

- $\epsilon$ is the electrical permitivity of the semiconductor material,
- $q$ is the elementary charge of a proton,
- $N$ is the net active impurity density, or impurity doping profile (the density of ionized donor impurities minus the density of ionized acceptor impurities),
\[ \begin{align*}
D_n, D_p & \quad \text{are the electron and hole diffusion coefficients,} \\
\mu_n, \mu_p & \quad \text{are the electron and hole mobilities,} \\
G & \quad \text{is the electron-hole generation rate,} \\
R & \quad \text{is the electron-hole recombination rate,} \\
\overrightarrow{J}_n, \overrightarrow{J}_p & \quad \text{are the electron and hole currents.}
\end{align*} \]

We assume that the Einstein relations, below, hold:

\[ \begin{align*}
D_n &= U_T \mu_n, \quad D_p &= U_T \mu_p \quad \text{with} \quad U_T = \frac{k_B T}{q},
\end{align*} \]

where \( k_B \) is the Boltzmann constant and \( T \) is the absolute temperature.

In general, \( \mu_n \) and \( \mu_p \) are field dependent, that is, \( \mu_n = \mu_n(x, \nabla u) \) and \( \mu_p = \mu_p(x, \nabla u) \). Different simplifying assumptions are often made, such as setting \( \mu_n = \mu_n(x) \), \( \mu_p = \mu_p(x) \) or even just equating them to constants.

The generation-recombination rate term models the energy transition process in the material. Recombination occurs when a conduction electron becomes a valence electron and neutralizes a hole, and generation occurs when a valence electron becomes a conduction electron and leaves a hole. Generation requires energy, while recombination releases energy. The most basic generation-recombination process is described by the Shockley–Read–Hall model (see, e.g., [9], [12], [13]), below:

\[ \begin{align*}
G - R &= - \frac{np - n_i^2}{\tau_n(n + n_i) + \tau_p(p + n_i)},
\end{align*} \]

where \( \tau_n = \tau_n(x) \), \( \tau_p = \tau_p(x) \) are the electron and hole lifetimes, and \( n_i \) (constant) is the intrinsic carrier density. Note that, at thermal equilibrium, there holds \( np = n_i^2 \). Thus, roughly speaking, the Shockley–Read–Hall model simply assumes that generation-recombination is governed by the deviation of the system from thermal equilibrium. We remark that this is the case only when no external energy source is present. There are also other models for the generation-recombination rate; for instance, the Anger model and the avalanche model (see, e.g., [9], [12], [13]).

Next, we consider the boundary conditions for (2.1). Usually, the boundary \( \partial \Omega \) is divided into two disjoint parts, denoted by \( \Sigma_N \) and \( \Sigma_D \). \( \Sigma_N \) represents the insulated or artificially cut-off boundaries, while \( \Sigma_D \) represents the ohmic contact boundaries where the contact resistance is assumed to be negligible. Hence, on \( \Sigma_N \) the following homogeneous Neumann boundary conditions hold for \( u, n, \) and \( p \):

\[ \begin{align*}
\frac{\partial u}{\partial \nu} &= \frac{\partial n}{\partial \nu} = \frac{\partial p}{\partial \nu} = 0 \quad \text{on} \quad \Sigma_N
\end{align*} \]

(\( \nu \) is the normal direction on \( \partial \Omega \)), and on \( \Sigma_D \) we have

\[ \begin{align*}
u &= U_T \ln \frac{N(x) + \sqrt{N^2(x) + 4n_i^2}}{2n_i} + U_0(t, x),
\end{align*} \]

\[ \begin{align*}
&= \frac{N(x) + \sqrt{N^2(x) + 4n_i^2}}{2},
\end{align*} \]

\[ \begin{align*}
&= \frac{N(x) + \sqrt{N^2(x) + 4n_i^2}}{2}.
\end{align*} \]
where $U_0(t, x)$ represents the externally applied potentials on $\Sigma_D$. The Dirichlet conditions for $n$ and $p$ are determined by assuming thermal equilibrium and vanishing space charge to hold on the contacts, and the condition for $u$ is chosen so that the system is in thermal equilibrium if all externally applied potentials are zero. See [9] and [12] for more details and other types of boundary conditions.

Thus, the standard drift-diffusion semiconductor equations are given in (2.1) with boundary conditions (2.4), (2.5). Note that initial conditions for the dependent variables must be prescribed to complete the description of this model. In the steady state, the dependent variables and externally applied potentials are time independent, and setting $\partial n/\partial t = 0$ and $\partial p/\partial t = 0$ in (2.1) with boundary conditions (2.4), (2.5) leads to the stationary version of the basic semiconductor equations.

In thermal equilibrium, we have the equilibrium densities $n_e = n_i e^{ue/UT}$ and $p_e = n_i e^{-ue/UT}$, where the thermal equilibrium potential $ue$ satisfies

$$\nabla \cdot (\varepsilon \nabla ue) = q(n_i e^{ue/UT} - n_i e^{-ue/UT} - N) \quad \text{in } \Omega,$$

$$\frac{\partial u_e}{\partial u} = 0 \text{ on } \Sigma_N \quad \text{and} \quad u_e = UT \ln \frac{N + \sqrt{N^2 + 4n_i^2}}{2n_i} \text{ on } \Sigma_D.$$

In the following, we turn to the modeling for LBIC technique. The schematic configuration of the technique is shown in Fig. 2.1. The focused laser beam is applied at one position, and the induced steady current flowing between the two ohmic contacts is measured. Such measurements are repeated for different position of the laser beam to obtain the LBIC image of the sample; that is, the measured current as a function of the laser beam position.

For this model, we assume that we have a stationary system for each measurement. Hence, $u = u(x)$, $n = n(x)$, and $p = p(x)$.

First, consider the presence of the focused laser beam in the material. In the localized area where the beam is directed, the laser supplies energy to generate conduction electrons, hence, also equal number of holes (electron-hole pairs); thus charges can be separated, and a current is induced to flow if there is an electric field gradient between the injection area and the contacts (see [1], [2]). On the other hand, some of these generated pairs will recombine and induce no current if there is no electric field gradient between. Therefore, the presence of the laser implies an externally applied...
generation source in a localized area. Hence, we modify the generation-recombination rate in (2.1) as follows:

\[ G - R = g(x) - \frac{np - n_i^2}{r_n(n + n_i) + r_p(p + n_i)}, \]

where we use the Shockley–Read–Hall generation-recombination term (2.3) in this model, and \( g(x) \) represents the generation rate due to the applied laser beam. We assume that \( g \geq 0 \), that \( \text{Maxag} \) is proportional to the applied laser intensity, and that the support of \( g \) is the focused area of the laser beam. In a typical application of the LBIC technique, the dimension of the domain \( d = 2 \), and

\[ g(x) = \alpha \chi_{B_S(x_0)}(x), \]

where the laser beam intensity \( \alpha > 0 \), \( \chi_S \) is the characteristic function of \( S \) and \( B_S(x_0) \) is the disk of radius \( \delta \) centered at \( x_0 \) with \( \delta \) small. Typical radii \( \delta \) of the laser beam spots in the laboratory are 1.5 microns on square samples with sides ranging from 5 mm to 1 cm [2].

The doping profile \( N \) measures the electrically active deposits of impurities, and it determines the performance of the semiconductor device. Any local variation in the doping profile will cause a nonzero electric field gradient locally. (See [11] for an example of a \( p-n \) junction in one dimension.) The LBIC technique is used to detect electrically active defects in a semiconductor. In our model, we use the doping profile as a measure of the electrically active defects or inhomogeneities. A typical example of such defects is a small \( p \)-type area embedded in a uniformly doped \( n \)-type semiconductor (see [1], [2]). In such a case, the corresponding doping profile takes a negative value in the \( p \)-type area and a positive value in the rest of the \( n \)-type area. A perfect or homogeneous semiconductor has a constant doping profile.

We assume that the two ohmic contacts in the LBIC technique are made on the boundary of \( \Omega \). More precisely, we assume that \( \Gamma_1 \cup \Gamma_2 = \Sigma_D \) and \( \Gamma_1 \cap \Gamma_2 = \emptyset \), where \( \Gamma_1 \) and \( \Gamma_2 \) are the locations of the two contacts. In the LBIC technique, no external potential is applied on the contacts. Hence the boundary conditions for our LBIC model are (2.4) and (2.5) with \( U_0 \equiv 0 \). We recall that a fraction of the separating charges, due to the applied laser beam and the existence of a local nonzero electric field gradient, redistributes by flowing through the contacts (the external circuit), and it is this component that produces the LBIC signal. Therefore, for each applied laser beam location (fixed \( g \)), the LBIC signal can be represented in terms of \( u, n, \) and \( p \) as follows:

\[ I = \int_{\Gamma_1} (\vec{J}_n + \vec{J}_p) \cdot \nu \, ds \]

\[ = q \int_{\Gamma_1} [\mu_n(U_T \nabla n - n \nabla u) - \mu_p(U_T \nabla p + p \nabla u)] \cdot \nu \, ds, \]

where the Einstein relations (2.2) are used. We remark that in the steady-state case that we are considering, the total current through both \( \Gamma_1 \) and \( \Gamma_2 \) is zero, i.e.,

\[ \int_{\Gamma_1 \cup \Gamma_2} (\vec{J}_n + \vec{J}_p) \cdot \nu \, ds = 0. \]

Hence, the total current flowing out through \( \Gamma_1 \) is equal to the total current flowing in through \( \Gamma_2 \).
Therefore, our model for the LBIC technique consists of the stationary version of (2.1) with \( G - R \) given by (2.7), the boundary conditions (2.4) and (2.5) with \( U_0 = 0 \), and the LBIC measurement given by (2.9). Note that \( I \) is dependent on \( g \); i.e., \( I \) is a functional of \( g \). Hence, in this model, the relationship between the defect of a semiconductor material and its LBIC image can be described as an inverse problem: Recover the doping profile \( N(x) \) from the LBIC measurement \( I \) for some \( g \) from a certain class of functions defined in \( \Omega \).

For convenience of analysis, we introduce the following transformations, which are similar to the ones used by Jerome [8] \((-v \text{ and } w \text{ are the so-called quasi-Fermi potentials})

\[
(2.10) \quad n = n_i e^{v+u/U_T}, \quad p = n_i e^{w-u/U_T},
\]

and we perform the rescalings

\[
\tilde{u} = \frac{u}{U_T}, \quad \tilde{N} = \frac{N}{n_i}, \quad \tilde{\varepsilon} = \frac{U_T}{n_i q} \varepsilon, \quad \tilde{\mu}_n = U_T \mu_n, \quad \tilde{\mu}_p = U_T \mu_p \quad \text{and} \quad \tilde{g} = \frac{g}{n_i}, \quad \tilde{I} = \frac{I}{q n_i}.
\]

Then our model equations and boundary conditions can be rewritten in terms of the new dependent variables \((\tilde{u}, v, w)\) and the scaled parameters. For convenience, we use the same symbols for the new variables and scaled parameters and obtain the following equations:

\[
(2.11) \quad \nabla \cdot (\varepsilon \nabla u) = e^{v+u} - e^{w-u} - N \quad \text{in } \Omega,
\]

\[
(2.12) \quad \nabla \cdot (\mu_n e^{v+u} \nabla v) - Q(u, v, w)(e^{v+w} - 1) + g = 0 \quad \text{in } \Omega,
\]

\[
(2.13) \quad \nabla \cdot (\mu_p e^{w-u} \nabla w) - Q(u, v, w)(e^{v+w} - 1) + g = 0 \quad \text{in } \Omega,
\]

with

\[
(2.14) \quad \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = \frac{\partial w}{\partial \nu} = 0 \quad \text{on } \Sigma_N,
\]

\[
(2.15) \quad u = \ln \frac{N(x) + \sqrt{N^2(x) + 4}}{2} \quad \text{on } \Sigma_D,
\]

and

\[
(2.16) \quad v = w = 0 \quad \text{on } \Sigma_D,
\]

where \( \Sigma_D = \Gamma_1 \cup \Gamma_2 \) and \( \Gamma_1 \cap \Gamma_2 = \emptyset \) and

\[
(2.17) \quad Q(u, v, w) = [\tau_n (e^{v+u} + 1) + \tau_p (e^{w-u} + 1)]^{-1}
\]

from the Shockley–Read–Hall model (2.3).

The LBIC measurement becomes

\[
(2.18) \quad I(g) = \int_{\Gamma_1} (\mu_n e^{v+u} \nabla v - \mu_p e^{w-u} \nabla w) \cdot \nu \, ds.
\]
Note that, in these new variables, the thermal equilibrium of the system is 
\((u, v, w) = (u_e, 0, 0)\) with \(u_e\) being the solution to
\[
\nabla \cdot (\varepsilon \nabla u) = e^u - e^{-u} - N \quad \text{in } \Omega,
\]
\[
\frac{\partial u}{\partial \nu} = 0 \quad \text{on } \Sigma_N \quad \text{and} \quad u = \ln \frac{N + \sqrt{N^2 + 4}}{2} \quad \text{on } \Sigma_D.
\]

In summary, our model for the LBIC technique is formulated as follows: The LBIC image consists of the current measurement given by (2.18), where \((u, v, w)\) is the solution to the nonlinear system of elliptic problems (2.11)-(2.16) with each given \(g\) from a certain class of functions, and this image is used to determine the doping profile \(N(x)\) that represents the structure of electrically active defects in the semiconductor. That is, the relation between the LBIC image and the defects is described by the relation between \(N\) and \(I(g)\) with \(g\) chosen from a class of functions.

Finally, we note that in this model we assume that \(N\) is the only function to be identified from the LBIC image. Although this is the major source of defects in many applications, other sources of defects may be also important, and they may be appropriately described by other parameters in this model, such as the lifetimes for electrons and holes \((\tau_n\) and \(\tau_p)\), the mobilities for electrons and holes \((\mu_n\) and \(\mu_p)\). These problems need further investigation in semiconductor modeling, and, consequently, they may lead to other interesting identification problems.

3. Wellposedness of the model equations. Although the model equations (2.11)-(2.16) differ from those studied by [8] only by the addition of the generation term \(g\), the method used by Jerome to establish the existence of solutions cannot be easily extended to cover our model equations. Since the LBIC technique is a direct consequence of this source term, which represents the laser input energy, it is important to analyze its effects in detail. Consequently, in this section, we show the existence of solutions even under more general assumptions on the data and parameters than we proposed in the previous section, but with the restriction that \(g\) is small enough. We also establish under appropriate conditions that the solution is unique.

We first state our assumptions.

Assumption 1. \(\Omega\) is an open bounded domain in \(\mathbb{R}^d (d \leq 3)\) of class \(C^{0,1}\).

Assumption 2. \(\partial \Omega = \Sigma_D \cup \Sigma_N\) with \(\Sigma_D\) being relatively closed.

Assumption 3. The boundary data \((\tilde{u}, \tilde{v}, \tilde{w}) \in H^1(\Omega)^3\) are Lipschitz continuous in a neighborhood of \(\Sigma_D\). The doping profile \(N \in L^\infty(\Omega)\) and the generation rate \(g \in L^2(\Omega)\) with \(g \geq 0\) almost everywhere in \(\Omega\).

Assumption 4. The coefficient \(Q(x, u, v, w)\) in the generation-recombination rate \(Q(x, \cdot): \mathbb{R}^3 \to \mathbb{R}\) is locally Lipschitz continuous (uniformly in \(x \in \Omega\)) and \(Q \geq 0\).

Assumption 5. The permittivity \(\varepsilon = \varepsilon(x) \in L^\infty(\Omega)\) is bounded below by a positive constant \(\varepsilon_0\). The mobilities \(\mu_n = \mu_n(x, \nabla u)\) and \(\mu_p = \mu_p(x, \nabla u) : \Omega \times \mathbb{R}^d \to \mathbb{R}\) satisfy (i) \(0 < \mu \leq \mu_n, \mu_p \leq \mu\) in \(\Omega \times \mathbb{R}^d\) for some positive constants \(\mu\) and (ii) \(|\mu_n(x, y) - \mu_n(x, \tilde{y})| + |\mu_p(x, y) - \mu_p(x, \tilde{y})| \leq L|y - \tilde{y}|\) for all \(x \in \Omega, y, \tilde{y} \in \mathbb{R}^d\).

Note that Assumption 4 contains the Shockley–Read–Hall model (2.17) as a special case.

We denote the usual \(L^2(\Omega)\) inner product by \(\langle \cdot, \cdot \rangle\), and \(L^p(\Omega)\)-norm and \(H^1(\Omega)\)-norm by \(|\cdot|_p, |\cdot|_{H^1(\Omega)}\), respectively.

The variational formulation of (2.11)-(2.16) is
\[
\langle \varepsilon \nabla u, \nabla \zeta \rangle + \langle e^{v+u} - e^{w-u} - N, \zeta \rangle = 0,
\]
\begin{align}
(3.1b) \quad & \langle \mu_n e^{v+u} \nabla v, \nabla \phi \rangle + \langle Q(e^{v+w} - 1) - g, \phi \rangle = 0, \\
(3.1c) \quad & \langle \mu_p e^{w-u} \nabla w, \nabla \psi \rangle + \langle Q(e^{v+w} - 1) - g, \psi \rangle = 0
\end{align}

for all \( (\zeta, \phi, \psi) \in Y \) and \( (u, v, w) \) and \( (\mu, \nu, \omega) \).

Existence of solutions in \( H^1 \cap L^\infty \) to (3.1) has been obtained when the generation rate \( g = 0 \) in [8], [9]. Our existence result here includes the case when \( g \neq 0, g \in L^2(\Omega) \) and is established using the Schauder fixed-point theory.

Given a positive constant \( m \), consider the following closed convex set \( K \) of \( L^2(\Omega)^2 \):

\[
K = \{ (v, w) \in L^2(\Omega)^2 : |v(x)| \leq m, |w(x)| \leq m \text{ a.e. in} \ \Omega \}.
\]

Define the solution map \( T \) on \( K \) as follows. For \( (\bar{v}, \bar{w}) \in K \), let \( u \in Y + \bar{u} \) satisfy

\begin{align}
(3.2) \quad & \langle e \nabla u, \nabla \zeta \rangle + \langle e^{u+u} - e^{\bar{w}-u} - N, \zeta \rangle = 0 \\
\end{align}

for all \( \zeta \in Y \), and then let \( T(\bar{v}, \bar{w}) = (v, w) \) be the solution to

\begin{align}
(3.3a) \quad & \langle \mu_n e^{v+u} \nabla v, \nabla \phi \rangle + \langle Q(e^{v+w} - 1) - g, \phi \rangle = 0, \\
(3.3b) \quad & \langle \mu_p e^{w-u} \nabla w, \nabla \psi \rangle + \langle Q(e^{v+w} - 1) - g, \psi \rangle = 0
\end{align}

for all \( \phi, \psi \in Y \), where \( (v, w) \in Y^2 + (\bar{v}, \bar{w}), \mu_n = \mu_n(x, \nabla u), \mu_p = \mu_p(x, \nabla u), \) and \( Q = Q(x, u, \bar{v}, \bar{w}) \). From [8, Lemma 3.1], we have the following lemma.

**Lemma 3.1.** Given \( (\bar{v}, \bar{w}) \in K \), there exists a unique solution \( u \in Y + \bar{u} \) to (3.2) satisfying \( u_{\text{min}} \leq u \leq u_{\text{max}} \), where \( u_{\text{max}} = \max\{\sup_{\Sigma_D} \bar{u}, \delta\} \), \( u_{\text{min}} = \min\{\inf_{\Sigma_D} \bar{u}, \gamma\} \) and \( \delta \) and \( \gamma \) are determined by

\begin{align}
(3.4a) \quad & e^{\delta-m} - e^{-(\delta-m)} = \sup_{\Omega} N, \\
(3.4b) \quad & e^{\gamma+m} - e^{-(\gamma+m)} = \inf_{\Omega} N.
\end{align}

From this lemma, we can easily see that

\begin{align}
(3.5) \quad & -U - m \leq \frac{e^{U+m}}{2} < 0,
\end{align}

where

\[
U = \sup_{\Sigma_D} |\bar{u}| + \ln \frac{N_0 + \sqrt{N_0^2 + 4}}{2} \quad \text{with} \quad N_0 = \sup_{\Omega} |N|.
\]

Note that \( |\nabla \phi|_{2, \Omega} \) defines an equivalent norm on \( H^1(\Omega) \) for \( \phi \in Y \), and the continuous embedding from \( H^1(\Omega) \) into \( L^6(\Omega) \) gives

\begin{align}
(3.6) \quad & |\phi|_{6, \Omega} \leq c|\nabla \phi|_{2, \Omega}
\end{align}

for all \( \phi \in Y \). Here the constant \( c > 0 \) depends only on \( \Omega \) and \( \Sigma_D \).

**Lemma 3.2.** Let \( u \) be as in Lemma 3.1. Then there exists a unique solution \( (v, w) \in Y^2 + (\bar{v}, \bar{w}) \) to (3.3) satisfying

\begin{align}
(3.7) \quad & |v(x)|, |w(x)| \leq \bar{m} + \frac{4c^2|\Omega|^{1/6}}{\mu} e^{U+2m} |g|_{2, \Omega},
\end{align}

where

\[
\bar{m} = \max_{\Sigma_D} |\bar{u}| + \ln \frac{N_0 + \sqrt{N_0^2 + 4}}{2} \quad \text{with} \quad N_0 = \sup_{\Omega} |N|.
\]
where $\bar{m} = \max\{\sup_{\partial D} |\bar{v}|, \sup_{\partial D} |\bar{w}|\}$, and $c$ is the embedding constant given by (3.6).

Proof. For existence of $(v, w) \in Y^2 + (\bar{v}, \bar{w})$ to (3.3), we consider the minimization problem

$$
\text{Min} \Phi(v, w) = \frac{1}{2} \langle \mu_n e^{u+u} \nabla v, \nabla v \rangle + \frac{1}{2} \langle \mu_p e^{\bar{v}+u} \nabla w, \nabla w \rangle + \langle \tilde{Q}, f(v + w) \rangle - \langle g, v + w \rangle
$$

over $(v, w) \in Y^2 + (\bar{v}, \bar{w})$, where $f(s) = \int_0^s (e^s - 1) ds = e^s - s - 1$ is convex. Note that $\mu_n e^{u+u}$ and $\mu_p e^{\bar{v}+u}$ are bounded from below by $\mu e^{-2m-U} > 0$ from (3.5) and $0 \leq \tilde{Q} \leq \bar{Q}$ for some $\bar{Q} > 0$. Hence $\Phi$ is convex, coercive and also lower-semicontinuous. Therefore there is a unique minimizing element $(v, w) \in Y^2 + (\bar{v}, \bar{w})$ of $\Phi(v, w)$ (see, e.g., [14] or [8]), and it satisfies (3.3).

Next, we show the bounds in (3.7) for this solution. Let

$$
v_k = (v - k)^+ \quad \text{and} \quad w_k = (-k - w)^+,
$$

where we use the notation $t^+ = \max\{t, 0\}$ for $t \in \mathbb{R}^1$, then $v_k$ and $w_k$ $\in Y$ when $k \geq \bar{m}$. Setting $\phi = v_k$ and $\psi = w_k$ in (3.3) leads to

$$
\langle \mu_n e^{u+u} \nabla v_k, \nabla v_k \rangle + \langle \mu_p e^{\bar{v}+u} \nabla w_k, \nabla w_k \rangle + \langle \bar{Q}(e^{u+w} - 1), v_k - w_k \rangle = \langle g, v_k - w_k \rangle.
$$

Noting that $(e^s - 1)s \geq 0$ for $s \in \mathbb{R}^1$ and $Q \geq 0$, we can easily show that $\langle \bar{Q}(e^{u+w} - 1), v_k - w_k \rangle \geq 0$. Hence

$$
(3.8) \quad \mu e^{-2m-U}(|\nabla v_k|^2 + |\nabla w_k|^2) \leq \langle g, v_k \rangle
$$

since $g \geq 0$. That is,

$$
\mu e^{-2m-U}|\nabla v_k|^2 \leq |g|_{2,\Omega} \cdot |v_k|_{2,\Omega}.
$$

If we set $\Omega_k = \{x \in \Omega : v > k\}$, then we also have

$$
|v_k|_{2,\Omega} = |v_k|_{2,\Omega_k} \leq |\Omega_k|^{1/3} |v_k|_{6,\Omega} \leq c |\Omega_k|^{1/3} |\nabla v_k|_{2,\Omega},
$$

where the last estimate is by (3.6). Therefore

$$
\mu e^{-2m-U} |\nabla v_k|^2 \leq c |\Omega_k|^{1/3} |g|_{2} |\nabla v_k|_{2};
$$

that is,

$$
|v_k|_6 \leq c |\nabla v_k|_2 \leq c e^{2m+U} |g|_2 |\Omega_k|^{1/3}.
$$

On the other hand, for $h > k \geq \bar{m}$, we have

$$
|v_k|^6_6 = \int_{\Omega_k} |v - k|^6 dx \geq \int_{\Omega_h} |v - k|^6 dx \geq |\Omega_h|(h - k)^6.
$$

Hence

$$
|\Omega_h| \leq C^6 (h - k)^{-6} |\Omega_k|^2,
$$

where $C = (c^2/\mu) e^{2m+U} |g|_2$. By applying Lemma 2.9 in [14] to $|\Omega_h|$ as a nonincreasing function in $h$, we conclude that

$$
|\Omega_h| = 0 \quad \text{for} \quad h = \bar{m} + 4C|\Omega|^{1/6}.
$$
Hence

\[ v(x) \leq h = \bar{m} + \frac{4c^2}{\mu} |\Omega|^{1/6}e^{2m+U}|g|_2, \]

and, from (3.8) with \( k = h \), we have

\[ w(x) \geq -h = - \left( \bar{m} + \frac{4c^2}{\mu} |\Omega|^{1/6}e^{2m+U}|g|_2 \right). \]

The lower bound for \( v \) and upper bound for \( w \) can be established similarly by choosing \( (-k v)^+ \) and \( (w k)^+ \) in (3.3). Thus (3.7) is proved. \( \square \)

Let

\[ \tau = \frac{\mu}{8c^2 |\Omega|^{1/6}e^{-U-2\bar{m}-1}}. \]

Then from (3.7) it is elementary to show the following lemma.

**Lemma 3.3.** For each \( g \in L^2(\Omega) \) with \( |g|_{2,\Omega} \leq \tau \), \( T \) defined by (3.2), (3.3) maps \( K \) into \( K \) when \( m \) is chosen as

\[ m = \begin{cases} \bar{m} + \frac{1}{2} + \frac{1}{2} \ln \frac{\tau}{|g|_{2,\Omega}} & \text{if } |g|_{2,\Omega} > 0, \\ \bar{m} & \text{if } |g|_{2,\Omega} = 0. \end{cases} \]

To apply the Schauder fixed-point theorem to \( T \), we also need the next lemma.

**Lemma 3.4.** The mapping \( T \) is continuous from \( K \subset L^2(\Omega)^2 \) into \( H^1(\Omega)^2 \).

**Proof.** The same arguments in [8, Lemmas 4.2-4.6] can be applied to our case, except that the identity in [8, Lemma 4.5 and eq. (4.3)] must be replaced by

\[ \langle \mu^1 e^{v_1+u_1} \nabla(v_1-v_2), \nabla(v_1-v_2) \rangle + \langle \mu^2 e^{w_1+u_1} \nabla(w_1-w_2), \nabla(w_1-w_2) \rangle + \langle (\mu^1 - \mu^2) e^{v_1+u_1} \nabla(v_1-v_2), \nabla(w_1-w_2) \rangle + \langle (\mu^2 - \mu^1) e^{w_1+u_1} \nabla(w_1-w_2), \nabla(v_1-v_2) \rangle \]

\[ = - \langle \mu_2 e^{v_1+u_1} - e^{w_2+u_2}, v_1-w_1 + v_2 - w_2 \rangle, \]

where the fact that the right-hand side, above, is a nonpositive quantity. Here \( \mu^i_n = \mu_n(x, \nabla u_i), \mu^p_n = \mu_p(x, \nabla u_i) \), and \( \bar{Q}_i = \bar{Q}(x, u_i, \bar{v}_i, \bar{w}_i) \) for \( i = 1, 2 \). \( \square \)

Noting that the embedding from \( H^1(\Omega) \) into \( L^2(\Omega) \) is compact, we conclude our existence result from the above lemmas and the Schauder fixed-point theorem (see, e.g., [5]), which we state in the following theorem.

**Theorem 3.5.** Under Assumptions 1-5, for \( g \in L^2(\Omega) \) with \( |g|_{2,\Omega} \leq \tau \), there exists a solution \( (u, v, w) \in (H^1(\Omega) \cap L^\infty(\Omega))^3 \) to (3.1), and it is a fixed point of the mapping \( T \) defined by (3.2), (3.3).

**Remark 3.6.** When \( g = 0, m = \bar{m}, \) and, from Lemma 3.1, we recover the maximum principle for the nonlinear elliptic system (3.1) that is described in [8] and [9]. Moreover, in our proof, the monotonicity assumption on \( P = Q(e^{v+w} - 1) \) by [8] is not necessary.

**Remark 3.7.** For \( H^1 \)-weak solution of (3.1), the validity of the expression \( I(g) \) given by (2.18) must be justified. First, note that, when the solution is in \( H^2 \),

\[ \bar{J} = \mu_n e^{v+u} \nabla v - \mu_p e^{w-u} \nabla w \]
has a trace on $\partial \Omega$ in $H^{1/2}(\partial \Omega)$ satisfying $\mathbf{J} \cdot \mathbf{\nu} = 0$ on $\Sigma_N$. Hence (2.18) can be expressed as

$$I(g) = \int_{\partial \Omega} \eta \cdot (\mathbf{J} \cdot \mathbf{\nu}) \, ds$$

for all $\eta \in X = \{ \eta \in H^{1/2}(\partial \Omega) : \eta = 1$ on $\Gamma_1$ and $\eta = 0$ on $\Gamma_2 \}$. $X$ is not empty because $\Gamma_1$, $\Gamma_2$ are closed and $\Gamma_1 \cap \Gamma_2 = \emptyset$. For the case of $H^1$-weak solution, since $\mathbf{J}$ is divergent-free in $\Omega$, we have $\mathbf{J} \cdot \mathbf{\nu} \in H^{-1/2}(\partial \Omega)$ (see, e.g., [6]). From Green’s formula and (3.1b), (3.1c),

$$\langle \mathbf{J} \cdot \mathbf{\nu}, \phi \rangle_{\partial \Omega} = \langle \mathbf{J}, \nabla \phi \rangle_{\Omega} = 0 \quad \text{for all } \phi \in Y,$$

where the pair $\langle \cdot, \cdot \rangle_{\partial \Omega}$ is the dual pair of $H^{1/2}(\partial \Omega)$ and $H^{-1/2}(\partial \Omega)$. Therefore, the numerical value $\langle \mathbf{J} \cdot \mathbf{\nu}, \eta \rangle_{\partial \Omega}$ for $\eta \in X$ is independent of $\eta \in X$, and the expression for $I(g)$ in this case can be understood as $I(g) = \langle \mathbf{J} \cdot \mathbf{\nu}, \eta \rangle_{\partial \Omega}$ for some $\eta \in X$.

Next, we turn to uniqueness of solutions. In general, there is no uniqueness for the solutions to this nonlinear elliptic system even for the simple case when $Q = 0$ and $g = 0$; see, e.g., [9], [10], [12]. Here we consider the uniqueness for the case when the boundary conditions on $\Sigma_D$ for $(u, v, w)$ are defined by (2.15), (2.16). Then, if $g = 0$, $u_e$ and $v = w = 0$ is a solution to (3.1), where $u_e \in Y + \bar{u}$ is the so-called equilibrium potential that satisfies

$$\langle \varepsilon \nabla u, \nabla \zeta \rangle + \langle e^u - e^{-u} - N, \zeta \rangle = 0$$

for all $\zeta \in Y$.

We show that the equilibrium solution $(u_e, 0, 0)$ is locally unique and that there exists a constant $\tau_0 > 0$ such that, if $|g| < \tau_0$, then (3.1) has a locally unique solution $(u, v, w)(g) \in H^2(\Omega)$, where the map $g \in L^2 \rightarrow (u, v, w) \in H^2$ is $C^1$ when Assumptions 4 and 5 are replaced by the following assumptions.

Assumption 4’. $Q(x, \cdot) : \mathbb{R}^3 \rightarrow \mathbb{R}$ is $C^1$ (uniformly in $x \in \Omega$) and $Q \geq 0$.

Assumption 5’. $\varepsilon = \varepsilon(x)$, $\mu_n = \mu_n(x)$, and $\mu_p = \mu_p(x) \in W^{1,\infty}(\Omega)$ are bounded below by a positive constant $c_0$.

We also assume the strong regularity of solutions to the Laplace equation.

Assumption 6. The solution $u$ of $-\Delta u = f, f \in L^2(\Omega)$ with $u|_{\Sigma_D} = 0$, and $(\partial u/\partial v)|_{\Sigma_N} = 0$ satisfies $|u|_{H^2(\Omega)} \leq M |f|_{2,\Omega}$ for some $M > 0$.

Assumption 6 is satisfied, for example, when $\Omega = (0, 1) \times (0, 1)$ and $\Sigma_D = [0, y] \cup [1, y]$, where $y \in [0, 1]$. In general, Assumption 6 represents a restriction on the boundary segment $\Sigma_D$.

Suppose that $(u, v, w) \in H^2(\Omega)^3$. Then (3.1) is equivalent to the strong form of the equations

$$-\varepsilon \Delta u - \nabla \varepsilon \cdot \nabla u + e^{v+u} - e^{-u} - N = 0,$$

(3.10)

$$-e^{v+u}(\mu_n(\Delta v + (\nabla v + \nabla u) \cdot \nabla v) + \nabla \mu_n \cdot \nabla v) + Q(e^{v+w} - 1) - g = 0,$$

$$-e^{w-u}(\mu_p(\Delta w + (\nabla w - \nabla u) \cdot \nabla w) + \nabla \mu_p \cdot \nabla w) + Q(e^{v+w} - 1) - g = 0,$$

with boundary conditions (2.14)–(2.16), which can be regarded as an operator equation $F(u, v, w; g) = 0$ in $L^2(\Omega)^3$ for $(u, v, w) \in Z^3 + (u_e, 0, 0)$, where

$Z = \{ \phi \in H^2(\Omega) : \phi|_{\Sigma_D} = 0, (\partial/\partial v)\phi|_{\Sigma_N} = 0 \}$. 


Then we have the following result.

**Theorem 3.8.** Let us assume Assumptions 1–3, 4′, 5′, and 6 and that \( u_e \in H^2(\Omega) \cap W^{1,\infty}(\Omega) \). Then, if \( |g|_{L^2(\Omega)} < \tau_0 \) for sufficiently small \( \tau_0 > 0 \), (3.10) has a locally unique solution \((u, v, w)(g) \in H^2(\Omega)^3\), which satisfies \((u, v, w)(0) = (u_e, 0, 0)\), and the mapping \( g \in L^2(\Omega) \rightarrow (u, v, w)(g) \in H^2(\Omega)^3 \) is \( C^1 \).

**Proof.** First, since the imbedding of \( H^2(\Omega) \) into \( W^{1,4}(\Omega) \) is continuous, we know that \( F \) is \( C^1 \). Moreover, \( F'(u_e, 0, 0, 0)(U, V, W) = (f_1, f_2, f_3) \) can be written as

\[
\begin{align*}
-\varepsilon \triangle U - \nabla \cdot \nabla U + (e^{u_e} + e^{-u_e})U + e^{u_e}V - e^{-u_e}W &= f_1, \\
-\nabla \cdot (\mu_n e^{u_e} \nabla V) + Q_0(V + W) &= f_2, \\
-\nabla \cdot (\mu_p e^{-u_e} \nabla W) + Q_0(V + W) &= f_3,
\end{align*}
\]

where \((U, V, W) \in Z^3\) and \( Q_0 = Q(x, u_e, 0, 0) \in L^\infty(\Omega) \) with \( Q_0 \geq 0 \) almost everywhere in \( \Omega \). The variational formulation of the last two equations of (3.11) is given by

\[
\begin{align*}
\langle \mu_n e^{u_e} \nabla V, \nabla \phi \rangle + \langle Q_0(V + W) - f_2, \phi \rangle &= 0, \\
\langle \mu_p e^{-u_e} \nabla W, \nabla \psi \rangle + \langle Q_0(V + W) - f_3, \psi \rangle &= 0
\end{align*}
\]

for all \( \phi, \psi \in Y \). Equation (3.12) has a unique solution \((V, W) \in Y \times Y\), which depends continuously on \((f_2, f_3) \in L^2(\Omega)^2\), since the quadratic form corresponding to (3.12) is bounded below by \( \varepsilon \mu_n e^{-2m_U} (|\nabla V|^2 + |\nabla W|^2) \). However, since \( \mu_n e^{u_e} \) and \( \mu_p e^{-u_e} \in W^{1,\infty}(\Omega) \), \((V, W) \in H^2_{loc}(\Omega)^2\) satisfies

\[
-\mu_n \triangle V = \mu_n \nabla u_e \cdot \nabla V + \nabla \mu_n \cdot \nabla V + e^{-u_e}(f_2 - Q_0(V + W)) \quad \text{in} \ \Omega,
\]

\[
-\mu_p \triangle W = -\mu_p \nabla u_e \nabla W + \nabla \mu_p \cdot \nabla W + e^{u_e}(f_3 - Q_0(V + W)) \quad \text{in} \ \Omega.
\]

Here the \( L^2 \)-norm of the right-hand sides of the above is bounded by \( |(f_2, f_3)|_{L^2 \times L^2} \) multiplied with a constant. Since \( e^{u_e} + e^{-u_e} \geq 2 \) in the first equation of (3.11), and the last two equations of (3.11) are decoupled from the first one, it follows from Assumption 6 that (3.11) has a unique solution \((U, V, W) \in Z^3\), which depends continuously on \( f = (f_1, f_2, f_3) \in L^2(\Omega)^3\). This means that \( F'(u_e, 0, 0, 0) : Z^3 \rightarrow L^2(\Omega)^3 \) has a bounded inverse. Thus, our theorem follows from the implicit function theorem.

Theorem 3.8 is the basis for a construction of our approximating model equation, which is described in the next section.

**4. Approximate model.** Due to the complexity of the basic semiconductor equations, simplifications are usually made in applications. Such assumptions ensure that the simplified system of equations is solvable or easier to analyze, but still preserves the desired properties of the original system in a particular application. The most common assumption is zero generation-recombination rate, which removes a complicated coupling from the system, and in many applications this proves to be an acceptable simplification (see, e.g., [13]). Another common simplification would be assuming that the holes (or electrons) remain in thermal equilibrium if the material considered is n-type (or p-type). This reduces the number of dependent variables in the system to two or even one (see, e.g., [11], [13]). We can see that the simplifications
mentioned above are not suitable for our problem, since the generation-recombination is a key factor in the model, and it is the distribution of $p$- and $n$-regions in the material that we are interested in; hence we must pursue a different path for simplifying the equations. Since our aim is to construct a model that will allow us to perform parameter identification based on the LBIC measurements $I(g)$, it is important that the simplified equations should yield a good approximation of these measurements. In this section, we obtain such a simplified model and derive estimates that show its validity.

Note that, in the laboratory uses of the LBIC method, the intensity of the applied laser power is small. If the laser beam were absent, the solution to the system would be the equilibrium states $(u, v, w) = (u_e, 0, 0)$, where $u_e$ is given by (2.19), and we have shown in Theorem 3.8 that $(u_e, 0, 0)$ is the unique solution of the system. Therefore it is natural that we look for perturbation solutions from the equilibrium $(u_e, 0, 0)$. That is, we assume the following approximations: $u \approx u_0 = u_e, v \approx v_0, w \approx w_0$, where $(v_0, w_0)$ is the solution to the linearized system of (2.12), (2.13) about the equilibrium $(0,0)$ with $u = u_e$.

More precisely, our approximate model consists of three dependent variables $(u_0, v_0, w_0)$, and the system of equations is

\begin{align}
\nabla \cdot (\varepsilon \nabla u_0) &= e^{u_0} - e^{-u_0} - N \quad \text{in } \Omega, \\
\nabla \cdot (\mu_n e^{u_0} \nabla v_0) - Q_0(u_0)(v_0 + w_0) + g &= 0 \quad \text{in } \Omega, \\
\nabla \cdot (\mu_p e^{-u_0} \nabla w_0) - Q_0(u_0)(v_0 + w_0) + g &= 0 \quad \text{in } \Omega,
\end{align}

with boundary conditions

\begin{align}
\frac{\partial u_0}{\partial v} = \frac{\partial v_0}{\partial v} = \frac{\partial w_0}{\partial v} &= 0 \quad \text{on } \Sigma_N, \\
u_0 &= \ln \frac{N + \sqrt{N^2 + 4}}{2} \quad \text{and} \quad v_0 = w_0 = 0 \quad \text{on } \Sigma_D,
\end{align}

where $Q_0(u_0) = Q(u_0, 0, 0)$; in the Shockley-Read-Hall model (2.7), we have

\begin{equation}
Q_0(u_0) = \left[ \tau_n (e^{u_0} + 1) + \tau_p (e^{-u_0} + 1) \right]^{-1}.
\end{equation}

In this case, we also simply assume that $\varepsilon, \mu_n, \mu_p, \tau_n,$ and $\tau_p$ are all smooth positive function in $\Omega$. Clearly, the LBIC measurement in this approximate model is given by

\begin{equation}
I_0(g) = \int_{\Gamma_1} (\mu_n e^{u_0} \nabla v_0 - \mu_p e^{-u_0} \nabla w_0) \cdot \nu \, ds.
\end{equation}

To study the wellposedness of this approximate model, we first state the variational formulation of (4.1) as follows:

\begin{align}
\langle \varepsilon \nabla u_0, \nabla \zeta \rangle + \langle e^{u_0} - e^{-u_0} - N, \zeta \rangle &= 0, \\
\langle \mu_n e^{u_0} \nabla v_0, \nabla \phi \rangle + \langle Q_0(u_0)(v_0 + w_0) - g, \phi \rangle &= 0, \\
\langle \mu_p e^{-u_0} \nabla w_0, \nabla \psi \rangle + \langle Q_0(u_0)(v_0 + w_0) - g, \psi \rangle &= 0.
\end{align}
for all $\chi, \phi, \psi \in Y$ and $(u_0 - \bar{u}, v_0, w_0) \in Y^3$. Then we have the following theorem.

**THEOREM 4.1.** Under Assumptions 1–3, 4’, and 5’ of the previous section, (4.5) has a unique solution. Furthermore, if Assumption 6 is assumed, the weak solution is in $H^2(\Omega)^3$, and $(v_0, w_0)$ is continuously dependent on $g \in L^2(\Omega)$.

**Proof.** Note that in the approximate model, $u_0$ is decoupled from the system and can be solved first from (4.5a). The unique existence of $u_0 \in V + \bar{u}$ is a special case of Lemma 3.1. With this $u_0$, the system for $(v_0, w_0)$, i.e., (4.5b), (4.5c), is a weakly coupled, linear elliptic system. Since $u_0 \in L^\infty(\Omega)$, we can easily construct a coercive, continuous sesquilinear form on $Y^2$ from (4.5b) and (4.5c) to establish the unique existence of $(v_0, w_0) \in Y^2$ by following the standard Hilbert space method. The $H^2$-regularity of the solution and the continuous dependence of $g$ are obvious under Assumption 6.

The comment we made in Remark 3.7 also applies to the validity of the current expression (4.4) in this approximate model.

The question arises of how this ad hoc approximation is related to the exact solution. We address this question in the following theorem, which shows that the approximation is, in fact, valid and produces a solution that approaches the exact solution as the $L^2$ norm of $g$ approaches zero, and the error in the LBIC measurements of the original and simplified model approaches zero as the square of the energy in the applied laser beam.

**THEOREM 4.2.** Assume that Assumptions 1–3, 4’, 5’, and 6 and $u_0 \in H^2(\Omega) \cap W^{1,\infty}(\Omega)$ hold. Then, for small $|g|_{2,\Omega}$, there hold

$$
|u - u_0|_{H^2} = O(|g|_{2,\Omega}),
$$

$$
|(v, w) - (v_0, w_0)|_{H^2} = O(|g|_{2,\Omega}^2),
$$

$$
|I(g) - I_0(g)| = O(|g|_{2,\Omega}^2),
$$

where $(u, v, w)$ and $(u_0, v_0, w_0)$ are the unique $H^2$-solution to (3.10) and (4.1), respectively.

**Proof.** In the following, we use $C$ to denote universal constant independent of $g$. First, it follows directly from the continuity results in Theorems 3.8 and 4.1 that

$$
(u - u_0, v, w)|_{H^2} \leq C|g|_{2,\Omega} \quad \text{and} \quad (v_0, w_0)|_{H^2} \leq C|g|_{2,\Omega}.
$$

From (3.10) and (4.1), we have the following strong form system for $(v - v_0, w - w_0)$:

$$
\nabla \cdot (\mu_n e^{u_0} \nabla (v - v_0)) - Q_0(u_0)(v - v_0 + w - w_0) = f_1,
$$

$$
\nabla \cdot (\mu_p e^{-u_0} \nabla (w - w_0)) - Q_0(u_0)(v - v_0 + w - w_0) = f_2
$$

with boundary conditions

$$
\frac{\partial (v - v_0)}{\partial \nu} = \frac{\partial (w - w_0)}{\partial \nu} = 0 \text{ on } \Sigma_N \quad \text{and} \quad v - v_0 = w - w_0 = 0 \text{ on } \Sigma_D,
$$

where

$$
f_1 = -\nabla \cdot (\mu_n e^{u_0} (e^{v+u-u_0} - 1) \nabla v) + (Q(u, v, w) - Q(u_0, 0, 0))(e^{v+w} - 1)
$$

$$
+ Q(u_0, 0, 0)(e^{v+w} - 1 - v - w),
$$

$$
f_2 = -\nabla \cdot (\mu_p e^{-u_0} (e^{w-u+u_0} - 1) \nabla w) + (Q(u, v, w) - Q(u_0, 0, 0))(e^{v+w} - 1)
$$

$$
+ Q(u_0, 0, 0)(e^{v+w} - 1 - v - w).$$
Note that the embedding from $H^2(\Omega)$ to $L^\infty(\Omega)$ is bounded for $d \leq 3$. Then, from the estimates in (4.6) and Assumptions 4' and 5', we can easily see that
\[ |f_1|_{2,\Omega}, \ |f_2|_{2,\Omega} \leq C|g|_{2,\Omega}^2. \]
Since the system for $(v - v_0, w - w_0)$ above is the same system as (3.12) in the proof of Theorem 3.8, we can apply the same argument to conclude that
\[ |(v - v_0, w - w_0)|_{H^2} \leq C|g|_{2,\Omega}^2. \]
Moreover,
\[ \vec{J} - \vec{J}_0 = (\mu_n e^{u_0} \nabla v - \mu_p e^{w_0} \nabla w) - (\mu_n e^{u_0} \nabla v_0 - \mu_p e^{-w_0} \nabla w_0) \]
\[ = \mu_n e^{u_0} \nabla (v - v_0) + \mu_n e^{u_0} (e^{v_0 + u_0} - 1) \nabla v - \mu_p e^{-w_0} \nabla (w - w_0) \]
\[ - \mu_p e^{-w_0} (e^{w_0 + u_0} - 1) \nabla w, \]
and hence by the trace theorem
\[ |I(g) - I_0(g)| \leq C|\vec{J} - \vec{J}_0|_{H^1} \leq C|g|_{2,\Omega}^2. \]
Thus the estimates claimed in the theorem hold.

In this approximate model (4.1), (4.2), the relationship between the doping profile and the LBIC image is modeled by the linear functional $I(g)$ defined in (4.4). This inverse problem (identifying $N(x)$ from the functional $I$) is studied by the authors in a forthcoming paper.

5. Numerical examples. As shown experimentally, one of the distinguishing features that the LBIC technique possesses is the bimodal behavior of the measured current as the laser beam crosses a p-n-p junction (see [1], [2]). To illustrate that the approximate model we presented in the previous section preserves this feature, we numerically compute the LBIC signal with a prescribed doping profile. In this section, we describe the algorithm we apply and present some numerical results.

For ease of presentation, we describe our algorithm for the one-dimensional case. We also assume that $\varepsilon, \mu_n, \mu_p$ be constant and that $\tau_n = \tau_p = \frac{1}{2}$. Let $\Omega = (0, 1) \subset \mathbb{R}^1$, $\Sigma_N = \emptyset$, and $\Sigma_D = \{0, 1\}$. The equation for $u_0 = u_0(x)$ from (4.1a) becomes
\[ -\varepsilon u_0''(x) + 2 \sinh(u_0(x)) - N(x) = 0 \quad \text{for} \ x \in (0, 1) \]
with boundary conditions
\[ u_0(x) = \ln \frac{N(x) + \sqrt{N^2(x) + 4}}{2} \quad \text{at} \ x = 0 \text{ and } 1, \]
and the system for $(v_0, w_0)$ becomes
\[ -\mu_n (e^{u_0(x)} v_0'(x))' + \frac{v_0(x) + w_0(x)}{\cosh(u_0(x)) + 1} - g(x) = 0, \quad x \in (0, 1), \]
\[ -\mu_p (e^{-u_0(x)} w_0'(x))' + \frac{v_0(x) + w_0(x)}{\cosh(u_0(x)) + 1} - g(x) = 0, \quad x \in (0, 1) \]
with boundary conditions

\[(5.5) \quad v_0(x) = w_0(x) = 0 \quad \text{at } x = 0 \text{ and } 1.\]

Assume that the measurement for LBIC is made at \(x = 0\); so it is given by (from (4.4))

\[(5.6) \quad I(g) = -\mu_n e^{u_0(0)} v'_0(0) + \mu_2 e^{-u_0(0)} w'_0(0).\]

First, we solve the nonlinear boundary value problem \((5.1), (5.2)\) by Newton’s method. If the solution map of \((5.1), (5.2)\) is denoted by \(F(u_0) = 0\), the derivative of \(F\) is described by

\[F'(u_0)\eta = -\varepsilon \eta'' + 2 \cosh(u_0)\eta \quad \text{for } \eta \in H^1_0(0,1) \cap H^2(0,1).\]

Hence the Newton scheme gives the following iteration procedure:

\[(5.7) \quad u^{m+1} = u^m - (F'(u^m))^{-1} F(u^m), \quad m = 0, 1, 2, \ldots,\]

where \(u^m\) is the \(m\)th iteration for \(u_0\). The discretized version of (5.7) by central finite difference discretization is used to compute the numerical \(u_0(x)\).

With this obtained \(u_0(x)\), the weakly coupled linear system \((5.3), (5.4)\) for \((v_0, w_0)\) is discretized by central finite difference discretization to compute the numerical \(v_0\) and \(w_0\). The class of \(g\)'s we choose is

\[(5.8) \quad g_j(x) = \begin{cases} mx - j + 1, & x \in \left[\frac{j-1}{m}, \frac{j}{m}\right], \\ -mx + j + 1, & x \in \left[\frac{j}{m}, \frac{j+1}{m}\right], \\ 0, & \text{elsewhere in } [0,1] \end{cases}\]

for \(j = 1, 2, \ldots, m - 1\). The meshsize of the discretization is \(1/n\). For each \(g_j(x)\), \(v_0(x)\) and \(w_0(x)\) are computed, and then the LBIC measurement is approximated by

\[I\left(\frac{j}{m}\right) = I_j = -\mu_n \exp\left(\frac{u_0(0) + u_0(\frac{1}{n})}{2}\right) v_0\left(\frac{1}{n}\right) + \mu_p \exp\left(-\frac{u_0(0) + u_0(\frac{1}{n})}{2}\right) w_0\left(\frac{1}{n}\right).\]

In Fig. 5.1, we present the numerical result for \(I(x)\) when we set \(m = 40, n = 80\), and choose the doping profile (representing an \(n-p-n\) junction)

\[N(x) = \begin{cases} N_0, & x \in [0,0.4) \cup (0.6,1], \\ 0, & x = 0.4 \text{ or } 0.6, \\ -N_0, & x \in (0.4,0.6) \end{cases}\]

with a set of arbitrary chosen parameters (\(\varepsilon = 0.1, N_0 = 100, \mu_n = \mu_p = 0.01\)). The bimodal behavior of the LBIC is easily seen. A sharper bimodal signal is observed for smaller \(\mu_n\) and \(\mu_p\).

To illustrate the effect of changes in the laser beam profile, we also computed the case when the laser profile \(g_j\) is replaced by a charactereristic function (keeping the energy at the same level). The computed LBIC signal was almost identical with maximum deviation from the previous example of less than 2 percent. Because the results in these two cases are so similar, we do not include a figure for the second case here.
The same algorithm is used for the two-dimensional case, and the computed LBIC signals are presented in Fig. 5.2. In this example, we choose the doping profile $N(x, y) = -100$ in $(0.4,0.6) \times (0.4,0.6)$ and $N(x, y) = 100$ elsewhere in $\Omega$ to represent a $p$-type defect buried in an $n$-type material in $\Omega = (0,1) \times (0,1)$. The ohmic contacts are at $\Gamma_1 = \{(0,y) : 0 \leq y \leq 1\}$ and $\Gamma_2 = \{(1,y) : 0 \leq y \leq 1\}$, and the LBIC measurement is made at $\Gamma_1$. The laser beam profile is chosen as

$$g_{ij}(x,y) = 100g_i(x)g_j(y)$$

for $i, j = 1, 2, \ldots, m - 1$, and $g_j$ are defined as in (5.8). The parameters are $\varepsilon = 0.1$ and $\mu_n = \mu_p = 0.01$, and we set $m = 20$ and $n = 20$ (meshsize in both $x$ and $y$ directions is $1/n$). Central finite difference discritization is also used in this two-dimensional example. The contour line picture of the LBIC measurement (known as the LBIC image if false colors are applied (see [2])) is plotted in Fig. 5.3, where the level curves are equally spaced from $-0.0684$ to $0.0684$. Again, the numerical result shows the unique feature of LBIC signals that have been observed in experiments.

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FIG. 5.3. Contour lines of the two-dimensional LBIC measurements (equally spaced from −0.0684 to 0.0684).

REFERENCES