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COLORING PERMUTATION-GAIN GRAPHS

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ABSTRACT. Correspondence colorings of graphs were introduced in 2018 by Dvořák and Postle as a generalization of list colorings of graphs which generalizes ordinary graph coloring. Kim and Ozeki observed that correspondence colorings generalize various notions of signed-graph colorings which again generalizes ordinary graph colorings. In this note we state how correspondence colorings generalize Zaslavsky's notion of gain-graph colorings and then formulate a new coloring theory of permutation-gain graphs that sits between gain-graph coloring and correspondence colorings. Like Zaslavsky's gain-graph coloring, our new notion of coloring permutation-gain graphs has well defined chromatic polynomials and lifts to colorings of the regular covering graph of a permutation-gain graph.

1. INTRODUCTION

Dvořák and Postle very recently introduced [2] the concept of *correspondence colorings* (also called DP-colorings) of graphs as a generalization of list colorings of graphs which is a generalization of ordinary colorings of graphs. A *signed graph* is a pair (G, σ) in which $\sigma\colon E(G) \to \{+, -\}$. Zaslavsky [7] defined two notions of coloring signed graphs: one has colors corresponding to the elements of $\{0, \pm 1, \ldots, \pm k\}$ and the other with colors corresponding to the elements of $\{\pm 1, \ldots, \pm k\}$ (also see Máčajová, Raspaud, and Škoviera [6]). Call this notion of coloring signed graphs *integer coloring* of signed graphs. Kang and Steffen explored [3, 4] the analogous notion of coloring signed graphs with elements of the group $\mathbb{Z}_k$. We will call this *modular coloring* of signed graphs. The notions of modular coloring and integer coloring of signed graphs correspond when the number of colors is odd but not when the number of colors is even. Kim and Ozeki noted [5] that correspondence colorings generalize both modular coloring and integer coloring of signed graphs. Zaslavsky also developed [7, 9] a notion of gain-graph colorings which generalizes integer colorings of signed graphs. The purpose of this note is to define a concept of coloring permutation-gain graphs which generalizes gain-graph colorings and both notions of signed graph colorings and which is in turn generalized by correspondence colorings (see Figure 1).

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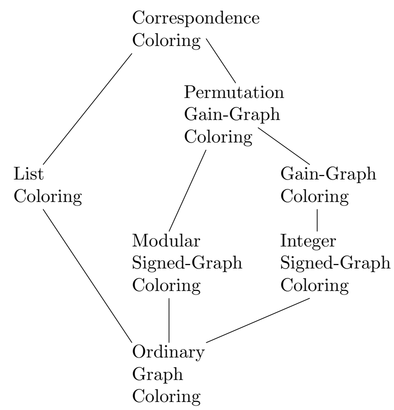

FIGURE 1. A Hasse diagram depicting generalization relations between various coloring concepts.

The main focus of [9] is the development of a rich theory of chromatic polynomials of gain graphs which are closely related to the very important Tutte polynomials of different matroids associated with gain graphs (see Theorems 5.1 and 5.2 in [9]). This connection with Tutte polynomials and its associated matroid invariants is one of the main attractions of Zaslavsky's concept of gain-graph colorings. In this note, we give two consequences of our definition of coloring permutation-gain graphs. In Section 3, we show that there is a well-defined chromatic polynomial for a permutation-gain graph along with a limited deletion-contraction relation. This chromatic polynomial, however, is in general not an invariant of matroids normally associated with the permutation-gain graph. In Section 4 we also show how proper colorings of permutation-gain graphs lift to proper colorings of their derived graphs (i.e., regular covering graphs). It is not clear whether or not correspondence colorings have chromatic polynomials or associations with colorings of covering graphs.

2. Formal Definitions

Graphs. In a graph G, an edge e with its ends on distinct endpoints is called a *link* and an edge with its ends on the same vertex is called a *loop*. An *oriented edge* is an edge e along with a choice of direction along e. We usually use e to refer to an oriented edge whose underlying edge is e; it usually causes no confusion. The reverse orientation of an oriented edge e is denoted by e^{-1}. The collection of oriented edges of G is denoted by $\vec{E}(G)$. The tail and head endpoints of an oriented edge e are sometimes denoted, respectively, by $\mathbf{t}(e)$ and $\mathbf{h}(e)$. A *walk* w is a product of oriented edges $e_1 \cdots e_n$ for which $\mathbf{h}(e_i) = \mathbf{t}(e_{i+1})$ for each $i \in \{1, \ldots, n-1\}$. The walk $w = e_1 \cdots e_n$ is *closed* when $\mathbf{h}(e_n) = \mathbf{t}(e_1)$. The *reverse walk* of w is $w^{-1} = e_n^{-1} \cdots e_1^{-1}$.

Correspondence Colorings. Given a graph G and a positive integer k, associate to each vertex $v \in V(G)$ the set $\{1, \ldots, k\} \times \{v\}$. A *$k$-correspondence assignment* is a function M assigning to each edge $e \in E(G)$ a partial matching M_e between the sets $\{1, \ldots, k\} \times \{u\}$ and $\{1, \ldots, k\} \times \{v\}$ where u and v are the endpoints of e. In the case that e is a loop, we can use two separate copies of $\{1, \ldots, k\} \times \{v\}$ for the matching M_e. An *M-coloring* of G is a k-coloring κ of G such that for each edge e with endpoints u and v, $(\kappa(u), u)$ and $(\kappa(v), v)$ are not connected in the matching M_e. Note that when the matchings of M are all complete matchings of k edges, then M corresponds to a gain function on G of some subgroup of the symmetric group S_k. This observation lead the author to consider coloring permutation-gain graphs as a special case of correspondence colorings.

Permutation-Gain Graphs. Let Γ be a subgroup of the symmetric group S_k. We will be looking at functions $\pi\colon \vec{E}(G) \to \Gamma$ and $\eta\colon V(G) \to \Gamma$ from G to Γ which will always be denoted with lowercase Greek letters; furthermore, the values of these functions at arguments e and v will be denoted by π_e and η_v rather than the far more common notation $\pi(e)$ and $\eta(v)$. The reason for this is because we will also be considering the action of $\Gamma \subseteq S_k$ on the set $\{1, \ldots, k\}$ and denoting the action of $\pi_e \in \Gamma$ on $i \in \{1, \ldots, k\}$ by $\pi_e(i)$ seems much more streamlined than writing $\pi(e)(i)$ or $\pi[e](i)$. Now, a Γ-*gain function* on a graph G is a function $\pi\colon \vec{E}(G) \to \Gamma$ for which $\pi_{e^{-1}} = \pi_e^{-1}$. A Γ-*gain graph* is the pair (G, π). Since Γ is a subgroup of S_k, we also refer to (G, π) a *permutation-gain graph.* Given a walk $w = e_1 \cdots e_n$, define π on w by $\pi_w = \pi_{e_1} \cdots \pi_{e_n}$. Note that $\pi_{w^{-1}} = \pi_w^{-1}$.

A *switching function* on (G, π) is a function $\eta\colon V(G) \to \Gamma$. The *switched* Γ-gain function π^η is defined by $\pi_e^\eta = \eta_{\mathbf{t}(e)} \pi_e \eta_{\mathbf{h}(e)}^{-1}$. For a walk $w = e_1 \ldots e_n$ we therefore get that $\pi_w^\eta = \eta_{\mathbf{t}(e_1)} \pi_w \eta_{\mathbf{h}(e_n)}^{-1}$. Hence if w is a closed walk (including a loop), then π_w^η is a conjugate of π_w in the group Γ.

Again, let Γ be a subgroup of the symmetric group S_l and consider its associated action on $\{1, \ldots, l\}$. For each non-identity element $\pi \in \Gamma$ let $\operatorname{fix}(\pi) \subseteq \{1, \ldots, l\}$ denote the set of fixed points of $\pi \in \Gamma$, that is, $\operatorname{fix}(\pi) = \{i : \pi(i) = i\}$. Now for any positive integer t, consider the canonical fixed-point free action of Γ on the set $\Gamma \times \{1, \ldots, t\}$ defined by right multiplication on the Γ-coordinate of each pair. Recall that all fixed-point free actions of Γ on a finite set are essentially of this type. So for each positive integer t, there is an isomorphic copy of Γ, again call it Γ, in the symmetric group S_k where $k = l + t|\Gamma|$ for which: the symbols $1, \ldots, l$ are as before, the action of Γ on $\{1, \ldots, l\}$ is as before, and the action of Γ on the remaining $t|\Gamma|$ elements is free. Thus $\operatorname{fix}(\pi)$ is the same subset of $\{1, \ldots, l\}$ no matter which $k = l + t|\Gamma|$ we consider.

A *k-coloring* of (G, π) (again where $k = l + t|\Gamma|$) is a function $\kappa\colon V(G) \to \{1, \ldots, k\}$. The k-coloring κ is *proper* when for each oriented edge e (including loops) with tail endpoint u and head endpoint v we have that

$\kappa(u) \neq \pi_e\kappa(v)$. Note that e satisfies the propriety condition if and only if e^{-1} does as well because $\kappa(u) \neq \pi_e\kappa(v)$ if and only if $\pi_{e^{-1}}\kappa(u) \neq \kappa(v)$. We now see how k-colorings of permutation-graph graphs generalizes proper-coloring concepts of gain graphs and signed graphs in [3, 4, 7, 9]. In gain-graph colorings and integer signed-graph colorings, the action of Γ either: acts freely on a color set of size $t|\Gamma|$ or acts freely on a color set of size $t|\Gamma|+1$ save at one color which is fixed by all of the elements of Γ. In modular signed-graph colorings with even color set $\mathbb{Z}_{2k}$, the action of $\Gamma = \{+,-\}$ fixes 0 and k and acts freely on $\mathbb{Z}_{2k} - \{0,k\}$.

If κ is a k-coloring of (G,π) and η switching function, then define the k-coloring $\eta\kappa$ by $\eta\kappa(v) = \eta_v\kappa(v)$. Note that $\eta\kappa(u) \neq \pi_e^\eta\eta\kappa(v)$ if and only if $\eta_u\kappa(u) \neq \eta_u\pi_e\eta_v^{-1}\eta_v\kappa(v)$ if and only if $\kappa(u) \neq \pi_e\kappa(v)$. This yields Proposition 2.1.

Proposition 2.1. *If $\eta\colon V(G) \to \Gamma$ is a switching function and κ is a k-coloring of (G,π), then κ is a proper k-coloring of (G,π) if and only if $\eta\kappa$ is a proper k-coloring of (G,π^η). Furthermore, $\kappa \mapsto \eta\kappa$ defines a bijection between the proper k-colorings of (G,π) and (G,π^η).*

3. A chromatic polynomial

In this section we show that there is a well-defined chromatic polynomial $p(G,\pi,k)$ for (G,π) where the value of $p(G,\pi,k)$ at any $k = l + t|\Gamma|$ is the number of proper k-colorings of (G,π). For example, if (G,π) is a graph in which every edge is a loop, then for each $v \in V(G)$ let $\mathrm{fix}(v) \subseteq \{1,\dots,l\}$ denote the union of all $\mathrm{fix}(e)$ in which e is a loop incident to v. We now get that

$$p(G,\pi,k) = \prod_{v_j \in V(G)} (k - |\mathrm{fix}(v_j)|)$$

This example is interesting in that it shows that the chromatic polynomial of a permutation-gain graph is dependent on the structure of fixed points in $\{1,\dots,l\} \subseteq \{1,\dots,k\}$ under the action of Γ. As such, this chromatic polynomial is not an invariant of any of the matroids normally associated with (G,π): the frame matroid, lift matroid, and complete lift matroid (see, for example, [8]).

Theorem 3.1. *Let $p(G,\pi,k)$ be the number of proper k-colorings of (G,π) for any $k = l + t|\Gamma|$.*

1. *If e is any link in G and η a switching function such that $\pi^\eta(e) = 1$ (here 1 is the identity permutation) then $p(G,\pi,k) = p(G,\pi^\eta,k) = p(G\backslash e,\pi^\eta,k) - p(G/e,\pi^\eta,k)$.*
2. *If (G,π) has no balanced loops (i.e., a loop e for which $\pi_e = 1$), then $p(G,\pi,k)$ is a well-defined, monic polynomial in k of degree $|V(G)|$.*

Proof. (1) By Proposition 2.1, the proper k-colorings of (G,π) correspond to the proper k-colorings of (G,π^η) by $\kappa \mapsto \eta\kappa$. Say that e has endpoints u and v. If κ is any proper k-coloring of $(G\backslash e,\pi^\eta)$, then κ is a proper k-coloring of

(G, π^η) if and only if $\kappa(u) \neq \kappa(v)$. Those proper colorings of $(G \backslash e, \pi^\eta)$ with $\kappa(u) = \kappa(v)$ correspond precisely to the proper colorings of $(G/e, \pi^\eta)$.

(2) Repeatedly applying the deletion-contraction identity in (1) reduces $p(G, \pi, k)$ to

$$p(G, \pi, k) = \sum_i (-1)^{|V(G)|-|V(G_i)|} p(G_i, \pi_i, k)$$

where each (G_i, π_i) is a linkless graph (that is, any edge of G_i is a loop) with $|V(G_i)| \leq |V(G)|$ and exactly one such term having $|V(G_i)| = |V(G)|$, that is, G with all of its links deleted, call it (G_0, π_0). Since (G, π) has no balanced loops, our calculation using Part (1) yields a polynomial in k of degree $|V(G)|$.

This polynomial is well-defined despite different ways of using part (1) to calculate it. The reason for this is as follows. The resulting polynomial of degree $|V(G)|$ counts the number of proper k-colorings for any $k = l + t|\Gamma|$ despite the way in which it is calculated. Now using polynomial interpolation over $|V(G)| + 1$ distinct values for $k = l + t|\Gamma|$ implies the uniqueness of the polynomial. □

4. Lifting colorings to covering graphs

Given a Γ-gain graph (G, π), its *derived graph* G^π is an ordinary graph defined as follows: $V(G^\pi) = V(G) \times \Gamma$ and $\vec{E}(G^\pi) = \vec{E}(G) \times \Gamma$ in which $\mathbf{t}(e, \gamma) = (\mathbf{t}(e), \gamma)$ and $\mathbf{h}(e, \gamma) = (\mathbf{h}(e), \gamma\pi_e)$. The derived graph G^π is known as a *regular covering graph* of the ordinary graph G. Now if κ is a proper k-coloring of (G, π), then κ induces a k-coloring, also call it κ, on G^π by $\kappa(v, \gamma) = \gamma\kappa(v)$.

Theorem 4.1. *If κ is a proper k-coloring of (G, π), then the k-coloring on the derived graph G^π induced by κ is a proper k-coloring of G^π.*

Proof. Given an oriented edge e (a link or a loop) in (G, π) we have $\kappa(\mathbf{t}(e)) \neq \pi_e\kappa(\mathbf{h}(e))$. Now for any $\gamma \in \Gamma$ we also have $\gamma\kappa(\mathbf{t}(e)) \neq \gamma\pi_e\kappa(\mathbf{h}(e))$. Also, $\kappa(\mathbf{t}(e, \gamma)) = \kappa(\mathbf{t}(e), \gamma) = \gamma\kappa(\mathbf{t}(e))$ and $\kappa(\mathbf{h}(e, \gamma)) = \kappa(\mathbf{h}(e), \gamma\pi_e) = \gamma\pi_e\kappa(\mathbf{h}(e))$ which implies our result. □

References

1. Anton Bernshteyn, Alexandr Kostochka, and Xuding Zhu, *DP-colorings of graphs with high chromatic number*, European J. Combin. **65** (2017), 122–129. MR 3679840
2. Zdeněk Dvořák and Luke Postle, *Correspondence coloring and its application to list-coloring planar graphs without cycles of lengths 4 to 8*, J. Combin. Theory Ser. B **129** (2018), 38–54. MR 3758240
3. Yingli Kang and Eckhard Steffen, *The chromatic spectrum of signed graphs*, Discrete Math. **339** (2016), no. 11, 2660–2663. MR 3518416
4. ———, *Circular coloring of signed graphs*, J. Graph Theory **87** (2018), no. 2, 135–148. MR 3742174
5. Seog-Jin Kim and Kenta Ozeki, *A sufficient condition for DP-4-colorability*, Discrete Math. **341** (2018), no. 7, 1983–1986. MR 3802151

6. Edita Máčajová, André Raspaud, and Martin Škoviera, *The chromatic number of a signed graph*, Electron. J. Combin. **23** (2016), no. 1, Paper 1.14, 10. MR 3484719
7. Thomas Zaslavsky, *Signed graph coloring*, Discrete Math. **39** (1982), no. 2, 215–228. MR 675866
8. ———, *Biased graphs. II. The three matroids*, J. Combin. Theory Ser. B **51** (1991), no. 1, 46–72. MR 1088626
9. ———, *Biased graphs. III. Chromatic and dichromatic invariants*, J. Combin. Theory Ser. B **64** (1995), no. 1, 17–88. MR 1328292

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