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Research Article

Odd solutions to systems of inequalities coming from regular chain groups

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Abstract

Hoffman's theorem on feasible circulations and Ghouila-Houry's theorem on feasible tensions are classical results of graph theory. Camion generalized these results to systems of inequalities over regular chain groups. An analogue of Camion's result is proved in which solutions can be forced to be odd valued. The obtained result also generalizes the results of Pretzel and Youngs as well as Slilaty. It is also shown how Ghouila-Houry's result can be used to give a new proof of the graph-coloring theorem of Minty and Vitaver.

Keywords: unimodular chain group; regular chain group; totally unimodular matrix; systems of inequalities; graph coloring.

2020 Mathematics Subject Classification: 05C15, 05A20, 05C50.

1. Introduction

Hoffman's theorem on feasible circulations [5] and Ghouila-Houry's theorem on feasible tensions [4] are classical results of graph theory. Within the ring of integers, Camion [2] generalized both of these results using the setting of regular chain groups (also known as unimodular modules).† Within the more general algebraic framework developed by Ghouila-Houry [5], Camion [3] again generalized both results. In this paper we will only be working within the ring of integers. Our idea might extend to the more general setting of [3, 4] by using an index-2 subgroup of the interval group.

Let E be a finite set. A $\mathbb{Z}$*-chain group* is a submodule $\mathcal{C}$ of $\mathbb{Z}^E$. The elements of $\mathcal{C}$ are called *chains*. A chain $c \in \mathcal{C}$ may be thought of as a function $c\colon E \to \mathbb{Z}$. The *support* of c is $|c| = \{e \in E : c(e) \neq 0\}$; furthermore, $|c|$ naturally partitions into subsets $|c|_+ = \{e \in E : c(e) > 0\}$ and $|c|_- = \{e \in E : c(e) < 0\}$. A chain $c \in \mathcal{C}$ is *elementary* when $|c|$ is nonempty and subset minimal among all non-zero chains in $\mathcal{C}$. A chain $c \in \mathcal{C}$ is *primitive* when $c(e) \in \{-1, 0, 1\}$ for all e. A $\mathbb{Z}$-chain group $\mathcal{C}$ is a *regular chain group* (also called a *unimodular module*) when every elementary chain satisfies $c = kc'$ where $c' \in \mathcal{C}$ is primitive and $k \in \mathbb{Z}$. It is known that the nullspace of a totally unimodular matrix is a regular chain group and a regular chain group is the nullspace of some totally unimodular matrix. The orthogonal complement of a regular chain group is also a regular chain group. The integer cycle and cocycle spaces of a graph G are orthogonal complements of each other and are both regular chain groups.‡

Let I be an assignment of closed $\mathbb{Z}$-intervals to each $e \in E$; that is, for each $e \in E$, $I(e) = [l(e), u(e)]$ is a closed interval in $\mathbb{Z}$ (possibly of length 0). Extend I to each $c \in \mathcal{C}$ by setting

$$l(c) = \sum_{e\in|c|_+} c(e)l(e) + \sum_{e\in|c|_-} c(e)u(e) \quad \text{and} \quad u(c) = \sum_{e\in|c|_+} c(e)u(e) + \sum_{e\in|c|_-} c(e)l(e).$$

For primitive chains this simplifies to

$$l(c) = \sum_{e\in|c|_+} l(e) - \sum_{e\in|c|_-} u(e) \quad \text{and} \quad u(c) = \sum_{e\in|c|_+} u(e) - \sum_{e\in|c|_-} l(e).$$

Given any chain $d \in \mathbb{Z}^E$, there is an associated homomorphism $\hat{d}\colon \mathcal{C} \to \mathbb{Z}$ defined by $\hat{d}(c) = \sum_{e\in|c|} d(e)c(e)$. Thus $\hat{d} = 0$ if and only if $d \in \mathcal{C}^*$ and $\hat{d}_1 = \hat{d}_2$ if and only if $d_1 - d_2 \in \mathcal{C}^*$.

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†Seymour proved a special case of Camion's result in [9, (14.5)] which he cites as a previously known folklore result.

‡Seymour famously proved [8] that regular chain groups decompose along small separations into cycle spaces of graphs, cocycle spaces of graphs, and copies of the chain group associated with the matroid R_{10}.

Camion's result may be stated as follows.

Theorem 1.1 (Camion [2]). *Let $\mathcal{C} \leq \mathbb{Z}^E$ be a regular chain group, let $f\colon \mathcal{C} \to \mathbb{Z}$ be a homomorphism, and let I be an assignment $\mathbb{Z}$-intervals to E. The following are equivalent.*

(1) $f(c) \in I(c)$ for each primitive chain $c \in \mathcal{C}$.

(2) $f(c) \in I(c)$ for each chain $c \in \mathcal{C}$.

(3) There is $d \in \mathbb{Z}^E$ satisfying $\hat{d} = f$ and $d(e) \in I(e)$ for each $e \in |d|$.

Define the *length* of $c \in \mathbb{Z}^E$ as $\text{length}(c) = \sum_{e\in|c|} |c(e)|$. This, of course, generalizes the notion of the length of a cycle in a graph. Theorem 1.2 is a "parity-respecting" version of Theorem 1.1. When $\mathcal{C}$ is the integer cycle space of a graph and all intervals are $[-k, k]$ for some odd k, Theorem 1.2 specializes to a result of Slilaty [10]. For $k = 1$, Slilaty's result specializes to a result of Pretzel and Youngs [7]. A notable consequence of this theorem is the chain d obtained in Part (3) is everywhere non-zero.

Theorem 1.2. *Let $\mathcal{C} \leq \mathbb{Z}^E$ be a regular chain group, let $f\colon \mathcal{C} \to \mathbb{Z}$ be a homomorphism for which $f(c) \equiv \text{length}(c) \pmod 2$ for each $c \in \mathcal{C}$, and let I be an assignment $\mathbb{Z}$-intervals to E for which both $u(e)$ and $l(e)$ are odd for each $e \in E$. The following are equivalent.*

(1) $f(c) \in I(c)$ for each primitive chain $c \in \mathcal{C}$.

(2) $f(c) \in I(c)$ for each chain $c \in \mathcal{C}$.

(3) There is $d \in \mathbb{Z}^E$ satisfying $\hat{d} = f$ and $d(e) \in I(e)$ with $d(e)$ odd for each $e \in |d|$.

Theorem 1.2 will be proven in Section 2. In relation to the discussion in this paper, it is relevant to note that Ghouila-Houry's Theorem (or Theorem 1.1) can be used to give a new proof in the non-trivial direction of the elegant Minty-Vitaver Theorem [6, 12] stated below. We will give this proof in Section 3. An acyclic orientation f of a graph G partitions the edges of any cycle C in G into two nonempty sets, C_+ and C_-. Let

$$\mathrm{ratio(C,f)} = \max\left\{\frac{|\mathrm{C}_+|}{|\mathrm{C}_-|}, \frac{|\mathrm{C}_-|}{|\mathrm{C}_+|}\right\}.$$

Theorem 1.3 (Minty [6] and Vitaver [12].). *A graph G is k-colorable if and only if there is an acyclic orientation f of G for which* $\mathrm{ratio(C,f)} \leq \mathrm{k}-1$ *for all cycles C in G.*

2. Proof of Theorem 1.2

Chain c' *conforms* to chain c when $|c'| \subseteq |c|$ and $c'(e)c(e) \geq 0$ for all $e \in E$. Tutte proved [11, (5.43)] that every c in a regular chain group $\mathcal{C}$ has a decomposition $c = c_1 + \cdots + c_m$ in which each $c_i \in \mathcal{C}$ is primitive and conforms to c. Our proof of Theorem 1.2 is much like Seymour's proof in [9, (14.5)] with the added concern of keeping track of parities.

Proof of Theorem 1.2. $(1 \to 2)$ Any $c \in \mathcal{C}$ decomposes as $c = c_1 + \cdots + c_m$ in which each c_i is primitive and conforms to c. Now because each c_i conforms to c and satisfies $f(c_i) \in I(c_i)$ we may add all of the associated inequalities to obtain $f(c) \in I(c)$.
$(2 \to 1)$ Follows trivially from the fact that primitive chains are chains.
$(3 \to 1)$ Suppose that $d \in \mathbb{Z}^E$ satisfies (3). Then, for any primitive $c \in \mathcal{C}$

$$f(c) = \sum_{e\in E} c(e)d(e) = \sum_{e\in|c|_+} d(e) - \sum_{e\in|c|_-} d(e) \leq u(c).$$

Similarly, $f(c) \geq l(c)$ and so $f(c) \in I(c)$, as required.
$(1 \to 3)$ Suppose that $f(c) \in I(c)$ for all $c \in \mathcal{C}$. We proceed by induction on the quantity $L(I) = \sum_{e\in E} \big(u(e) - l(e)\big)$. If $L(I) = 0$, then $u(e) = l(e)$ for all e and so $f(c) \in I(c)$ implies that $I(c) = [f(c), f(c)]$ for each c. Thus

$$\sum_{e\in E} u(e)c(e) = f(c),$$

which along with the fact that $u(e) = l(e)$ is odd for each $e \in E$ makes $u = l$ our desired element of $\mathbb{Z}^E$.

Now take interval assignment I such that $L(I) > 0$. Thus there is $e_0 \in E$ for which $u(e_0) - l(e_0) > 0$. Since $l(e_0)$ and $u(e_0)$ are both odd, $u(e_0) - l(e_0) \geq 2$. Let I' be an interval assignment such that $I'(e) = I(e)$ for all $e \neq e_0$ and

$$I'(e_0) = [l(e_0), u(e_0) - 2].$$

Also, let I'' be an interval assignment such that $I''(e) = I(e)$ for all $e \neq e_0$ and $I''(e_0) = [l(e_0) + 2, u(e_0)]$. Evidently $L(I') = L(I'') = L(I) - 2$ and for all $e \in E$ the intervals $I'(e)$ and $I''(e)$ have odd-valued endpoints. So, if either $f(c) \in I'(c)$ for all primitive $c \in \mathcal{C}$ or $f(c) \in I''(c)$ for all primitive $c \in \mathcal{C}$, then by induction we get $d \in \mathbb{Z}^E$ satisfying (3). If neither of these conditions hold, then there are primitive chains $c_1, c_2 \in \mathcal{C}$ with $e_0 \in |c_1|_+ \cap |c_2|_+$ such that $f(c_1) \notin I'(c_1)$ and $f(c_2) \notin I''(c_2)$. Because $f(c) \equiv \text{length}(c) \pmod 2$ for all $c \in \mathcal{C}$ and $l(e)$ and $u(e)$ are odd for each $e \in E$,

$$f(c) \equiv \text{length}(c) \equiv l(c) \equiv u(c) \pmod 2$$

for all $c \in \mathcal{C}$. So now because $f(c_1) \in I(c_1)$ and $f(c_2) \in I(c_2)$ we get that

$$f(c_1) = u(c_1) = \sum_{e \in |c_1|_+} u(e) - \sum_{e \in |c_1|_-} l(e) \quad \text{and}$$

$$f(c_2) = l(c_2) = \sum_{e \in |c_2|_+} l(e) - \sum_{e \in |c_2|_-} u(e).$$

Replace c_2 with $-c_2$ to obtain

$$f(c_2) = u(c_2) = \sum_{e \in |c_2|_+} u(e) - \sum_{e \in |c_2|_-} l(e)$$

where we now have $e_0 \in |c_1|_+ \cap |c_2|_-$. Now let $z = c_1 + c_2$. Since we have already proven $1 \longleftrightarrow 2$ we get the following in which $A = (|c_1|_+ \cap |c_2|_-) \cup (|c_1|_- \cap |c_2|_+)$:

$$\begin{aligned}
f(z) &\leq u(z) \\
&= \sum_{e \in |z|_+} z(e)u(e) + \sum_{e \in |z|_-} z(e)l(e) \\
&= \sum_{e \in |c_1|_+} u(e) - \sum_{e \in |c_1|_-} l(e) + \sum_{e \in |c_2|_+} u(e) - \sum_{e \in |c_2|_-} l(e) - \sum_{e \in A}(u(e) - l(e)) \\
&= f(c_1) + f(c_2) - \sum_{e \in A}(u(e) - l(e)) \\
&= f(z) - \sum_{e \in A}(u(e) - l(e)).
\end{aligned}$$

Now because $e_0 \in A$ we obtain $0 \leq -\sum_{e \in A}(u(e) - l(e)) \leq -2$, a contradiction. □

3. A new proof of the Minty-Vitaver Theorem

Say that f is an acyclic orientation of G for which $\mathrm{ratio(C, f)} \leq \mathrm{k} - 1$ for all cycles C. Let $\mathcal{C}$ be the integer cycle space of the directed graph (G, f) and assign to each e the interval $[1, k-1]$. Now and let $c \in \mathcal{C}$ be any primitive and also elementary chain. (We will consider all primitive chains later.) Thus $|c|$ corresponds to the edge set of a cycle, call it C, in G where without loss of generality $|c|_+ = C_+$ and $|c|_- = C_-$ with respect to f. By assumption we have that $|C_+|/|C_-|$ and $|C_-|/|C_+|$ are both at most $k-1$. Thus we have that

$$l(c) = \sum_{e \in |C_+|} 1 - \sum_{e \in |C_-|} (k-1) = |C_+| - (k-1)|C_-| \leq |C_+| - |C_+| = 0$$

and also that

$$u(c) = \sum_{e \in |C_+|} (k-1) - \sum_{e \in |C_-|} 1 = (k-1)|C_+| - |C_-| \geq |C_-| - |C_-| = 0.$$

Thus $l(c) \leq 0 \leq u(c)$ for any primitive and elementary $c \in \mathcal{C}$. Now if c is any primitive chain in $\mathcal{C}$, then because c is orthogonal to all of the chains in the cocycle space $\mathcal{C}^*$ (in particular, the vertex cocycles) $|c|$ corresponds to a subgraph of the directed graph (G, f) in which each vertex has equal in-degree and out-degree. Thus $|c|$ partitions into edge-disjoint directed cycles in (G, f), thus $c = c_1 + \cdots + c_n$ in which each c_i is primitive, elementary, and conforms to c. So now since we have proven that $l(c_i) \leq 0 \leq u(c_i)$ for each c_i, we now have that $l(c) \leq 0 \leq u(c)$. Thus Ghouila-Houry's Theorem (which is a special case of Theorem 1.1 within the ring of integers) implies that there is $d \in \mathcal{C}^*$ for which $d(e) \in [1, k-1]$ for each edge e in G. A chain in $\mathcal{C}^*$ is often called a *tension* in graph theory. Thus $d \pmod{k}$ is a nowhere-zero $\mathbb{Z}_k$-tension. It is well known that, for a connected graph G, the proper k-colorings of G using color set $\mathbb{Z}_k$ correspond k-to-1 to the nowhere-zero $\mathbb{Z}_k$-tensions of G. (See, for example, the discussion of potential functions in [1, Sec. 5.3].)

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