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Stochastic Mistuning Simulation of Integrally Bladed Rotors using Nominal and Non-Nominal Component Mode Synthesis Methods

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Stochastic Mistuning Simulation of Integrally Bladed Rotors using Nominal and Non-Nominal Component Mode Synthesis Methods

A thesis submitted in partial fulfillment of the requirements for the degree of Masters of Science in Engineering

by

Joseph A. Beck
B.S.M.E, United States Air Force Academy, 2006

2010
Wright State University
I HEREBY RECOMMEND THAT THE THESIS PREPARED UNDER MY SUPERVISION BY Joseph A. Beck ENTITLED Stochastic Mistuning Simulation of Integrally Bladed Rotors using Nominal and Non-Nominal Component Mode Synthesis Methods BE ACCEPTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF Master of Science in Engineering.

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ABSTRACT


Mistuning prediction in integrally bladed rotors is often done with reduced order models that minimize computational expenses. A common model reduction technique used for mistuning applications is the component mode synthesis method. In this work, two modern component mode synthesis methods are used to generate mistuned response distributions that will be used to determine if the two methods are statistically indistinguishable. The first method, nominal mode approximation, assumes an airfoil geometric perturbation alters only the corresponding substructure modal stiffnesses while its mode shapes remain unaffected. The mistuned response is then predicted by a summation of the nominal modes. The second method, non-nominal mode approximation, makes no simplifying assumptions of the dynamic response due to airfoil geometric perturbations, but requires recalculation of substructure matrices and mode shapes with each iteration. The number of retained fixed interface normal modes for the non-nominal method are increased until there is satisfactory accuracy compared to full finite element model results. Each approach is employed for calculating the mistuned response of a simple academic rotor and an advanced rotor with complex geometries. Three different veering regions are investigated in the advanced test case. Results indicate there is minimal difference between response distributions generated by the nominal and non-nominal methods for the academic rotor. Large differences were observed for the advanced rotor, where the nominal method typically predicted conservative response levels larger than non-nominal predictions.
List of Symbols

\( n \)  
Number of blades on the IBR

\( s \)  
Substructure reference number

\( M^{(s)} \)  
Substructure \( s \) physical coordinate mass matrix

\( K^{(s)} \)  
Substructure \( s \) physical coordinate stiffness matrix

\( u^{(s)} \)  
Substructure \( s \) physical coordinate displacement vector

\( f^{(s)} \)  
Substructure \( s \) physical coordinate applied force vector

\( \Psi^{(s)} \)  
Substructure \( s \) C-B component mode matrix

\( p^{(s)} \)  
Substructure \( s \) generalized coordinate displacement vector

\( \hat{M}^{(s)} \)  
Substructure \( s \) generalized coordinate mass matrix

\( \hat{K}^{(s)} \)  
Substructure \( s \) generalized coordinate stiffness matrix

\( \hat{f}^{(s)} \)  
Substructure \( s \) generalized coordinate applied force vector

\( \omega \)  
Natural frequency

\( \Lambda \)  
Diagonal matrix of eigenvalues

\( \phi \)  
Fixed interface normal mode

\( \Phi \)  
Complete set of fixed interface normal modes

\( \Psi_c \)  
Interface constraint mode matrix

\( S \)  
Substructure coupling matrix

\( \hat{K}_{CB} \)  
Assembled system stiffness matrix

\( \hat{M}_{CB} \)  
Assembled system mass matrix

\( N_T \)  
Total degrees of freedom of the assembled system

\( N_k^{(s)} \)  
Number of retained fixed interface normal modes for substructure \( s \)

\( N_b^{(s)} \)  
Number of degrees of freedom at substructure \( s \) interface

\( \delta^{(s,m)} \)  
Frequency mistuning perturbation coefficient

\( \Delta X \)  
Matrix of measured coordinate deviations

\( \bar{x} \)  
Variable mean

\( \Sigma \)  
First order covariance matrix of \( \Delta X \)

\( \Psi \)  
Matrix of eigenvectors

\( Z \)  
Principal component score matrix

\( \xi \)  
Fraction of critical damping

\( \Omega \)  
Imposed forcing frequency

\( F^{(s)} \)  
Substructure \( s \) imposed harmonic forcing function

\( C \)  
Engine order excitation number
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Dedicated to
Samuel Joseph Beck
Introduction

The following chapter introduces the primary subject of this research effort. First, the operation of a turbine engine is described and the integrally bladed rotor (IBR) component is introduced for further discussion. IBR modeling and implications of inherent dynamic characteristics are then described. Furthermore, IBR cyclic symmetry modeling and the need for reduced order models (ROM) are outlined. A synopsis of IBR forced response is given, as well as current IBR design aides that assess vulnerability to large amplitude vibration. High cycle fatigue (HCF), a problematic failure mode for IBRs, is then outlined, along with current design techniques to ensure IBR reliability. Mistuning, a vibrational phenomenon and principal driver of HCF, is finally described and established as the central focus of this research.

1.1 The Turbine Engine

The accomplishments of the turbine engine since its inception in the 1940’s rendered it one of the most well-recognized and versatile power systems. From generating thrust for commercial and military aerospace platforms to producing torque for electrical generators, the turbine engine represents a great achievement over a multitude of complex engineering problems requiring complex solutions. These problems encompass many different technical disciplines, each engulfing a vast amount of research. This work will use these past research efforts to model and better understand physical phenomena that will aid and im-
prove the design of turbine engine components from a structural dynamics standpoint.

1.1.1 Turbine Engine Components

The turbine engine is composed primarily of three constituents: a compressor, combustor, and turbine. Figure 1.1 on page 9 illustrates these primary modules with a cross-sectional view of a turbine engine. The process first begins with the compressor where rotating components exchange mechanical energy to an air stream. The airflow enters the engine and passes through a decreasing annulus where the flow is compressed by successive stages of stationary and rotating components. The stationary components are referred to as stators and diffuse the airflow. The rotating components are referred to as rotors, where each is comprised of airfoils attached to a disk. The blades are either inserted or the disk/blade structure is machined from a single metal forging or manufactured through welding airfoils to the outer diameter of the disk. The latter type is referred to as Integrally Bladed Rotors (IBR) or Blisks (Bladed Disks). The research conducted herein further focuses on IBR components.

After the air is compressed, fuel is added and the air is burned in the combustor. In this stage of the propulsion system, the thermal energy of the flow stream is increased by the exothermic chemical reaction between the fuel and oxygen in the airflow. The subsequent increase in energy in the flow is extracted via the turbine section. The turbine is fundamentally the reverse process of the compressor, where the garnered energy powers the compressor. The remaining amount of flow energy is either left to produce thrust for aircraft applications or torque through further turbine expansion.
1.1.2 Integrally Bladed Rotors

1.1.2.1 Modeling

IBRs based on nominal design parameters are a rotationally periodic structure. This implication mandates that each disk-blade sector is an exact replica of its neighboring sector and, consequently, the entire IBR rotor can be modeled by a single fundamental disk-blade segment. The significance of this cyclic symmetry assumption is utilized particularly in Finite Element Analysis (FEA) because the size of the model, and hence computational requirements, will be significantly reduced. As a result, solution time is greatly decreased and the design space can be explored to a much greater extent.

Unfortunately, there are small differences in the geometric and material characteristics of individual blades causing the cyclic symmetry assumption to break down. The ramifications of this is two-pronged. First, the model size and hence, solution time, is greatly increased because the entire rotor must now be solved in an FEA application. Second, the dynamic properties of the rotor change from the nominal case and individual blades can experience forced response levels greater than predicted in a tuned analysis. This second implication is referred to as mistuning and is further described in Sec. 1.1.4. Overcoming the resulting expansion in model size and the ability to effectively model mistuning in IBRs is of further focus for this research effort.

1.1.2.2 Forced Response

The forced response of individual blades alluded to in the previous section is driven by the interaction between successive rotor and stator stages. As the airflow passes through the stationary stators, the downstream air is distorted with regions of high and low pressure. Additionally, since the stators are spaced symmetrically around the circumference of the annulus, the regions of high-low pressure are also symmetrically spaced. As the spinning rotor passes through these alternating pockets of high and low pressure, a harmonic forcing
function is imparted on the blades that drives a harmonic forced response. The frequencies of the response are then functions of the rotational speed of the rotor. Not only will the characteristic frequency be an integer multiple of the speed, but there is also a characteristic shape since the forcing function is being simultaneously applied to all the blades. Furthermore, a given rotor stage can also be excited by downstream stators and struts, but the upstream stators are the principal excitation sources [1].

As the excitation frequency approaches or equals the airfoil resonant frequency, the forced response amplitude dramatically increases. The airfoil resonant frequency and mode shapes are inherent dynamic characteristics of the component. The dynamic characteristics of a bladed assembly differ from an individual blade due to the coupling of blades through the disk and/or shrouds. Hence, IBRs experience system modes of vibration which are influenced by individual airfoil modes, support structures, speed, temperature, and damping. Accurately modeling airfoil mode shapes is essential because they are used to determine IBR forced response amplitudes [2]. This fact is important to remember when assessing the reliability of models that rely on nominal airfoil mode shapes. Furthermore, correctly modeling material damping prevents unbounded forced response amplitudes in IBRs since there is no mechanical damping achieved at the blade-disk interface - common only to inserted-blade rotors. However, there is little material damping associated with monolithic metal alloys used to construct IBRs, so resonance avoidance is crucial to limiting large amplitude vibration [3].

1.1.2.3 Resonance Avoidance

The lack of damping available to an IBR makes resonance avoidance critical in blade design. Designers rely upon use of a Campbell diagram, shown in Fig. 1.2 on page 10. This diagram displays the concurrence of airfoil resonant frequencies and engine order excitations over a range of frequencies and rotational speeds. Where the engine order lines (sloping upward to the right from the origin) cross lines of airfoil resonant frequencies,
resonance will occur at the IBR rotational speed, i.e., RPM on the abscissa. The lines of airfoil resonant frequencies can either slope upward due to stress-stiffening with increased RPM or downward due to temperature effects. In addition, two closely spaced airfoil resonant frequency lines can indicate any asymmetry on the otherwise cyclically symmetric IBR [2].

It is an industry standard design practice to situate frequencies of integral order vibration in low engine orders (2E, 3E, and sometime 4E and 5E) either outside critical operating ranges or at lower RPMs in the engine operating range. Critical operating ranges of a turbine engine in aerospace applications are typically idle, cruise, and max speed. Unavoidable crossing of resonant blade frequencies with known engine excitations may occur for blade designs, but usually these are for presumed weaker, higher order modes. This presumption should be examined in certain cases, depending upon magnification factors, such as blade damping [2].

In addition, the need for high-performance turbomachinery has produced complex, highly swept blade geometries with low aspect ratios [4]. Such geometries are subject to high modal density that makes resonance avoidance a difficult task, often requiring many design iterations until a satisfactory design is produced [5]. Compounding the problem, IBRs seldom experience pure bending or pure torsional modes. Instead, torsional and bending oscillations become coupled in each mode of vibration with varying amounts of contributions from each type of deformation - thus signifying further importance of obtaining accurate mode shapes in mistuning analyses. Furthermore, the minimal material damping and lack of mechanical interface damping discussed in the previous subsection compounds the difficulty of resonance avoidance. Considering these points, blade vibration seems to be all but eliminated. As a result, failure modes such as high cycle fatigue (HCF) are a constant threat to the integrity of the IBR, and ultimately, the turbine engine and its platform.
1.1.3 High Cycle Fatigue

High cycle fatigue is responsible for a large amount of component failures in the modern gas turbine engine [1]. Furthermore, between 1982 and 1996 HCF accounted for 56% of Class A engine-related failures that cause loss of human life or at least one million dollars in damage. In 1994, the required maintenance man-hour expenditures for risk management inspections surpassed 850,000 - carrying a financial burden of $400 million per year across DoD aircraft [6]. The pervasiveness of this failure mode manifests from a lack of detailed information on vibratory loading and component dynamic response, as well as unknown material capabilities when subjected to HCF.

Implementation of a “damage tolerant” approach to managing HCF in a fleet of engines is impractical. In this approach, remaining component life is based on predictions of a crack propagation rate related to an inspectable flaw size. Such an approach to the HCF problem is unreliable because required inspection sizes are below state-of-the-art inspection techniques and the number of cycles to failure are extremely large due to the high frequencies entailed [7]. Moreover, a relatively large fraction of component life for crack initiation is required for HCF, resulting in a small fraction of life in which the crack actually propagates. This concept is illustrated in Fig. 1.3 on page 11. Considering that HCF cycles can rapidly accumulate due to forcing frequencies reaching into a kHz regime, engine components can fail in a matter of minutes - well before any precursors are detected. Thus, the more urgent issue related to HCF is the existence of vibratory stresses from unexpected drivers and forced response induced stresses that exceed material capabilities.

To deterministically design for HCF, a threshold at which the failure mode will not occur, usually $10^7$ cycles of oscillations, is necessary and accomplished through the use of a Modified Goodman Diagram, depicted in Fig. 1.4 on page 12. The abscissa is the steady stress value while the ordinate represents the alternating stress value. The threshold is constructed by drawing a straight line from the data point corresponding to fully reversed loading, $R = -1$, to the ultimate tensile strength of the material on the abscissa. Data to
create the diagram is empirically obtained, with scatter handled through statistical analysis to establish a lower bound of material capabilities. A factor of safety can also be appended to provide an additional margin from finite life, typically to account for the rather uncertain nature of peak vibration amplitudes. Additionally, a maximum allowable vibratory stress may also be imposed [7]. The infinite life region is then the remaining shaded portion falling below and to the left of established allowables. This area provides a permissible vibratory stress as a function of mean stress, where the latter is closely tied to the rotational velocity of the engine. Anything outside this threshold is considered finite life and will fail somewhere before $10^7$ cycles.

A deterministic approach to HCF avoidance can lead to overly conservative, heavy blade designs. This is a detriment to high performance blades because they require advanced geometries and reduced weight. Thus, the impact of vibrations must be quantified to reduce over conservatism introduced in the Goodman diagram. Quantification of the vibratory stresses cannot be accomplished empirically as this would require thousands of full IBRs at a cost of several thousand dollars apiece. Thus, advanced physics-based models are needed to predict and quantify forced response distributions for a population of mistuned IBRs. This probabilistic approach based on the statistics of vibration induced stresses provides a quantitative method to arrive at design allowables.

1.1.4 Mistuning Defined

As previously introduced in Sec. 1.1.2.1, mistuning is referred to as forced response amplifications due to perturbations of airfoil characteristics that break down the rotational periodicity of an IBR. These perturbations can arise from manufacturing deviations (even if the deviations are within established tolerances), material property disparity, or non-uniform component wear; geometric perturbations are therefore unavoidable and random. The amplification in forced response levels has a subsequent increase in stresses that are not predicted by a tuned analysis and can lead to rogue failures.
Mistuning is driven by system frequency splitting and double modes. Where a cyclically symmetric IBR will have repeated modes excited by a single frequency, loss of symmetry will cause a mode occur at two closely spaced frequencies. If there is a high modal density, a mode can occur at multiple frequencies. In addition, weak internal coupling gives rise to mode localization, where energy associated with a particular mode become localized in a specific region [8]. The phenomena occurs because modes dispersing from the energy source are reflected at boundaries of nearly similar (yet still dissimilar) disk-blade sectors making up the nearly cyclically symmetric IBR. The resulting confinement in energy causes amplitudes higher than predicted by an analysis with assumed periodicity, with possibly catastrophic consequences in HCF [9]. The mode localization phenomenon is illustrated and compared to the tuned response in Fig 1.5 on page 13, where red indicates large displacement and blue is minimal displacement. Note that blade modes shapes are symmetric around the rotor in the tuned response in Fig. 1.5(a), while the energy is localized to a single blade in the mistuned response in Fig. 1.5(b).

It is important to note that mistuning is not a deterministic process. Even if it were possible to manufacture blades that were, in fact, exactly design intent, non-uniform wear from fielded use would introduce random geometric perturbations. Due to the randomness of the geometric deviations and the subsequent sensitivity of mode localization to these deviations, mistuning is a stochastic process. Investigating mistuning as a strict deterministic process restricts the ability to look at full population statistics and derive confidence intervals of forced response amplitudes, and ultimately stress and fatigue life.
Figure 1.1: Industrial turbine engine
Figure 1.2: Typical airfoil Campbell diagram showing selected mode crossings [1]
Figure 1.3: Schematic showing HCF crack length as a function of fatigue life [7]
Figure 1.4: Schematic representation of a Modified Goodman Diagram [7]
Figure 1.5: Modes of tuned and mistuned rotors
Mistuning Research Review

This chapter outlines a review of published literature that researched how to model the mistuning phenomenon and its stochastic nature. In Sec. 2.1, a wide variety of developed models are outlined and the benefits and shortcomings of each for predicting IBR forced response are highlighted. These efforts are restricted to deterministic forced response predictions. Sec. 2.2 then describes research conducted on the stochastic nature of mistuning and the attempts to probabilistically quantify forced response distributions for a population of randomly mistuned IBRs. From this literature review, a need for additional research is proposed and discussed in Sec. 2.3 with an overview of the research presented in Sec. 2.4.

2.1 Reduced Order Model Research

The following subsections outline previous research efforts that have been conducted that have done two things:

1. Identified the mistuning phenomenon as an explanation for rogue blade failures

2. Developed various reduced order models (ROM) to alleviate the computational burden of full IBR finite element analysis

In the first subsection, models that assume lumped system parameters to model mistuning are described. The second subsection outlines a broad range of ROMs originating from finite element models. The methods in which mistuning is implemented further categorizes
these ROMs. Either mistuning is implemented in a modal stiffness perturbation or through geometric perturbations. Finally, the last subsection further describes past research efforts to ROMs developed with Craig-Bampton Component Mode Synthesis decomposition for geometric mistuning.

2.1.1 Early Mistuning Research

Fundamental mistuning research began with lumped parameter models that demonstrated blade response amplification due to minor dimensional variations [10, 11, 12, 13, 14]. These models varied from author to author but common to each is the coarse representation of disk and blade dynamic interaction through the use of lumped masses, stiffness springs, and dampers. The values assigned to these system variables have to be determined through difficult parameter identification, which becomes arduous as the number of model degrees of freedom (DOF) increases. Such a model is illustrated in Fig (2.1). While these papers effectively identified that mistuning could be the culprit of a few “rogue” blade failures, the magnitude of blade response amplification predicted by each author varied greatly. Since each author used a different model, these varying quantitative predictions are not an unexpected result. The strong dependence between a specific lumped parameter model and the mistuning amounts, i.e., variation in results from author to author, emphasized the need for more rigorous investigation that would come with the advent of finite element modeling and computational advancements.

2.1.2 Mistuning Research with the Advent of Finite Element Models

A multitude of current mistuning models are based on early work by Craig and Bampton that developed a methodology for treating a complex structure as an assemblage of components, or substructures [16, 17]. The established work became known as the Craig-Bampton Component Mode Synthesis (C-B CMS) method and uses a retained set of constraint modes
Figure 2.1: Lumped Parameter Mistuning Model [15]
and fixed-interface normal modes to model the dynamic behavior of a component. Here, normal modes refer to eigenvectors that are classified according to the imposed interface boundary conditions. Unfortunately, this method can result in large ROMs if the number of interface DOF are large.

Castanier, et al., used a component mode approach that modified the traditional CMS approach to reduce the size of the ROM through the use of disk induced constraint modes [18]. The disk motion is described by FEM mode shapes of a disk with massless blades attached. The blade motion is then represented as a summation of the blade displacements caused by disk modes and blade displacements resulting from cantilevered modes. This methodology does not require any interface constraint modes and still achieves disk-blade coupling. The intent sought to provide two advantages: first, a ROM that limits the large amount of interface DOF seen in traditional C-B CMS techniques to improve model solution time; and second, to provide direct access to blade frequencies (modal stiffnesses) for easy perturbation, i.e., intentional mistuning. As a corollary to blade frequency perturbation, this method assumes that the mistuned response can be represented by a linear combination of tuned modes. As such, the impact of geometric blade perturbations on blade mode shapes is neglected and provides a source of error. Approximations compared well with full finite element models subject to Young’s Modulus perturbations. It is important to note that this type of perturbation does not alter blade mode shapes, so the full models were only an approximation geometric perturbations. Initial results indicated overly stiff disk-blade interfaces that had to be alleviated by iteratively adjusting the blade modal stiffnesses. This method has been used extensively in modern industry mistuning efforts and lead to the software referred to as REDUCE [6]. The work would later be modified by Bladh, et al., to encompass IBRs with shrouds [19].

Around the same time the previous effort was being developed, Yang and Griffin developed a new ROM from a parent FEM based on the receptance method [20]. This method revolves around the knowledge that a substructure’s dynamic response is driven by envi-
ronmental interactions at the boundaries. With this in mind, the entire substructure DOF can be expressed in terms of those boundary DOF, thus reducing the model size. Once the boundary DOF are determined, substructure response can be calculated and ultimately the whole structure response. In a mistuned IBR, the substructures are composed of the blades and disk. The behavior of the blades and disks are then determined by modal analysis in terms of the interface DOF. It was assumed that the disk-blade interfaces were subjected to rigid body translations and rotations so blade vibration could be determined by a combination of blade base motion and cantilevered blade modes - similar to the REDUCE method. This method greatly reduced the model size (only 6 DOF at the disk-blade interface), but accuracy was limited when modes from two different families from the blade and disk were excited simultaneously.

Later, Yang and Griffin developed a new mistuning model that represents the mistuned modes in terms of a limited sum or subset of “nominal” system modes [21]. Likewise, the new method is referred to as the Subset of Nominal Modes (SNM) technique, but it has also been referred to as the Modal Domain Approach (MDA). This new approach was based on a previous paper by the same authors that illustrated that closely spaced modes in a perturbed system can be approximated as a sum of the closely spaced nominal modes [5]. In this effort there is no substructure as the entire IBR is treated as a single structure. Equations of motion are transformed from the physical domain into a domain of nominal system modes, and a ROM is built by limiting the amount of kept modes. A modal eigen-problem is formed and solved so the mistuned response is formed as a weighted sum of nominal modes and the eigenvectors determined in the analysis are the weighting coefficients. Mistuning is introduced by Young’s Modulus perturbations of the blades and an efficient procedure calculates the changes in the modal stiffness matrix. As in previously described efforts, validation is performed on a FEM that is only an approximation of geometric mistuning. This method is similar to work outlined by Lin and Mignolet [22], except that solution process solves for the mistuned forced response by an impedance approach and an adaptive
perturbation scheme.

Feiner and Griffin would later extend the SNM technique to take advantage of when the nominal modes used in the mistuning representation are limited to a single family with the strain energy primarily in the blades [23]. The method, referred to as the Fundamental Mistuning Model (FMM), requires only two sets of input parameters to predict mistuned modes and system natural frequencies: first, the nominal system frequencies; and second, deviations of individual blade frequencies from their tuned values. The theory behind the FMM method assumes that a mistuned system’s dynamic response depends only on blade frequency deviation and not the physical geometric perturbations that cause mistuning. Furthermore, the model has limited accuracy when more than a single family of modes are excited, the family’s frequencies are widely dispersed, and the vibrational strain energy does not coalesce in the blades. However, the simplicity of the model was used to identify fundamental parameters that control the mistuned response in later works [24, 25].

Shortly following the SNM method, Bladh, Castanier, and Pierre developed a ROM of mistuned IBRs based primarily on Craig-Bampton Component Mode Synthesis (C-B CMS) techniques reformulated for a cyclic symmetry description of the disk [26]. In the same work, a non-CMS technique was also formulated, along with a general idea of a secondary modal analysis reduction technique (SMART). The C-B CMS formulation decomposes the IBR into one disk and $n$ blades. The disk component is reduced to a cyclically symmetric component for two reasons: first, to achieve a smaller CMS model (and subsequent smaller solution time); and second, modal convergence is improved because disk component mode shapes resemble system mode shapes. Disk-blade interface stiffness is better represented than in the REDUCE technique, but there is a resulting increase in model size due to the retained interface DOF. SMART further reduces the model size by performing a full-scale secondary modal analysis on the C-B CMS system matrices. This method encompasses mistuning projection carried out in the low order modal domain. The C-B CMS obtained system matrices subjected to the SMART methodology were also compared
to a Young’s Modulus perturbed full FEM as in REDUCE and the inability to represent geometric perturbations are still apparent [27].

Petrov, Sanliturk, and Ewins presented a method for the dynamic analysis of a mistuned IBR based on an exact relationship between the tuned and mistuned system using a Sherman-Morrison-Woodbury identity [28]. The tuned sector is described by a sector model of the cyclic symmetry properties of the IBR. A key aspect of this method is the model size reduction to a “manageable” size without introducing any loss of accuracy during the reduction process, e.g. the loss of accuracy by retaining only a subset of substructure or system modes. An exact response of a mistuned system is obtained by considering a subset of DOF that encompass locations of applied mistuning and locations where forced response levels are of interest. If geometric perturbations occur over the blade, then a detrimental amount of DOF must be retained and the model size dramatically increases. In addition, the ROM corresponding to the retained DOF is as accurate as the tuned sector model and thus carries the same nominal mode assumptions previously described for REDUCE and SNM.

An investigation to include geometric perturbations was performed by Lim, et al [29]. A general ROM for mistuning was formulated by substructuring a mistuned IBR into two components: one tuned IBR and a set of virtual mistuning components that represent the differences of the mass and stiffness matrices between the tuned and mistuned blades. All the DOF in the mistuning components are then retained as interface DOF and a mistuning projection method is employed that does not necessarily require the tuned and mistuned mode shapes to be the same. The work considered large geometric deviations due to foreign object damage (FOD) of only a single blade. As the number of blades with geometric deviations increases, the number of retained interface DOF subsequently follow, resulting in large models. This work would be referred to as the Component Mode Mistuning (CMM) method.

A follow on effort to CMM was performed by Lim, Castanier, and Pierre to overcome
two primary CMM deficiencies: first, the resulting large model size for multiple blades with geometric deviations, and second, matrix ill-conditioning and numerical instability that occurs with the use of attachment modes [30]. The latter problem is caused by the much larger displacement values of the normal modes compared to the unit displacements of the attachment modes. To fix this discrepancy the number of retained normal modes must be decreased, but consequentially, so does the ROM accuracy. Instead of increasing the amount of attachment modes (equates to larger model size) to compensate for the loss of accuracy, the authors investigated a new, non-CMS technique using the mode-acceleration method with static mode compensation to account for geometric mistuning. A new set of basis vectors is established for the mistuned IBR by compensating the tuned normal modes with static modes. These static modes account for effects of mistuning as if they were generated by external forces. A much more expensive modal analysis is avoided with the new basis vectors and they approximately span the space of the mistuned system with better convergence than the tuned normal modes. This method is referred to as the Static Mode Condensation (SMC) method in the literature. Results compared well with a full FEM model, however, the SMC technique requires an abundant amount of analysis. Furthermore, the authors restricted the work to single geometrically perturbed blade and did not assume a general case of all blades having geometry deviations.

Sinha also developed techniques to account for changes in blade mode shapes due to airfoil geometric perturbations in the Modified Modal Domain Approach (MMDA) to mistuning [31]. The author addressed the inabilities of the SNM method in [21] to model mistuning in IBRs without a large number of tuned modes. This earlier method has been modified to include tuned modes of blades with geometric perturbations along Proper Orthogonal Decomposition (POD) features as basis functions. This POD method of representing the spatial statistics has been outlined by Garzon and Darmofal [32] and Sinha, et al. [33]. Results from a geometrically perturbed academic case study were excellent, however, the actual geometry of the airfoils was not used to generate the modal basis so the
technique is still an approximate basis for the airfoil mode shapes.

2.1.3 Geometric Mistuning C-B CMS Reduced Order Model Efforts

Brown would later address the limitations of the nominal-mode methods to model mistuning [3]. Nominal-mode approaches assume there is no change in airfoil mode shape and that mistuning can be modeled by introducing blade frequency deviations only. Airfoil geometric perturbations do, however, cause mode shapes to change in addition to blade frequencies. Ignoring this fact will introduce errors in the forced response prediction. To account for the effect of geometric perturbations, Brown developed two methods to predict the mistuned forced response of IBRs that are of primary focus of this research thesis.

Each developed technique requires a reduced order model that accurately represents airfoil geometry deviations. Measuring an airfoil surface with a Coordinate Measurement Machine (CMM) can result in thousands of node spatial coordinates requiring large computational costs if all physical DOF are retained. Brown proposed modeling geometric variations measured by a CMM through the use of Principal Component Analysis (PCA). Using this technique, geometric deviations of the example problem in Sec. 6 were reduced to 15 basis vectors that represent correlated geometry variations. These basis vectors can then be used to generate stochastic airfoil geometry models. Previous works have used PCA for geometric deviations [32, 33, 34, 3], where the theory is presented in Sec. 3.2 for application to an IBR with reference to Jolliffe [35].

The first mistuning method was the Nominal-Mode Approximation with geometrically perturbed FEM airfoil modal stiffnesses, referred to as NMA-λ_{FEM}. This method used C-B CMS as a reduction basis and the nominal-mode approximation to predict mistuned rotor response. Blade modal stiffnesses are obtained from geometrically perturbed FEMs and used as input into the C-B CMS technique. Using this method, Brown produced a mistuning prediction method similar to REDUCE, except it is not subject to overly stiff disk-induced constraint modes. Furthermore, previous C-B CMS approaches have not con-
sidered using geometrically perturbed FEMs for modal stiffness perturbations and have not been compared to mistuning levels obtained from a geometrically perturbed FEM. Results were of sufficient quality as compared to the geometrically perturbed FEM prediction and the method provides a good qualitative tool. The theory of the NMA-$\lambda_{FEM}$ method is presented in Sec. 3.1.2.

The second mistuning technique was the Non-Nominal Mode Approach with geometrically perturbed FEM mode shapes, referred to as NNMA. Again, a C-B CMS method is employed except with a non-nominal mode approximation with geometrically perturbed FEM mode shapes. While computational efficiency is diminished, this method will provide an exact solution in the limit of retained normal modes. Results proved very accurate and were almost exact to that of a geometrically perturbed FEM, while retaining 200 disk modes and 50 airfoil modes. The theory of the NNMA method is presented in Sec. 3.1.3.

### 2.2 Probabilistic Mistuning Research

As discussed in Sec. 1.1.4, mistuning is not a deterministic process and limiting research efforts to this assumption will provide only a single, specific instance in the forced response distribution. Thus, many mistuning research efforts have attempted to compute the statistics of forced response for a population of mistuned IBRs. Knowing the distribution of forced response for a given IBR type is important to a designer, particularly if a response maximum value will exceed some pre-determined critical value. This potentially could limit the amount of “rogue” HCF failures seen in fielded engines.

Sinha was one of the first to develop an analytical technique to calculate a mistuned IBR forced response statistics [36]. By assuming a Gaussian distribution of independent mass and stiffness matrices, this technique was established by modeling mistuned response as a combination of tuned and perturbed system matrices. The effort developed a closed form solution for the forced response distribution of a simple lumped parameter model that
yielded complete information about the probability density functions of a blade’s amplitude. However, the accuracy of the model was highly dependent upon the level of system damping.

Mignolet and Lin later developed a perturbation technique for approximation of forced response probability density functions (PDF) of a mistuned IBR [37]. This method, referred to as the combined closed-form perturbation approach or CFP, used an integral representation of the PDF of blade amplitudes under the assumptions that nonlinearities of the physical system are neglected. Deterministic perturbation techniques were then employed to generate the PDF. A simple lumped parameter model with only stiffness mistuning was used as a test case and results were compared to Monte Carlo simulations and the method proposed by Sinha [36]. Results proved to be more accurate than Sinha’s method and accurate for certain engine order excitations, however, accuracy was diminished for other engine orders, especially the zero engine order excitation condition. Furthermore, the method involved computationally expensive mathematical operations that were alleviated by assuming restrictive conditions on random blade characteristics.

The same authors would later develop and adaptive perturbation technique where the level of approximation can be varied to allow changes to the system parameters [22]. The results of this adaptive approach compared well for the same lumped parameter model. This technique would later be used by Mignolet, Lin, and LaBorde to develop a closed form blade response PDF in a mistuned IBR [38]. The accuracy of the resulting approach diminished when stated assumptions of blade-to-blade coupling were violated.

Bladh, et al., performed 1000 Monte Carlo simulations on the nominal mode REDUCE method to compute the statistics of the forced response and post-processed results to give blade stresses [39]. This simulation was used as a benchmark for accelerated Monte Carlo simulations using the theory of the statistics of extremes used later in the work. This method assumed a Weibull distribution to statistically determine the variation in forced response levels with 50 random mistuning patterns. Results compared well to the full Monte
Carlo simulations at the 5th, 50th, and 95th percentiles.

As an alternative to perturbation techniques and traditional Monte Carlo simulations, Bah, Nair, Bhaskar, and Keane developed a reduced basis approach for predicting the forced response statistics of randomly mistuned IBRs [40]. A linear combination of complex stochastic Krylov subspace basis vectors with undetermined coefficients is used to predict the system response. A stochastic Bubnov-Galerkin methodology is used to calculate the unknown coefficients of the Krylov vectors. The method predicted the mean and variance of the forced response amplitudes well compared to Monte Carlo benchmarks and proved better than classical perturbation methods. Accuracy deteriorated as the mistuning strength is increased, and it should be noted that the method does not generate the full distribution of forced response.

The following year Capiez-Lernout and Soize extended a new approach, a nonparametric model of random uncertainties, for vibration analysis of cyclically symmetric structures [41]. The new method accounts for both model and data uncertainties. This method begins with construction of the mean reduced matrix model, using Craig-Bampton component mode synthesis [16, 17] to calculate each uncertain substructure. Random uncertainties in each blade reduced matrices are modeled by a nonparametric probabilistic approach, where dispersion parameters control the nonparametric model. A non-physical parameter is introduced to account for model parameters that makes interpretation of model accuracy difficult to quantify. A follow-on effort by the same authors, with contributions of Lombard, Dupont, and Seinturier, would later estimate the dispersion parameters as a function of airfoil geometric tolerances [42].

Sinha would compute the statistics of the peak maximum amplitude of forced response over a range of inputs consisting of blade-to-blade structural coupling and standard deviation of mistuning [43]. A relationship between these two inputs and the forced response distribution was developed using Hermite polynomial coefficients and a multi-layer neural network. A simple lumped parameter model is used as an application. Results showed the
the distribution of peak maximum amplitudes does not follow a Weibull distribution, however, the neural network approach provided limited results. Using the same lumped parameter model, Sinha would later compute the statistics of forced response with blade modal stiffness mistuning using polynomial chaos [44]. Using this approach, a non-Gaussian distribution can be represented as an expansion of polynomial chaoses. Symmetry properties of IBRs ensure the number of polynomial chaoses required is reduced. Results for a third-order polynomial chaos expansion had sufficient accuracy when compared to Monte Carlo simulations over a typical range of mistuning and damping. Earlier work by Cha and Sinha attempted to compute the response statistics of mistuned rotors excited by white noise and narrow band excitation [45].

Scarselli and Lecce would also use a multi-layered neural network to efficiently predict the statistics of forced response [46]. In this case a 20-blade FEM is used with perturbations in blade Young’s modulus to introduce mistuning. Randomly mistuned models were then used to train the network for forced response prediction of a separate set of random mistuned models. In the first test, 67 mistuned rotors were used to train the net, which presented poor results. A later test with 920 training sets presented accurate results for the lowest responding mode, while forced response predictions in higher modes were still poor. The authors performed additional work using Genetic algorithms to predict the worst mistuning patterns in a population of rotors. This work is similar to that performed by Choi, Lentz, Rivas-Guerra, and Mignolet a year earlier [47].

Lee, Castanier, and Pierre investigated the accuracy and efficiency of numerous probabilistic approaches to predicting forced response distributions [48]. Two distributions were considered: forced response of a specific blade at a specific frequency and the peak rotor forced response amplitude over a range of frequencies. For most probable point (MPP)-based methods - including the first order reliability method (FORM), second order reliability method (SORM), and advance mean value (AMV+) - accurate predictions of the specific blade distributions were obtained, but the peak rotor response was not accurately predicted.
A response surface method (RSM) yielded similar results. A radius-based importance sampling method had poor results in both the specific blade and peak rotor responses. Finally, an accelerated Monte Carlo simulation method proved to calculate results accurately and efficiently. This probabilistic method proved to be superior to the reliability-based methods.

An experimental Monte Carlo mistuning simulation was performed by Li, Castanier, Pierre, and Ceccio that simulated mistuning on an experimental rotor by varying the external forcing function [49]. An actual bladed disk was manufactured and mistuning identification was performed to update the component mode mistuning (CMM) model described in [29]. The updated CMM model was used to determine the varied external forces based on random mistuning patterns. The experimental results were compared to numerical calculations of the CMM model. Initial tests showed sensitivity to environmental factors and experimentally forced response distributions were inaccurate. Follow-up tests with controlled environmental factors provided greater accuracy as the number of experimental simulations increased. For 80 rotors, results matched well with analytical results.

2.3 Research Requirement

Advanced physics-based models have been developed to predict mistuned response in a non-cyclically symmetric IBR. These models have the challenge of being computationally efficiently while simultaneously predicting forced response levels accurately. To remain efficient, the overall model size must be reduced from the upper-limit of a converged full finite element model. Many early works accomplished this by using lumped parameter models, but results only simulated the mistuning phenomenon. Finite element models produced a variety of advanced models, but the majority relied on the assumption that blade mode shapes do not change with mistuning. This assumption provides computational efficiency by allowing the mistuned response to be determined by a sum of tuned modes. This assumption is questionable because mode shapes are directly used in calculating the modal
force, and ultimately, the forced response level. However, geometric perturbations do cause changes in airfoil mode shapes and geometric mistuning models have been developed that do not make the tuned mode assumption.

While the physics-based models described above predict mistuning, they are deterministic. They have a mistuning pattern as an input and give a forced response level as an output, and with each input a different output will be generated. Much work has been done to develop probabilistic models that don’t predict individual IBR response levels, but response levels for a population of IBRs. Since mistuning is a stochastic process, a single mistuning pattern cannot quantify a distribution of forced response levels. The response distribution is needed quantify a probability of “rogue” blade failure resulting from mistuning induced HCF. These probabilistic models have variable accuracy, and many have been based on simple lumped-parameter models. Some have used Monte Carlo simulations of a single deterministic model to generate response distributions and as statistical model validation. However, the work in the literature stopped short of comparing Monte Carlo generated response distributions determined from different deterministic models.

With these facts in mind, there exists a need to assess the nominal mode assumption currently used as an industry standard. While there have been many cases of comparisons for an individual rotor, there has been no probabilistic comparison of forced response distributions generated by each method. Since mistuning is a stochastic process, limiting the comparison to a single mistuning case will provide only a specific instance in a response distribution. This restricts the ability to look at full population statistics and derive confidence intervals of forced response amplitudes, and ultimately stress and fatigue life. The work contained in this thesis will evaluate the nominal mode assumption over a population of mistuned rotors.
2.4 Research Overview

This work will rely on two deterministic mistuning models developed by Brown [3]: the NMA-λ_{FEM} and NNMA mistuning models. Since NNMA predictions approach the full FEA solution as the number of retained fixed-interface normal modes increases, this approach will be assumed the correct response. The number of normal modes will be determined so results converge to the full FEM solution. This will provide a valid baseline to compare the nominal mode assumption against. A Monte Carlo simulation will be conducted where the forced response level determined by each deterministic model is calculated for the same geometrically perturbed IBR over a population of mistuned IBRs. The two models are based on a Craig-Bampton component mode synthesis reduction technique and are described in the following chapter. A reduced order geometric model based on Principal Component Analysis is also presented that is used to generate additional random, geometrically perturbed IBRs. Two test cases are presented: a simple academic IBR and an advanced geometry IBR typically seen in turbine engine applications. The academic case study provides a qualitative overview of the nominal mode assumption. The advanced geometry IBR provides a rigorous test case that can fully determine the difference in forced response distributions calculated by Monte Carlo simulations of the NMA-λ_{FEM} and NNMA mistuning models.
3.1 Component Mode Synthesis

Large, complex structural systems subject to dynamic excitation are often too computationally expensive to analytically determine full-model forced response solutions. Simply reducing mesh density of the finite element model (FEM) is not always a viable option for decreasing the number of degrees of freedom (DOF), as this negatively impacts solution convergence. As an alternative to model reduction, some form of substructure coupling method, such as component mode synthesis (CMS) can be employed. The term component modes refer to Ritz Vectors, or assumed modes, that are basis vectors that describe nodal displacements within a substructure or component; e.g., eigenvectors are component normal modes that are just one category of component modes [50]. Each specific CMS method is characterized by the types of modes retained in the solution. The CMS approach used herein is referred to as the Craig-Bampton (C-B) method, which employs a combination of fixed-interface normal modes and interface constraint modes described in Section 3.1.1.

Three basic steps are performed in C-B CMS approach: division of a system FEM into components, definition and calculation of component modes, and coupling of the component modes to form the reduced-order model (ROM) of the system. The nature of these steps provides a natural fit for CMS application to turbomachinery FEMs, particularly in probabilistic mistuning studies. Rotors can be divided into components consisting of $n$ blades and the disk, totaling $n + 1$ components. By substructuring in this fashion
mistuning can be applied directly to blade component matrices and the disk component matrices need be computed only once - an attractive feature for probabilistic mistuning studies. Furthermore, C-B CMS is capable of providing the exact solution if, in the limit, all of the fixed-interface normal modes are retained for each component. These aspects provide ample justification for used of C-B CMS in this probabilistic mistuning study. The following section is derived from [51] and outlines the C-B CMS method as it pertains to turbomachinery.

### 3.1.1 Craig-Bampton Component Mode Synthesis

For turbomachinery, the disk and blade components share a common, redundant interface that is somewhat arbitrary for Integrally Bladed Rotors (IBRs), but is typically defined where the blade root can clearly be characterized. DOF falling on this interface are referred to as boundary coordinates while the remaining DOF are referred to as interior coordinates. Fig. 3.1 on page 45 illustrates a partitioning of a disk-blade sector into subcomponents and interface DOF. Note that only a sector of the disk is shown, where in the following derivations, the entire disk is a single substructure. The equation of motion (EOM) for a single, undamped component $s$ is of the form

$$M^{(s)} \ddot{u}^{(s)} + K^{(s)} u^{(s)} = f^{(s)}$$

(3.1)

where $M^{(s)}$, $K^{(s)}$, and $u^{(s)}$ are derived in the original physical coordinate system. The component’s physical displacement coordinates $u$ are transformed to component generalized coordinates $p$ by a Ritz coordinate transformation

$$u^{(s)} = \Psi_{CB}^{(s)} p^{(s)}$$

(3.2)

where $\Psi^{(s)}$ is the C-B component mode matrix composed of fixed-interface normal modes.
and interface constraint modes. The component modal model is then subject to the following EOM

$$\hat{M}^{(s)} \ddot{p}^{(s)} + \hat{K}^{(s)} p^{(s)} = \hat{f}^{(s)}$$  \hspace{1cm} (3.3)$$

where the component mass matrix, stiffness matrix, and force vector are given by

$$\hat{M}^{(s)} = \Psi^{(s)T} M_{CB} \Psi^{(s)}, \quad \hat{K}^{(s)} = \Psi^{(s)T} K^{(s)} \Psi^{(s)}, \quad \hat{f}^{(s)} = \Psi^{(s)T} f^{(s)}$$  \hspace{1cm} (3.4)$$

To derive the component modes used in the C-B method, Eq. 3.1 is partitioned according to

$$\begin{bmatrix} M_{ii} & M_{ib} \\ M_{bi} & M_{bb} \end{bmatrix} \begin{bmatrix} \ddot{u}_i \\ \ddot{u}_b \end{bmatrix} + \begin{bmatrix} K_{ii} & K_{ib} \\ K_{bi} & K_{bb} \end{bmatrix} \begin{bmatrix} u_i \\ u_b \end{bmatrix} = \begin{bmatrix} 0 \\ f_b \end{bmatrix}$$  \hspace{1cm} (3.5)$$

where \( i \) refers to interior coordinates and \( b \) refers to boundary coordinates.

Fixed-interface normal modes required for the C-B method are obtained by restraining all boundary DOF and solving the classical eigen-problem

$$\begin{bmatrix} K_{ii} - \omega_j^2 M_{ii} \end{bmatrix} \{ \phi_i \}_j = 0 \quad j = 1, 2, \ldots, N_i$$  \hspace{1cm} (3.6)$$

where \( N_i \) is the number of interior DOF. Combining the complete set of fixed-interface normal modes yields the matrix \( \Phi_{ii} \) and is assembled in the modal matrix

$$\Phi_i = \begin{bmatrix} \Phi_{ii} \\ 0_{bi} \end{bmatrix}$$  \hspace{1cm} (3.7)$$

If the fixed-interface normal modes are normalized with respect to the interior partition of the mass matrix, \( M_{ii} \), they satisfy
\[ \Phi_{ii}^T M_{ii} \Phi_{ii} = I_{ii} \quad \Phi_{ii}^T K_{ii} \Phi_{ii} = \Lambda_{ii} = \text{diag} \left( \omega_j^2 \right) \]  

(3.8)

Constraint modes are ascertained by statically deforming a structure by applying a unit displacement to one coordinate of an established set of constraint coordinates while the remaining coordinates of the set are restrained, and the remaining DOF of the component are force-free. Interface constraint modes are prescribed by using the boundary DOF as the established set of constraint coordinates, applying successive unit displacements on the boundary DOF, and leaving all of the interior DOF of the component force-free. This is given by

\[
\begin{bmatrix}
K_{ii} & K_{ib} \\
K_{bi} & K_{bb}
\end{bmatrix}
\begin{bmatrix}
\Psi_{ib} \\
I_{bb}
\end{bmatrix}
= 
\begin{bmatrix}
0_{ib} \\
R_{bb}
\end{bmatrix}
\]  

(3.9)

Solving Eq. 3.9 for \( \Psi_{ib} \) allows for the interface constraint mode matrix to be given by

\[
\Psi_c = 
\begin{bmatrix}
\Psi_{ib} \\
I_{bb}
\end{bmatrix}
= 
\begin{bmatrix}
-K_{ii}^{-1} K_{ib} \\
I_{bb}
\end{bmatrix}
\]  

(3.10)

As a check, these constraint modes are stiffness-orthogonal to all of the fixed interface normal modes.

From Eq. 3.3 and 3.4, the C-B transformation matrix is a combination of Eq. 3.7 and 3.10 given by

\[
\Psi_{CB}^{(s)} = 
\begin{bmatrix}
\Phi_i & \Psi_c
\end{bmatrix}^{(s)}
\begin{bmatrix}
\Phi_{ik} & \Psi_{ib} \\
0_{bi} & I_{bb}
\end{bmatrix}^{(s)}
\]  

(3.11)

and Eq. 3.12 is expanded as
\[ u^{(s)} = \begin{cases} u_i \\ u_b \end{cases}^{(s)} = \begin{bmatrix} \Phi_{ik} & \Psi_{ib} \\ 0 & I_{bb} \end{bmatrix}^{(s)} \begin{cases} p_k \\ p_b \end{cases}^{(s)} \]  

(3.12)

where the subscript \( k \) on \( \Phi_{ik} \) represents the number of kept fixed-interface modes. When the component fixed-interface constraint modes are normalized according to Eq. 3.8, the mass and stiffness matrices of the component modal EOM have the form

\[
\begin{align*}
\hat{M}_{CB}^{(s)} &= \begin{bmatrix} I_{kk} & \hat{M}_{kb} \\ \hat{M}_{bk} & \hat{M}_{bb} \end{bmatrix}^{(s)} \\
\hat{K}_{CB}^{(s)} &= \begin{bmatrix} \Lambda_{kk} & 0_{kb} \\ 0_{bk} & \hat{K}_{bb} \end{bmatrix}^{(s)}
\end{align*}
\]  

(3.13)

The coupling of components begins with the bottom row of Eq. 3.12, which implies that \( u_b^{(s)} = p_b^{(s)} \). Thus for a two component system, \( p_b^{(1)} = p_b^{(2)} = u_b \), and the coupled component modal coordinates \( p \) are transformed to a set of independent modal coordinates \( q \) through the component coupling matrix \( S \)

\[
\begin{align*}
\begin{bmatrix} p_k^{(1)} \\ p_b^{(1)} \\ p_k^{(2)} \\ p_b^{(2)} \end{bmatrix} &= S q = \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & I \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} q_k^{(1)} \\ q_k^{(2)} \\ u_b \end{bmatrix}
\end{align*}
\]  

(3.14)

that guarantees equality of DOF between components at the interface. Eq. 3.14 can be expanded for a multiple interface system, such as the \( n \) blade-disk interfaces of a rotor. The reduced system coupled EOM is then given by

\[ M_q \ddot{q} + K_q q = f_q \]  

(3.15)

where

\[ M_q = S^T M_p S, \quad K_q = S^T K_p S, \quad f_q = S^T f_p \]  

(3.16)
The assembled system stiffness matrix in Eq. 3.15 for a rotor with a disk and \( n \) blade components in C-B space is given by

\[
\hat{K}_{CB} = \begin{bmatrix}
\Lambda_{k_1 k_1}^{(1)} & 0_{k_1 k_2} & \cdots & 0_{k_1 b} \\
0_{k_2 k_1} & \Lambda_{k_2 k_2}^{(2)} & \cdots & \\
\vdots & \ddots & \ddots & \ddots \\
0_{b k_1} & \Lambda_{k_n k_n}^{(n)} & \ddots & \Lambda_{k_D k_D}^{(D)} \\
\end{bmatrix}
\] (3.17)

where

\[
\hat{K}_{bb} = \sum_{s=0}^{n} S_{s}^{T} \hat{K}_{bb}^{(s)} S_{s}
\] (3.18)

where \( \hat{K}_{bb}^{(s)} \) are the boundary portions of the disk and blade stiffness matrices in C-B space calculated from Eq. 3.13. The summation occurs over each of the \( s \) components, with \( s = 0 \) corresponding to the disk component or superscript \( D \) seen first in Eq. 3.17.

The assembled system mass matrix in Eq. 3.15 for a rotor with a disk and \( n \) blade components in C-B space is given by

\[
\hat{M}_{CB} = \begin{bmatrix}
I_{k_1 k_1} & 0_{k_1 k_2} & \cdots & \hat{M}_{k_1 b}^{(1)} \\
0_{k_2 k_1} & I_{k_2 k_2} & \cdots & \\
\vdots & \ddots & \ddots & \ddots \\
0_{b k_1} & \hat{M}_{k_n k_n}^{(n)} & \cdots & \hat{M}_{k_D b}^{(D)} \\
\end{bmatrix}
\] (3.19)

where
\[
\hat{M}_{bs}^{(s)} = \hat{M}_{bs}^{(T)} = \hat{M}_{bs}^{(s) T} S^{(s)} \quad s = 0, 1, 2, \ldots, n
\] (3.20)

and

\[
\hat{M}_{bb} = \sum_{s=0}^{n} S^{(s) T} \hat{M}_{bb}^{(s)} S^{(s)}
\] (3.21)

where \(\hat{M}_{bs}^{(s)}\) and \(\hat{M}_{bb}^{(s)}\) are the off-diagonal portions and boundary portions, respectively, of the disk and blade mass matrices in C-B space calculated from Eq. 3.13. The summation occurs over each of the \(s\) components, with \(s = 0\) corresponding to the disk component or superscript \(D\). This completes the assembly of the global system matrices. Each matrix will be of the size

\[
N_T = \sum_{s=0}^{n} N_k^{(s)} + \sum_{i=1}^{n} N_b^{(l)}
\] (3.22)

where \(N_k^{(s)}\) is the number of fixed-interface normal modes retained for each component and \(N_b^{(l)}\) is the number of DOF at a single interface. The first summation occurs over each of the \(m\) components, with \(m = 0\) corresponding to the disk component or superscript \(D\). The second summation occurs over each of the \(n\) disk-blade interfaces.

Solution of the system eigen-problem

\[
\left[ \hat{K}_{CB} - \omega_j^2 \hat{M}_{CB} \right] \{ \phi_i \}_j = 0 \quad j = 1, 2, \ldots, N_T
\] (3.23)

yields the C-B CMS system natural frequencies and eigenvectors in independent modal coordinates, \(q\). Combining the complete set of eigenvalues and eigenvectors yields \(\Lambda_{CMS}\) and \(\Phi_{CMS}\), respectively. Results can be expanded to physical space, \(u\), with the following transformation

\[
u^{(s)} = \Psi^{(s)} C^{(s)} q^{(s)}
\] (3.24)
3.1.2 Nominal Mode Approach with Geometrically-Perturbed FEM

Airfoil Modal Stiffnesses

The Nominal Mode Approach (NMA-\(\lambda_{FEM}\)) to mistuning from Brown [3] is implemented by using geometrically-perturbed FEMs to obtain the fixed-interface normal modes for \(\Lambda_{k_s, k_s}^{(s)}\), i.e., cantilevered airfoil natural frequencies,

\[
\Lambda_{k_s}^{(s,k_s)} = \lambda_{FEM}^{(s,k_s)} \quad s = 1, 2, \ldots, n \\
k_s = 1, 2, \ldots, N_k^{(s)}
\]  

(3.25)

where the second superscript \(k_s\) is the \(k^{th}\) retained mode of the \(s^{th}\) airfoil and \(\lambda_{FEM}^{(s,k_s)}\) is the FEM obtained \(k^{th}\) mode of the \(s^{th}\) airfoil. The CMS reduction maintains the use of the nominally-obtained \(\Psi_{CB}^{(s)}\), or C-B matrix of Eq. 3.11, where for the nominal IBR case

\[
\Psi_{CB}^{(1)} = \Psi_{CB}^{(2)} = \ldots = \Psi_{CB}^{(n)}
\]  

(3.26)

where \(n\) is the number of blades. Of course, this only holds if each substructure has the same and same number (e.g. \(k_1 = k_2 = \ldots = k_n\)) of retained fixed-interface normal modes and boundary constraint modes. In other words, the interface of each substructure must have the same mesh density as other airfoil substructures so the boundary constraint modes are the same. Furthermore, the same fixed interface normal modes must be retained for each airfoil, e.g., all airfoils retained the first ten normal modes. For the example problems in Chpt. 4, this is assumed. The C-B matrix is then

\[
\Psi_{CB-NMA} = \begin{bmatrix}
[\Phi_{ik}]^0 & [\Psi_{ib}]^0 \\
0_{bi} & I_{bb}
\end{bmatrix}
\]  

(3.27)

where the superscript 0 refers to the nominal substructure model. The airfoil substructure mass and stiffness matrices in the Craig-Bampton space from Eq. 3.4 are then found by
\[ \hat{M} = \Psi_{CB-NMA}^T \hat{M}_{CB-NMA} \Psi_{CB-NMA}, \quad \hat{K} = \Psi_{CB-NMA}^T \hat{K}_{CB-NMA} \Psi_{CB-NMA} \]  

(3.28)

where original mass and stiffness matrices \((M, K)\) are determined from the geometrically perturbed FEM. Solution of the system eigen-problem

\[ \left[ \hat{K}_{CB-NMA} - \omega_j^2 \hat{M}_{CB-NMA} \right] \{\phi_i\}_j = 0 \quad j = 1, 2, \ldots, N_T \]  

(3.29)

yields the NMA-\(\lambda_{FEM}\) C-B CMS system natural frequencies and eigenvectors in independent modal coordinates, \(q\). Combining the complete set of eigenvalues and eigenvectors yields \(\Lambda_{CMS}\) and \(\Phi_{CMS}\), respectively for NMA-\(\lambda_{FEM}\). Results can be expanded to physical space, \(u\), with the following transformation

\[ u^{(s)} = \Psi_{CB-NMA} S^{(s)} q^{(s)} \]  

(3.30)

Techniques such as REDUCE implement mistuning by perturbing the nominal airfoil component modal stiffnesses, i.e., cantilevered airfoil natural frequencies, \(\Lambda^{(s)}_{k_s k_s}\) from Eq. 3.17 according to

\[ \Lambda^{(s,m)}_{k_s k_s} = \left(1 + \delta^{(s,m)}\right) \Lambda^{(s,m)}_{k_s k_s} \quad s = 1, 2, \ldots, n \]

\[ m = 1, 2, \ldots, N^{(s)}_k \]  

(3.31)

where the second subscript \(m\) allows perturbation on the \(m^{th}\) retained mode of the \(s^{th}\) airfoil. The perturbation coefficient, \(\delta^{(s,m)}\), is a mistuning parameter that is a percentage of frequency mistuning and is usually prescribed or obtained by differences from average in experimental airfoil frequency measurements. The remaining portions of the reduced system stiffness matrix, \(\hat{K}_{CB}\), are unchanged. Furthermore, nominal mode approaches assume that only the nominal, cantilevered natural frequencies are perturbed while the re-
duced system mass matrix, \( \hat{M}_{CB} \), is derived from the nominal geometries. Expansion from independent modal coordinates, \( q \), is carried out according to Eq. 3.34, however, \( \Psi^{(s)}_{CB} \) from Eq. 3.11 is also derived from nominal geometry.

Since mistuning is only applied through modal stiffness perturbation, nominal mode approaches assume that mistuned modes of a blade can be accurately estimated by linear combination of tuned or nominal blade modes. Although the application of mistuning in NMA-\( \lambda_{FEM} \) is done with relative ease, neglecting the changes caused by geometric alterations in \( \hat{M}_{CB} \), \( \Psi^{(s)}_{CB} \), and remaining portions of \( \hat{K}_{CB} \) can negatively impact the accuracy of the approach. This will be shown in Chpt. 4.

3.1.3 Non-Nominal Mode Approach with Geometrically-Perturbed FEM

Mode Shapes

The Non-Nominal Mode Approach (NNMA) proposed by Brown [3] makes no assumptions about the lack of impact of geometric perturbations on system matrices. For the example problems of Chpt. 4, each airfoil substructure has the same number of retained modes, i.e., \( k_1 = k_2 = \ldots = k_n \), in the C-B matrix of Eq. 3.11. However, each mode is perturbed from the corresponding nominal mode so each airfoil substructure will have a different C-B matrix

\[
\Psi^{(1)}_{CB} \neq \Psi^{(2)}_{CB} \neq \ldots \neq \Psi^{(n)}_{CB} \neq \Psi_{CB-NMA}
\]  

(3.32)

where \( n \) is the number of blades. Solution of the system eigen-problem

\[
\left[ \hat{K}_{CB-NNMA} - \omega_j^2 \hat{M}_{CB-NNMA} \right] \{ \phi_i \} = 0 \quad j = 1, 2, \ldots, N_T
\]  

(3.33)

yields the NNMA C-B CMS system natural frequencies and eigenvectors in indepen-
dent modal coordinates, \( q \). Combining the complete set of eigenvalues and eigenvectors yields \( \Lambda_{CMS} \) and \( \Phi_{CMS} \), respectively for NNMA. Results can be expanded to physical space, \( u \), with the following transformation

\[
    u^{(s)} = \Psi_{CB-NNMA}^{(s)} S^{(s)} q^{(s)}
\]  

(3.34)

This method requires recalculation of C-B CMS matrices, beginning with Eq. 3.1, with each iteration of geometric perturbations for each airfoil substructure. Considering if a large percentage of DOF of an IBR FEM are in the disk, effective substructuring will contain a large amount of DOF to the disk. Furthermore, since the disk substructure is not altered during airfoil geometry perturbations, the disk portions of \( \hat{K}_{CB-NNMA} \) and \( \hat{M}_{CB-NNMA} \) of Eq. 3.33 do not change resulting in faster successive calculations of Eq. 3.33. This fact is important for probabilistic mistuning analysis where many mistuned IBRs will be successively evaluated for mistuned forced response.

3.2 Principal Component Analysis

3.2.1 Overview

Rotor dynamic response is highly sensitive to small geometric deviations and, as a result, gross-measurement quality control methods do not provide the quantitative details needed for mistuning studies. One requisite approach is the use of coordinate measurement machines (CMMs) that obtain a surface map of geometric locations through the use of a transversing probe. Consequently, thousands of data points are created but the computational expense of assessing the sensitivity of mistuning to each geometric perturbation at every location is impracticable. The use of such a measurement device warrants the need of reduced order models (ROMs) of blade geometry that still retain geometric deviations. An attractive approach is Principal Component Analysis (PCA) because it is based on fun-
Fundamental idea of reducing the dimensionality of interrelated data sets while the user can choose the amount of variation in the data set that is retained. By transforming the original, correlated data set to a new set of uncorrelated Principal Components (PCs), the first few PCs bear the majority of the variation in all the initial variables. Equations 3.35 - 3.37 below outline the covariance method of PCA, derived from [35] and explained for turbomachinery.

### 3.2.2 Principal Component Analysis Theory

As applied to blade geometry, suppose that \( x \) is a vector of \( p \) three-dimensional coordinate data points, where \( x \in \mathbb{R}^p \). If the variances of and the covariances between the \( p \) data points are of interest, CMM measurement data will dictate that \( p \) variances and \( \frac{1}{2}p(p - 1) \) covariances be reviewed, where \( p \) measures in the thousands. Furthermore, a set of \( n \) blades increases the original data size and results in a matrix \( X \in \mathbb{R}^{p \times n} \). An alternative approach to processing thousands of pieces of geometric information is to find a few (\( \ll p \)) derived variables that maintain the majority of geometrical variances and covariances.

Implementation occurs by computing \( \Delta X \), an \( (p \times n) \) matrix of measured deviations with the \((i, j)\)th element \((x_{i,j} - \bar{x}_j)\), where

\[
\bar{x}_j = \frac{1}{n} \sum_{i=1}^{n} x_{i,j} \quad j = 1, 2, \ldots, n
\]  

(3.35)

This process of computing \( \Delta X \) measures the \( j \)th variable about its mean \( \bar{x}_j \) for the \( i \)th observation, or in blade terminology, the variation around the average blade - which is not necessarily the original design specification. The first order covariance matrix of \( \Delta X \), known as \( \Sigma \), mandates that a total of \( (p \times n) \) pieces of information be reviewed, which is unacceptable for CMM data. To provide the set of uncorrelated PCs, \( \Sigma \) is cast into the standard eigen-problem formulation, where the PCs or weights are the eigenvectors \( \Psi \) and \( \Lambda \) is a diagonal matrix of eigenvalues.
\[ \Sigma \Psi = \Psi \Lambda \] (3.36)

In general, the eigenvalues are the variances of the PCs that give an indication of the amount of variance of the original data captured by the PCs. Furthermore, due to the nature of the eigen-problem the PCs are also orthogonal, meaning they are statistically uncorrelated.

The measured deviations, \( \Delta X \), is linearly transformed to Principal Component space by the following equation, where \( Z \) is the score matrix and \( Z \in \mathbb{R}^{m,n} \) with \( m \) as the number of retained PCs

\[ Z = [\Delta X] \Psi \] (3.37)

The scores are fundamentally regression coefficients for the PC space and explain the participation of each PC in the CMM data. Thus, if \( m < p \) PCA transforms a large set of interrelated data to a much smaller set of \( m \) uncorrelated parameters.

If the PCs and scores are known, the measured deviations can be determined by

\[ \Delta X = Z \Psi^T \] (3.38)

Although this might seem unorthodox, it provides an opportunity to perturb each score \( z_i \) in the score matrix and achieve new blade geometries while retaining the variation of the original blade geometries. This is done by

\[ \tilde{z}_i = \xi_i z_i \quad i = 1, 2, \ldots, m \] (3.39)

where \( \xi_i \) is a randomly drawn scalar from the distribution of \( z_i \). Eqs. 3.38 and 3.39 are the manner in which new IBR blade geometries are determined for geometric mistuning applications of an industrial IBR of Sec. 6.
3.3 Forced Response Calculations

Once the system eigen-problem of Eq. 3.29 or Eq. 3.33 has been solved for the eigenvalues and eigenvectors, $\Lambda_{CMS}$ and $\Phi_{CMS}$, respectively, the forced response of the rotor can be calculated. This is done by the mode superposition method, where the natural frequencies (eigenvalues) and mode shapes (eigenvectors) from the modal analysis are used to characterize the dynamic response of the rotor to harmonic engine order excitations. This method reduces the solution time of Eq. 3.23 because only $N_F$ eigenvalues and eigenvectors are solved instead of the full solution. The EOM is converted to modal form

$$\ddot{u}_m + 2\omega_m\xi_m\dot{u}_m + \omega_m^2 u_m = f_m \quad m = 1, 2, \ldots, N_F$$

(3.40)

where $m$ is the mode number of the $N_F$ system modes retained in $\Phi_{CMS}$, $u_m$ is the modal coordinate in physical space from Eq. 3.34, $\omega_m$ is the natural frequency of mode $m$, $\xi_m$ is the fraction of critical damping for mode $m$, and $f_m$ is the modal force. For a steady sinusoidal vibration the modal force as a function of time $t$ has the form

$$f_m = f_{m,c} e^{i\Omega t}$$

(3.41)

where $f_{m,c}$ is the complex force amplitude for mode $m$ and $\Omega$ is the imposed forcing frequency. The harmonic forcing function on each blade is calculated by

$$F(s) = F_{max} \cos \left[ \frac{2\pi C (s - 1)}{n} \right] + F_{max} i \sin \left[ \frac{2\pi C (s - 1)}{n} \right] \quad s = 1, 2, \ldots, n$$

(3.42)

where $F_{max}$ is the magnitude, $C$ is the engine order excitation number, the quantity $(s - 1)$ is the appropriate phase shift for blade $s$, and $n$ is the number of blades. The modal force in physical DOF on the $s$th blade is then calculated by the inner product of the loading vector with the mode shape vector
\begin{equation}
    f_m = \{u_m\}^T \{F\} \quad m = 1, 2, \ldots, N_F
\end{equation}

Likewise, \( u_m \) must have the same form as Eq. 3.41

\begin{equation}
    u_m = u_{m,c} e^{i\Omega t}
\end{equation}

Differentiating this equation and substituting back into the modal EOM, Eq. 3.40, the complex amplitude of the modal coordinate of mode \( m \) is solved

\begin{equation}
    u_{m,c} = \frac{f_{m,c}}{\left(\omega_m^2 - \Omega^2 \right) - i \left(2\omega_m \Omega \xi_m\right)}
\end{equation}

where it is easy to see that the magnitude of forced response is a function of modal loading, excitation frequency, and damping.

The contribution from each mode in the forced response amplitude is

\begin{equation}
    \{C_m\} = \{u_m\} u_{m,c}
\end{equation}

and the total complex displacement is determined by

\begin{equation}
    \{u_c\} = \sum_{m=1}^{N_F} \{C_m\}
\end{equation}
Figure 3.1: Disk-blade sector substructuring process for C-B CMS application
Reduced Order Model Applications

4.1 Overview

In the work that follows, the NMA-$\lambda_{FEM}$ approach will be simply referred to as the Nominal method since it relies upon nominal mode expansions and provides a concise reference to the method. Likewise, the NNMA approach will be referred to as the Non-nominal or Geometric method since it relies upon geometrically perturbed modal expansions. Two different finite element models were used in this probabilistic assessment of the Nominal Mode Approach (NMA-$\lambda_{FEM}$) and Non-Nominal Mode Approach (NNMA) as proposed by Brown [3]:

1. A simple, academic model with few DOF that allows for computational efficiency and proof of concepts, depicted in Fig. 5.3(a) on page 68

2. An IBR used in the Augmented Damping Low Aspect Ratio Fan (ADLARF) program [52] that provides airfoil geometries typically seen in modern IBR applications, depicted in Fig. 6.1(a) on page 102

Each model is meshed with eight-node linear solid (brick) elements with translations in the $x$-, $y$-, and $z$-directions at each node. The mesh of the academic model are rather course and may not accurately represent the actual behavior of the solid model. However, this is of little concern since this mesh density is used for both full FEM, Nominal, and Geometric
mistuning prediction methods. Each model considers material damping only, accounted for as a constant damping ratio, \( \xi_m = 0.002 \), in Eq. 3.45 on page 44.

Each model was geometrically perturbed and then solved by both Geometric and Nominal methods. A Monte Carlo simulation was conducted that solved 1000 different geometrically perturbed IBRs. A full FEM solution was conducted for a single mistuned IBR of both models to determine the forced response predictions. Since the Geometric method approaches the exact full FEM solution as the number of fixed-interface normal modes are retained, the least number of modes that needed to be included to match the exact solution were determined. The resulting simulations were then performed with this amount of fixed-interface normal modes so the Geometric solution could be assumed the exact solution. This alleviated the need to calculate 1000 full FEM solutions and provided an accurate solution to compare the nominal mode assumption against.

A breakdown of the tasks involved for solving a population of IBR forced response for the Nominal and Geometric methods can be seen in Fig. 4.1 on page 50. The disk substructure mass and stiffness matrices are obtained and the substructure fixed interface normal modes and constraint modes are calculated and stored for later use. A set of \( n \) geometrically perturbed blades are built and the Non-nominal system CMS matrices are solved. The fixed interface normal modes are saved and used in the Nominal system CMS stiffness matrix. The Nominal system is then solved and expanded into physical space using the nominal C-B matrix of Eq. 3.27 on page 37. The physical space coordinates of the Nominal and Non-nominal methods are then used in the forced response calculations. Once this is done, a new set of geometrically perturbed blades are built and the process loops for a pre-determined number of rotors. This work investigated a population of 1,000 rotors, where worldwide aircraft fleets typically number in the few thousands.
4.2 Importance of Results

Results for the models investigated in this chapter are divided into the following four categories for each respective veering region studied:

1. A population of peak airfoil forced responses
2. A population of blade-to-blade Nominal prediction errors
3. A population of peak rotor forced responses
4. A population of rotor-to-rotor Nominal prediction errors

The importance of peak airfoil and rotor predictions is established below and provides justification for investigating the listed airfoil and rotor responses.

4.2.1 Importance of Peak Airfoil Response

Assessing the ability of the Nominal method to predict peak airfoil forced response is critical for testing demonstrator engines. Engine diagnostics rely upon strain gages strategically placed on only a small subset of airfoils on each rotor. If gages are placed on every airfoil, incomplete results often arise due to gage malfunction, destruction, etc... If all airfoil geometries were nominal, only one strain gage would be needed to predict airfoil stresses since each blade would be responding at the same levels. Any additional strain gage placement would be redundant for gage survivability. However, each airfoil geometry is in fact unique and mistuning becomes significant. This presents a predicament because the limited placement of strain gages increases the likelihood that the peak responding blade is not being measured. Accurate monitoring of airfoil vibratory stresses would then be at an impasse. Reduced order models attempt to bridge this impediment by using the measured responses of a few airfoils. Given only the airfoil frequency deviations and the response...
level of a single instrumented airfoil, it is important to know if the Nominal method can accurately describe the response levels of the non-instrumented airfoils.

Distributions of response predictions are also important for calculating distributions of blade vibratory stresses for a probabilistic calculations. Given the random nature of airfoil input parameters (geometry, material properties, etc...), a distribution of blade stresses will be generated. Knowing the full distribution accurately is important because the deterministic assumption of all airfoils responding at the maximum mistuning level is conservative. This conservatism negatively impacts weight, performance, and cost. Use of the full distributions, including low and high responding airfoils, allows a more accurate assessment of risk and optimal design of components.

4.2.2 Importance of Peak Rotor Response

Ultimately, the greatest concern lies with the maximum response of a rotor. The high responding blade will act as the weakest link since failure of a single blade mandates that the rotor be removed and subsequently repaired or discarded. Conservatism can be applied by then assuming all blades are responding at the maximum mistuned level. While this conservatism negatively impacts weight, performance, and cost, it provides a robust estimate of peak blade stresses, and ultimately rotor HCF life. Furthermore, some studies have shown that mistuning ROMs are better at predicting maximum rotor mistuning than airfoil-to-airfoil mistuning. In other words, the ROMs can predict tuned absorber conditions, i.e., high mistuning versus low mistuning conditions.
Figure 4.1: Monte Carlo simulation of the Nominal and Geometric comparison flowchart
Academic Rotor Case Study

5.1 Model Overview

The academic model in Fig. 5.3 on page 68 was used originally for its low number of DOF. The model is similar to that used in previous mistuning studies [27]. The model consists of 12-60 mm long blades with a base width of approximately 7.5 deg. Each blade tapers from the 5 mm uniform disk thickness to 2 mm at the blade tip. The blades exhibit a 30 deg radial “twist” to promote coupling between bending and torsion modes typical of realistic IBR geometries. The disk has an outer radius of 100 mm and an inner radius of 20 mm, where all inner radius DOF are fix-clamped.

A harmonic forcing function calculated from Eq. 3.42 on page 43 was loaded at each airfoil’s leading edge tip node. While such point loads are not representative of in-flight airfoil loading, it does provide typical harmonic loading replicated by bench testing procedures [3]. The applied force was $F_{max} = \left[\frac{1}{3}\right]$ in z-direction, representative of an axial engine order excitation force. Fig. 5.6 on page 70 illustrates the applied force vector at each blade tip for the academic model and the zero-displacement boundary conditions applied in the x-, y-, and z-direction at the disk inner radius.

A frequency veering versus nodal diameters plot was constructed for the Nominal model and is shown in Fig. 5.1 on page 66. This plot characterizes the free vibration of the rotor, with nearly horizontal lines representing blade dominated modes, the off-horizontal lines representing disk-dominated modes, and frequency veering regions. It is
well established in the literature that these veering regions are indicative of large disk-blade interaction and forced response sensitivity to mistuning. Here the veering region of interest occurs at a nodal diameter of zero, corresponding to a zero-order engine excitation, at a frequency range of 6000 \( HZ \) to 6800 \( HZ \). In this range, the 54\textsuperscript{th} and 59\textsuperscript{th} system modes are excited, corresponding to the 6\textsuperscript{th} and 7\textsuperscript{th} blade modes at the zero engine order excitation condition. These modes are depicted in Fig. 5.2 on page 67.

### 5.2 Geometric Perturbation Implementation

For application of C-B CMS, the model is represented with 13 substructures: one disk and 12 airfoils depicted in Fig. 5.3(b). Basic model size information can be seen in Table 5.1 on page 65. Geometric mistuning is introduced in a manner similar to previous studies where PCA modes are assumed to be of a particular shape [31]. Each blade has four radial locations numbered one to four in Fig. 5.4 on page 69 with two nodes located on the front and back of the airfoil in the x-y plane at each numbered location. The nodal coordinates in the z-direction are \( \pm q \ mm \), where \( 2q \ mm \) is the nominal blade thickness. The blade thickness is changed at each circumferential location by multiplying the \( z \)-coordinate of each node by some factor. Each blade of an IBR can have a different, uniform thickness by multiplying all nodal \( z \)-coordinates by

\[
1 + \delta_{1i} \quad i = 1, 2, \ldots, 12
\]

(5.1)

where \( \delta_1 \) is a random variable generated from a Gaussian distribution of zero mean and any chosen standard deviation. This variable represents PCA mode \#1 in Fig. 5.5(a) on page 69, and \( \delta_{1i} \) is the value for the \( i \)th blade. The uniform thickness perturbation can be represented in a normalized vector of \( z \)-coordinate locations.
A second assumed PCA mode in Fig. 5.5(b) creates a linear variation in the blade thickness with a generated random variable in Eq. 5.1

\[ u_2 = \begin{bmatrix} -1 & -1/3 & 1/3 & 1 \end{bmatrix} \]  \hspace{1cm} (5.3)

The two PCA modes were combined in Fig. 5.5(c) to create a mistuning pattern used in this analysis, where the blade thickness is represented as

\[ u_{\text{blade}} = u_1 + \delta_{1i}u_1 + \delta_{2i}u_2; \quad i = 1, 2, \ldots, 12 \]  \hspace{1cm} (5.4)

For this academic study, the random mistuning variables, \( \delta_{ki} \), were drawn from Gaussian distribution with a mean of zero and a standard deviation of 2%, i.e., a \( N(0, 2\%) \) distribution. It is important to note that Figs. 5.4 and 5.5 depict the geometric mistuning pattern for an “untwisted” blade to easily describe the PCA modes, however, the blades are actually “twisted” radially. The \( z \)-direction in which the blade thickness is varied is then the direction normal to the blade surface at the nominal “twisted” blade nodal location.

Blade perturbations calculated by Eq. 5.4 were restricted to a maximum thickness variation of \( \pm 5\% \) of the original blade thickness. This was done to ensure the Nominal approach mistuning prediction errors would not arise from large geometric perturbations. Since the nominal mode assumption was established only for small geometric perturbations, restricting the blade thickness deviations ensures a fair comparison between the Nominal and Geometric approaches.
5.3 Academic Case Study Results

The number of retained airfoil and disk fixed-interface normal modes were determined by a simple quantitative comparison between the Geometric approach and full FEA predicted forced response amplitudes of a mistuned rotor, seen in Fig. 5.7 on page 71 for 40 airfoil and 150 disk fixed-interface normal modes. These amplitudes are the euclidean distance of tip displacement measured at a single tip node where point forces were applied in Fig. 5.6. As shown, there is good agreement between the Geometric approach and full FEA solutions methods for both a single blade frequency response function (FRF) and the maximum rotor displacement FRF. There is a 0.03% difference at the peak blade prediction and a 0.14% difference at the peak rotor prediction between the Geometric approach and full FEA solution techniques. Since the non-nominal methodology approaches the full FEA solution as the number of retained normal modes increases, and since there is good agreement between the Geometric method and full FEA peak predictions for the 40 airfoil and 150 disk fixed-interface normal modes, the Geometric approach is assumed to be the correct response. For this retained amount of fixed-interface normal modes, the resulting ROM system size is $774 \times 774$. This represents a $\sim 61\%$ decrease in problem size from the full $2016 \times 2016$ system model. In fact, requiring this many blade modes is not entirely necessary for the specific forcing region of interest since mainly the $6^{th}$ and $7^{th}$ blade modes are excited.

By reducing the retained fixed-interface normal modes of the blades to seven, the ROM system size is $378 \times 378$ and represents a $\sim 81\%$ decrease in problem size. Fig. 5.8 on page 72 depicts another simple quantitative comparison between the Geometric approach and full FEA predicted FRFs for seven airfoil and 150 disk fixed-interface normal modes. As shown, there is good agreement between the Geometric approach and full FEA solutions methods for both a single blade frequency response function (FRF) and the maximum rotor displacement FRF. There is a 0.37% difference at the peak blade prediction and a 0.21% difference at the peak rotor prediction between the Geometric approach and full
FEA solution techniques. For this study however, 40 blade modes are used in case future investigation requires additional blade modes.

The solution times of calculating the eigen-problem for first 70 eigen-pairs in each method can be seen in Fig. 5.9 on page 73. Solution times are normalized by the full FEM solution time. Only the time required to solve the eigen-problem of the system matrices are considered, i.e., time required to construct the CMS system matrices with substructure fixed-interface normal modes and constraint modes are ignored. Particular inclusion of the disk substructure generation in the total solution time would make C-B CMS unattractive for a single mistuning analysis. Instead, for multiple mistuning studies the disk portion would be solved once and can be re-used with different airfoil substructures. This fact is utilized in this study, as shown in Fig. 4.1 on page 50, where the computational cost of calculating the disk substructure can be distributed among the 1000 mistuned rotors considered. The burden of calculating the disk substructure can also be further reduced by modeling a cyclically symmetric disk sector. With the disk substructure calculated initially, each subsequent iteration will only require the solution of the airfoil substructures and the coupled system matrices. The time required to solve the airfoil substructures can be reduced by appropriately selecting the required number of fixed-interface normal modes needed for required system level accuracy. As previously shown, the ROM considering only seven airfoil fixed-interface normal modes will require less computational cost for accurate system level response.

Fig. 5.9 on page 73 highlights the potential downfall of the C-B CMS method: solution times can be greater for the ROM than for the full FEA solution. The reduced system matrices will have a fully populated boundary portion of the system matrices, referenced in Eqs. 3.18 and 3.21 on page 36, while the full system matrices will be sparsely populated. As a result, there comes a critical point when the number of boundary DOF preclude the use of the C-B CMS for reduced computational effort. However, for models representing actual IBR geometries with fine mesh densities, the size of the full model can require more
solution time the the C-B CMS reduced system.

As discussed above, an in-phase load was applied over a forcing frequency range of 6000 $Hz$ to 6800 $Hz$, corresponding to a zero engine order excitation or zero nodal diameter load. In this range, two closely spaced nominal modes are excited and can have combined mistuning effects. Frequency response functions (FRFs) were calculated to give displacement at each of the twelve airfoil tips. Results were categorized into peak blade displacement and peak rotor displacement. Peak blade displacement corresponds to the maximum displacement of the airfoil tip over the range of excitation frequency. This peak amplitude is computed for each of the 12 airfoils on the rotor, over a population of 1,000 rotors. This yields 12,000 blade FRFs from which the peak forced response statistics of the Nominal and Geometric methods will be computed. Likewise, peak rotor displacement considers only the largest tip displacement experienced by the rotor, or a subset of the population of the peak blade displacements. From the 1,000 rotor population, peak forced response statistics will also be calculated.

5.3.1 Peak Airfoil Forced Response

Probability mass functions of the peak airfoil tip displacement were examined in Fig. 5.10 on page 74. The maximum amplitude has been normalized and is represented by the ratio of the peak mistuned response to peak tuned response. Response amplification due to mistuning will then appear greater than one on the abscissa. The range of peak amplitudes has been divided into 20 bins, and the number of occurrences of displacements in the population of 12,000 airfoils are counted on the ordinate. Even though the Geometric and Nominal bars are not located at the same places along the abscissa, they correspond to the same segment of maximum amplitudes. As shown, there is close agreement between Nominal and Geometric predictions of the peak displacement PMF. The first two statistical moments of the PMF can be viewed in Table 5.2 on page 65, where it can be seen the two approaches compared quite well when predicting the statistics of the population of peak
airfoil response. A Kolmogorov-Smirnov statistical test of the two distributions failed to reject the null hypothesis that the peak airfoil response distributions generated from each method are from the same continuous distribution for a significance level of $\alpha = 0.05$. That is, the data does not give strong support to the claim that the response distributions for each method are different.

Next, when considering that the majority of the responses do not have mistuning amplification, the cumulative distribution function depicted in Fig. 5.11 on page 75 shows that approximately 18.49% of the responses did have amplification predicted by the Geometric method, while the Nominal method predicted 18.27%. It is expected that the majority of the blade population does not exhibit mistuning because mode localization occurs in a single or subset of blades on the IBR, while the remaining blades have forced response levels equal to or less than the tuned value. The largest amplification predicted by the Geometric method was 1.83 and the Nominal method maximum prediction was 1.82, providing an error of only 0.49%. Note that this error is a measure of the population maximum, and not of error between the two methods for a single blade. The small error seen in the Nominal prediction of the peak airfoil forced response suggests the nominal mode assumption has little impact on the population response. Using distributions generated from the Nominal approach for this academic model do not introduce any false conservatism and can be used in a probabilistic calculation discussed in Sec. 4.2.1.

Now that it has been established that the Nominal approach predicts the distribution accurately, it is important to assess how the same method predicts the airfoil response on a specific rotor. The error between the Nominal and Geometric predictions of an individual airfoil’s peak response was calculated as a measure of how accurately the Nominal approach can predict individual blade stresses. This standard percent error calculated by

$$Error = \frac{FRF_{peak}^{Geometric} - FRF_{peak}^{Nominal}}{FRF_{peak}^{Geometric}}$$

(5.5)

where $FRF_{peak}^{Geometric}$ and $FRF_{peak}^{Nominal}$ are the peak airfoil FRF predictions of a single air-
foil for each method. If the Nominal approach over-predicted the true response the percent error will be negative and positive for under-predictions. The PMF of the percent error calculated by Eq. 5.5 can be seen in Fig. 5.12(a) on page 76. Here the percent error of the 12,000 blades has been divided into 30 bins to show the relative shape of the distribution. Results showed that the Nominal method produces larger error when predicting individual blade peak response levels. The maximum Nominal over-prediction was -27.7% error, slightly more than the maximum under-prediction of 25.2% error. However, the mean of this distribution was 0.496% and the standard deviation was 5.1% so these maximum errors are approximately five standard deviations from the mean. Ignoring whether the Nominal predictions are either over or under-predictions (absolute value of the error), the data follows a Weibull distribution quite well. This is shown in Fig. 5.12(b) and illustrates minimal curvature in the data and follows the linear Weibull line over the majority of the probability. The errors only deviate at the extreme lower tail of the distribution, corresponding to low Nominal approach error. The estimated parameters to the probability density function of a Weibull distribution given by

\[
f_X (x) = \frac{\alpha x^{\alpha-1}}{\beta^\alpha} \exp \left[ -\left( \frac{x}{\beta} \right)^\alpha \right]
\]  

(5.6)

are \( \alpha = 1.1454 \) and \( \beta = 4.016 \).

The blade-to-blade results illustrate larger errors than seen for the distribution of peak airfoil response. However, the accuracy of Nominal method blade-to-blade predictions is still considered good. Considering the maximum over and under-predictions given above portrays a sense of inaccuracy, but these errors are over five standard deviations away from the mean. Fig. 5.13(a) on page 77 illustrates the average blade-to-blade error seen on a single rotor for the entire population. This figure provides evidence that the largest blade-to-blade errors seen in Fig. 5.12(b) on page 76 do not all necessarily come from the same rotor because the means of the blade-to-blade error for a specific rotor are not large enough.
to reflect that. The possibility still exists that the largest over-predictions and largest under-
predictions this figure happen to fall on the same rotor, thus providing a small mean error 
and a corresponding false sense of accuracy of the Nominal method for a specific rotor. 
The rotor corresponding to error closest to zero in Fig. 5.13(a) is shown in the stem plot 
in Fig. 5.13(b), and shows that this is not the case. The blade-to-blade error is plotted for 
each blade according to the axis on the right and shows that the largest over and under-
predictions from Fig. 5.12(b) do not occur for this specific rotor. Evidence suggests the 
Nominal method shows potential for predicting airfoil displacements after calibration from 
a few known displacements in a instrumented rotor test.

Lastly, the PMF of the fixed interface normal modes, i.e., cantilevered blade natural 
frequencies, was constructed in Fig. 5.14 on page 78 to determine the distribution of the 
frequency deviations. Since 40 fixed interface normal modes were retained, each natural 
frequency was normalized to its corresponding tuned value. For a population of 12,000 
blades, this provided 480,000 frequencies for comparison. For 40 equally spaced bins, 
Fig. 5.14 shows there were small frequency deviations. Restricting the maximum blade 
perturbations to 5% results with a maximum frequency deviation of 5.74% and a coefficient 
of variation of 0.01.

5.3.2 Peak Rotor Forced Response

Depicting only the single, peak response of a rotor gives insight to the number of rotors 
in a population that have a responding blade that is responding above predicted, tuned lev-
els. Such information is important to determining the possibility of rogue blade failures on 
a rotor resulting from mistuning. This information cannot be determined from the popu-
lation of blade responses since the number of large responding blades cannot be directly 
tied to any specific rotor. Probability mass functions (PMF) of the Geometric and Nominal 
approaches can be seen in Fig 5.15 on page 79. These amplitudes are a subset of the pop-
ulation of blades depicted in Fig. 5.10 on page 74 - only the maximum peak displacement
of the rotor is selected. The maximum amplitude has been normalized by the ratio of the peak mistuned response to peak tuned response. Response amplification due to mistuning will then appear greater than one on the abscissa. The range of peak amplitudes has been divided into 20 bins, and the number of occurrences of displacements in the population of 1,000 rotors are counted on the ordinate. For this specific frequency range, the CDF of peak rotor response in Fig. 5.16 on page 80 shows that 97.9% and 97.4% of the rotors exhibit mistuning for Geometric and Nominal methods, respectively. The agreement between Geometric and Nominal methods is excellent over the range of the CDF, as well as the first two statistical moments calculated in Table 5.3 on page 65. As found earlier, the maximum mistuning amplification will be the same as the maximum blade amplification, with a Geometric method prediction of 1.83 and a Nominal prediction of 1.82.

A Kolmogorov-Smirnov statistical test of the two distributions failed to reject the null hypothesis that the peak airfoil response distributions generated from each method are from the same continuous distribution for a significance level of $\alpha = 0.05$. That is, the data does not give strong support to the claim that the response distributions for each method are different. Considering this and the excellent Nominal method accuracy for predicting the distribution of peak rotor response, the nominal mode assumption has little impact on calculating the correct forced response distributions for the Academic model.

Now that it has been established that the Nominal approach predicts the peak rotor response distribution accurately, it is important to assess how the same method predicts the maximum response on a specific rotor. The error between the Nominal and Geometric predictions of an individual rotor’s peak response was calculated as a measure of how accurately Nominal methods can predict individual rotor peak stresses. This standard percent error calculated by Eq. 5.5 on page 57, where the peak FRFs are for the peak rotor response for this case. The CDF of the Nominal error for each individual rotor is shown in Fig. 5.17 on page 81. As shown, the Nominal approach over-predicted the an individual rotor peak response 46.9% of the time, with a mean of 0.174% and a standard deviation
of 3.238%. The maximum Nominal error found was 12.61% for the peak individual rotor prediction. For individual rotors, the ability of the Nominal method to predict the correct peak response is diminished and produces a large coefficient of variation of 18.61 for a population of rotors.

5.4 Academic Conclusions and Further Discussion

The Nominal method predicted the correct distribution of peak airfoil forced response quite well. A qualitative comparison of the forced response CDF showed excellent agreement over entire range of probabilities. A quantitative comparison of the first two statistical moments also showed excellent agreement. However, when considering a Nominal forced response prediction on a blade-to-blade comparison with the Geometric method, accuracy was diminished. The Nominal method produced errors as large as 27%.

The Nominal approach also predicted the correct distribution of peak rotor forced response accurately. This is expected since the peak rotor forced response is actually a subset of the peak airfoil responses. Nonetheless, this peak rotor amplitudes can be used to quantify the life of an IBR since the peak responding blade represents the weakest link for rotor failure. There was excellent quantitative and qualitative agreements between the Nominal prediction and the Geometric baseline peak rotor forced response distributions. The accuracy also diminished for prediction of rotor-to-rotor peak displacements between the two methods. Poor blade-to-blade accuracy of the Nominal method suggests rotor-to-rotor accuracy will also be poor, but the maximum error of the peak rotor-to-rotor displacement was roughly half of the blade-to-blade accuracy. Considering that the peak rotor-to-rotor predictions were mistuned for $\sim 97\%$ of the rotors, as shown in the CDF of Fig. 5.16 on page 80, this provided larger (i.e., $> 1$) maximum mistuning numbers to be used in the error calculation. Blade-to-blade predictions, however, were mistuned for only approximately $\sim 18\%$ of the blades so $\sim 82\%$ of the blades provided smaller (i.e., $< 1$) maximum
misting numbers to be used in the error calculation. This is a likely source for the large
discrepancy between blade-to-blade and rotor-to-rotor error.

It is also important to note that rotor-to-rotor error can also be smaller than blade-to-
blade error because the maximum responding blade on a specific rotor predicted by the
Geometric method is not necessarily the Nominally predicted maximum responding blade
on the same rotor. For example, if the Geometric approach predicts a peak rotor response
on blade 12 with an amplified response of 1.51, while the Nominal method predicts 1.42
on that same blade, a blade-to-blade error of 5.70% is produced. However, the Nominal
approach is predicting a peak rotor response of 1.45 on blade 7. Thus, the rotor-to-rotor
peak response error will be 3.78%, which is much smaller. This specific case can be seen
in Fig. 5.18(a) on page 82. Thus, the rotor-to-rotor error can be smaller than the blade-to-
blade error because on a peak rotor prediction the blade on which the maximum occurs is
neglected.

For this academic rotor, only 89 of the 1000 rotors had predictions where the peak
rotor predictions were found on different blades by each approach. Of these 89, it fol-
lows that all had a maximum responding blade with a blade-to-blade error greater than the
rotor-to-rotor error. This is somewhat obvious because the blade-to-blade error of the max-
imum responding blade cannot be less than rotor-to-rotor error on a specific rotor when the
Nominal method correctly identified the maximum responding blade - at most they can be
equal. This is the case for the rotor-to-rotor error for the remaining 911 rotors because the
Nominal method correctly identified the maximum responding blade on the rotor. These
errors can be summarized in by the four following cases:

1. Of the 89 rotors where the Nominal method incorrectly identified the maximum re-
sponding blade, Case 1 in Fig. 5.19 on page 83 identifies the Nominal blade-to-blade
error on the actual (Geometric method prediction) maximum responding blade. A
single instance of this error corresponds to the Nominal prediction error on blade 12
in Fig. 5.18(a) on page 82.
2. Of the 911 rotors where the Nominal method correctly identified the maximum responding blade, *Case 2* in Fig. 5.19 identifies the Nominal blade-to-blade error on the actual (Geometric method prediction) maximum responding blade. This error is the same as *Case 3* below. A single instance of this error corresponds to the Nominal prediction rotor error on rotor 1 in Fig. 5.18(b).

3. Of the 911 rotors where the Nominal method correctly identified the maximum responding blade, *Case 3* in Fig. 5.19 identifies the Nominal rotor-to-rotor error. A single instance of this error corresponds to the Nominal prediction rotor error on rotor 1 in Fig. 5.18(b).

4. Of the 89 rotors where the Nominal method incorrectly identified the maximum responding blade, *Case 4* in Fig. 5.19 identifies the Nominal rotor-to-rotor error. A single instance of this error corresponds to the Nominal prediction rotor error on rotor 538 in Fig. 5.18(a), where the error is calculated from the Geometric and Nominal peak rotor predictions from blade 12 and seven, respectively.

The 89 rotors of *Case 1* were removed from the entire population of rotor-to-rotor error in Fig. 5.17 on page 81 to determine if the large difference between the maximum rotor-to-rotor error maximum blade-to-blade error occurs because the peak rotor response does not account for the Nominal method incorrectly identifying the maximum responding blade. The remaining 911 rotors of *Case 2* should then be statistically distinguishable from the entire population of 1000 rotors because the rotors with smaller rotor-to-rotor error are removed. A Kolmogorov-Smirnov statistical test of the *Case 2* distribution and the entire population failed to reject the null hypothesis that the distributions are different for a significance level of $\alpha = 0.05$. That is, the data does not give strong support to the claim that the smaller rotor-to-rotor error of the 89 rotors of Case 1 had any significant impact on the total rotor-to-rotor error population. Thus, the larger observed blade-to-blade error is not likely due to the peak rotor response neglecting the instances where the Nominal method
incorrectly identified the maximum responding blade. The large discrepancy is more likely due to dividing in the error calculation of Eq. 5.5 on page 57 with a majority of numbers below a magnitude of one. Similarly, the Case 4 distribution and the entire rotor-to-rotor error distribution are also not statistically distinguishable from each other, thus providing more evidence that neglecting the incorrect maximum responding blade prediction has little impact on the overall distribution.

These results suggest that the Nominal approach predicts the peak responding blade 91.1% of the time, but with a varying degree accuracy resulting from over and under-predictions. Fig. 5.19 highlights in Case 2 and Case 3 that this error had a range of approximately ±12%. For the remaining 8.9% of rotors when the Nominal approach incorrectly identified the maximum responding blade, it also under-predicted the response or the true maximum responding blade on 76 of the 89 rotors (∼85%) of the time. These results, along with error Cases 1-4 from above, become relevant when considering the importance of blade-to-blade predictions outlined in Sec. 4.2.1. Applications of this investigation are given in Chpt. 7 for the ADLARF test case.

The excellent agreement between both the blade and rotor forced response populations warrants further investigation of the nominal mode assumption of the Nominal formulation. The rather simple geometry of this academic model could result in relatively similar nominal and non-nominal mode shapes, thus giving an unfair advantage to the Nominal method considering this model is not representative of fielded IBRs. An advanced geometry IBR typical of real world applications will be considered next to address this lopsided Nominal advantage.
Table 5.1: Basic size data for the academic model

<table>
<thead>
<tr>
<th>Component</th>
<th>Elements</th>
<th>Nodes</th>
<th>DOF</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cantilevered Airfoil</td>
<td>4</td>
<td>16</td>
<td>48</td>
</tr>
<tr>
<td>Disk</td>
<td>240</td>
<td>432</td>
<td>1296</td>
</tr>
<tr>
<td>Disk-Airfoil Interface</td>
<td>n/a</td>
<td>4</td>
<td>12</td>
</tr>
<tr>
<td>Full Model</td>
<td>288</td>
<td>672</td>
<td>2016</td>
</tr>
</tbody>
</table>

Table 5.2: Peak academic blade displacement population statistics

<table>
<thead>
<tr>
<th></th>
<th>Mean</th>
<th>Error</th>
<th>Standard Deviation</th>
<th>Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>Geometric</td>
<td>0.809</td>
<td>–</td>
<td>0.223</td>
<td>–</td>
</tr>
<tr>
<td>Nominal</td>
<td>0.805</td>
<td>0.519%</td>
<td>0.224</td>
<td>-0.152%</td>
</tr>
</tbody>
</table>

Table 5.3: Peak academic rotor displacement population statistics

<table>
<thead>
<tr>
<th></th>
<th>Mean</th>
<th>Error</th>
<th>Standard Deviation</th>
<th>Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>Geometric</td>
<td>1.262</td>
<td>–</td>
<td>0.154</td>
<td>–</td>
</tr>
<tr>
<td>Nominal</td>
<td>1.259</td>
<td>0.249%</td>
<td>0.151</td>
<td>1.937%</td>
</tr>
</tbody>
</table>
Figure 5.1: Frequency veering plot of the Nominal academic rotor
Figure 5.2: Academic blade modes for the frequency range of interest
Figure 5.3: Finite element mesh for the academic IBR example
Figure 5.4: Discretization of an academic airfoil into a finite element mesh

Figure 5.5: Academic model assumed PCA modes
Figure 5.6: Academic model tip node force vector components
Figure 5.7: Academic forced response level for 40 blade and 150 disk retained fixed-interface normal modes
Figure 5.8: Academic forced response level for 7 blade and 150 disk retained fixed-interface normal modes
Figure 5.9: Academic eigen-problem solution time for 70 eigen-pair extraction
Figure 5.10: PMF of peak academic airfoil displacement
Figure 5.11: CDF of peak academic airfoil displacement
Figure 5.12: Distribution of Nominal prediction error for individual academic blades
Figure 5.13: Normal probability plot of average blade-to-blade errors on a single rotor for the population of Academic Rotors
Figure 5.14: PMF of academic fixed interface normal modes
Figure 5.15: PMF of peak academic rotor displacement
Figure 5.16: CDF of peak academic rotor displacement
Figure 5.17: CDF of the Nominal prediction error of individual academic rotor displacement

1000 Rotors: $\mu = 0.17408, \sigma = 3.2375$
Figure 5.18: Academic rotor predicted airfoil displacements
Figure 5.19: CDFs of error for cases when Nominal and Geometric methods predict peak rotor response on different blades
ADLARF Rotor Case Study

6.1 Model Overview

The ADLARF rotor in Fig. 6.1(a) on page 102 is used in this probabilistic study offers geometries typically seen in modern IBRs. While the previous academic model had simpler cantilevered bar-type airfoils with simple mode shapes, this ADLARF model will have a high modal density with interaction between complex mode shapes. This provides a sound model to compare probabilistic Nominal and Geometric predictions since tuned modes can vary greatly from mistuned modes and support (or oppose) the validity of the Nominal tuned mode assumption for a population of rotors. This is the same rotor model used by Brown [3] in his Nominal and Geometric formulations.

A harmonic forcing function calculated from Eq. 3.42 on page 43 was loaded at each airfoil’s leading edge tip node. While such point loads are not representative of in-flight airfoil loading, it does provide typical harmonic loading replicated by bench testing procedures [3]. Replication of in-flight loading would not only require more sophisticated aerodynamic loading, but also aerodynamic damping. Application of this type of loading is usually done in fluid-structural interaction problems and is too computational expensive for probabilistic analyses. The applied force vector was $F_{max} = \left[ \frac{1}{3} \right]$ for the axial $z$-direction. Fig. 6.2 on page 103 illustrates the applied force vector at each blade tip for the ADLARF model. The IBR is constrained by imposing zero-displacement boundary conditions for all DOF on the aft flange, seen in Fig. 6.3 on page 104.
A frequency veering plot was constructed for the nominal ADLARF rotor and is shown in Fig. 6.4 on page 105. A frequency range of 4.6 kHz to 5.0 kHz at the zero nodal diameter was applied to encompass possible complex modal interaction between system modes 205 and 219. This encompasses the 15\textsuperscript{th} and 16\textsuperscript{th} airfoil modes. A second frequency range is investigated between 2.5 kHz and 3 kHz at the second nodal diameter that encompasses a veering region at ~2766 Hz corresponding to repeated system modes 114/115 and 121/122 and airfoil modes eight and nine. A final frequency range of 6.3 kHz to 6.7 kHz at the second nodal diameter was investigated around system mode 296 with airfoil modes 19 and 20. The six airfoil modes used in these frequency ranges and engine order excitations can be seen in Fig. 6.5 on page 106.

6.2 Geometric Perturbation Implementation

Airfoil geometry measurements were not available for the ADLARF rotor, so measured deviations from an industrial IBR fan were used. These deviations were used in previous works by Brown [34, 3]. A reduced-order geometry model also proposed by Brown in the same works used the deviations and the nominal geometry of the ADLARF model to provide as-measured geometries. This Geometric ROM used Principal Component Analysis (PCA) outlined in Sec. 3.2.2 to provide a reduced basis in which to develop perturbed airfoils. PCA of the 16 industrial fan blades used to generate airfoil geometry deviations generated 15 principal components, where Fig. 6.6 on page 107 illustrates the variance explained by each principal component (PC) in the CMM data. The PCs are ordered such that each subsequent PC explains less spatial variation, until 100\% of the variation is explained by all 15 PCs. This represents a significant reduction in model size, where if all physical nodal locations were retained, thousands of DOF would need to be perturbed.

After using PCA to obtain the score and principal components of the measured CMM data, Eqs. 3.38 on page 42 was used to determine random airfoil surface deviations. A new
score matrix is determined by using the inverse transform method since the cumulative distribution function (CDF) of the score is empirically obtained. This method is summarized graphically in Fig. 6.7 on page 108 and described below:

1. A random number generator is produces a random number between 0 and 1 from a uniform distribution based on a arbitrary seed

2. From the random number, the corresponding uniform distribution CDF probability is obtained and transferred to the score CDF

3. Use 1-D interpolation to evaluate a new target value at the random probability on the score’s CDF

Fig. 6.8 on page 109 illustrates an empirical cumulative distribution function for a single score where a randomly generated probability given on the ordinate will yield a random scalar factor on the abscissa used to perturb the score. Once this is done for each score and a new, perturbed score matrix $Z$ is obtained and an airfoil perturbed geometry can be obtained. Fig. 6.9 on page 110 depicts a single airfoil surface with the nominal ADLARF geometry and geometric deviations obtained using the methods outlined above. It is apparent that surface deviations correlate across the airfoil and that PCA accounts for this spatial correlation. Fig. 6.10 on page 111 outlines a 100 times magnification of the distorted airfoil geometry generated using PCA methods that are otherwise unnoticeable to the naked eye.

A five mil (five thousandths of an inch) geometric tolerance limit was imposed on blade deviations. This is consistent with modern blade manufacturing tolerance limits. Random blades generated in the previously described PCA method were screened for deviations exceeding the established tolerance. This was determined by computing the euclidean norm of each blade perturbation according to

$$\sqrt{(\Delta x_i)^2 + (\Delta y_i)^2 + (\Delta z_i)^2} \leq 0.005 \text{ in} \quad i = 1, 2, \ldots, N_p$$  \hspace{1cm} (6.1)
where \( N_p \) is the number of spatial points describing the airfoil surface. If the above inequality was violated, a new random vector norm was generated less than five mils. The individual euclidean norm components were set equal to each other in magnitude and their corresponding signs were set to the original component signs so the euclidean norm would equal the randomly selected vector norm. Furthermore, in-plane deviations were retained in the geometry perturbations.

### 6.3 ADLARF Case Study Results

The number of retained airfoil and disk fixed-interface normal modes were determined by a simple quantitative comparison between the Geometric approach and full FEA predicted forced response amplitudes of a mistuned rotor, seen in Fig. 6.12 on page 113 for 50 airfoil and 200 disk fixed-interface normal modes. These amplitudes are the euclidean distance of tip displacement measured at a single tip node where point forces were applied in Fig. 6.2 on page 103. As shown, there is good agreement between the Geometric and full FEA solutions methods for both a single blade frequency response function (FRF) and the maximum rotor displacement FRF. There is a 1.40% difference at the peak blade prediction and a 0.58% difference at the peak rotor prediction between the Geometric and full FEA solution techniques. Since the Geometric methodology approaches the full FEA solution as the number of retained normal modes increases, and since there is good agreement between the Geometric and full FEA peak predictions for the 50 airfoil and 200 disk fixed-interface normal modes, the Geometric approach is assumed to be the correct response. The resulting ROM system size is \( 2632 \times 2632 \) for this amount of retained modes. This represents a \( \sim 96\% \) decrease in problem size from the full \( 64080 \times 64080 \) system model.

The solution time for the full FEA and the coupled substructure ROM methods can be seen in Fig. 6.11 on page 112. Times are normalized by the full FEA eigenvalue problem solution time for extracting the first 100 eigen-pairs. The computational costs associated
with the C-B CMS method include:

1. Solution time of the coupled substructure eigenvalue problem for extracting the first 100 eigen-pairs

2. Solution time of computing 50 fixed interface normal modes and all constraint modes for 16 airfoil substructures

3. One thousandth (i.e., 0.001) of the solution time of computing 200 fixed interface normal modes and all constraint modes for the disk substructure where only small portion of the disk substructure computational burden is included because it only occurs once at the beginning of the Monte Carlo analysis. Thus, any solution time pertaining to the disk can be distributed evenly amongst the 1,000 rotors calculated. This cost can be reduced even further by performing a cyclic sector analysis of the disk. If only a single mistuned rotor is needed for analysis, this illustrates that the unattractiveness of the C-B CMS method since the disk substructure computational burden cannot be dispersed.

The C-B CMS ROM will then have a much higher computational cost than the full FEA method. However, for this instance the FEA solution time will come at a cost to the storage requirements of the machine. Fig. 6.11 also shows that the storage requirements of the full FEA method system matrices are larger than the C-B CMS matrices. These storage requirements are normalized by the full FEA sparse matrix size, where any zero components are removed from the storage organization. The ROM then requires \( \sim 30\% \) less storage space than the full solution.

As discussed above, loads were applied over different forcing frequency ranges and different engine order excitations. Frequency response functions were calculated to give displacement at each of the 16 airfoil tips. Results were categorized into peak blade displacement and peak rotor displacement. Peak blade displacement corresponds to the maximum displacement of the airfoil tip over the range of excitation frequency. This peak amplitude is computed for each of the 16 airfoils on the rotor, over a population of 1,000
rotors. This yields 16,000 blade FRFs from which the peak forced response statistics of the Nominal and Geometric approaches will be computed. Likewise, peak rotor displacement considers only the largest tip displacement experienced by the rotor, or a subset of the population of the peak blade displacements. From the 1,000 rotor population, peak forced response statistics will also be calculated.

### 6.3.1 Peak Airfoil Forced Response

Probability mass functions of the peak airfoil tip displacement has been calculated at three system modes, M114, M205, M296, and can be examined in Figs. 6.13 - 6.15. The maximum amplitude has been normalized and is represented by the ratio of the peak mistuned response to peak tuned response. Response amplification due to mistuning will then appear larger than one on the abscissa. The range of peak amplitudes has been divided into 20 bins, and the number of occurrences of displacements in the population of 16,000 airfoils are counted on the ordinate. Even though the Geometric and Nominal bars are not located at the same places along the abscissa, they correspond to the same segment of maximum amplitudes. Fig. 6.14 corresponds to a zero order engine excitation for the 205th system mode, while Fig. 6.13 and 6.15 were generated from a second order engine excitation at M114 and M296, respectively. M114 and M296 do not show large mistuning for their respective veering regions. The agreement between Nominal and Geometric methods for M114 is generally good, however, there is a tendency for the Nominal method to under-predict the number of occurrences for normalized amplitudes less than 0.8 and over-predict for normalized amplitudes greater than 0.8. However, the accuracy of the Nominal approach deteriorates as the frequency range of interest increases to encompass higher order modes, such as M205 in Fig. 6.14. The Nominal formulation grossly under-predicts the number of occurrences for normalized amplitudes less than 0.5 and then over predicts over a range of 0.5 to 1.0. There is good agreement for amplitudes larger than 1.0. M296 in Fig. 6.31 shows poor Nominal accuracy for the entire normalized amplitude range. In each
of the three modes considered, the Nominal method had a tendency to under-predict the number of occurrences for amplitudes in ranges less than the tuned frequency, or a normalized amplitude of one. However, the upper-tails of each distribution show good qualitative agreement between the two methods.

A Kolmogorov-Smirnov statistical test of the Geometric and Nominal response distributions was performed for all three modes considered. In each case, the test rejected the null hypothesis that the peak airfoil response distributions generated from each method are from the same continuous distribution for a significance level of $\alpha = 0.05$. That is, the data gives strong support to the claim that the response distributions for each method are different for all three modes. In addition, the deterioration of the Nominal method’s accuracy for higher order modes can be seen in Table 6.2 on page 100 by comparing the first two statistical moments of each distribution, giving further support that the distributions are indeed different.

The cumulative distribution functions of M114, M205, M296 can be depicted in Figs. 6.16 - 6.18. Forced response levels are again normalized by the peak tuned response. Each plot will give the probability of a forced response level exceeding the tuned response. The CDFs also depict a qualitative level of inaccuracy of the Nominal approach over probabilities of zero to one. Table 6.3 on page 100 summarizes the CDFs and outlines the percentage of blades in the population that have mistuning amplification. It is expected that the majority of the blade population does not exhibit mistuning because mode localization occurs in a single or subset of blades on the IBR, while the remaining blades have forced response levels equal to or less than the tuned value. The maximum forced response predicted by each method is also shown, with the maximum of all three shown modes occurring at M205 at the zero engine order excitation for a $2.4x$ mistuning amplification. The rotor with the predicted maximum responding blade is also given. It is important to note that for a given mode, each ROM predicted the maximum blade response on different rotors.
The error between the Nominal and Geometric predictions of an individual airfoil’s peak response was calculated as a measure of how accurately the Nominal method can predict blade-to-blade displacements. This standard percent error was calculated by Eq. 5.5 on page 57. If the Nominal approach over-predicted the true response the percent error will be negative and positive for under-predictions. Table 6.4 on page 100 outlines the maximum over and under-predictions as well as the first two statistical moments of the error in the Nominal airfoil forced response prediction. The mean of the error shows that as the system mode number increases, the Nominal method’s accuracy diminishes, generally over-predicting the correct forced response level. Further evidence of this is given by the large over-prediction errors in excess of 200% for all modes, while the under-predictions are less than 70%. The normalized probability plots are shown in Figs. 6.19(a), 6.20(a), and 6.21(a). The large Nominal method over-predictions for the distribution can be easily viewed. The data has large curvature from the normal probability line, depicting that the individual blade error does not follow a normal distribution. The first two statistical moments for the absolute value of the error are also shown in the plots. If the over/under-predictions are ignored, i.e., only the absolute value of the error is considered, the error more closely follows a Weibull distribution. Figs. 6.19(b), 6.20(b), and 6.21(b) illustrate the collinearity of the error data and the linear Weibull distribution line. A perfect Weibull distribution will follow this line. Population statistics and parameter estimates are listed in the respective figures for each mode.

Figures 6.22 - 6.24 depict airfoil forced response levels for the rotor that contains the Nominal method’s largest airfoil over and under-prediction error corresponding to M114, M205, and M296, respectively. These displacement plots contain the normalized forced response levels against the blade number for a specific rotor. The airfoil representing the largest Nominal method over and under-predictions from Table 6.4 on page 100 are outlined by a dashed box. It is important to note that these are the maximum errors and the remaining predictions have error that fall somewhere between these values for the respective modes.
Lastly, the probability mass function of the fixed interface normal modes, i.e., cantilevered blade natural frequencies, was constructed in Fig. 6.25 to determine the distribution of the frequency deviations. Since 50 fixed interface normal modes were retained, each natural frequency was normalized to its corresponding tuned value. For a population of 16,000 blades, this provided 800,000 frequencies for comparison. Only the blade frequencies for the zero engine excitation mistuning case were considered. For 40 equally spaced bins, Fig. 6.25 shows there were small frequency deviations. Restricting the maximum blade perturbations to \( \pm 0.005 \) in results with a maximum frequency deviation of 1.97% and a coefficient of variation of 0.0038.

### 6.3.2 Peak Rotor Forced Response

Probability mass functions of the Geometric and Nominal peak rotor forced response predictions for M114, M205, and M296 can be seen in Figs. 6.26 - 6.28, respectively. These amplitudes are a subset of the population of blades depicted in Figs. 6.13 - 6.15 for the same three modes. Here, only the single peak displacement of the 16 airfoils on each rotor is selected. The maximum amplitude has been normalized by the ratio of the peak mistuned response to peak tuned response. Response amplification due to mistuning will then appear greater than one on the abscissa. The range of peak amplitudes has divided into 20 bins, and the number of occurrences of displacements in the population of 1,000 rotors are counted on the ordinate. Depicting only the single, peak response of a rotor gives insight to the number of rotors in a population that have a responding blade above predicted, tuned levels. Such information is important to determining the possibility of rogue blade failures on a rotor resulting from mistuning. This information cannot be determined from the population of blade responses since the number of large responding blades cannot be directly tied to a specific rotor. The first two statistical moments of these distributions can be seen in Table 6.5 on page 101. The error between these two distribution statistics is relatively small, but generally increases with increasing mode number.
While the statistical error between the Geometric and Nominal approaches is generally small, the actual shape of the PMFs show a much larger qualitative disparity of the methods. This is also highlighted by the CDFs of maximum forced response for the three selected modes shown in Figs. 6.29 - 6.31. These CDFs also indicate the percentage of rotors in the population that have a response greater than the tuned case. Roughly 100 percent of the rotors for M114 and M205 have mistuning amplification predicted by both Nominal and non-nominal methods. For M296, the non-nominal method predicted that 84.8% of the rotors were mistuned, while the Nominal method over-predicted this number by 15% with 97.6% mistuned rotors. A Kolmogorov-Smirnov statistical test of the Geometric and Nominal response distributions was performed for all three modes considered. In each case, the test rejected the null hypothesis that the peak rotor response distributions generated from each method are from the same continuous distribution for a significance level of $\alpha = 0.05$. That is, the data gives strong support to the claim that the response distributions for each method are different for all three modes. However, when considering only the upper-tail in the CDFs, Table 6.6 on page 101 outlines that there is excellent agreement between the two ROM approaches for M114 and M296. M205 has larger error overall in the upper-percentiles, but the agreement is still good.

Also highlighted on the Nominal CDFs of Figs. 6.29 - 6.31 are the locations of Nominally predicted peak rotor response for rotors with a non-nominal predicted peak response in the upper $97th$ percentile of the Geometric method’s CDF. If the Nominal peak rotor response is correct, then the locations of the markers will also be in the upper $97th$ percentile of the Nominal CDFs. As shown however, this does not take place for M114, M205, and M296. The rotors in the upper tail of the non-nominal CDF are dispersed throughout the Nominal distribution and show that the Nominal method does not predict the upper tail of the distribution with the same rotors as the non-nominal method. However, 22 of the 29 rotors in the upper $97th$ percentile of the non-nominal CDF for M205 are in the upper $90th$ percentile of the Nominal distribution, but the remaining seven rotors are shown
to be dispersed throughout the remaining probability range.

Figures 6.32 - 6.34 highlight the peak blade responses for the maximum responding rotors predicted by the Nominal and non-nominal methods for the three modes considered. The ordinate represents the normalized airfoil amplitude against the blade number on the abscissa. The predicted maximum responding rotor is also listed in the figures. Figs. 6.32(a), 6.33(a), and 6.34(a) give the peak blade displacements for the non-nominally predicted maximum responding rotors, for M114, M205, and M296, respectively. The corresponding Nominal peak blade predictions for the same rotor are also plotted. Likewise, Figs. 6.32(b), 6.33(b), and 6.34(b) give the peak blade displacements for the Nominally predicted maximum responding rotors, for M114, M205, and M296, respectively. The corresponding non-nominal peak blade predictions for the same rotor are also plotted. For each mode considered, these peak displacement plots illustrate that a maximum responding rotor predicted by one method is not necessarily the maximum responding rotor predicted by the other method. Furthermore, the comparison between the two methods for the same rotor illustrates that while one method predicts the maximum responding rotor in the upper tail of the peak rotor response distribution, the other method predicts the rotor falls somewhere else in the distribution. This provides an example of the conclusion found earlier that the upper tails of the peak rotor responses are generated with different rotors for each method. The peak rotor response and the rotor on which it was predicted will be the same as the peak blade amplification, shown as the Maximum and Rotor headings in Table 6.3 on page 100.

The error between the Nominal and Geometric predictions of an individual rotor’s peak response was calculated to measure Nominal approach accuracy from rotor-to-rotor. This percent error calculated by Eq. 5.5 on page 57, where the peak FRFs are from the peak rotor response for this case. The Normal probability plot of the Nominal error for each individual rotor is shown in Figs. 6.35(a), 6.36(a), and 6.37(a) for M114, M205, and M296, respectively. In each case, the error will fall along the dashed, red line if it is
normally distributed. As shown, the tails of the distributions exhibit curvature from the normal distribution line, highlighting that the error is approximately normally distributed. The first two statistical moments of each distribution are outlined in Table 6.7. The negative mean errors indicate that the Nominal approach generally over-predicts the peak rotor-to-rotor response in this population of 1000 rotors. A similar qualitative conclusion can be drawn from the CDFs of Figs. 6.29 - 6.31 by noting the Nominal data lies to the right of the Geometric data over a majority of the probabilities, illustrating a larger forced response prediction. This difference can be attributed to the nominal mode assumption and how the nominal modes do not represent the non-nominal modes well. Geometric perturbations could have a more profound effect on these higher order blade modes, making drastic changes from the corresponding nominal mode. A modal assurance criterion test could be performed to provide quantitative support to this argument. M114 and M296, which are subject to second order engine excitation, have a much larger over-prediction error than under-prediction as shown in Table 6.7. A zero order engine excitation for M205 has similar maximum and minimum errors.

Figures 6.35(b), 6.36(b), and 6.37(b) correspond to the maximum Nominal over-predictions found in Table 6.7 on page 101 and the illustrated normalized probability plot of Nominal method error of the respective figure. These plots outline the predicted airfoil forced response levels, with the peak rotor-to-rotor displacements given by the horizontal lines. It is to these values which the rotor-to-rotor Nominal error is calculated. As shown in each case, the peak Nominal rotor prediction does not always correspond to the same blade predicted by the Geometric approach to be the maximum responding blade.

6.4 ADLARF Conclusions and Further Discussion

A two-sample Kolmogorov-Smirnov test of the distribution of airfoil forced response illustrated the Nominal approach did not predict the correct distributions for the three modes
considered. As the mode number increased to M205 and M296, this became qualitatively apparent from the histograms in Figs. 6.14 and 6.15. The accuracy of the Nominal method to predict blade-to-blade displacements was also poor. There was a general tendency for Nominal over-prediction, with errors in excess of 200% for all three modes. Errors for under-prediction were found to be a maximum of 70%.

The rotor-to-rotor Nominal method error were found to be as large as 52%, with a tendency for over-prediction for M114 and M296. These errors are significantly less than the blade-to-blade errors between the two methods. Considering that the peak rotor-to-rotor predictions were mistuned for a significantly large portion of rotors, as shown in the rotor response CDFs of Figs. 6.29 - 6.31, this provided larger (i.e., > 1) maximum mistuning numbers to be used in the error calculation. Blade-to-blade predictions, however, were mistuned for a much smaller percent of the blades, shown in Figs. 6.16 - 6.18, and provided smaller, (i.e., < 1) maximum mistuning numbers to be used in the error calculation. This is a likely source for the large discrepancy between blade-to-blade and rotor-to-rotor error.

Another source of this discrepancy can result from the fact that rotor-to-rotor error can also be smaller than blade-to-blade error because the maximum responding blade on a specific rotor predicted by the Geometric method is not necessarily the Nominally predicted maximum responding blade on the same rotor. This was first noted for the Academic case study, and was found to be applicable for the ADLARF case study as well for all three modes. Fig. 6.38(a) for M205 illustrates the Geometric approach predicts a peak rotor response on blade 16 with an amplified response of 1.89, while the Nominal method predicts 0.78 on that same blade, a blade-to-blade error of 58.73% is produced. However, the Nominal approach is predicting a peak rotor response of 2.06 on blade 9. Thus, the rotor-to-rotor peak response error will be -8.99%. Thus, the magnitude of the rotor-to-rotor error can be smaller than the blade-to-blade error because the blade on which the maximum occurs is neglected for a peak rotor prediction.

Table 6.8 on page 101 outlines the number of rotors in the population where the Nom-
inal method incorrectly identified the maximum responding blade. A shown, the Nominal approach missed the maximum responding blade up to $\sim 67\%$ of the time. Of these rotors where the Nominal method incorrectly identified the maximum responding blade, it follows that all had a maximum responding blade with a blade-to-blade error greater than the rotor-to-rotor error. This is somewhat obvious because the blade-to-blade error of the maximum responding blade cannot be less than rotor-to-rotor error on a specific rotor when the Nominal method correctly identified the maximum responding blade - at most they can be equal. This is the case for the rotor-to-rotor error for the rotors where the Nominal method correctly identified the maximum responding blade on the rotor. These errors can be summarized in by the four following cases:

1. Of the rotors listed in column two of Table 6.8 where the Nominal method incorrectly identified the maximum responding blade, *Case 1* in Figs. 6.39 - 6.41 identify the Nominal blade-to-blade error on the correct maximum responding blade predicted by the Geometric method. A single instance of this error corresponds to the Nominal prediction error on blade 16 in Fig. 6.38(a) on page 139.

2. Of the rotors listed in column one of Table 6.8 where the Nominal method correctly identified the maximum responding blade, *Case 2* in Figs. 6.39 - 6.41 identify the Nominal blade-to-blade error on the correct maximum responding blade predicted by the Geometric method. This error is the same as *Case 3* below. A single instance of this error corresponds to the Nominal prediction rotor error on rotor 1 in Fig. 6.38(b).

3. Of the rotors listed in column one of Table 6.8 where the Nominal method correctly identified the maximum responding blade, *Case 3* in Figs. 6.39 - 6.41 identify the Nominal rotor-to-rotor error. A single instance of this error corresponds to the Nominal prediction rotor error on rotor 1 in Fig. 6.38(b).

4. Of the rotors listed in column two of Table 6.8 where the Nominal method incorrectly identified the maximum responding blade, *Case 4* in Figs. 6.39 - 6.41 identify
the Nominal rotor-to-rotor error. A single instance of this error corresponds to the Nominal rotor prediction error on rotor 17 in Fig. 6.38(a), where the error is calculated from the Geometric and Nominal peak rotor predictions from blade 16 and nine, respectively.

5. Of the rotors listed in column two of Table 6.8 where the Nominal method incorrectly identified the maximum responding blade, Case 5 in Figs. 6.39 - 6.41 identify the Nominal blade-to-blade error on the incorrect maximum responding blade predicted by the Nominal method. A single instance of this error corresponds to the Nominal prediction error on blade 9 in Fig. 6.38(a).

These results for the ADLARF case illustrate that the Nominal method incorrectly identifies the maximum responding blade quite often with a varying degree of accuracy due to over and under-predictions. The blade-to-blade error in Case 1 from above illustrates that when the nominal method incorrectly identifies the maximum responding blade, it most often under-predicts the response of the maximum responding blade. The number of rotors with a Nominal under-prediction is listed in the last column of Table 6.8. This can also be seen from the CDF of Case 1 in Figs. 6.39 - 6.41. The rotor-to-rotor error of Case 4 above calculated from the same rotors in Case 1, illustrates that when the Nominal method incorrectly identifies the maximum responding blade, the maximum rotor-to-rotor error seen is larger than Case 2 and 3 where the maximum responding blade was correctly identified. This highlights that the largest error typically seen in a population of rotor-to-rotor error can likely be attributed to rotors where the Nominal method incorrectly identified the maximum responding blade. The significance of these results will be highlighted in Chpt. 7 where they will be applied to mistuning scenarios.

Peak rotor displacements predicted by the Nominal method also deteriorated as the system modes increased. In each case, the Nominal method lacked the ability to provide an accurate representation of the peak rotor forced response distributions. Of further interest is the in-depth look at the specific rotors that populated the upper 97th percentile of the Ge-
ometric method predicted forced response distributions. The Nominal prediction of these exact same rotors did not provide forced response levels restricted to the upper 97th percentile of the Nominally predicted distribution. In fact, these rotors were distributed over the full range of probabilities. M205 had a larger percentage of the Nominal predictions in the upper 90th percentile, but there were still seven rotors that were at lower probabilities. This illustrates that the upper percentiles of the forced response distributions for each method are composed with different rotors.
### Table 6.1: Basic size data for the ADLARF model

<table>
<thead>
<tr>
<th>Component</th>
<th>Elements</th>
<th>Nodes</th>
<th>DOF</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cantilevered Airfoil</td>
<td>320</td>
<td>680</td>
<td>2040</td>
</tr>
<tr>
<td>Disk</td>
<td>8080</td>
<td>9936</td>
<td>29808</td>
</tr>
<tr>
<td>Disk-Airfoil Interface</td>
<td>n/a</td>
<td>34</td>
<td>102</td>
</tr>
<tr>
<td>Full Model</td>
<td>13200</td>
<td>21360</td>
<td>64080</td>
</tr>
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</table>

### Table 6.2: ADLARF Peak blade displacement population statistics

<table>
<thead>
<tr>
<th></th>
<th>Mean</th>
<th>Error</th>
<th>Standard Deviation</th>
<th>Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>M114</td>
<td>Geometric</td>
<td>0.856</td>
<td>–</td>
<td>0.232</td>
</tr>
<tr>
<td></td>
<td>Nominal</td>
<td>0.883</td>
<td>3.259%</td>
<td>0.228</td>
</tr>
<tr>
<td>M205</td>
<td>Geometric</td>
<td>0.808</td>
<td>–</td>
<td>0.427</td>
</tr>
<tr>
<td></td>
<td>Nominal</td>
<td>0.850</td>
<td>5.094%</td>
<td>0.407</td>
</tr>
<tr>
<td>M296</td>
<td>Geometric</td>
<td>0.736</td>
<td>–</td>
<td>0.195</td>
</tr>
<tr>
<td></td>
<td>Nominal</td>
<td>0.834</td>
<td>13.385%</td>
<td>0.183</td>
</tr>
</tbody>
</table>

### Table 6.3: ADLARF blades with mistuning amplification

<table>
<thead>
<tr>
<th></th>
<th>% of Population</th>
<th>Error</th>
<th>Maximum</th>
<th>Error</th>
<th>Rotor #</th>
</tr>
</thead>
<tbody>
<tr>
<td>M114</td>
<td>Geometric</td>
<td>27.41%</td>
<td>–</td>
<td>1.663</td>
<td>190</td>
</tr>
<tr>
<td></td>
<td>Nominal</td>
<td>31.11%</td>
<td>13.42%</td>
<td>1.613%</td>
<td>920</td>
</tr>
<tr>
<td>M205</td>
<td>Geometric</td>
<td>24.81%</td>
<td>–</td>
<td>2.427</td>
<td>564</td>
</tr>
<tr>
<td></td>
<td>Nominal</td>
<td>23.11%</td>
<td>6.851</td>
<td>0.124%</td>
<td>1</td>
</tr>
<tr>
<td>M296</td>
<td>Geometric</td>
<td>10.14%</td>
<td>–</td>
<td>1.514</td>
<td>687</td>
</tr>
<tr>
<td></td>
<td>Nominal</td>
<td>18.48%</td>
<td>82.31%</td>
<td>0.799%</td>
<td>239</td>
</tr>
</tbody>
</table>

### Table 6.4: Statistics of individual ADLARF airfoil Nominal method prediction error data

<table>
<thead>
<tr>
<th>Mode</th>
<th>Mean</th>
<th>Std. Dev.</th>
<th>Max Over-Pred.</th>
<th>Max Under-Pred.</th>
<th>% Over-Pred.</th>
</tr>
</thead>
<tbody>
<tr>
<td>M114</td>
<td>-6.12%</td>
<td>24.15%</td>
<td>-310.96%</td>
<td>63.17%</td>
<td>56.21%</td>
</tr>
<tr>
<td>M205</td>
<td>-11.96%</td>
<td>31.66%</td>
<td>-253.68%</td>
<td>69.62%</td>
<td>66.06%</td>
</tr>
<tr>
<td>M296</td>
<td>-17.56%</td>
<td>27.55%</td>
<td>-230.15%</td>
<td>53.28%</td>
<td>74.95%</td>
</tr>
</tbody>
</table>
Table 6.5: ADLARF maximum rotor displacement population statistics

<table>
<thead>
<tr>
<th></th>
<th>Mean</th>
<th>Error</th>
<th>Standard Deviation</th>
<th>Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>M114</td>
<td>Geometric</td>
<td>1.281</td>
<td>–</td>
<td>0.105</td>
</tr>
<tr>
<td></td>
<td>Nominal</td>
<td>1.300</td>
<td>1.535%</td>
<td>0.091</td>
</tr>
<tr>
<td>M205</td>
<td>Geometric</td>
<td>1.908</td>
<td>–</td>
<td>0.209</td>
</tr>
<tr>
<td></td>
<td>Nominal</td>
<td>1.961</td>
<td>2.793%</td>
<td>0.255</td>
</tr>
<tr>
<td>M296</td>
<td>Geometric</td>
<td>1.120</td>
<td>–</td>
<td>0.119</td>
</tr>
<tr>
<td></td>
<td>Nominal</td>
<td>1.178</td>
<td>5.195%</td>
<td>0.094</td>
</tr>
</tbody>
</table>

Table 6.6: ADLARF maximum rotor displacement CDF Percentile Data

<table>
<thead>
<tr>
<th></th>
<th>97th %</th>
<th>Error</th>
<th>98th %</th>
<th>Error</th>
<th>99th %</th>
<th>Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>M114</td>
<td>Geometric</td>
<td>1.492</td>
<td>–</td>
<td>1.512</td>
<td>–</td>
<td>1.541</td>
</tr>
<tr>
<td></td>
<td>Nominal</td>
<td>1.490</td>
<td>0.132%</td>
<td>1.501</td>
<td>0.753%</td>
<td>1.532</td>
</tr>
<tr>
<td>M205</td>
<td>Geometric</td>
<td>2.252</td>
<td>–</td>
<td>2.280</td>
<td>–</td>
<td>2.301</td>
</tr>
<tr>
<td>M296</td>
<td>Geometric</td>
<td>1.367</td>
<td>–</td>
<td>1.406</td>
<td>–</td>
<td>1.435</td>
</tr>
<tr>
<td></td>
<td>Nominal</td>
<td>1.377</td>
<td>-0.783%</td>
<td>1.408</td>
<td>1.221%</td>
<td>1.412</td>
</tr>
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Table 6.7: ADLARF maximum rotor Nominal displacement error statistics

<table>
<thead>
<tr>
<th></th>
<th>Mean</th>
<th>Standard Deviation</th>
<th>Max Under-Prediction</th>
<th>Max Over-Prediction</th>
</tr>
</thead>
<tbody>
<tr>
<td>M114</td>
<td>-2.026</td>
<td>9.113</td>
<td>23.863</td>
<td>-34.103</td>
</tr>
<tr>
<td>M205</td>
<td>-3.295</td>
<td>12.324</td>
<td>42.119</td>
<td>-40.347</td>
</tr>
<tr>
<td>M296</td>
<td>-6.068</td>
<td>11.516</td>
<td>24.787</td>
<td>-51.781</td>
</tr>
</tbody>
</table>

Table 6.8: ADLARF maximum responding blades identified by the Nominal Method

<table>
<thead>
<tr>
<th></th>
<th>Rotors Correctly Identified</th>
<th>Rotors Incorrectly Identified</th>
</tr>
</thead>
<tbody>
<tr>
<td>No.</td>
<td>No.</td>
<td>No. with Nominal over-prediction</td>
</tr>
<tr>
<td>M114</td>
<td>394</td>
<td>610</td>
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<tr>
<td>M205</td>
<td>617</td>
<td>383</td>
</tr>
<tr>
<td>M296</td>
<td>334</td>
<td>666</td>
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(a) Normal Probability Plot

(b) Rotor With Maximum NMA Over-Prediction

Figure 6.36: Distribution of Nominal prediction error for individual ADLARF rotors for M205
Figure 6.37: Distribution of Nominal prediction error for individual ADLARF rotors for M296
<table>
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<th>Normalized Amplitude</th>
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<td>16</td>
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</tbody>
</table>

(a) Nominal Incorrect Prediction of Max Responding Blade

(b) Nominal Correct Prediction of Max Responding Blade

Figure 6.38: ADLARF rotor predicted airfoil displacements for M205
Figure 6.39: CDFs of error for cases when Nominal and Geometric methods predict peak ADLARF rotor response on different blades for M114
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7.1 Verification and Validation of the Nominal Mode Assumption in Academic Models

Academic models are used throughout the literature as validation cases to show how reduced order models accurately predict mistuning. The Nominal method also did a good job of predicting population statistics of the Academic case study in this work. However, a comparison of Nominal method accuracy between the Academic and ADLARF test cases reveals that models with geometries representative of actual engine hardware present much larger prediction errors. This result highlights that Academic case studies provide a verification that the mistuning code is capable of mathematically reproducing the mistuning phenomenon, but validation of the assumptions made in the Nominal ROM are not accomplished. The simplicity of Academic models prevent an accurate grasp of the impacts associated with the nominal mode assumption. While a comparison of tuned and mistuned
mode shapes was not accomplished in this work, evidence suggests that the tuned modes of simplistic geometries accurately represent mistuned modes resulting from geometric perturbations. Modes of complex airfoil geometries appear to be more sensitive to geometric perturbations and cannot be accurately represented as a sum of the tuned modes. Typically, higher order modes of complex geometries are more spatially complex and have localized displacements along airfoil tips and edges, as shown in Fig. 6.5. Any geometric perturbations in these areas can alter these displacements so the mode shape is not recognizable from the corresponding nominal geometry modes. As the mode number increases, Nominal method accuracy diminishes. This is supported by noticing that Nominal prediction errors increase with increasing mode numbers M114, M205, and M296.

7.2 Determination of Airfoil Response after an Engine Test

As discussed in Sec. 4.2.1, airfoil-to-airfoil response is critical in any testing scenario where a limited number of strain gages are instrumented on a rotor. Such cases often arise in demonstrator engine tests or tests where strain gage malfunction prevents capturing stress levels in all gaged locations. Due to the localization phenomenon of mistuning, limited strain gage placement may not fully describe the peak airfoil stresses observed in the rotor. If the airfoil frequency deviations are known along with a measured displacement of a single blade, it is important to assess if the Nominal reduced order model can effectively predict the displacements of the un-instrumented blades. This is accomplished by the following steps for M205:

1. The natural frequencies of each airfoil on a specific rotor are obtained and used as input into the Nominal ROM. The engine test is conducted and a single blade is instrumented to measure the displacement, as shown as a blue line for blade 16 in Fig. 7.1. Here the engine test response is assumed to be the Geometric method prediction.
2. The Nominal method predicts the airfoil responses of the rotor, as shown in red circles in Fig. 7.2.

3. Scale the Nominal predictions to the measured response of blade 16, as shown as green stars in Fig. 7.3.

4. Compare the scaled Nominal predictions to the actual airfoil responses if they were actually known through instrumentation. Here the actual airfoil responses are the Geometric method predictions, shown in Fig. 7.4.

Figure 7.4 shows that scaling the Nominal predictions to that of a single measured blade does not increase the accuracy of predicting the response levels of the remaining blades. The Nominal prediction error on blade 16 (the blade used to scale the remaining Nominal predictions) is 60.5%. Using this blade to scale the Nominal predictions will reduce the accuracy of the approach. In fact, such an approach can inflate the prediction of the maximum responding blade. Note that the scaled prediction of blade four is not shown in Fig. 7.4 because the maximum response is 6.33. This provides an error of $\sim 200\%$, while the un-scaled Nominal prediction error on the maximum responding blade is 32.4%. Since large airfoil-to-airfoil errors were seen in Section 6.3.1, using the above approach will produce inaccurate predictions of the remaining blades.

Further evidence of this can be seen in Fig. 7.5(a). Here, the above process is repeated using each single airfoil to scale the Nominal predictions. Thus, for a single rotor there are 16 different scaled Nominal predictions to be made and compared to the actual response, as in Fig. 7.4. Doing this for the population of rotors then yields 16,000 scaled Nominal predictions. Taking the mean of the absolute value of the airfoil-to-airfoil error of the scaled Nominal predictions will then provide a quantitative measure for comparison to the mean airfoil-to-airfoil error of the un-scaled Nominal predictions. Rotor one with blade 16 used to scale the Nominal predictions used in Figures 7.1 - 7.4 is shown in Fig. 7.5(a). Note in the un-scaled Nominal prediction mean errors that there will only be a population of
1,000, and is shown in Fig. 7.5(b). The much larger errors shown in Fig. 7.5(a) further suggest that using an airfoil with a large Nominal prediction error to scale the Nominal airfoil predictions further reduces the accuracy of the remaining blades.

### 7.3 Determining Strain Gage Placement before Engine Test

A scenario similar to the previous section is using a ROM to select the high responding blade before installing instrumentation and running an engine test. This would provide three cases of error: first, the maximum responding blade is incorrectly selected; second, the incorrectly predicted maximum responding blade will have error on the prediction; and third, the actual maximum responding blade will have error on the prediction. As shown in Table 6.8 of Sec. 6.4, the Nominal method incorrectly identifies the maximum responding blade in over 60% of the predictions for M114 and M296 in the ADLARF case study. For M205, this value is still large at 38.3%. Furthermore, of these blades incorrectly labeled as the maximum responding blade by the Nominal method, Fig. 6.40 for Case 5 shows the Nominal method had over-predictions 97.4% of the time. Thus, the placement of a strain gage determined by the Nominal method would give an overall probability of an over-prediction of 37.3% of the time. This would give an overly-conservative estimate of the maximum blade stress. If an additional strain gage were fortuitously placed on the actual maximum responding blade (fortuitous because the correct maximum responding blade is not known \textit{a priori}), \textit{Case 3} illustrates that the response is under-predicted 97.0% of the time. Thus, for the actual maximum responding blade the Nominal method presents an unsafe estimate on stress. However, this is subject to chance that a gage is placed on the actual responding blade. Peak rotor response provides a comparison of how the Nominal over-prediction estimate on the incorrect maximum responding blade compares with the Nominal under-prediction on the actual responding blade to present a “whole” rotor life estimate.
7.4 Failure Analysis Using Airfoil Population Results

A Stress vs. Life curve, or S-N curve, outlines the maximum alternating or vibratory stress a given specimen can withstand for a set number of cycles. If a specimen is desired to last to the infinite life point ($10^9$ cycles for Titanium), then the alternating stress cannot be larger than that displayed on the S-N curve. In reality, there is variability in the alternating stress that specimens of the same material can withstand at a given life, i.e., variability in fatigue limits. This is outlined in Fig. 7.6, where each $X$ is a different specimen that is characterized by a distribution. If individual blades are forged/milled and welded onto a disk to create an IBR, each individual blade can be treated as one of these specimens that has a different fatigue limit. If the vibratory stress is larger than this fatigue limit the blade will fail.

Each airfoil is assigned a random fatigue limit drawn from a normal distribution of different means and a standard deviation of 4.7. A tuned alternating stress is also selected and the mistuned stress is calculated by the magnitude of mistuning amplification determined by both the Nominal and Geometric methods. If this mistuned stress is larger than the randomly assigned fatigue limit, the airfoil will fail. Figure 7.7 outlines the number of Type I and II error defined by the following:

1. **Type I Error**: False Acceptance. The Nominal approach predicts response below the amplification limit when the blade is actually above the limit. This error will lead to blade failure.

2. **Type II Error**: False Rejection. The Nominal approach predicts response above the amplification limit when the blade is actually below the limit. This error leads to unnecessary and costly blade rejection.

Note that the results are blade specific and are not tied to a specific rotor. Results illustrate that the Nominal method has larger Type II error than Type I over a range of fatigue limits.
Furthermore, the Nominal method also predicts blade stresses above the fatigue limit significantly more often than the actual response. As a result, using the Nominal method can lead to system failures for falsely accepting a failed blade. This approach can also lead to excess costs for rejecting blades that are still safe.

7.5 Use of the Nominal Method for Manufactured Rotor Certification

In design practice, mistuning predictions determine rotor peak response and whether it lies below defined mistuning amplification limits. All rotors with predictions above limits would be discarded or require modification. Clearly, Nominal prediction errors can lead to the following two cases:

1. Type I Error: False Acceptance. The Nominal approach predicts response below the amplification limit when the rotor is actually above the limit. This error will lead to rotor failure.

2. Type II Error: False Rejection. The Nominal approach predicts response above the amplification limit when the rotor is actually below the limit. This error leads to unnecessary and costly rotor rejection or rework.

M205 of the ADLARF test case is used as case study for the above Type I and II errors. Existence of Type I error is evident in Fig. 6.30 on page 131 of Sec. 6.3.2 where the rotors in the 97\text{th} percentile of the Geometric predictions do not always fall in the 97\text{th} of the Nominal prediction. A specific case of Type II error can be viewed in Figs. 6.36(b) on page 137 and 6.38(a) on page 139 if the mistuning cutoff were set 2.0. Fig. 6.38(b) also illustrates Type II error if the cutoff were 2.1. The Nominal prediction in these three responses illustrates that the rotor should be rejected since it responds beyond the established
threshold. The actual response, however, is predicted will below this threshold.

Fig. 7.8 outlines different mistuning cutoff levels and the corresponding number of rotors in the population of 1,000 that satisfy either False Acceptance or False Rejection error types. Geometric predictions above the cutoff outline the total number of rotors the Geometric method predicted to be above the mistuning cutoff. Nominal predictions above the cutoff follow the same logic for Nominal method predictions. As expected, as the mistuning cutoff increases in value, the total number of Geometric and Nominal predictions decreases because the maximum mistuning limit is approached for the population. Furthermore, the number of Nominal predictions above each cutoff exceeds that of the Geometric predictions. This can be verified in the PMF of the peak rotor response in Fig. 6.27 of Sec. 6.3.2.

The number of Type I and Type II errors are superimposed over the number of Geometric and Nominal predictions above the cutoff, respectively, to give a qualitative assessment of the percentage of rotors predicted by each method to be above the cutoff with error. Type II error occurrences exceeded Type I in each cutoff limit considered. This highlights over-conservatism in the Nominal assessment because many rotors, 22.5% of the population for a cutoff level of 2.1, are falsely classified as exceeding established mistuning levels. Since the mistuning amplification of a specific rotor is not known \textit{a priori} to manufacturing, falsely rejecting 225 rotors at an approximate cost of $50K$ will total $11.25M$ if a population of 1,000 rotors is needed. At the same time, 60 or 6% of the rotors are classified as safe when they are not. These rotors essentially \textit{blow up} and can cause catastrophic failure and loss of life.

Fig. 7.8 also shows that a cutoff of 2.4 has a limited number of Type I and II errors. It would be easy to set established mistuning cutoffs at higher levels with minimal Type I and II errors, but the added robustness would come at a cost. An increased mistuning cutoff limit requires added strength so the airfoil can withstand larger deflections and stresses. As established in Sec. 1.1.3, this overly-conservative design can limit the performance
capabilities of the airfoil and engine. Ultimately, the nominal assumption in mistuning studies can lead to excess conservatism in designs at a cost to performance and falsely rejected parts.
Figure 7.1: Experimental measurement of a single blade response
Figure 7.2: Nominal predictions of all airfoils
Figure 7.3: Scaled Nominal predictions of all airfoils
Figure 7.4: Comparison of scaled Nominal predictions
Figure 7.5: Normal probability plot of average blade-to-blade errors on a single rotor
Figure 7.6: Example Stress vs. Life curve

Figure 7.7: Nominal prediction Type I and Type II peak blade-to-blade error
Figure 7.8: Nominal prediction Type I and Type II peak rotor-to-rotor error
Conclusions

The Nominal method was shown to accurately predict peak airfoil and rotor mistuned response distributions well for the single veering region of interest in the academic test rotor. Statistical tests found that evidence suggests the distributions were not statistically indistinguishable. The maximum amplitudes seen in the distributions were also accurately predicted. Thus, response distributions generated with the nominal mode assumption are representative of the actual response distribution and can be used for probabilistic calculations used in a design sense to increase performance and lower weight and cost. Accuracy was slightly diminished for peak blade-to-blade and peak rotor-to-rotor responses, but the maximum errors seen were over five standard deviations away from the mean. It was also determined that rotors with a small average blade-to-blade mean does not suggest that the rotor has accurate accuracy for all the blade-to-blade predictions. For all results considered, the Nominal approach is considered to predict the response distributions accurately, but the simplicity of the model suggests that it is not a good test case to determine the validity of the nominal mode assumption.

The complex geometries present in the ADLARF model provided a much more realistic test case. For the three veering regions investigated, it was found that the Nominal method did not accurately predict the correct response distributions. Furthermore, the blade-to-blade and rotor-to-rotor errors were found to be quite large. The largest responding blades in the population were often not predicted on the correct rotor. As a result, the maximum responding rotors in the population were often incorrectly identified by the
Nominal approach. Accuracy was further deteriorated as veering regions included higher order modes. Further work is needed to quantify a modal assurance criterion (MAC) for rotors with large error and determine if there is a relationship between MAC values and rotors with large Nominal prediction error. Nominal method predictions for this test case limit the applicability of the nominal mode assumption for probabilistic calculations. Established scenarios suggest a conservative, Nominal method over-prediction in a majority of applications. Using the conservative results can negatively impact performance, weight, and cost of an IBR.
Bibliography


