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An Investigation of Group Developed Weighing Matrices

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AN INVESTIGATION OF
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I HEREBY RECOMMEND THAT THE THESIS PREPARED UNDER MY SUPERVISION BY Jeff R. Hollon ENTITLED An Investigation of Group Developed Weighing Matrices BE ACCEPTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF Master of Science.

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Abstract

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A weighing matrix is a square matrix whose entries are 1, 0 or -1 and has the property that the matrix times its transpose is some integer multiple of the identity matrix. We examine the case where these matrices are said to be developed by an abelian group. Through a combination of extending previous results and by giving explicit constructions we will answer the question of existence for 318 such matrices of order and weight both below 100. At the end, we are left with 98 open cases out of a possible 1,022. Further, some of the new results provide insight into the existence of matrices with larger weights and orders.

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Chapter 1

Introduction

1.1 Group Developed Weighing Matrices

A weighing matrix $W = W(n, k)$ is a square matrix, of order n, whose entries are in the set $w_{i,j} \in \{-1, 0, +1\}$. This matrix satisfies $WW^t = kI_n$ where t denotes the matrix transpose, k is a positive integer known as the weight, and I_n is the identity matrix of size n. A $W = W(n, k)$ is said to be G-developed (or G-invariant) if the entries of W may be generated only by knowing the first row of the matrix. Here we assume that the group G is abelian. To see how a matrix W is developed we label the first row of the matrix by the elements in G. If $G = \{g_0, g_2 \ldots, g_{n-1}\}$ then we can define

$$P = \{g_i | W(1, i) = +1\} \tag{1.1}$$

$$N = \{g_i | W(1, i) = -1\}$$

From these definitions we can see the $|P|+|N|=k$ and given only the first row of W allows for the remaining elements to be developed by

$$w_{i,j} = \begin{cases} +1 & if\, g_i g_j^{-1} \in P \\ -1 & if\, g_i g_j^{-1} \in N \\ 0 & otherwise \end{cases}$$

Theorem 1. *For any G-developed $W(n,k)$ the following are true:*

1. $k = s^2$ for some positive integer s,
2. $|P| = \frac{s^2+s}{2}$,
3. $|N| = \frac{s^2-s}{2}$.

Remark. Interchanging +1s and -1s in a weighing matrix does not change any of the properties of the matrix. Thus, it is only by convention that $|P|$ and $|N|$ are chosen as such. (i.e. The orders of P and N can be switched.) For further information on these facts see [20].

Definition 2. The support of a G-developed $W(n,k)$ is the set of elements which are non-zero and is denoted by $supp(W)$. It is clear that $supp(W) = P \bigcup N$.

From the definition of support comes the natural idea of disjoint matrices.

Definition 3. Let W and X be two G-developed $W(n,k)$ matrices. Also, let

$$\begin{aligned} P_W &= \{g_i | W(1,i) = +1\} \\ N_W &= \{g_i | W(1,i) = -1\} \end{aligned}$$

and

$$\begin{aligned} P_X &= \{g_i | X(1,i) = +1\} \\ N_X &= \{g_i | X(1,i) = -1\} \end{aligned}$$

Then we have that $supp(W) = P_W \bigcup N_W$ and $supp(X) = P_X \bigcup N_X$. The two matrices are said to be disjoint if $supp(W) \bigcap supp(X) = \emptyset$.

A very useful notation for G-developed weighing matrices is the use of the group ring $Z[G]$.

Definition 4. Let G be a finite group and R a ring where $G = \{g_0, g_1, g_2, \ldots, g_{n-1}\}$. Then the group ring of G over R is the set denoted by $R[G]$ defined as

$$R[G] = \left\{ \sum_{i=0}^{n-1} a_i g_i \mid a_i \in R \right\}$$

with multiplication and addition defined as usual.

Any G-developed weighing matrix can be written in the group ring by using the sets P and N defined by Equation 1.1 and using the following formula:

$$W = \sum_{i=0}^{n-1} a_i g_i \; where \; a_i = \begin{cases} 1 & if \; g_i \in P \\ -1 & if \; g_i \in N \\ 0 & otherwise \end{cases}$$

1.2 Examples of G-developed Weighing Matrices

Definition 5. A circulant weighing matrix of order n and weight k, denoted by $CW(n, k)$, is a G-developed $W(n, k)$ weighing matrix where the group G is cyclic.

It should be clear that the identity matrix I_n is a (Z_n)-developed $W(n, 1)$ and that this should be considered the trivial example of such a matrix. Here are some non-trivial examples of group weighing matrices.

Example 6. A (Z_7)-developed $W(7, 4)$

Let W be the matrix given by

$$\begin{pmatrix} - & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & - & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & - & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & - & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & - & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & - & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 & - \end{pmatrix}$$

Figure 1.1: (Z_7)-developed $W(7,4)$

It can be shown that W is a weighing matrix with $WW^t = 4I_7$. Further, by labeling the first row with $\{1, x, x^2, x^3, x^4, x^5, x^6\}$, where $G = Z_7 = \langle x \rangle$, the remaining rows may be developed. This means that that $P = \{x, x^2, x^4\}$ and $N = \{1\}$. W may also be represented in $Z[Z_7]$ by $W = -1 + x + x^2 + x^4$.

Example 7. A $(Z_2 \times Z_2 \times Z_2)$-developed $W(8,4)$

Such a W is given by the matrix

$$\begin{pmatrix} - & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & - & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & - & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & - & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & - & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & - & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & - & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & - \end{pmatrix}$$

Figure 1.2: $(Z_2 \times Z_2 \times Z_2)$-developed $W(8,4)$

Direct computation shows that $WW^t = 4I_8$. W may also be developed by labeling the first row with the elements $\{1, z, y, yz, x, xz, xy, xyz\}$ in $G = Z_2 \times Z_2 \times Z_2 = \langle x \rangle \times \langle y \rangle \times \langle z \rangle$ then filling in the remaining rows by developing them. From this labeling, W may also be written as an element of $Z[Z_2 \times Z_2 \times Z_2]$ by $W = -1 + y + xz + xyz$. In this case $P = \{y, xz, xyz\}$ and $N = \{1\}$.

Chapter 2

Preliminaries

The study of G-developed weighing matrices is closely related to the study of difference sets. Since these are similar topics, we will give a brief explanation of the tools commonly used to study both. The tools needed here are group rings and character theory.

2.1 Group Rings and Character Theory

By definition of a G-developed $W(n, k)$ we know that $WW^t = kI_n$. So, if the matrix W can also be represented as an element of a group ring then we must define how its transpose is to be handled.

Definition 8. Let W be a G-developed $W(n, k)$. Further, let W be written as an element in $Z[G]$ as defined above in Definition 4. Then

$$WW^{(-1)} = k$$

where

$$\begin{aligned} W^{(-1)} &= \left(\sum_{i=0}^{n-1} a_i g_i\right)^{(-1)} \\ &= \sum_{i=0}^{n-1} a_i g_i^{-1} \end{aligned}$$

and g_i^{-1} denotes the usual group element inverse in G.

For more information on group rings in this context one may refer to [7, 21].

Let us take a quick look at an example with this notation.

Example 9. Take the W from Example 6. Written in $Z[Z_7]$, $W = -1 + x + x^2 + x^4$ and we see that

$$\begin{aligned} W^{(-1)} &= -1^{-1} + x^{-1} + x^{-2} + x^{-4} \\ &= -1 + x^6 + x^5 + x^3 \end{aligned}$$

Then direct computation shows that

$$\begin{aligned} WW^{(-1)} &= (-1 + x + x^2 + x^4)(-1 + x^6 + x^5 + x^3) \\ &= 1 - x - x^2 - x^3 + 3x^7 + x^8 + x^9 + x^{10} \\ &= 4 \end{aligned}$$

where $x^7 = 1$.

Definition 10. A homomorphism from an abelian group G to the field $\mathbb{C}$ of complex numbers is called a character of G and is denoted by χ.

Remark. The set of all characters of the abelian group G is represented by $\widehat{G}$ and it can be shown that $\widehat{G}$ is isomorphic to G. The principal character of G is defined to be the homomorphism that maps each element g of G to 1. This character is denoted χ_o. The character homomorphism can be extended linearly to the group rings defined above. Also, any non-principal character is denoted by χ.

While we are restricting our topic to weighing matrices here, these tools are used to study difference sets.

Definition 11. Let D be an element of $Z[G]$ where D has coefficients from {0,1}. D is said to be a $(v, k, \lambda) - difference$ set in G if

$$DD^{(-1)} = k - \lambda + \lambda G \, in \, Z[G].$$

There is much information in the literature about difference sets and the use of character theory and group rings. See any of [22, 15] for more information on such topics.

2.2 Integer-G-developed Weighing Matrices

An interesting fact with G-developed $W(n, k)$ matrices is that the existence of one implies the existence of an integer-G-developed $W(n, k)$ matrix.

Definition 12. An integer-G-developed weighing matrix is a weighing matrix whose entries are not limited to {-1, 0, 1}. The entries may be any integer. Clearly, any G-developed $W(n, k)$ is also an integer-G-developed $W(n, k)$.

By taking an existing G-developed $W(n, k)$ we can create an integer-($^G/_H$)-developed $W(m, k)$ by folding.

Definition 13. Let $\sigma : G \longrightarrow H$ be some homomorphism between the groups G and H. Then σ can be extended linearly as a ring homomorphism from $Z[G] \longrightarrow Z[H]$. In particular, if $\sigma : G \longrightarrow {}^G/_H$ is the canonical homomorphism between groups, then we refer to the linearly extended form of σ between the rings $Z[G] \longrightarrow Z[{}^G/_H]$ as folding.

Theorem 14. *Let W be a G-developed $W(n, k)$ and let H be a subgroup of G. Define $\sigma : Z[G] \longrightarrow Z[{}^G/_H]$ as the canonical homomorphism between group rings (i.e. folding). Then W^σ is a (${}^G/_H$)-developed $W(|{}^G/_H|, k)$. Further, the coefficients of W^σ are confined by $[-|H|, |H|]$. Thus, meaning that W^σ is an integer-(${}^G/_H$)-developed $W(|{}^G/_H|, k)$.*

2.3 Hadamard Matrices

There is much known about a special type of weighing matrix known as a Hadamard matrix. These are weighing matrices in which the weight and order are the same. That is to say that there are only non-zero entries in the matrix.

Definition 15. A Hadamard matrix is a weighing matrix which has all non-zero entries. i.e. A $W(n,k)$ where $n = k$.

Theorem 16. *(Hadamard) If a Hadamard matrix exists then the order of the matrix must be 1, 2, or a multiple of 4.*

Remark. This fact eliminates all G-developed $W(n,k)$ where $|G| = k$ and k is not a multiple of 4. i.e. All groups of order 9, 25, 49, or 81 may not contain Hadamard matrices.

Conjecture 17. *(Hadamard) A Hadamard matrix exists iff n is a multiple of 4.*

Remark 18. This conjecture has been verified for orders up to n=668 as mentioned in [13]. There are only 13 integers for which a Hadamard matrix is not known for orders $4m$ and m<500. These values are m = 167, 179, 223, 251, 283, 311, 347, 359, 419, 443, 479, 487, and 491.

An interesting fact about Hadamard matrices is that there is only one non-trivial circulant Hadamard matrix known to exist. This matrix is the (Z_4)-developed $W(4,4)$ and is given by

$$\begin{pmatrix} - & 1 & 1 & 1 \\ 1 & - & 1 & 1 \\ 1 & 1 & - & 1 \\ 1 & 1 & 1 & - \end{pmatrix}$$

Figure 2.1: The Circulant Hadamard Matrix

or equivalently written as $W = -1 + x + x^2 + x^3$ in $Z[Z_4]$.

Conjecture 19. *(Hadamard) A circulant, non-trivial, Hadamard matrix (Z_n)-developed $W(n,k)$ exists iff $n = k = 4$.*

Remark. While this conjecture is so far unproven it has been verified for orders up to 4×10^{26} with fewer than 1,600 exceptions [19].

Theorem 20. *(McFarland) If a Hadamard matrix of order $4p^2$ exists then p is prime and $p = 2$ or $p = 3$.*

See [18] for more information.

Chapter 3

Some Previously Known Results

3.1 Characterizations of cyclic-G-developed $W(n,k)$ of Weights 4, 9, and 16

For weights 4, 9, and 16 the set of G-developed weighing matrices is fully classified if the group is cyclic. The following theorems illustrate this fact.

Theorem 21. *A $CW(n,4)$ exists iff $n \geq 4$ is even or a multiple of 7.*

See [14] for more information.

Theorem 22. *A $CW(n,9)$ exists iff n is a multiple of 13 or 24.*

See [2] for more information.

Theorem 23. *A $CW(n,16)$ exists iff $n \geq 21$ is a multiple of 14 or 21 or 31.*

See [6] for details.

3.2 Embedding and Construction Theorems

This next result comes from work by Arasu, Dillon, Jungnickel, and Pott [3]. This produces an infinite family of minimal circulant weighing matrices.

Theorem 24. *For each prime power q and positive integer d, there exists a $CW(\frac{q^{2d+1}-1}{q-1}, q^{2d})$.*

The following are other useful constructions for group weighing matrices.

Theorem 25. *(Kronecker) If there exists both a G_1-developed $W(n_1, k_1)$ and a G_2-developed $W(n_2, k_2)$ then there exists a $(G_1 \times G_2)$-developed $W(n_1 n_2, k_1 k_2)$.*

See [8] for more details on this well known result.

Theorem 26. *If there exists a G-developed $W(n, k)$ then there exists a H-developed $W(m, k)$ for all groups H containing a subgroup isomorphic to G.*

This can also be seen in [8].

Theorem 27. *(Arasu and Dillon) Let H be an abelian group and let $D_i \in Z[H]$ for $i = 0, 1, 2, \ldots, (n-1)$. Assume that these three conditions are satisfied:*

1. The coefficients of each D_i are in {-1, 0, 1},
2. $\sum_{i=0}^{n-1} D_i D_i^{(-1)} = n|H|$,
3. $D_i D_j^{(-1)} = 0$ for all $i \neq j$.

Further, let G be an abelian group containing H as a subgroup of index $l > n$. Then there exists a G-developed $W(|G|, n|H|)$.
This result can also be found in [8].

Theorem 28. *(Patching) Let M and N be two disjoint G-developed $W(n, k)$ matrices as in Definition 3. Then $(1+t)M + (1-t)N$ is a $(Z_2 \times G)$-developed $W(2n, 4k)$, where $Z_2 = \langle t \rangle$.*

Proof. Let $W = (1+t)M + (1-t)N$ in $Z[Z_2 \times G]$. Then by direct calculation and noting that $(1+t)(1-t) = 1-t^2 = 0$ we see that

$$
\begin{aligned}
WW^{(-1)} &= [(1+t)M + (1-t)N)][(1+t)M + (1-t)N]^{(-1)} \\
&= [(1+t)M + (1-t)N][(1+t)M^{(-1)} + (1-t)N^{(-1)}] \\
&= (1+t)^2 MM^{(-1)} + (1+t)(1-t)[MN^{(-1)} + M^{(-1)}N] + (1-t)^2 NN^{(-1)} \\
&= (1+t)^2 MM^{(-1)} + (1-t)^2 NN^{(-1)} \\
&= (1+t)^2 k + (1-t)^2 k \\
&= 4k.
\end{aligned}
$$

□

Remark. This patching scheme is a well known result and, among many different sources, can be found in [4].

Theorem 29. *(Arasu and Linthicum) A $(Z_2 \times Z_p \times Z_p)$-developed $W(2p^2, p^2)$ exists for all p prime and can be constructed as such:*

$$W = \langle x, y\rangle - \langle x\rangle - \langle y\rangle - \sum_{n=1}^{p-3} x^n \left\langle x^{-n} y^{-n-2}\right\rangle + z[\langle xy\rangle - \left\langle xy^{-1}\right\rangle]$$

where $x^p = y^p = z^2 = 1$.

This result can be found in [8].

Remark. Take this W, from Theorem 29, and write it as $W = A + zB$ where

$$
\begin{aligned}
A &= \langle x, y\rangle - \langle x\rangle - \langle y\rangle - \sum_{n=1}^{p-3} x^n \left\langle x^{-n} y^{-n-2}\right\rangle \\
B &= \langle xy\rangle - \left\langle xy^{-1}\right\rangle
\end{aligned}
$$

Then A and B have the following properties:

1. $AA^{(-1)} + BB^{(-1)} = k$,

2. $AB^{(-1)} = 0$,

3. The coefficients of A and B are all in {-1, 0 ,1}.

A and B are said to form an orthogonal pair. Further, note that this is a well known idea and has been used often by Craigen and others as in [12].

3.3 Other Key Theorems

The following few theorems are critical to the development of the new results found in this thesis.

Theorem 30. *(McFarland) For every positive integer m, there exists an integer $M(m)$ such that if G is a finite abelian group whose order v is relatively prime to $M(m)$, then the only solutions of $A \in Z[G]$ satisfying $AA^{(-1)} = m^2$ in $Z[G]$ are $A = \pm mg$ for some $g \in G$. The value $M(m)$ is defined as follows: $M(1) = 1$, $M(2) = 2 \times 7$, $M(3) = 2 \times 3 \times 11 \times 13$, $M(4) = 2 \times 3 \times 7 \times 31$, and for any $m > 4$ let $M(m)$ be the product of distinct prime factors of m and*

$$M\left(\frac{m^2}{p^{2e}}\right),\ p-1,\ p^2-1, \ldots,\ p^{u(m)-1}$$

where p is a prime dividing m such that $p^e | m$ but $p^{e+1} \nmid m$, and we have $u(m) = \frac{m^2-m}{2}$.

This can be found in [8].

Theorem 31. *(Inversion Formula [11]) Let G be a finite abelian group and $\hat{G}$ be the group of all characters of G. If $A = \sum_{g \in G} \alpha_g g \in \mathbb{C}[G]$ then*

$$\alpha_g = \frac{1}{|G|} \sum_{\chi \in \hat{G}} \chi(A)\chi(g^{-1}).$$

The following are key theorems by Turyn and Ma that are required to generate the main result of this thesis (The mentioned result is Theorem 37).

Theorem 32. *(Turyn [22]) Let p be a prime and $G = H \times P$ an abelian group, where P is the Sylow p-subgroup of G. Assume that there exists and integer f such that $p^f \equiv$*

$1\,(mod\,exp(H))$. *Let* χ *be a non-principal character of* G *and let* a *be a positive integer. Suppose* $A \in Z[G]$ *satisfies* $\chi(A)\overline{\chi(A)} \equiv 0\,(mod\,p^{2\alpha})$. *Then* $\chi(A) \equiv 0\,(mod\,p^{\alpha})$.

Lemma 33. *(Ma [16]) Let* p *be a prime and* G *an abelian group with a cyclic Sylow* p*-subgroup. If* $A \in Z[G]$ *satisfies* $\chi(A) \equiv 0\,(mod\,p^{\alpha})$ *for all non-principal characters of* G*, then there exist* $x_1, x_2 \in Z[G]$ *such that*

$$A = p^{\alpha}x_1 + Qx_2$$

where Q *is the unique subgroup of order* p.

Theorem 34. *(Arasu, Seberry [10]) Suppose that a* $CW(n,k)$ *exists. Let* p *be a prime such that* $p^{2t}|k$ *for some positive integer* t. *Assume that* m *is a divisor of* n *and write* $m = m'p^u$ *where* $gcd(p,m') = 1$. *Also, assume there exists an integer* f *such that* $p^f \equiv -1\,(mod\,m')$.

Then

1. $\frac{2n}{m} \geq p^t$ if $p|m$,
2. $\frac{n}{m} \geq p^t$ if $p \nmid m$.

Chapter 4

New Results

In this section we answer the question of existence for G-developed $W(n,k)$ for a total of 318 matrices. Specifically, we exclude the existence of 211 by use of Theorem 37, and give affirmation of the existence of 107 by constructions (70 by Kronecker products, 27 by patching, and 10 by the section on Other Constructions). Recall that only orders and weights under 100 are being examined and thus some of these results can be extended directly to higher orders and weights. In total we are left with 98 open cases out of the 1,022 cases investigated. This first item is an extension of the result by Eades and Haine. See Theorem 21.

4.1 The Complete Characterization of G-developed $W(n,4)$

Theorem 35. *An abelian-G-developed $W(n,4)$ exists iff $n \geq 4$ and $2|n$ or $7|n$.*

Proof. Suppose that a G-developed $W(n,4)$ exists for some abelian group G of order n. Then by Theorem 30 and using $p = 2$ we must have that $gcd(n, M(2)) = gcd(n, 14) \neq 1$. Thus, either $2|n$ or $7|n$ as otherwise only trivial examples exist and they have a coefficient of 2 (Note that clearly, $n \geq 4$ since n must be greater than $k = 4$).

Conversely, suppose that $2|n$ and $n \geq 4$. Then let $W = -1 + g + h + gh$ for $g, h \in G$

where $g^2 = 1$, $h \neq 1 \, or \, g$. Then

$$\begin{aligned}
WW^{(-1)} &= (-1+g+h+gh)(-1+g+h+gh)^{(-1)} \\
&= (-1+g+h+gh)(-1+g+h^{-1}+gh^{-1}) \\
&= 1-g-h-gh^{-1}-g+1+gh^{-1}+h^{-1}-h+gh+1+g-gh+h+g+1 \\
&= 4.
\end{aligned}$$

So, W is a G-developed $W(n,4)$.

On the other hand, if $7|n$, then let $W = -1 + x + x^2 + x^4$. By Example 6, W is a Z_7-developed $W(7,4)$. Further, since $7|n$ we know that there exists a subgroup of order 7 in G which is isomoprhic to Z_7 and thus we can use Theorem 26 to see that a G-developed $(n,4)$ exists. $\square$

Conjecture 36. *If a G-developed $W(n,9)$ exists then $n \geq 13$ and one the following must be true:*

1. 13 divides n,
2. 24 divides n,
3. G contains a subgroup isomorphic to $Z_3 \times Z_3$.

Remark. Even though this remains unproven, the table of known matrices in this thesis supports the conjecture. Note that the converse is not true as a $(Z_2 \times Z_3 \times Z_4)$-developed $W(24,9)$ does not exist by Theorem 37. The characterization of all G-developed $W(n,9)$ should probably follow from the analogous result given by Ang, Arasu, Ma and Strassler in [2], for the case when G is a cyclic group of order n.

The following theorem is the main result of this thesis. It is an extension of the result by Arasu and Seberry for cyclic groups. See Theorem 34 above. Later, this extension is used to exclude numerous examples from existence.

4.2 Main Result

Theorem 37. *Suppose an abelian-G-developed $W(n,k)$ exists and let p be a prime such that $p^{2t}|k$ for some $t \in \mathbb{N}$. Further, let H be a subgroup of G, of order $|H| = \frac{n}{m}$, where ${}^{G}/_{H} = J \times P$ for P the cyclic-Sylow-p-subgroup of ${}^{G}/_{H}$ and $gcd(p, |J|) = 1$. Assume also that there exists an $f \in \mathbb{Z}$ such that $p^f \equiv -1\,(mod\, exp(J))$. Then*

1. If $p|m$ then $\frac{2n}{m} \geqslant p^t$,
2. If $p \nmid m$ then $\frac{n}{m} \geqslant p^t$.

Proof. For $W = G - developed\, W(n,k)$ we know that $WW^t = kI_n$, or by Equation 1.1 that

$$(A - B)(A - B)^{(-1)} = k$$

in $Z[G]$ for some abelian group G. Now, let $H \leq G$ where $|{}^{G}/_{H}| = m$. Define $\sigma : G \longrightarrow {}^{G}/_{H}$ to be the canonical homomorphism. Then we get

$$(A^\sigma - B^\sigma)(A^\sigma - B^\sigma)^{(-1)} = k$$

in $Z[{}^{G}/_{H}]$. Now, for each non-principal character of ${}^{G}/_{H}$, as in Definition 10, we have

$$\chi(A^\sigma - B^\sigma)\overline{\chi(A^\sigma - B^\sigma)} = k \equiv 0\,(mod\, p^{2t}).$$

Next, since ${}^{G}/_{H}$ contains a Sylow-p-subgroup and we have self-conjugacy, we may now use Theorem 32 to yield

$$\chi(A^\sigma - B^\sigma) \equiv 0\,(mod\, p^t).$$

Further, since the Sylow-p-subgroup of ${}^{G}/_{H}$ is cyclic, we may invoke Lemma 33 to obtain

$$A^\sigma - B^\sigma = p^t x_1 + Qx_2$$

for some $x_1, x_2 \in Z[{}^{G}/_{H}]$ and $Q = \langle h \rangle$ is the unique subgroup of ${}^{G}/_{H}$ with $|Q| = p$. Next, we show that:

$$(A^\sigma - B^\sigma)(1-h) \equiv 0 \, (mod\, p^t) \tag{4.1}$$

This is true since:

$$\begin{aligned}
(A^\sigma - B^\sigma)(1-h) &= (p^t x_1 + Q x_2)(1-h) \\
&= p^t x_1(1-h) + Q x_2(1-h) \\
&= p^t x_1(1-h) + x_2(Q - Qh) \\
&= p^t x_1(1-h) + x_2(0) \\
&= p^t x_1(1-h) \\
&\equiv 0 \, (mod\, p^t).
\end{aligned}$$

Now, notice that the coefficients of $A^\sigma - B^\sigma$ are in $[-\frac{n}{m}, \frac{n}{m}]$ as $A^\sigma - B^\sigma$ is an integer-G-developed $W(n,k)$ (see Theorem 14), and so $(A^\sigma - B^\sigma)(1-h)$ must have coefficients which are bounded by $[-\frac{2n}{m}, \frac{2n}{m}]$. Finally, from Equation 4.1 and since $(A^\sigma - B^\sigma)(1-h)$ must contain at least one non-zero coefficient (as otherwise we have a violation of the weight of W being k) we have shown that $\frac{2n}{m} \geq p^t$.

For the second part, suppose that $p \nmid m$ (i.e. P is trivial) and let χ_0 be the principal character of $Z[G/H]$, then we get that $\chi_0(A^\sigma - B^\sigma) = k \equiv 0 \, (mod\, p^{2t})$ and Theorem 32 again implies $\chi(A^\sigma - B^\sigma) \equiv 0 \, (mod\, p^t)$. Finally, use Theorem 31 to obtain the fact that $\alpha_g = \frac{1}{|G/H|} \sum_{\chi \in \widehat{G/H}} \chi(A)\chi(g^{-1})$ where each term in the sum has a factor of p^t. Thus, the coefficients of $A^\sigma - B^\sigma$ are in $[-\frac{n}{m}, \frac{n}{m}]$ and so $\frac{n}{m} \geq p^t$ as needed, since at least one coefficient of W must be non-zero. □

4.3 Non-Existence by Theorem 37

Here are a few examples of how to use Theorem 37.

Example 38. A $(Z_3 \times Z_9)$-developed $W(27, 25)$ does not exist.

Proof. Let $G = Z_3 \times Z_9$ and $p = 5$. Then we see first of all that $p^{2t} = 5^{2t}$ must divide 25 and so $t = 1$. Next, we must be able to write G/H in the form $J \times P$. Let $H = 1$ and then

$$\begin{aligned} G/H &= Z_3 \times Z_9 \\ &= (Z_3 \times Z_9) \times (1) \end{aligned}$$

and so $J = Z_3 \times Z_9$ as we only have a trivial cyclic 5-Sylow subgroup. Now we can readily compute the value of $exp(J) = exp(Z_3 \times Z_9) = 9$. Further, doing a quick calculation shows that $p^f = 5^3 = 125 \equiv -1 \, (mod \, 9)$, meaning that we have self-conjugacy. So, we must have that

$$\begin{aligned} \frac{n}{m} &\geq p^t \\ 1 &\geq 5, \end{aligned}$$

which is a contradiction. Therefore, W does not exist. □

Example 39. A $(Z_3 \times Z_5 \times Z_5)$-developed $W(75, 49)$ does not exist.

Proof. Let $G = Z_3 \times Z_5 \times Z_5$ and $p = 7$. Then we see first of all that $p^{2t} = 7^{2t}$ must divide 49 and so $t = 1$. Next, we must be able to write G/H in the form $J \times P$. Let $H = Z_3$ and then

$$\begin{aligned} G/H &= Z_5 \times Z_5 \\ &= (Z_5 \times Z_5) \times (1) \end{aligned}$$

and so $J = Z_5 \times Z_5$ as we only have a trivial cyclic 7-Sylow subgroup. Now we can readily compute the value of $exp(J) = exp(Z_5 \times Z_5) = 5$. Further, doing a quick calculation shows that $p^f = 7^2 = 49 \equiv -1 \, (mod \, 5)$, meaning that we have self-conjugacy. So, we must have that

$$\begin{aligned} \frac{n}{m} &\geq p^t \\ \frac{75}{25} &\geq 7 \\ 3 &\geq 7, \end{aligned}$$

which is a contradiction. Therefore, W does not exist. $\square$

Example 40. A$(Z_2 \times Z_4 \times Z_9)$-developed $W(72, 64)$ does not exist.

Proof. Let $G = Z_2 \times Z_4 \times Z_9$ and $p = 2$. Then we see first of all that $p^{2t} = 2^{2t}$ must divide 64 and so $t = 3$. Next, we must be able to write ${}^{G}/_{H}$ in the form $J \times P$. Let $H = Z_2$ and then

$$\begin{aligned} {}^{G}/_{H} &= Z_4 \times Z_9 \\ &= (Z_9) \times (Z_4) \end{aligned}$$

and so $J = Z_9$ and $P = Z_4$. Now we can readily compute the value of $exp(J) = exp(Z_9) = 9$. Further, doing a quick calculation shows that $p^f = 2^3 = 8 \equiv -1 \ (mod\, 9)$, meaning that we have self-conjugacy. So, we must have that

$$\begin{aligned} \frac{2n}{m} &\geq p^t \\ \frac{2 \times 72}{36} &\geq 2^3 \\ 4 &\geq 8, \end{aligned}$$

which is a contradiction. Therefore, W does not exist. $\square$

The following tables, ordered by the weight of the matrices, show the G-developed $W(n, k)$ matrices found not to exist by Theorem 37 for the abelian group G. The necessary parameters to exclude their existence are given as well.

Table 4.1: Non-Existence for $k = 9$, $p = 3$, $t = 1$

G	n	m	$\exp(J)$	f
$Z_2 \times Z_6$	12	12	2	0
$Z_4 \times Z_4$	16	16	4	1
$Z_2 \times Z_8$	16	8	4	1
$Z_2 \times Z_2 \times Z_4$	16	16	4	1
$Z_2 \times Z_2 \times Z_2 \times Z_2$	16	16	2	0
$Z_2 \times Z_2 \times Z_5$	20	20	10	2
$Z_2 \times Z_3 \times Z_4$	24	24	4	1
$Z_2 \times Z_2 \times Z_2 \times Z_3$	24	24	2	0
$Z_5 \times Z_5$	25	25	5	2
$Z_2 \times Z_2 \times Z_7$	28	28	14	3
$Z_2 \times Z_4 \times Z_4$	32	32	4	1
$Z_2 \times Z_2 \times Z_2 \times Z_4$	32	32	4	1
$Z_2 \times Z_2 \times Z_2 \times Z_2 \times Z_2$	32	32	2	0
$Z_2 \times Z_2 \times Z_8$	32	16	4	1
$Z_4 \times Z_8$	32	16	4	1
$Z_2 \times Z_2 \times Z_9$	36	36	2	0
$Z_2 \times Z_4 \times Z_5$	40	20	10	2
$Z_2 \times Z_2 \times Z_2 \times Z_5$	40	40	10	2
$Z_2 \times Z_2 \times Z_3 \times Z_4$	48	48	4	1
$Z_2 \times Z_2 \times Z_2 \times Z_2 \times Z_3$	48	48	2	0
$Z_3 \times Z_4 \times Z_4$	48	48	4	1
$Z_7 \times Z_7$	49	49	7	3
$Z_2 \times Z_5 \times Z_5$	50	50	10	2
$Z_2 \times Z_2 \times Z_2 \times Z_7$	56	56	14	3
$Z_2 \times Z_4 \times Z_7$	56	56	28	3
$Z_2 \times Z_2 \times Z_3 \times Z_5$	60	60	10	2
$Z_2 \times Z_2 \times Z_2 \times Z_8$	64	32	4	1
$Z_2 \times Z_2 \times Z_2 \times Z_2 \times Z_4$	64	64	4	1
$Z_2 \times Z_2 \times Z_2 \times Z_2 \times Z_2 \times Z_2$	64	64	2	0
$Z_2 \times Z_2 \times Z_4 \times Z_4$	64	64	4	1
$Z_4 \times Z_4 \times Z_4$	64	64	4	1
$Z_2 \times Z_4 \times Z_8$	64	32	4	1
$Z_2 \times Z_2 \times Z_{17}$	68	68	34	8
$Z_2 \times Z_4 \times Z_9$	72	72	4	1
$Z_2 \times Z_2 \times Z_2 \times Z_9$	72	72	2	0
$Z_3 \times Z_5 \times Z_5$	75	75	5	2
$Z_2 \times Z_2 \times Z_{19}$	76	76	38	9
$Z_2 \times Z_2 \times Z_4 \times Z_5$	80	40	10	2
$Z_2 \times Z_2 \times Z_2 \times Z_2 \times Z_5$	80	80	10	2
$Z_2 \times Z_2 \times Z_3 \times Z_7$	84	84	14	3
$Z_2 \times Z_2 \times Z_2 \times Z_3 \times Z_4$	96	96	4	1
$Z_2 \times Z_2 \times Z_2 \times Z_2 \times Z_2 \times Z_3$	96	96	2	0

G	n	m	$\exp(J)$	f
$Z_2 \times Z_3 \times Z_4 \times Z_4$	96	96	4	1
$Z_2 \times Z_7 \times Z_7$	98	98	14	3
$Z_2 \times Z_2 \times Z_{25}$	100	100	50	10
$Z_2 \times Z_2 \times Z_5 \times Z_5$	100	100	10	2
$Z_4 \times Z_5 \times Z_5$	100	50	10	2

Table 4.2: Non-Existence for $k = 16$, $p = 2$, $t = 2$

G	n	m	$\exp(J)$	f
$Z_2 \times Z_3 \times Z_3$	18	18	3	1
$Z_5 \times Z_5$	25	25	5	2
$Z_3 \times Z_9$	27	27	9	3
$Z_3 \times Z_3 \times Z_3$	27	27	3	1
$Z_3 \times Z_3 \times Z_4$	36	36	3	1
$Z_2 \times Z_5 \times Z_5$	50	50	5	2
$Z_2 \times Z_3 \times Z_3 \times Z_3$	54	54	3	1
$Z_2 \times Z_3 \times Z_9$	54	54	9	3
$Z_3 \times Z_3 \times Z_8$	72	72	3	1
$Z_3 \times Z_{27}$	81	81	27	9
$Z_3 \times Z_3 \times Z_9$	81	81	9	3
$Z_3 \times Z_3 \times Z_3 \times Z_3$	81	81	3	3
$Z_9 \times Z_9$	81	81	9	3
$Z_3 \times Z_3 \times Z_{11}$	99	99	33	5
$Z_4 \times Z_5 \times Z_5$	100	100	5	2

Table 4.3: Non-Existence for $k = 25$, $p = 5$, $t = 1$

G	n	m	$\exp(J)$	f
$Z_3 \times Z_9$	27	27	9	3
$Z_3 \times Z_3 \times Z_3$	27	27	3	1
$Z_2 \times Z_2 \times Z_7$	28	28	14	3
$Z_2 \times Z_4 \times Z_4$	32	8	2	0
$Z_2 \times Z_2 \times Z_2 \times Z_4$	32	16	2	0
$Z_2 \times Z_2 \times Z_2 \times Z_2 \times Z_2$	32	32	2	0
$Z_2 \times Z_2 \times Z_8$	32	8	2	0
$Z_2 \times Z_2 \times Z_9$	36	36	18	3
$Z_2 \times Z_2 \times Z_3 \times Z_3$	36	36	6	1
$Z_3 \times Z_3 \times Z_4$	36	18	6	1
$Z_2 \times Z_4 \times Z_5$	40	20	2	0
$Z_2 \times Z_2 \times Z_2 \times Z_5$	40	40	2	0
$Z_3 \times Z_3 \times Z_5$	45	45	3	1
$Z_2 \times Z_3 \times Z_8$	48	12	6	1
$Z_2 \times Z_2 \times Z_3 \times Z_4$	48	24	6	1
$Z_2 \times Z_2 \times Z_2 \times Z_2 \times Z_3$	48	48	6	1
$Z_3 \times Z_4 \times Z_4$	48	12	6	1
$Z_7 \times Z_7$	49	49	7	3
$Z_2 \times Z_2 \times Z_{13}$	52	52	26	2
$Z_2 \times Z_3 \times Z_3 \times Z_3$	54	54	6	1
$Z_2 \times Z_3 \times Z_9$	54	54	18	3
$Z_2 \times Z_2 \times Z_2 \times Z_7$	56	56	14	3
$Z_2 \times Z_4 \times Z_7$	56	28	14	3
$Z_2 \times Z_2 \times Z_3 \times Z_5$	60	60	6	1
$Z_3 \times Z_3 \times Z_7$	63	63	21	3
$Z_2 \times Z_2 \times Z_2 \times Z_8$	64	16	2	0
$Z_2 \times Z_2 \times Z_2 \times Z_2 \times Z_4$	64	32	2	0
$Z_2 \times Z_2 \times Z_2 \times Z_2 \times Z_2 \times Z_2$	64	64	2	0
$Z_2 \times Z_2 \times Z_4 \times Z_4$	64	16	2	0
$Z_2 \times Z_2 \times Z_{17}$	68	68	34	8
$Z_2 \times Z_4 \times Z_9$	72	36	18	3
$Z_2 \times Z_2 \times Z_2 \times Z_9$	72	72	18	3
$Z_2 \times Z_2 \times Z_2 \times Z_3 \times Z_3$	72	72	6	1
$Z_3 \times Z_3 \times Z_8$	72	18	6	1
$Z_2 \times Z_3 \times Z_3 \times Z_4$	72	36	6	1
$Z_2 \times Z_2 \times Z_4 \times Z_5$	80	40	2	0
$Z_2 \times Z_2 \times Z_2 \times Z_2 \times Z_5$	80	80	2	0
$Z_3 \times Z_{27}$	81	81	27	9
$Z_3 \times Z_3 \times Z_9$	81	81	9	3
$Z_3 \times Z_3 \times Z_3 \times Z_3$	81	81	3	1

G	n	m	$\exp(J)$	f
$Z_9 \times Z_9$	81	81	9	3
$Z_2 \times Z_2 \times Z_3 \times Z_7$	84	84	42	3
$Z_2 \times Z_3 \times Z_3 \times Z_5$	90	90	6	1
$Z_2 \times Z_2 \times Z_{23}$	92	92	46	11
$Z_2 \times Z_2 \times Z_3 \times Z_8$	96	24	6	1
$Z_2 \times Z_2 \times Z_2 \times Z_3 \times Z_4$	96	48	6	1
$Z_2 \times Z_2 \times Z_2 \times Z_2 \times Z_2 \times Z_3$	96	96	6	1
$Z_2 \times Z_3 \times Z_4 \times Z_4$	96	24	6	1
$Z_2 \times Z_7 \times Z_7$	98	98	14	3
$Z_2 \times Z_2 \times Z_{25}$	100	100	2	0

Table 4.4: Non-Existence for $k = 36$, $p = 2$, $t = 1$

G	n	m	$\exp(J)$	f
$Z_3 \times Z_{27}$	81	81	27	9
$Z_3 \times Z_3 \times Z_9$	81	81	9	3
$Z_3 \times Z_3 \times Z_3 \times Z_3$	81	81	3	1
$Z_9 \times Z_9$	81	81	9	3
$Z_3 \times Z_3 \times Z_{11}$	99	99	33	5

Table 4.5: Non-Existence for $k = 36$, $p = 3$, $t = 1$

G	n	m	$\exp(J)$	f
$Z_2 \times Z_2 \times Z_9$	36	36	2	0
$Z_2 \times Z_4 \times Z_5$	40	20	10	2
$Z_2 \times Z_2 \times Z_2 \times Z_5$	40	40	10	2
$Z_2 \times Z_2 \times Z_3 \times Z_4$	48	48	4	1
$Z_2 \times Z_2 \times Z_2 \times Z_2 \times Z_3$	48	48	2	0
$Z_3 \times Z_4 \times Z_4$	48	48	4	1
$Z_7 \times Z_7$	49	49	7	3
$Z_2 \times Z_5 \times Z_5$	50	50	10	2
$Z_2 \times Z_2 \times Z_2 \times Z_7$	56	56	14	3
$Z_2 \times Z_4 \times Z_7$	56	56	28	3
$Z_2 \times Z_2 \times Z_3 \times Z_5$	60	60	10	2
$Z_2 \times Z_2 \times Z_2 \times Z_8$	64	32	4	1
$Z_2 \times Z_2 \times Z_2 \times Z_2 \times Z_4$	64	64	4	1
$Z_2 \times Z_2 \times Z_2 \times Z_2 \times Z_2 \times Z_2$	64	64	2	0
$Z_2 \times Z_2 \times Z_4 \times Z_4$	64	64	4	1
$Z_4 \times Z_4 \times Z_4$	64	64	4	1
$Z_2 \times Z_4 \times Z_8$	64	32	4	1
$Z_2 \times Z_2 \times Z_{17}$	68	68	34	8
$Z_2 \times Z_4 \times Z_9$	72	72	4	1
$Z_2 \times Z_2 \times Z_2 \times Z_9$	72	72	2	0
$Z_3 \times Z_5 \times Z_5$	75	75	5	2
$Z_2 \times Z_2 \times Z_{19}$	76	76	38	9
$Z_2 \times Z_2 \times Z_4 \times Z_5$	80	40	10	2
$Z_2 \times Z_2 \times Z_2 \times Z_2 \times Z_5$	80	80	10	2
$Z_2 \times Z_2 \times Z_3 \times Z_7$	84	84	14	3
$Z_2 \times Z_2 \times Z_2 \times Z_3 \times Z_4$	96	96	4	1
$Z_2 \times Z_2 \times Z_2 \times Z_2 \times Z_2 \times Z_3$	96	96	2	0
$Z_2 \times Z_3 \times Z_4 \times Z_4$	96	96	4	1
$Z_2 \times Z_7 \times Z_7$	98	98	14	3
$Z_2 \times Z_2 \times Z_{25}$	100	100	50	10
$Z_2 \times Z_2 \times Z_5 \times Z_5$	100	100	10	2
$Z_4 \times Z_5 \times Z_5$	100	50	10	2

Table 4.6: Non-Existence for $k = 49$, $p = 7$, $t = 1$

G	n	m	$\exp(J)$	f
$Z_2 \times Z_5 \times Z_5$	50	50	10	2
$Z_2 \times Z_2 \times Z_{13}$	52	52	26	6
$Z_2 \times Z_2 \times Z_2 \times Z_7$	56	56	2	0
$Z_2 \times Z_4 \times Z_7$	56	56	4	1
$Z_2 \times Z_2 \times Z_3 \times Z_5$	60	20	10	2
$Z_2 \times Z_{32}$	64	16	8	1
$Z_2 \times Z_2 \times Z_{16}$	64	32	8	1
$Z_2 \times Z_2 \times Z_2 \times Z_8$	64	64	8	1
$Z_2 \times Z_2 \times Z_2 \times Z_2 \times Z_4$	64	64	4	1
$Z_2 \times Z_2 \times Z_2 \times Z_2 \times Z_2 \times Z_2$	64	64	2	0
$Z_2 \times Z_2 \times Z_4 \times Z_4$	64	64	4	1
$Z_4 \times Z_4 \times Z_4$	64	64	4	1
$Z_4 \times Z_{16}$	64	32	8	1
$Z_2 \times Z_4 \times Z_8$	64	64	8	1
$Z_8 \times Z_8$	64	64	8	1
$Z_2 \times Z_2 \times Z_{17}$	68	68	34	8
$Z_3 \times Z_5 \times Z_5$	75	25	5	2
$Z_2 \times Z_5 \times Z_8$	80	20	10	2
$Z_2 \times Z_2 \times Z_4 \times Z_5$	80	40	10	2
$Z_2 \times Z_2 \times Z_2 \times Z_2 \times Z_5$	80	80	10	2
$Z_4 \times Z_4 \times Z_5$	80	20	10	2
$Z_2 \times Z_2 \times Z_3 \times Z_7$	84	28	2	0
$Z_2 \times Z_4 \times Z_{11}$	88	88	44	5
$Z_2 \times Z_2 \times Z_2 \times Z_{11}$	88	88	22	5
$Z_2 \times Z_2 \times Z_{23}$	92	92	46	11
$Z_2 \times Z_2 \times Z_3 \times Z_8$	96	32	8	1
$Z_2 \times Z_2 \times Z_2 \times Z_3 \times Z_4$	96	32	4	1
$Z_2 \times Z_2 \times Z_2 \times Z_2 \times Z_2 \times Z_3$	96	32	2	0
$Z_3 \times Z_4 \times Z_8$	96	32	8	1
$Z_2 \times Z_3 \times Z_4 \times Z_4$	96	32	4	1
$Z_2 \times Z_2 \times Z_{25}$	100	100	50	2
$Z_2 \times Z_2 \times Z_5 \times Z_5$	100	100	10	2
$Z_4 \times Z_5 \times Z_5$	100	50	10	2

Table 4.7: Non-Existence for $k = 64$, $p = 2$, $t = 3$

G	n	m	$\exp(J)$	f
$Z_2 \times Z_2 \times Z_{17}$	68	34	17	4
$Z_2 \times Z_4 \times Z_9$	72	36	9	3
$Z_3 \times Z_3 \times Z_8$	72	72	3	1
$Z_2 \times Z_3 \times Z_3 \times Z_4$	72	36	3	1
$Z_3 \times Z_5 \times Z_5$	75	25	5	2
$Z_2 \times Z_2 \times Z_{19}$	76	38	19	9
$Z_2 \times Z_5 \times Z_8$	80	40	5	2
$Z_3 \times Z_{27}$	81	81	27	9
$Z_3 \times Z_3 \times Z_9$	81	81	9	3
$Z_3 \times Z_3 \times Z_3 \times Z_3$	81	81	3	1
$Z_9 \times Z_9$	81	81	9	3
$Z_2 \times Z_4 \times Z_{11}$	88	44	11	5
$Z_2 \times Z_3 \times Z_{16}$	96	48	3	1
$Z_3 \times Z_3 \times Z_{11}$	99	99	33	5
$Z_2 \times Z_2 \times Z_{25}$	100	50	25	10
$Z_2 \times Z_2 \times Z_5 \times Z_5$	100	50	5	2
$Z_4 \times Z_5 \times Z_5$	100	100	5	2

Table 4.8: Non-Existence for $k = 81$, $p = 3$, $t = 2$

G	n	m	$\exp(J)$	f
$Z_2 \times Z_2 \times Z_3 \times Z_7$	84	84	14	3
$Z_2 \times Z_3 \times Z_3 \times Z_5$	90	30	10	2
$Z_2 \times Z_3 \times Z_{16}$	96	24	4	1
$Z_2 \times Z_2 \times Z_3 \times Z_8$	96	48	4	1
$Z_2 \times Z_2 \times Z_2 \times Z_3 \times Z_4$	96	96	4	1
$Z_2 \times Z_2 \times Z_2 \times Z_2 \times Z_2 \times Z_3$	96	96	2	0
$Z_3 \times Z_4 \times Z_8$	96	48	4	1
$Z_2 \times Z_3 \times Z_4 \times Z_4$	96	96	4	1
$Z_2 \times Z_7 \times Z_7$	98	98	14	3
$Z_2 \times Z_2 \times Z_{25}$	100	100	50	10
$Z_2 \times Z_2 \times Z_5 \times Z_5$	100	100	10	2
$Z_4 \times Z_5 \times Z_5$	100	50	10	2

4.4 Existence by Patching or Kronecker Product

4.4.1 Patching

The following examples are constructed using a given group weighing matrix M and N written as group ring elements. Using the patching method from Theorem 28 one can compute the sought after G-developed $W(n,k)$.

A $(Z_2 \times Z_{2s})$-developed $W(4s,16)$ Exists for $s \geq 4$.

Let $M = -1 + x + x^s + x^{s+1}$ and $N = x^2 M$ where $M, N \in (Z_{2s})$-developed $W(2s,4)$. Then M and N have disjoint supports. Thus, by patching, $(1+t)M + (1-t)N$ is a $(Z_2 \times Z_{2s})$-developed $W(4s,16)$.

A $(Z_2 \times Z_2 \times Z_{2s})$-developed $W(8s,64)$ Exists for $s \geq 8$.

Let $A = -1 + x + x^s + x^{s+1}$, $B = x^2 A$, $C = x^2 B$, and $D = x^2 C$, where $A, B, C, D \in (\mathbb{Z}_{2s})$-developed $W(2s,4)$ and have pairwise disjoint supports by construction. Then $M = (1+t)\,A + (1-t)\,B$ and $N = (1+t)\,C + (1-t)\,D$ are both $(Z_2 \times Z_{2s})$-developed $W(4s,16)$ and also are disjoint. Continue by patching M and N to get a $(Z_2 \times Z_2 \times Z_{2s})$-developed $W(8s,64)$.

A $(Z_2 \times Z_2 \times Z_3 \times Z_3)$-developed $W(36,36)$ Exists.

Let $M = 1 + y + x + xy - x^2y^2 + z(1 - y^2 - x^2 + x^2y^2)$ and $N = y^2 + xy^2 - x^2 - x^2y + z(y + x - xy + xy^2 + x^2y)$ where both are $(Z_2 \times Z_3 \times Z_3)$-developed $W(18,9)$ and disjoint. Then patch to obtain the sought after matrix.

A $(Z_2 \times Z_3 \times Z_3 \times Z_3)$-developed $W(54,36)$ Exists.

Let $M = 1 + y + x + xy - x^2y^2 + z(1 - y^2 - x^2 + x^2y^2)$ and $N = y^2 + xy^2 - x^2 - x^2y + z(y + x - xy + xy^2 + x^2y)$ where both are $(Z_3 \times Z_3 \times Z_3)$-developed $W(27,9)$ and are disjoint. Then patch to obtain the sought after matrix.

4.4.2 Kronecker Product

The following tables list all G-developed $W(n,k)$ matrices which can be generated by use of the Kronecker product as in Theorem 26. The left columns give the matrix to be generated and the right columns give the products needed. The Kronecker product is denoted by $\otimes$

and the trivial weighing matrix $CW(n,1)$ is denoted by I_n. For formatting, the G-developed $W(n,k)$ matrices are being written in the form (G,n,k).

Table 4.9: Kronecker Products for $k = 9$

G-developed $W(n,k)$ Generated	Construction
$(Z_2 \times Z_2 \times Z_3 \times Z_3, 36, 9)$	$I_2 \otimes (Z_2 \times Z_3 \times Z_3, 18, 9)$
$(Z_2 \times Z_3 \times Z_8, 48, 9)$	$I_2 \otimes (Z_{24}, 24, 9)$
$(Z_2 \times Z_2 \times Z_{13}, 52, 9)$	$I_2 \otimes (Z_{26}, 26, 9)$
$(Z_2 \times Z_3 \times Z_3 \times Z_3, 54, 9)$	$I_2 \otimes (Z_3 \times Z_3 \times Z_3, 27, 9)$
$(Z_2 \times Z_3 \times Z_9, 54, 9)$	$I_2 \otimes (Z_3 \times Z_9, 27, 9)$
$(Z_2 \times Z_2 \times Z_2 \times Z_3 \times Z_3, 72, 9)$	$I_2 \otimes I_2 \otimes (Z_2 \times Z_3 \times Z_3, 18, 9)$
$(Z_2 \times Z_3 \times Z_3 \times Z_4, 72, 9)$	$I_4 \otimes (Z_2 \times Z_3 \times Z_3, 18, 9)$
$(Z_3 \times Z_3 \times Z_3 \times Z_3, 81, 9)$	$I_3 \otimes (Z_3 \times Z_3 \times Z_3, 27, 9)$
$(Z_2 \times Z_3 \times Z_3 \times Z_5, 90, 9)$	$I_5 \otimes (Z_2 \times Z_3 \times Z_3, 18, 9)$
$(Z_2 \times Z_3 \times Z_{16}, 96, 9)$	$I_2 \otimes (Z_{48}, 48, 9)$
$(Z_2 \times Z_2 \times Z_3 \times Z_8, 96, 9)$	$I_2 \otimes I_2 \otimes (Z_{24}, 24, 9)$
$(Z_3 \times Z_4 \times Z_8, 96, 9)$	$I_4 \otimes (Z_{24}, 24, 9)$

Table 4.10: Kronecker Products for $k = 16$

G-developed $W(n,k)$ Generated	Construction
$(Z_4 \times Z_4, 16, 16)$	$(Z_4, 4, 4) \otimes (Z_4, 4, 4)$
$(Z_2 \times Z_2 \times Z_4, 16, 16)$	$(Z_2 \times Z_2, 4, 4) \otimes (Z_4, 4, 4)$
$(Z_2 \times Z_3 \times Z_4, 24, 16)$	$(Z_6, 6, 4) \otimes (Z_4, 4, 4)$
$(Z_2 \times Z_2 \times Z_2 \times Z_3, 24, 16)$	$(Z_6, 6, 4) \otimes (Z_2 \times Z_2, 4, 4)$
$(Z_2 \times Z_4 \times Z_4, 32, 16)$	$I_2 \otimes (Z_4, 4, 4) \otimes (Z_4, 4, 4)$
$(Z_2 \times Z_2 \times Z_2 \times Z_4, 32, 16)$	$I_2 \otimes (Z_2 \times Z_2, 4, 4) \otimes (Z_4, 4, 4)$
$(Z_2 \times Z_2 \times Z_2 \times Z_2 \times Z_2, 32, 16)$	$I_2 \otimes (Z_2 \times Z_2, 4, 4) \otimes (Z_2 \times Z_2, 4, 4)$
$(Z_2 \times Z_2 \times Z_8, 32, 16)$	$(Z_2 \times Z_2, 4, 4) \otimes (Z_8, 8, 4)$
$(Z_4 \times Z_8, 32, 16)$	$(Z_4, 4, 4) \otimes (Z_8, 8, 4)$
$(Z_2 \times Z_2 \times Z_3 \times Z_3, 36, 16)$	$(Z_6, 6, 4) \otimes (Z_6, 6, 4)$
$(Z_2 \times Z_2 \times Z_2 \times Z_5, 40, 16)$	$(Z_2 \times Z_2, 4, 4) \otimes (Z_{10}, 10, 4)$
$(Z_2 \times Z_2 \times Z_3 \times Z_4, 48, 16)$	$I_2 \otimes (Z_6, 6, 4) \otimes (Z_4, 4, 4)$
$(Z_2 \times Z_2 \times Z_2 \times Z_2 \times Z_3, 48, 16)$	$I_2 \otimes (Z_2 \times Z_2, 4, 4) \otimes (Z_6, 6, 4)$
$(Z_3 \times Z_4 \times Z_4, 48, 16)$	$I_3 \otimes (Z_4, 4, 4) \otimes (Z_4, 4, 4)$
$(Z_7 \times Z_7, 49, 16)$	$(Z_7, 4, 4) \otimes (Z_7, 4, 4)$
$(Z_2 \times Z_2 \times Z_2 \times Z_7, 56, 16)$	$(Z_2 \times Z_2, 4, 4) \otimes (Z_{14}, 14, 4)$
$(Z_3 \times Z_3 \times Z_7, 63, 16)$	$I_3 \otimes (Z_{21}, 21, 16)$
$(Z_2 \times Z_2 \times Z_{16}, 64, 16)$	$(Z_2 \times Z_2, 4, 4) \otimes (Z_{16}, 16, 4)$
$(Z_2 \times Z_2 \times Z_2 \times Z_8, 64, 16)$	$I_2 \otimes (Z_2 \times Z_2, 4, 4) \otimes (Z_8, 8, 4)$
$(Z_2 \times Z_2 \times Z_2 \times Z_2 \times Z_4, 64, 16)$	$I_2 \otimes I_2 \otimes (Z_2 \times Z_2, 4, 4) \otimes (Z_4, 4, 4)$
$(Z_2 \times Z_2 \times Z_2 \times Z_2 \times Z_2 \times Z_2, 64, 16)$	$I_2 \otimes I_2 \otimes (Z_2 \times Z_2, 4, 4) \otimes (Z_2 \times Z_2, 4, 4)$
$(Z_2 \times Z_2 \times Z_4 \times Z_4, 64, 16)$	$I_2 \otimes (Z_4, 4, 4) \otimes (Z_4, 4, 4)$
$(Z_4 \times Z_4 \times Z_4, 64, 16)$	$I_4 \otimes (Z_4, 4, 4) \otimes (Z_4, 4, 4)$
$(Z_4 \times Z_{16}, 64, 16)$	$(Z_4, 4, 4) \otimes (Z_{16}, 16, 4)$
$(Z_2 \times Z_4 \times Z_8, 64, 16)$	$I_2 \otimes (Z_4, 4, 4) \otimes (Z_8, 8, 4)$
$(Z_8 \times Z_8, 64, 16)$	$(Z_8, 8, 4) \otimes (Z_8, 8, 4)$
$(Z_2 \times Z_2 \times Z_2 \times Z_9, 72, 16)$	$(Z_2 \times Z_2, 4, 4) \otimes (Z_{18}, 18, 4)$
$(Z_2 \times Z_2 \times Z_2 \times Z_3 \times Z_3, 72, 16)$	$I_2 \otimes (Z_6, 6, 4) \otimes (Z_6, 6, 4)$
$(Z_2 \times Z_3 \times Z_3 \times Z_4, 72, 16)$	$(Z_6, 6, 4) \otimes (Z_{12}, 12, 4)$
$(Z_2 \times Z_2 \times Z_4 \times Z_5, 80, 16)$	$I_2 \otimes (Z_4, 4, 4) \otimes (Z_{10}, 10, 4)$
$(Z_2 \times Z_2 \times Z_2 \times Z_2 \times Z_5, 80, 16)$	$I_2 \otimes (Z_2 \times Z_2, 4, 4) \otimes (Z_{10}, 10, 4)$
$(Z_4 \times Z_4 \times Z_5, 80, 16)$	$I_5 \otimes (Z_4, 4, 4) \otimes (Z_4, 4, 4)$
$(Z_2 \times Z_4 \times Z_{11}, 88, 16)$	$(Z_4, 4, 4) \otimes (Z_{22}, 22, 4)$
$(Z_2 \times Z_2 \times Z_2 \times Z_{11}, 88, 16)$	$(Z_2 \times Z_2, 4, 4) \otimes (Z_{22}, 22, 4)$
$(Z_2 \times Z_2 \times Z_3 \times Z_8, 96, 16)$	$I_6 \otimes (Z_6, 6, 4) \otimes (Z_8, 8, 4)$
$(Z_2 \times Z_2 \times Z_2 \times Z_3 \times Z_4, 96, 16)$	$I_2 \otimes I_2 \otimes (Z_4, 4, 4) \otimes (Z_6, 6, 4)$
$(Z_2 \times Z_2 \times Z_2 \times Z_2 \times Z_2 \times Z_3, 96, 16)$	$I_2 \otimes I_2 \otimes (Z_2 \times Z_2, 4, 4) \otimes (Z_6, 6, 4)$
$(Z_3 \times Z_4 \times Z_8, 96, 16)$	$I_3 \otimes (Z_4, 4, 4) \otimes (Z_8, 8, 4)$
$(Z_2 \times Z_3 \times Z_4 \times Z_4, 96, 16)$	$I_2 \otimes I_3 \otimes (Z_4, 4, 4) \otimes (Z_4, 4, 4)$
$(Z_2 \times Z_7 \times Z_7, 98, 16)$	$I_2 \otimes (Z_7, 7, 4) \otimes (Z_7, 7, 4)$
$(Z_2 \times Z_2 \times Z_5 \times Z_5, 100, 16)$	$(Z_{10}, 10, 4) \otimes (Z_{10}, 10, 4)$

Table 4.11: Kronecker Products for $k = 25$

G-developed $W(n,k)$ Generated	Construction
$(Z_3 \times Z_3 \times Z_{11}, 99, 25)$	$I_3 \otimes (Z_{33}, 33, 25)$

Table 4.12: Kronecker Products for $k = 36$

G-developed $W(n,k)$ Generated	Construction
$(Z_2 \times Z_2 \times Z_{13}, 52, 36)$	$(Z_2 \otimes Z_2, 4, 4) \otimes (Z_{13}, 13, 9)$
$(Z_2 \times Z_2 \times Z_2 \times Z_3 \times Z_3, 72, 36)$	$(Z_2 \otimes Z_2, 4, 4) \otimes (Z_2 \times Z_3 \times Z_3, 18, 9)$
$(Z_2 \times Z_3 \times Z_3 \times Z_4, 72, 36)$	$(Z_4, 4, 4) \otimes (Z_2 \times Z_3 \times Z_3, 18, 9)$
$(Z_2 \times Z_2 \times Z_3 \times Z_8, 96, 36)$	$(Z_2 \otimes Z_2, 4, 4) \otimes (Z_{24}, 24, 9)$
$(Z_3 \times Z_4 \times Z_8, 96, 36)$	$(Z_4, 4, 4) \otimes (Z_{24}, 24, 9)$

Table 4.13: Kronecker Products for $k = 64$

G-developed $W(n,k)$ Generated	Construction
$(Z_2 \times Z_2 \times Z_2 \times Z_8, 64, 64)$	$(Z_2 \times Z_2, 4, 4) \otimes (Z_2 \times Z_8, 16, 16)$
$(Z_2 \times Z_2 \times Z_2 \times Z_2 \times Z_4, 64, 64)$	$(Z_2 \times Z_2, 4, 4) \otimes (Z_2 \times Z_2, 4, 4) \otimes (Z_4, 4, 4)$
$(Z_2 \times Z_2 \times Z_2 \times Z_2 \times Z_2 \times Z_2, 64, 64)$	$(Z_2 \times Z_2, 4, 4) \otimes (Z_2 \times Z_2, 4, 4) \otimes (Z_2 \times Z_2, 4, 4)$
$(Z_2 \times Z_2 \times Z_4 \times Z_4, 64, 64)$	$(Z_2 \times Z_2, 4, 4) \otimes (Z_4, 4, 4) \otimes (Z_4, 4, 4)$
$(Z_4 \times Z_4 \times Z_4, 64, 64)$	$(Z_4, 4, 4) \otimes (Z_4, 4, 4) \otimes (Z_4, 4, 4)$
$(Z_2 \times Z_4 \times Z_8, 64, 64)$	$(Z_4, 4, 4) \otimes (Z_2 \times Z_8, 16, 16)$
$(Z_2 \times Z_2 \times Z_2 \times Z_2 \times Z_5, 80, 64)$	$(Z_2 \times Z_2, 4, 4) \otimes (Z_2 \times Z_2 \times Z_5, 20, 16)$
$(Z_2 \times Z_2 \times Z_3 \times Z_7, 84, 64)$	$(Z_2 \times Z_2, 4, 4) \otimes (Z_{21}, 21, 16)$
$(Z_2 \times Z_2 \times Z_2 \times Z_3 \times Z_4, 96, 64)$	$(Z_2 \times Z_2, 4, 4) \otimes (Z_4, 4, 4) \otimes (Z_6, 6, 4)$
$(Z_2 \times Z_2 \times Z_2 \times Z_2 \times Z_2 \times Z_3, 96, 64)$	$(Z_2 \times Z_2, 4, 4) \otimes (Z_2 \times Z_2, 4, 4) \otimes (Z_6, 6, 4)$
$(Z_2 \times Z_3 \times Z_4 \times Z_4, 96, 64)$	$(Z_6, 6, 4) \otimes (Z_4, 4, 4) \otimes (Z_4, 4, 4)$

4.5 Other Constructions

The following constructions come from the G-developed $W(n,k)$ given in Theorem 29 by Arasu and Linthicum where $G = Z_2 \times Z_p \times Z_p$ and $k = p^2$.

Theorem 41. *A $(L \times Z_p \times Z_p) - developed W(rp^2, p^2)$ exists for all p prime and ALL groups L of order r.*

Proof. By the construction in Theorem 29 the matrix $W = \langle x, y\rangle - \langle x\rangle - \langle y\rangle - \sum_{n=1}^{p-3} x^n \left\langle x^{-n}y^{-n-2}\right\rangle + z[\langle xy\rangle - \left\langle xy^{-1}\right\rangle]$ can be written as $W = A + zB$ where $A = \langle x, y\rangle - \langle x\rangle - \langle y\rangle - \sum_{n=1}^{p-3} x^n \left\langle x^{-n}y^{-n-2}\right\rangle$ and $B = \langle xy\rangle - \left\langle xy^{-1}\right\rangle$. These two blocks form an orthogonal pair. Now, simply let

$x^p = y^p = 1$ and z be some element in L. Then we see that by computing $WW^{(-1)}$ without restriction on the nature of z:

$$
\begin{aligned}
WW^{(-1)} &= (A + zB)(A + zB)^{(-1)} \\
&= (A + zB)(A^{(-1)} + z^{-1}B^{(-1)}) \\
&= AA^{(-1)} + z^{-1}AB^{(-1)} + zA^{(-1)}B + BB^{(-1)} \\
&= AA^{(-1)} + BB^{(-1)} \\
&= k.
\end{aligned}
$$

Here z is canceled out by the orthogonality of the two blocks A and B. Thus, z can be any element of any group and the lifted group weighing matrix exists. $\square$

Theorem 42. *A $(Z_p \times Z_{p^2})$-developed $W(p^3, p^2)$ exists for all prime p.*

Proof. Take $W = A + zB$ as before. Note that $A, B \in Z[\langle x, y\rangle]$ (i.e. they contain no z elements). Here $x^p = y^p = 1$. Now, we need to create new blocks $\overline{A}, \overline{B} \in Z[\langle x, t\rangle]$. This can be done by simply using the homomorphic mapping $y \longrightarrow t$, where $t = y^p$ in both A and B. In this new group we allow for $x^p = t^p = 1$ where $t = y^p$. Next, examine the new $(Z_p \times Z_{p^2})$-developed matrix $\overline{W} = \overline{A} + y\overline{B}$.

$$
\begin{aligned}
\overline{WW}^{(-1)} &= (\overline{A} + y\overline{B})(\overline{A} + y\overline{B})^{(-1)} \\
&= (\overline{A} + y\overline{B})(\overline{A}^{(-1)} + y^{-1}\overline{B}^{(-1)}) \\
&= \overline{AA}^{(-1)} + \overline{BB}^{(-1)} + y\overline{BA}^{(-1)} + y^{-1}\overline{B}^{(-1)}\overline{A} \\
&= k
\end{aligned}
$$

by the orthogonality of $\overline{A}$ and $\overline{B}$.

Further, the coefficients of $\overline{W}$ lie in {-1, 0 ,1} still as A and B were already disjoint and that we now have $\overline{A}, \overline{B} \in Z[\langle x, t\rangle]$ where t has the same order as y. Meaning that

t is really the same element as y in the original group. Therefore, $\overline{W}$ is a $(Z_p \times Z_{p^2}) - developed\, W(p^3, p^2)$. $\square$

Chapter 5

Table of abelian-G-developed Weighing Matrices

The following is the table for group weighing matrices. It includes the updated Strassler Table from Alex Gutman's master's thesis [9] plus the added material above. The labels are as follows: '.' = previously known non-existence, 'N' = non-existence given in this thesis, '+' = previously known existence, 'Y' = existence given in this thesis, 'K' = existence given by Kronecker product, and blank spaces remain open cases. (Note the obviously omitted $k = 1$ set since this represents only trivial weighing matrices.)

Table 5.1: Table of G-developed $W(n,k)$

G	k=4	9	16	25	36	49	64	81	100
Z_4	+	.	.	.	.	.	.	.	.
$Z_2 \times Z_2$	Y	.	.	.	.	.	.	.	.
Z_5	.	.	.	.	.	.	.	.	.
Z_6	+	.	.	.	.	.	.	.	.
Z_7	+	.	.	.	.	.	.	.	.
Z_8	+	.	.	.	.	.	.	.	.
$Z_2 \times Z_4$	Y	.	.	.	.	.	.	.	.
$Z_2 \times Z_2 \times Z_2$	Y	.	.	.	.	.	.	.	.
Z_9	.	.	.	.	.	.	.	.	.
$Z_3 \times Z_3$	N	.	.	.	.	.	.	.	.
Z_{10}	+	.	.	.	.	.	.	.	.
Z_{11}	.	.	.	.	.	.	.	.	.
Z_{12}	+	.	.	.	.	.	.	.	.
$Z_2 \times Z_6$	Y	N	.	.	.	.	.	.	.
Z_{13}	.	+	.	.	.	.	.	.	.
Z_{14}	+	.	.	.	.	.	.	.	.
Z_{15}	.	.	.	.	.	.	.	.	.
Z_{16}	+	.	.	.	.	.	.	.	.
$Z_4 \times Z_4$	Y	N	K	.	.	.	.	.	.
$Z_2 \times Z_8$	Y	N	Y	.	.	.	.	.	.
$Z_2 \times Z_2 \times Z_4$	Y	N	K	.	.	.	.	.	.
$Z_2 \times Z_2 \times Z_2 \times Z_2$	Y	N	+	.	.	.	.	.	.
Z_{17}	.	.	.	.	.	.	.	.	.
Z_{18}	+	.	.	.	.	.	.	.	.
$Z_2 \times Z_3 \times Z_3$	Y	+	N	.	.	.	.	.	.
Z_{19}	.	.	.	.	.	.	.	.	.
Z_{20}	+	.	.	.	.	.	.	.	.
$Z_2 \times Z_2 \times Z_5$	Y	N	Y	.	.	.	.	.	.
Z_{21}	+	.	+	.	.	.	.	.	.
Z_{22}	+	.	.	.	.	.	.	.	.
Z_{23}	.	.	.	.	.	.	.	.	.
Z_{24}	+	+	.	.	.	.	.	.	.
$Z_2 \times Z_3 \times Z_4$	Y	N	K	.	.	.	.	.	.
$Z_2 \times Z_2 \times Z_2 \times Z_3$	Y	N	K	.	.	.	.	.	.
Z_{25}	.	.	.	.	.	.	.	.	.
$Z_5 \times Z_5$	N	N	N	.	.	.	.	.	.
Z_{26}	+	+	.	.	.	.	.	.	.
Z_{27}	.	.	.	.	.	.	.	.	.
$Z_3 \times Z_9$	N	Y	N	N	.	.	.	.	.
$Z_3 \times Z_3 \times Z_3$	N	Y	N	N	.	.	.	.	.
Z_{28}	+	.	+	.	.	.	.	.	.
$Z_2 \times Z_2 \times Z_7$	Y	N	Y	N	.	.	.	.	.

G	k=4	9	16	25	36	49	64	81	100
Z_{29}	.	.	.	.	.	.	.	.	.
Z_{30}	+	.	.	.	.	.	.	.	.
Z_{31}	.	.	+	+	.	.	.	.	.
Z_{32}	+	.	.		.	.	.	.	.
$Z_2{\times}Z_{16}$	Y		Y		.	.	.	.	.
$Z_2{\times}Z_4{\times}Z_4$	Y	N	K	N	.	.	.	.	.
$Z_2{\times}Z_2{\times}Z_2{\times}Z_4$	Y	N	K	N	.	.	.	.	.
Z_2^5	Y	N	K	N	.	.	.	.	.
$Z_2{\times}Z_2{\times}Z_8$	Y	N	K	N	.	.	.	.	.
$Z_4{\times}Z_8$	Y	N	K		.	.	.	.	.
Z_{33}	.	.	.	+	.	.	.	.	.
Z_{34}	+	.	.	.	.	.	.	.	.
Z_{35}	+	.	.	.	.	.	.	.	.
Z_{36}	+	.	.	.	.	.	.	.	.
$Z_2{\times}Z_2{\times}Z_9$	Y	N	Y	N	N	.	.	.	.
$Z_2{\times}Z_2{\times}Z_3{\times}Z_3$	Y	K	K	N	+	.	.	.	.
$Z_3{\times}Z_3{\times}Z_4$	Y	Y	N	N	+	.	.	.	.
Z_{37}	.	.	.	.	.	.	.	.	.
Z_{38}	+	.	.	.	.	.	.	.	.
Z_{39}	.	+	.	.	.	.	.	.	.
Z_{40}	+	.	.		.	.	.	.	.
$Z_2{\times}Z_4{\times}Z_5$	Y	N	Y	N	N	.	.	.	.
$Z_2{\times}Z_2{\times}Z_2{\times}Z_5$	Y	N	K	N	N	.	.	.	.
Z_{41}	.	.	.	.	.	.	.	.	.
Z_{42}	+	.	+	.	.	.	.	.	.
Z_{43}	.	.	.	.	.	.	.	.	.
Z_{44}	+	.	.	.	.	.	.	.	.
$Z_2{\times}Z_2{\times}Z_{11}$	Y		Y			.	.	.	.
Z_{45}	.	.	.	.	.	.	.	.	.
$Z_3{\times}Z_3{\times}Z_5$	N	Y		N		.	.	.	.
Z_{46}	+	.	.	.	.	.	.	.	.
Z_{47}	.	.	.	.	.	.	.	.	.
Z_{48}	+	+	.	.	.	.	.	.	.
$Z_2{\times}Z_3{\times}Z_8$	Y	K	Y	N		.	.	.	.
$Z_2{\times}Z_2{\times}Z_3{\times}Z_4$	Y	N	K	N	N	.	.	.	.
$Z_2^4{\times}Z_3$	Y	N	K	N	N	.	.	.	.
$Z_3{\times}Z_4{\times}Z_4$	Y	N	K	N	N	.	.	.	.
Z_{49}	+	.	.	.	.	.	.	.	.
$Z_7{\times}Z_7$	Y	N	K	N	N	.	.	.	.
Z_{50}	+	.	.	.	.	.	.	.	.
$Z_2{\times}Z_5{\times}Z_5$	Y	N	N	+	N	N	.	.	.
Z_{51}	.	.	.	.	.	.	.	.	.
Z_{52}	+	+	.	.	+	.	.	.	.

G	k=4	9	16	25	36	49	64	81	100
$Z_2\times Z_2\times Z_{13}$	Y	K	Y	N	K	N	.	.	.
Z_{53}	.	.	.	.	.	.	.	.	.
Z_{54}	+	.	.	.	.		.	.	.
$Z_2\times Z_3\times Z_3\times Z_3$	Y	K	N	N	Y		.	.	.
$Z_2\times Z_3\times Z_9$	Y	K	N	N			.	.	.
Z_{55}	.	.	.	.	.	.	.	.	.
Z_{56}	+	.	+	.	.	.	.	.	.
$Z_2\times Z_2\times Z_2\times Z_7$	Y	N	K	N	N	N	.	.	.
$Z_2\times Z_4\times Z_7$	Y	N	Y	N	N	N	.	.	.
Z_{57}	.	.	.	.	.	+	.	.	.
Z_{58}	+	.	.	.	.	.	.	.	.
Z_{59}	.	.	.	.	.	.	.	.	.
Z_{60}	+	.	.	.		.	.	.	.
$Z_2\times Z_2\times Z_3\times Z_5$	Y	N	Y	N	N	N	.	.	.
Z_{61}	.	.	.	.	.	.	.	.	.
Z_{62}	+	.	+	+	.	.	.	.	.
Z_{63}	+	.	+	.	.		.	.	.
$Z_3\times Z_3\times Z_7$	Y	Y	K	N			.	.	.
Z_{64}	+	.	.		.		.	.	.
$Z_2\times Z_{32}$	Y		Y			N		.	.
$Z_2\times Z_2\times Z_{16}$	Y		K			N	Y	.	.
$Z_2\times Z_2\times Z_2\times Z_8$	Y	N	K	N	N	N	K	.	.
$Z_2^4\times Z_4$	Y	N	K	N	N	N	K	.	.
Z_2^6	Y	N	K	N	N	N	K	.	.
$Z_2\times Z_2\times Z_4\times Z_4$	Y	N	K	N	N	N	K	.	.
$Z_4\times Z_4\times Z_4$	Y	N	K		N	N	K	.	.
$Z_4\times Z_{16}$	Y		K			N		.	.
$Z_2\times Z_4\times Z_8$	Y	N	K		N	N		.	.
$Z_8\times Z_8$	Y		K			N		.	.
Z_{65}	.	+	.	.	.	.	.	.	.
Z_{66}	+	.	.	+	.	.	.	.	.
Z_{67}	.	.	.	.	.	.	.	.	.
Z_{68}	+	.	.	.	.	.	.	.	.
$Z_2\times Z_2\times Z_{17}$	Y	N	Y	N		N	N	.	.
Z_{69}	.	.	.	.	.	.	.	.	.
Z_{70}	+	.	+	.		.	.	.	.
Z_{71}	.	.	.	+	.	.	.	.	.
Z_{72}	+	+	.	.	.		.	.	.
$Z_2\times Z_4\times Z_9$	Y	N	Y	N	N		N	.	.
$Z_2\times Z_2\times Z_2\times Z_9$	Y	N	K	N	N		Y	.	.
$Z_2^3\times Z_3\times Z_3$	Y	K	K	N	K			.	.
$Z_3\times Z_3\times Z_8$	Y	Y	N	N			N	.	.
$Z_2\times Z_3\times Z_3\times Z_4$	Y	K	K	N	K		N	.	.

G	k=4	9	16	25	36	49	64	81	100
Z_{73}	.	.	.	.	.	.	+	.	.
Z_{74}	+	.	.	.	.	.	.	.	.
Z_{75}	.	.	.	.	.	.	.	.	.
$Z_3 \times Z_5 \times Z_5$	N	N		Y	N	N	N	.	.
Z_{76}	+	.	.	.	.		.	.	.
$Z_2 \times Z_2 \times Z_{19}$	Y	N	Y		N		N	.	.
Z_{77}	+	.	.	.	.	.	.	.	.
Z_{78}	+	+	.	.	+	.	.	.	.
Z_{79}	.	.	.	.	.	.	.	.	.
Z_{80}	+	.	.		.		.	.	.
$Z_2 \times Z_5 \times Z_8$	Y		Y			N	N	.	.
$Z_2 \times Z_2 \times Z_4 \times Z_5$	Y	N	K	N	N	N	Y	.	.
$Z_2^4 \times Z_5$	Y	N	K	N	N	N	K	.	.
$Z_4 \times Z_4 \times Z_5$	Y		K			N		.	.
Z_{81}	.	.	.	.	.	.	.	.	.
$Z_3 \times Z_{27}$	N	Y	N	N	N		N	.	.
$Z_3 \times Z_3 \times Z_9$	N	Y	N	N	N		N	.	.
$Z_3 \times Z_3 \times Z_3 \times Z_3$	N	K	N	N	N		N	.	.
$Z_9 \times Z_9$	N	Y	N	N	N		N	.	.
Z_{82}	+	.	.	.	.	.	.	.	.
Z_{83}	.	.	.	.	.	.	.	.	.
Z_{84}	+	.	+	.	.	.	+	.	.
$Z_2 \times Z_2 \times Z_3 \times Z_7$	Y	N	Y	N	N	N	K	N	.
Z_{85}	.	.	.	.	.	.	.	.	.
Z_{86}	+	.	.	.	.	.	.	.	.
Z_{87}	.	.	.	.	.	+	.	.	.
Z_{88}	+	.	.		.	.	.	.	.
$Z_2 \times Z_4 \times Z_{11}$	Y		K			N	N		.
$Z_2 \times Z_2 \times Z_2 \times Z_{11}$	Y		K			N	Y		.
Z_{89}	.	.	.	.	.	.	.	.	.
Z_{90}	+	.	.	.	.			.	.
$Z_2 \times Z_3 \times Z_3 \times Z_5$	Y	K		N				N	.
Z_{91}	+	+	.	.	+	.	.	+	.
Z_{92}	+	.	.	.	.	.	.		.
$Z_2 \times Z_2 \times Z_{23}$	Y		Y	N		N			.
Z_{93}	.	.	+	+	.	.	.	.	.
Z_{94}	+	.	.	.	.	.	.	.	.
Z_{95}	.	.	.	.	.		.	.	.
Z_{96}	+	+	.		+		.	.	.
$Z_2 \times Z_3 \times Z_{16}$	Y	K	Y				N	N	.
$Z_2 \times Z_2 \times Z_3 \times Z_8$	Y	K	K	N	K	N	Y	N	.
$Z_2^3 \times Z_3 \times Z_4$	Y	N	K	N	N	N	K	N	.
$Z_2^5 \times Z_3$	Y	N	K	N	N	N	K	N	.

G	k=4	9	16	25	36	49	64	81	100
$Z_3 \times Z_4 \times Z_8$	Y	K	K		K	N		N	.
$Z_2 \times Z_3 \times Z_4 \times Z_4$	Y	N	K	N	N	N	K	N	.
Z_{97}	.	.	.	.	.	.	.	.	.
Z_{98}	+	.	+	.	.	.	.	.	.
$Z_2 \times Z_7 \times Z_7$	Y	N	K	N	N	+		N	.
Z_{99}	.	.	.	+	.	.	.	.	.
$Z_3 \times Z_3 \times Z_{11}$	N	Y	N	K	N		N		.
Z_{100}	+	.	.	.	.	.	.	.	.
$Z_2 \times Z_2 \times Z_{25}$	Y	N	Y	N	N	N	N	N	.
$Z_2 \times Z_2 \times Z_5 \times Z_5$	Y	N	K		N	N	N	N	.
$Z_4 \times Z_5 \times Z_5$	Y	N	N	Y	N	N	N	N	.

Chapter 6

Future Work

The research done for this thesis was partly due to the open questions posed by Alex Gutman in his master's thesis [9], where he looked at studying the case of cyclic-G-developed weighing matrices. This thesis has studied the case for abelian-G-developed weighing matrices. Clearly, there is a very significant amount of work that has been done in this field - simply look at all of the other publications out there involving this topic. However, even with so much done there are still many unanswered questions that require work. From the above data alone it appears that there are up to 98 abelian-G-developed weighing matrices which need to be checked out. Remember too that only groups, and weights, under 100 were studied. Effort should certainly be placed into answering the open cases above. There are many different approaches that could be attempted. For example, computer searches to construct examples or possibly extending theorems to help fill in the gaps. Also, attempting to generate similar data for non-abelian groups might be an interesting endeavor.

Chapter 7

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