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Intersections of Deleted Digits Cantor Sets With Their Translates

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INTERSECTIONS OF DELETED DIGITS CANTOR SETS WITH THEIR TRANSLATES

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By

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I HEREBY RECOMMEND THAT THE THESIS PREPARED UNDER MY SUPERVISION BY Jason D. Phillips ENTITLED Intersections of Deleted Digits Cantor Sets With Their Translates BE ACCEPTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF Master of Science.

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Abstract

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We define a family of deleted digits Cantor sets which satisfy specific constraints on the generating set of digits. We explore the structure and dimension of the intersection of a deleted digits Cantor set with its translate by a real value t. These results apply directly to the traditional Middle Thirds Cantor set as well as regular and uniform Cantor sets. We show that this family includes certain irregular sets which have not been previously analyzed. Our methods not only reveal the upper and lower bounds for the Minkowski dimension, but also uncover a formula for calculating the dimension of these intersections when specific conditions are met.

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Part 1. **Basic Formulas**

1. Introduction

This paper has been divided into three parts. Part 1 includes preliminary results and definitions essential to the understanding of later sections. While the results in Part 1 are well known, they form a vital foundation for the methods used throughout the paper.

Part 2 studies the specific case of the traditional Middle Thirds Cantor set. This particular case has been studied previously in [DaHu95] and [NeLi02]. More generally, the conclusions of [DaTi08] can be directly applied to the Middle Thirds case. The primary purpose of Part 2 is to introduce our method in a simple and familiar setting and demonstrate consistency with previous findings.

Part 3 studies a family of self-similar sets which includes the Middle Thirds Cantor set, as well as uniform sets as shown in [DaTi08]. The results of Part 3 are also applied to a family of irregular fractals which are not uniform. This allows us to generalize the results of previous findings to a much larger class of sets.

Since Part 3 also applies to the Middle Thirds Cantor set, a number of results and definitions are duplicated from Part 2. This repetition is intentional for the continuity and readability of both Part 2 and Part 3. Both Parts are structured similarly to further assist with readability.

Finally, the upcoming paper [PePh11] contains a proof that the Hausdorff dimension is equal to the lower Minkowski dimension with respect to this family of fractals. Our calculations focus primarily on the Minkowski dimension for this reason.

We refer the reader to [Fal90] and [Hut81] for a background in the study of self-similar sets.

2. Real Numbers Base n

Let $n \geq 2$ be a natural number and $r \in \mathbb{R}$ be arbitrary. Since there exist $j \in \mathbb{Z}$ and $t \in [0, 1)$ such that $r = j + t$, it is sufficient to consider real values $t \in [0, 1)$.

Let $\lfloor \cdot \rfloor$ denote the floor function so that for any $x \in \mathbb{R}$, $\lfloor x \rfloor$ represents the unique integer satisfying $\lfloor x \rfloor \leq x < \lfloor x \rfloor + 1$. Define the integer sequence (t_k) based on t such that

$$
\begin{aligned}
t_1 &:= \left\lfloor (t - 0)\, n^1 \right\rfloor \\
t_k &:= \left\lfloor \left(t - \sum_{j=1}^{k-1} \frac{t_j}{n^j} \right) n^k \right\rfloor \quad \forall k \geq 2.
\end{aligned}
\tag{2.1}
$$

Lemma 2.1. *For any natural number $n \geq 2$ and real $t \in [0, 1)$, the sequence (t_k) determined by Equation (2.1) is such that $0 \leq t_k < n$ for all $k \in \mathbb{N}$.*

Proof. Since $t \in [0, 1)$ by assumption then $t_1 = \left\lfloor (t - 0)\, n^1 \right\rfloor = \lfloor t \cdot n \rfloor \leq t \cdot n < n$. Furthermore, $t \geq 0$ and $n \geq 2$ implies $t \cdot n \geq 0$ and thus $t_1 = \lfloor t \cdot n \rfloor \geq 0$.

Suppose $t_k \geq n$ for some $k \geq 2$. Then

$$
\begin{aligned}
n \leq t_k &= \left\lfloor \left(t - \sum_{j=1}^{k-1} \frac{t_j}{n^j} \right) n^k \right\rfloor \\
&\leq \left(t - \sum_{j=1}^{k-1} \frac{t_j}{n^j} \right) n^k \\
&= \left(t - \sum_{j=1}^{k-2} \frac{t_j}{n^j} \right) n^k - \left(\frac{t_{k-1}}{n^{k-1}} \right) n^k.
\end{aligned}
$$

Therefore, $\left(t - \sum_{j=1}^{k-2} \frac{t_j}{n^j} \right) n^{k-1} - \left(\frac{t_{k-1}}{n^{k-1}} \right) n^{k-1} \geq 1$ so that

$$
\left(t - \sum_{j=1}^{k-2} \frac{t_j}{n^j} \right) n^{k-1} \geq 1 + t_{k-1}. \tag{2.2}
$$

However, by definition of the floor function there exists $0 \leq \delta < 1$ such that

$$
\left(t - \sum_{j=1}^{k-2} \frac{t_j}{n^j} \right) n^{k-1} = \left\lfloor \left(t - \sum_{j=1}^{k-2} \frac{t_j}{n^j} \right) n^{k-1} \right\rfloor + \delta = t_{k-1} + \delta. \tag{2.3}
$$

We can combine Equation (2.2) with Equation (2.3) so that $\delta + t_{k-1} \geq 1 + t_{k-1}$, which contradicts that $0 \leq \delta < 1$. Hence, $t_k < n$ for all k.

We need to show that $0 \leq t_k$ for all k. Define the rational sequence (s_k) such that for all $k \in \mathbb{N}$,

$$s_k := \sum_{j=1}^{k} \frac{t_j}{n^j}. \tag{2.4}$$

If $k = 1$, then $s_1 = \frac{t_1}{n^1} = \frac{\lfloor t \cdot n \rfloor}{n} \leq \frac{t \cdot n}{n} = t$. Suppose $s_k \leq t$ for some k. Then $s_{k+1} - s_k = \sum_{j=1}^{k+1} \frac{t_j}{n^j} - \sum_{j=1}^{k} \frac{t_j}{n^j} = \frac{t_{k+1}}{n^{k+1}}$. By definition of (t_k),

$$\begin{aligned}
\frac{t_{k+1}}{n^{k+1}} &= \frac{1}{n^{k+1}} \cdot \left\lfloor \left(t - \sum_{j=1}^{k} \frac{t_j}{n^j} \right) n^{k+1} \right\rfloor \\
&= \frac{1}{n^{k+1}} \cdot \left\lfloor (t - s_k)\, n^{k+1} \right\rfloor \\
&\leq \frac{(t - s_k)\, n^{k+1}}{n^{k+1}} \\
&= (t - s_k)\,.
\end{aligned}$$

Hence, $s_{k+1} - s_k \leq t - s_k$ so that $s_{k+1} \leq t$. By induction, $s_k \leq t$ for all k.

Thus, $n^k > 0$ and $\sum_{j=1}^{k-1} \frac{t_j}{n^j} \leq t$ for all k so that $\left(t - \sum_{j=1}^{k-1} \frac{t_j}{n^j} \right) n^k \geq 0$. Therefore, $t_k = \left\lfloor \left(t - \sum_{j=1}^{k-1} \frac{t_j}{n^j} \right) n^k \right\rfloor \geq 0$ for all k. □

Lemma 2.1 shows that $t_k \in \{0, 1, \ldots, n-1\}$ for all k. We now show the importance of the sequence (s_k) defined in Equation (2.4).

Proposition 2.2. *For any $t \in [0,1)$, let (t_k) be the integer sequence determined by Equation (2.1). Let (s_k) be the rational sequence $s_k = \sum_{j=1}^{k} \frac{t_k}{n^k}$ for all k as in Equation (2.4). Then (s_k) converges to t.*

Proof. Suppose $t - s_k \geq \frac{1}{n^k}$ for some k. Then, by definition of (t_k),

$$\begin{aligned}
t_{k+1} &= \left\lfloor \left(t - \sum_{j=1}^{k} \frac{t_j}{n^j} \right) n^{k+1} \right\rfloor \\
&= \left\lfloor (t - s_k)\, n^{k+1} \right\rfloor \\
&\geq \left\lfloor \left(\frac{1}{n^k} \right) n^{k+1} \right\rfloor
\end{aligned}$$

$$= \lfloor n \rfloor = n.$$

This contradicts Lemma 2.1 and since k is arbitrary, then $t - s_k < \frac{1}{n^k}$ for all $k \in \mathbb{N}$. Furthermore, $t - s_k \geq 0$ according to the proof of Lemma 2.1.

Let $1 > \varepsilon > 0$ be given. Choose $N \in \mathbb{N}$ such that $N > -\log_n(\varepsilon) > 0$. Then $|t - s_k| < \frac{1}{n^k} \leq \frac{1}{n^{-\log_n(\varepsilon)}} = \varepsilon$ for all $k \geq N$. □

Given any real value $t \in [0, 1)$, if (x_k) is an integer sequence such that $0 \leq x_k < n$ and $t = \sum_{k=1}^{\infty} \frac{x_k}{n^k}$, then we say that (x_k) is a *base n representation* for t. Proposition 2.2 shows that Equation (2.1) detemines a base n representation (t_k). However, it remains to show that (t_k) is determined uniquely.

Corollary 2.3. *The sequence (t_k) determined by Equation (2.1) represents the largest integer such that $s_{k-1} + \frac{t_k}{n^k} \leq t < s_{k-1} + \frac{t_k+1}{n^k}$ for any $k \in \mathbb{N}$.*

Proof. Let $k \in \mathbb{N}$ be arbitrary. If t_k is too small, then $s_{k-1} + \frac{t_k+1}{n^k} \leq t$ so that $t - s_k \geq \frac{1}{n^k}$. Since this is a contradiction by Proposition 2.2, then, t_k is sufficiently large so that one of the following inequalities holds:

$$(a)\ \ s_{k-1} + \frac{t_k}{n^k} \leq t < s_{k-1} + \frac{t_k + 1}{n^k}$$
$$(b)\ \ s_{k-1} + \frac{t_k}{n^k} > t$$

However, case (b) implies that $s_k = s_{k-1} + \frac{t_k}{n^k} > t$, which is a contradiction by Lemma 2.1. □

Corollary 2.3 shows that t_k represents the unique value such that $s_{k-1} + \frac{t_k}{n^k} \leq t < s_{k-1} + \frac{t_k+1}{n^k}$ for each $k \in \mathbb{N}$. Thus, the base n representation (t_k) determined by Equation (2.1) is unique for any $t \in [0, 1)$.

However, if we do not choose the largest value according to the proof of Corollary 2.3, then it may be possible to generate a distinct base n representation (τ_k) such that $\tau_k \neq t_k$ for some k. We are able to classify cases when this alternate representation can occur.

Proposition 2.4. *Let (t_k) and (s_k) be the sequences determined by Equation (2.1) and Equation (2.4) respectively. Let $(\tau_k) \subset \mathbb{Z}$ and $(\sigma_k) \subset \mathbb{Q}$ be sequences such that $0 \leq \tau_k < n$ for all k, $\sigma_k = \sum_{j=1}^{k} \frac{\tau_k}{n^k}$, and (σ_k) converges to t. Then either $\tau_k = t_k$ for all k or $t = s_N$ for some N.*

Proof. Suppose $\tau_k \neq t_k$ for some $k \in \mathbb{N}$. Without loss of generality, let k be the least element of the set $\{i \in \mathbb{N} \mid \tau_i \neq t_i\}$.

If $\tau_k > t_k$, then $\sigma_k = \sigma_{k-1} + \frac{\tau_k}{n^k} = s_{k-1} + \frac{\tau_k}{n^k} > t$ by definition of (s_k) and Corollary 2.3. However, since (σ_k) is an increasing sequence, then $\sigma_k > t$ contradicts that (σ_k) converges to t. Therefore, $t_k > \tau_k$ and $s_k - \sigma_k = \frac{t_k - \tau_k}{n^k} \geq \frac{1}{n^k}$.

Since $t \geq s_k$ by Lemma 2.1, then suppose we have strict inequality and let $\beta = t - s_k > 0$. Then

$$t - \sigma_k = s_k - \sigma_k + \beta \geq \frac{1}{n^k} + \beta > \frac{1}{n^k}.$$

However, $\tau_h \leq n-1$ for any $h \in \mathbb{N}$, so that $\sum_{j=h+1}^{\infty} \frac{\tau_j}{n^j} \leq \sum_{j=h+1}^{\infty} \frac{n-1}{n^j} = \frac{n-1}{n^{h+1}} \cdot \frac{1}{1-\frac{1}{n}} = \frac{1}{n^h}$. Thus, $0 \leq \frac{1}{n^k} - \sum_{j=k+1}^{\infty} \frac{\tau_j}{n^j}$ and

$$t - \sigma_k - \sum_{j=k+1}^{\infty} \frac{\tau_j}{n^j} > \frac{1}{n^k} - \sum_{j=k+1}^{\infty} \frac{\tau_j}{n^j} \geq 0.$$

Since σ_k converges to t, then $\sigma_k + \sum_{j=k+1}^{\infty} \frac{\tau_j}{n^j} = \sum_{j=1}^{\infty} \frac{\tau_j}{n^j} = t$ and we have the contradiction $t - \sigma_k - \sum_{j=k+1}^{\infty} \frac{\tau_j}{n^j} = t - t > 0$. Therefore $t - s_k = 0$.

Note that if $t - \sigma_k = s_k - \sigma_k > \frac{1}{n^k}$, then we generate the same contradiction by the argument above. Thus, $t - \sigma_k = \sum_{j=k+1}^{\infty} \frac{\tau_j}{n^j} = \frac{1}{n^k}$ which only occurs if $\tau_j = n-1$ for all $j > k$. $\square$

Corollary 2.5. *The sequence (t_k) uniquely determined by Equation (2.1) is a base n representation for $t \in [0,1)$. If $t \neq \sum_{j=1}^{N} \frac{t_j}{n^j}$ for any $N \in \mathbb{N}$, then (t_k) is the only base n representation of t. Otherwise, if $t = \sum_{j=1}^{N} \frac{t_j}{n^j}$ for some $N \in \mathbb{N}$, then the only two base n*

representations of t are (t_k) and the sequence (τ_k) determined by

$$\tau_k = \begin{cases} t_k & k \leq N \\ t_k - 1 & k = N \\ n-1 & k > N \end{cases} \cdot$$

Corollary 2.5 determines all possible base n representations for a given $t \in [0,1)$. Since Equation (2.1) determines the sequence (t_k) uniquely, we will refer to $t = \sum_{k=1}^{\infty} \frac{t_k}{n^k}$ as the *n-ary expansion* of t. In general, we will also denote the n-ary expansion for a given t as $t = 0._n t_1 t_2 \ldots$.

For any $k \in \mathbb{N}$ and any $t \in [0,1]$, define the *truncation* of t to k places

$$\lfloor t \rfloor_k := 0._n t_1 t_2 \cdots t_k = \sum_{j=1}^{k} \frac{t_j}{n^j} = \frac{\lfloor n^k t \rfloor}{n^k}.$$

Part 2. The Middle Thirds Cantor Set

3. The Cantor Set

Let $n = 3$. For a given $t \in [0, 1]$ define the sequence (t_k) as in Equation (2.1). As a result, $t_k \in \{0, 1, 2\}$ for all $k \in \mathbb{N}$ and

$$t = \sum_{k=1}^{\infty} \frac{t_k}{3^k}.$$

This sequence represents the *ternary expansion* of t and will be equivalently denoted $t = 0._3 t_1 t_2 t_3 \cdots = 0.t_1 t_2 t_3 \cdots$. For the purposes of Part 2 we will forego the notation $0._3$ to denote the base 3 system and assume that all expansions are of triadic base.

Since Equation (2.1) determines the sequence (t_k) uniquely, we will use the finite representation whenever $t = s_N$ as in Proposition 2.4. That is, we assume that the ternary expansion of t does not terminate in repeating 2's.

The traditional Middle Thirds Cantor set can be defined in multiple equivalent ways. Define the continuous contraction mappings $f_1(x) := \frac{x}{3}$ and $f_2(x) := \frac{(x+2)}{3}$. Let $C_0 = [0, 1]$ and for all $k \in \mathbb{N}$ define $C_{k+1} = f_1(C_k) \cup f_2(C_k)$. Then C_k consists of 2^k closed intervals each of length $\frac{1}{3^k}$. Specifically,

$$C_k = \{0.t_1 t_2 \cdots \mid t_j \in \{0, 2\} \text{ for } j \leq k\}.$$

The *Middle Thirds Cantor set* is the compact set determined by

$$C = \bigcap_{k=0}^{\infty} C_k = \left\{ \sum_{k=1}^{\infty} \frac{t_k}{3^k} \mid t_k \in \{0, 2\} \right\}.$$

Define $f(K) := f_1(K) \cup f_2(K)$ for any compact $K \subseteq \mathbb{R}$. Then C is the unique nonempty, compact subset of $\mathbb{R}$ which satisfies $f(C) = C$ [Hut81].

4. An Analysis of Construction

For any bounded $X \subseteq \mathbb{R}$ and any $r \in \mathbb{R}$, define the *translation* of X by r, $(X + r) := \{x + r \mid x \in X\}$. Similarly, we will use the notation $r \cdot X := \{r \cdot x \mid x \in X\}$, and when X is a finite set we denote the cardinality of X by $\#X$. We are interested in the structure of the intersection of the Middle Thirds Cantor set C with its translation by a real value $t \in [0,1]$

$$C \cap (C+t).$$

Trivially, $C \cap (C+0) = C$ and $C \cap (C+1) = \{1\}$, and in general we will assume $t \in (0,1)$. It is important to identify the values of t resulting in non-empty intersections.

Lemma 4.1. *For any $t \in [-1,1]$, $C \cap (C+t) \neq \varnothing$.*

Proof. Let $t \in [-1,1]$ be arbitrary and define $z := \frac{(t+1)}{2}$. Then $z \in [0,1]$ has ternary expansion $z = 0.z_1 z_2 \cdots$ such that $z_k \in \{0,1,2\}$ for all $k \in \mathbb{N}$. Construct the ternary digits of $x, y \in [0,1]$ as follows:

$$\begin{cases} \text{if } z_k = 0 & \text{define } x_k = 0,\ y_k = 2 \\ \text{if } z_k = 1 & \text{define } x_k = 0,\ y_k = 0 \\ \text{if } z_k = 2 & \text{define } x_k = 2,\ y_k = 0. \end{cases}$$

Thus, $x, y \in C$ by construction so that $(x_k - y_k) \in \{-2, 0, 2\}$ for all k. Specifically, $z_k = 1 + \frac{(x_k - y_k)}{2}$. Therefore,

$$\frac{x-y}{2} = \sum_{k=1}^{\infty} \frac{z_k}{3^k} - \sum_{k=1}^{\infty} \frac{1}{3^k} = z - \frac{1}{2} = \frac{t}{2}.$$

Hence, $x = y + t$ and $C \cap (C+t) \neq \varnothing$. $\square$

Lemma 4.1 shows that any choice of $t \in [0,1]$ will result in a nonempty intersection $C \cap (C+t)$ and thus we can investigate the structure of the set. However, $C \cap (C+t)$ may be a finite set, as with the case $t = 1$.

We begin our investigation of $C \cap (C+t)$ by considering the method of construction. As shown in Section 3, $C = \bigcap_{k=0}^{\infty} C_k$ and similarly, $C + t = \left(\bigcap_{k=0}^{\infty} C_k \right) + t = \bigcap_{k=0}^{\infty} (C_k + t)$. Thus,

$$C \cap (C+t) = \bigcap_{k=0}^{\infty} \left(C_k \cap (C_k + t) \right)$$

We will consider the properties of $C_k \cap (C_k + t)$ by analyzing the related sets $C_k \cap (C_k + \lfloor t \rfloor_k)$. In particular, for each $k \in \mathbb{N}$ we are interested in the relationship between

$C_k \cap (C_k + \lfloor t \rfloor_k)$ and $C_{k+1} \cap \left(C_{k+1} + \lfloor t \rfloor_{k+1}\right)$. For any interval $I = [a, b]$ with length $c = b - a$, we refer to $\left[a, a + \frac{c}{3}\right] \cup \left[b - \frac{c}{3}, b\right]$ as the *refinement* of I.

Given the construction of C, all intervals of C_k (and its translate $C_k + \lfloor t \rfloor_k$) will have similar properties: intervals are closed with length $\frac{1}{3^k}$ where the left endpoint is $\frac{h}{3^k}$ for some $h \in \mathbb{N}$. In general, for any $k \in \mathbb{N}$ and any $h \in \mathbb{Z}$, we refer to a closed interval $I = \left[\frac{h}{3^k}, \frac{h+1}{3^k}\right]$ as a *ternary* interval. As ternary intervals are the basis of our analysis, we will often use the terms interval and ternary interval interchangeably, and state explicitly when an interval is not ternary.

Note that C_{k+1} is generated by refining each interval contained in C_k. Similarly, $C_{k+1} + \lfloor t \rfloor_{k+1}$ is generated by first refining each interval of $C_k + \lfloor t \rfloor_k$ and then translating the result by an additional $\frac{t_{k+1}}{3^{k+1}}$. We say that $C_k \cap (C_k + \lfloor t \rfloor_k)$ *transitions* to $C_{k+1} \cap \left(C_{k+1} + \lfloor t \rfloor_{k+1}\right)$ by first generating the sets C_{k+1} and $C_{k+1} + \lfloor t \rfloor_{k+1}$ and then taking the intersection.

In this way there are only four different types of interactions between any two ternary intervals $J = [\alpha, \beta] \subset C_k$ and $I = [a, b] \subset C_k + \lfloor t \rfloor_k$. In general, we say that a ternary interval $J \subset C_k$ of length $\frac{1}{3^k}$ is in the *interval case* if there exists a ternary interval $I \subset C_k + \lfloor t \rfloor_k$ such that $I = J$. Similarly, J is in the *potential interval case* if $I + \frac{1}{3^k} = J$ and the *potentially empty case* if $I - \frac{1}{3^k} = J$. We say J is in the *empty case* if $I \cap J = \varnothing$ for all $I \subset C_k + \lfloor t \rfloor_k$.

We will examine the interactions between I and J after transitioning to $C_{k+1} \cap C_{k+1} + \lfloor t \rfloor_{k+1}$.

4.1. **The empty case $I \cap J = \varnothing$.** This occurs if $b < \alpha$ or $a > \beta$. Since I and J are both ternary intervals, then the distance between I and J is $\min\{|\alpha - b|, |a - \beta|\} \geq \frac{1}{3^k}$. Note that $\frac{t_{k+1}}{3^{k+1}} < \frac{1}{3^k}$ for any choice of $0 \leq t_{k+1} < n$. Thus, the refinement and translation of I will not intersect any refinement of J. Therefore, any subintervals of I or J will also have empty intersection in $C_{k+1} \cap \left(C_{k+1} + \lfloor t \rfloor_{k+1}\right)$.

In general, since t does not terminate in repeating twos by assumption, then for any remaining digits t_j for $j > k$,

$$t - \lfloor t \rfloor_k = \sum_{j=k+1}^{\infty} \left(\frac{t_j}{3^j}\right) < \frac{1}{3^k}.$$

Thus, the interval $(I + (t - \lfloor t \rfloor_k))$ is contained in $C_k + t$ such that $J \cap (I + (t - \lfloor t \rfloor_k)) = \varnothing$.

4.2. **The potentially empty case $I - \frac{1}{3^k} = J$.** Figure 4.1 depicts an arbitrary potentially empty case before translating by $\frac{t_{k+1}}{3^k}$. The figure shows the refinement of J on top and the refinement of I on bottom. Retained subintervals are denoted as black bands, while deleted subintervals are shown as a thin black line to provide better visual spacing. Ternary intervals other than I and J have also been removed for clarity.

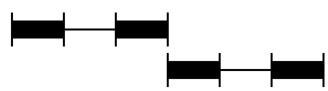

FIGURE 4.1. A potentially empty case in the Middle Thirds Cantor set.

If $t_{k+1} = 0$, then we refine both I and J and do not translate I further. This transitions to one potentially empty case in $C_{k+1} \cap \left(C_{k+1} + \lfloor t \rfloor_{k+1}\right)$ which also shares the endpoint $\beta = a$.

If $t_{k+1} > 0$, then we translate the refinement of I by an additional amount $\frac{t_{k+1}}{3^{k+1}} > 0$. Therefore, β is not be contained in the refinement and translation of I, and the intersection will be empty in $C_{k+1} \cap \left(C_{k+1} + \lfloor t \rfloor_{k+1}\right)$.

4.3. **The interval case $I = J$.** If $t_k = 0$, then we refine both I and J and do not translate I further. This transitions to two interval cases in $C_{k+1} \cap \left(C_{k+1} + \lfloor t \rfloor_{k+1}\right)$.

If $t_k = 1$, then we translate the refinement of I by $\frac{1}{3^{k+1}}$. Thus, the left-most subinterval of I shares an endpoint with the right-most subinterval of J and we transition to one potential interval case and one potentially empty case. Figure 4.2 depicts an arbitrary interval case such that $t_{k+1} = 1$. The figure shows the refinement of J on top and the refinement and translation of I on bottom.

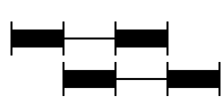

FIGURE 4.2. An interval case translated by $\frac{1}{3^{k+1}}$ in the Middle Thirds Cantor set.

If $t_k = 2$, then we translate the refinement of I by $\frac{2}{3^{k+1}}$. Thus, the left-most subinterval of I is equal to the right-most subinterval of J and we transition to one interval case.

4.4. **The potential interval case $I + \frac{1}{3^k} = J$.** Figure 4.3 depicts an arbitrary potential interval case before translating by $\frac{t_{k+1}}{3^k}$.

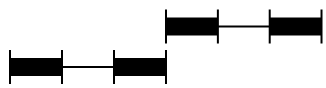

FIGURE 4.3. A potential interval case in the Middle Thirds Cantor set.

If $t_k = 0$, then we refine both I and J and do not translate I further. Thus, we transition to one potential interval case in $C_{k+1} \cap \left(C_{k+1} + \lfloor t \rfloor_{k+1}\right)$ which also shares the endpoint $b = \alpha$.

If $t_k = 1$, then we translate the refinement of I by $\frac{1}{3^{k+1}}$. Thus, the left-most subinterval of I is equal to the right-most subinterval of J and we transition to one interval case.

If $t_k = 2$, then we translate the refinement of I by $\frac{2}{3^{k+1}}$. Thus, the left-most subinterval of I shares an endpoint with the left-most subinterval of J. Also, the right-most subinterval of I will share an endpoint with the right-most subinterval of J. Therefore, we transition to two potential interval cases.

We will say that the set $C_k \cap (C_k + \lfloor t \rfloor_k)$ is in the *interval case* if there exists at least one $J \subset C_k$ in the interval case and for any intervals such that $I \cap J \neq \varnothing$, then I and J are in the interval case except for a finite number of potentially empty cases. Similarly, we will say that $C_k \cap (C_k + \lfloor t \rfloor_k)$ is in the *potential interval case* if there exists at least one $J \subset C_k$ in the potential interval case and for any intervals such that $I \cap J \neq \varnothing$, then I and J are in the potential interval case except for a finite number of potentially empty cases.

The four cases outlined above give a complete description for the transition from $C_k \cap (C_k + \lfloor t \rfloor_k)$ to $C_{k+1} \cap \left(C_{k+1} + \lfloor t \rfloor_{k+1}\right)$. Any subsets of $C_k \cap (C_k + \lfloor t \rfloor_k)$ are covered by the interval and potential interval cases, except for at most a finite number of isolated points resulting from potentially empty cases. By construction, $C_0 = [0, 1]$ where $C_0 \cap (C_0 + \lfloor t \rfloor_0) =$

C_0 so that we begin in the interval case. We can derive $C_k \cap (C_k + \lfloor t \rfloor_k)$ for any $k \in \mathbb{N}$ using induction.

As we are only interested in interval and potential interval cases, it is important to note that each interval case can only lead to one interval case, one potential interval case, or two interval cases. It is not possible for an interval case to transition to both intervals and potential intervals at the next step. Similarly, each potential interval case can only lead to one potential interval case, one interval case, or two potential interval cases. Thus, for any $k \in \mathbb{N}$ and any $t \in \mathbb{R}$, $C_k \cap (C_k + \lfloor t \rfloor_k)$ contains either intervals or potential intervals and cannot have both simultaneously.

5. Counting Intervals

According to the analysis in Section 2, if $t_{k+1} = 1$, then interval cases in $C_k \cap (C_k + \lfloor t \rfloor_k)$ transition to potential interval cases in $C_{k+1} \cap \left(C_{k+1} + \lfloor t \rfloor_{k+1}\right)$. In the same way, if $t_{k+1} = 1$, then potential interval cases transition to interval cases. Since the digit $t_k = 1$ flips between the two cases, define the function $\sigma_k : \mathbb{R} \to \{-1, 1\}$ for all k such that

$$\sigma_0(t) := 1$$
$$\sigma_k(t) := \prod_{j=1}^{k} (-1)^{t_j} \ \forall k > 0.$$

Since $C_0 = C_0 + \lfloor t \rfloor_0 = [0, 1]$ is in the interval case and interval and potential interval cases cannot occur simultaneously, then $\sigma_k(t) = 1$ iff $C_k \cap (C_k + \lfloor t \rfloor_k)$ is in the interval case and $\sigma_k(t) = -1$ iff $C_k \cap (C_k + \lfloor t \rfloor_k)$ is in the potential interval case. Furthermore, $\sigma_k(t) = -1$ if and only if the set $\{h \leq k \mid t_h = 1\}$ contains an odd number of elements.

Since we assume that the ternary representation of t does not terminate in repeating 2's, then $t - \lfloor t \rfloor_k < \frac{1}{3^k}$ for any $k \in \mathbb{N}$. Let $J \subset C_k$ and $I \subset C_k + \lfloor t \rfloor_k$ be ternary intervals of length $\frac{1}{3^k}$. If $C_k \cap (C_k + \lfloor t \rfloor_k)$ is in the interval case, then there exists at least one pair of intervals such that $I = J$ by definition. Thus, $I + (t - \lfloor t \rfloor_k)$ is an interval contained in $(C + t)$ such that $(I + (t - \lfloor t \rfloor_k)) \cap J$ is a closed interval of length $\frac{1}{3^k} - (t - \lfloor t \rfloor_k) > 0$. Since the pair (I, J) is arbitrary, then each n-ary interval contained in $C_k \cap (C_k + \lfloor t \rfloor_k)$ corresponds to an interval of $C_k \cap (C_k + t)$. Therefore, the number of interval cases contained in $C_k \cap (C_k + \lfloor t \rfloor_k)$ is equal to the number of intervals in $C_k \cap (C_k + t)$.

By the same argument, if $C_k \cap (C_k + \lfloor t \rfloor_k)$ is in the potential interval case, then there exists at least one pair of intervals such that $I + \frac{1}{3^k} = J$. Thus, $(I + (t - \lfloor t \rfloor_k)) \cap J$ is nonempty. If $t = \lfloor t \rfloor_k$, then we do not further translate I so that $I + (t - \lfloor t \rfloor_k) \cap J$ is the isolated endpoint $\left\{\frac{h}{3^k}\right\}$ for some $h \in \mathbb{N}$. However, if there exists $i > k$ such that $t_i \neq 0$, then $I + (t - \lfloor t \rfloor_k) \cap J$ is a closed interval of length $(t - \lfloor t \rfloor_k) > 0$. Therefore, if $(t - \lfloor t \rfloor_k) > 0$, then the number of potential interval cases in $C_k \cap (C_k + \lfloor t \rfloor_k)$ is equal to the number of intervals in $C_k \cap (C_k + t)$.

Thus, for any t, we can accurately count the number of intervals contained in $C_k \cap (C_k + t)$ by analyzing the related sets $C_k \cap (C_k + \lfloor t \rfloor_k)$. It is important to note that there are only two possible ways to increase the number of intervals in a given transition. If $C_k \cap (C_k + \lfloor t \rfloor_k)$ is in the interval case and $t_{k+1} = 0$, then each interval case $I = J$ transitions to two subinterval cases in $C_{k+1} \cap \left(C_{k+1} + \lfloor t \rfloor_{k+1}\right)$. Similarly, if $C_k \cap (C_k + \lfloor t \rfloor_k)$ is in the potential interval case and $t_{k+1} = 2$, then each potential interval case $I + \frac{1}{3^k} = J$ transitions to two potential subinterval cases in $C_{k+1} \cap \left(C_{k+1} + \lfloor t \rfloor_{k+1}\right)$. Any other combination results in the same number of total interval cases (resp. potential interval cases), or flips between these two cases. In order to track these increases, define

$$A_t := \{k \mid \sigma_k(t) = 1 \text{ and } t_{k+1} = 0\}$$

$$B_t := \{k \mid \sigma_k(t) = -1 \text{ and } t_{k+1} = 2\}.$$

Where A_t contains values of k resulting in increases in the interval case and B_t contains values resulting in increases in potential interval cases. Since interval and potential interval cases cannot occur simultaneously, then A_t and B_t are disjoint subsets of $\mathbb{N}$ for all $t \in [-1, 1]$. These sets are potentially infinite, and since we are interested in tracking the number of increases for a finite value k, define

$$\nu_k(t) := \#\{j < k \mid j \in A_t \cup B_t\}.$$

Note that the total number of interval or potential interval cases remains the same for any $k \notin A \cup B$. Thus, $\nu_k(t)$ counts the number of transitions which double the total number of intervals up to step k. Therefore, the number of intervals or potential intervals in $C_k \cap (C_k + \lfloor t \rfloor_k)$ is equal to $2^{\nu_k(t)}$. Whenever $(t - \lfloor t \rfloor_k) > 0$, then $2^{\nu_k(t)}$ also corresponds to the number of intervals in $C_k \cap (C_k + t)$.

We are only interested in intervals of $C_k \cap (C_k + t)$ which contain points of $C \cap (C + t)$, so it is important to show when this occurs.

Proposition 5.1. *Let $t \in [0,1]$ be fixed and let $J = \left[\frac{h}{3^k}, \frac{h+1}{3^k}\right] \subset C_k$. If there exists an interval $I \subset (C_k + \lfloor t \rfloor_k)$ such that either $I = J$ or $I + \frac{1}{3^k} = J$, then the corresponding closed set $J \cap (I + (t - \lfloor t \rfloor_k)) \subset C_k \cap (C_k + t)$ contains at least one point of $C \cap (C + t)$.*

Proof. Let $J = \left[\frac{h}{3^k}, \frac{h+1}{3^k}\right] \subset C_k$ be arbitrary. Let $I \subset C_k + \lfloor t \rfloor_k$ be the interval such that $J \cap I \neq \varnothing$ and the pair (I, J) is not in the potentially empty case. Since intervals are at least $\frac{1}{3^k}$ apart, then I is unique for each J. Note that $I + (t - \lfloor t \rfloor_k) \subset (C_k + t)$ and $I + (t - \lfloor t \rfloor_k) \cap J$ is nonempty according to the analysis in Section 4. The proof is divided into two cases:

1. Suppose $\sigma_k(t) = 1$ so that $C_k \cap (C_k + \lfloor t \rfloor_k)$ is in the interval case. Then $I = J$ and $0 \leq (t - \lfloor t \rfloor_k) \leq \frac{1}{3^k}$.

Since $J \subset C_k$ is ternary, then J refines to the small Cantor set $\frac{1}{3^k}C + h$ based on the construction of C. Let $J_0 = J$ and let J_{i+1} be the refinement of J_i for all $i \in \mathbb{N}$. Since $J_0 \subset C_k$ and C_{k+1} is the refinement of all intervals of C_k, then $J_i \subset C_{k+i}$ for all i. Thus, $3^k \cdot J_0 - h = [0,1] = C_0$ and since $3^k \cdot J_{i+1} - h$ is the refinement of $3^k \cdot J_i - h = C_i$, then $3^k \cdot J_i - h = C_i$ for all i. Therefore,

$$C = \bigcap_{i=0}^{\infty} \left(3^k \cdot J_i - h\right) = 3^k \cdot \bigcap_{i=0}^{\infty} (J_i) - h.$$

Define $\widetilde{t} := 3^k (t - \lfloor t \rfloor_k)$. Since $0 \leq \widetilde{t} \leq 1$ and $I = J$, then for refinements $I_i = J_i$,

$$C + \widetilde{t} = 3^k \cdot \bigcap_{i=0}^{\infty} (I_i + (t - \lfloor t \rfloor_k)) - h.$$

Furthermore, because $\widetilde{t} \in [0,1]$, then $C \cap \left(C + 3^k \left(t - \lfloor t \rfloor_k\right)\right) \neq \varnothing$ according to Lemma 4.1. Let $r \in C \cap \left(C + \widetilde{t}\right)$. We need to show that $\frac{r+h}{3^k} \in C \cap (C+t)$.

Then $r \in 3^k \cdot \bigcap_{i=0}^{\infty} (J_i) - h$ such that $\frac{r+h}{3^k} \in \bigcap_{i=0}^{\infty} (J_i) \subset J$. Since $\bigcap_{i=0}^{\infty} (J_i)$ represents the refinement of J in the construction of C, then $\bigcap_{i=0}^{\infty} (J_i) \subset \bigcap_{i=0}^{\infty} (C_{k+i}) = C$ so that $\frac{r+h}{3^k} \in C$.

Similarly, $r \in 3^k \cdot \bigcap_{i=0}^{\infty} \left(I_i + \left(t - \lfloor t \rfloor_k\right)\right) - h$ such that $\frac{r+h}{3^k} \in \bigcap_{i=0}^{\infty} \left(I_i + \left(t - \lfloor t \rfloor_k\right)\right)$. Since $\left(I_i + \left(t - \lfloor t \rfloor_k\right)\right)$ are refinements of $\left(I + \left(t - \lfloor t \rfloor_k\right)\right) \subset (C_k + t)$, then $\bigcap_{i=0}^{\infty} \left(I_i + \left(t - \lfloor t \rfloor_k\right)\right) \subset \bigcap_{i=0}^{\infty} (C_{k+i} + t) = (C+t)$ and $\frac{r+h}{3^k} \in (C+t)$.

Therefore, $\frac{r+h}{3^k} \in C \cap (C+t)$ such that $\frac{r+h}{3^k} \in J \cap \left(I + \left(t - \lfloor t \rfloor_k\right)\right)$. Hence, $J \cap \left(I + \left(t - \lfloor t \rfloor_k\right)\right)$ contains at least one point of $C \cap (C+t)$.

2. Suppose $\sigma_k(t) = -1$ so that $C_k \cap \left(C_k + \lfloor t \rfloor_k\right)$ is in the potential interval case. Then $I + \frac{1}{3^k} = J$ and $-\frac{1}{3^k} \leq \left(t - \lfloor t \rfloor_k\right) - \frac{1}{3^k} \leq 0$ such that

$$I + \left(t - \lfloor t \rfloor_k\right) = I + \frac{1}{3^k} + \left[\left(t - \lfloor t \rfloor_k\right) - \frac{1}{3^k}\right].$$

Define $\widetilde{t} := 3^k \cdot \left(t - \lfloor t \rfloor_k\right) - 1 \in [-1, 0]$. Thus, we can use the same argument above so that $J \cap \left(I + \left(t - \lfloor t \rfloor_k\right)\right)$ contains at least one point of $C \cap (C+t)$. $\square$

Proposition 5.1 shows that all intervals and potential intervals for any $C_k \cap \left(C_k + \lfloor t \rfloor_k\right)$ will generate points in $C \cap (C+t)$. Furthermore, this method shows that for any $k \in \mathbb{N}$ we can choose the collection of intervals $J \subset C_k$ in the interval or potential interval cases and that this collection will cover all but a finite number of points of $C \cap (C+t)$. That is, we can cover all points except for potentially empty cases, if they exist.

6. Minkowski-Bouligand Dimension

Given a compact set K and $k \in \mathbb{N}$ let $N_k(K)$ denote the smallest number of closed intervals of length $\frac{1}{3^k}$ needed to cover K. The *upper Minkowski dimension* of K is

$$\overline{\dim}_M (K) = \limsup_{k \to \infty} \frac{\log N_k(K)}{\log 3^k}.$$

Similarly, the *lower Minkowski dimension* is $\underline{\dim}_M (K) = \liminf_{k \to \infty} \frac{\log N_k(K)}{\log 3^k}$ and if the upper and lower Minkowski dimensions are equal then $\dim_M (K) = \overline{\dim}_M (K) = \underline{\dim}_M (K)$ is the

Minkowski dimension of K. The Minkowski dimension is also referred to as the Minkowski-Bouligand dimension or box-counting dimension. Based on the construction of C it is natural to use this dimension for analysis.

Since $N_k(C) = 2 \cdot N_{k-1}(C)$ for all $k \in \mathbb{N}$ and $N_0(C) = 1$ by construction of C, then $N_k(C) = 2^k$ for all k. Thus, the Minkowski dimension exists and

$$\dim_M(C) = \lim_{k\to\infty} \frac{\log\ N_k(C)}{\log 3^k} = \lim_{k\to\infty} \frac{k \cdot \log 2}{k \cdot \log 3} = \frac{\log 2}{\log 3}.$$

This allows us to determine the constraints on the Minkowski dimension of $C \cap (C+t)$.

Proposition 6.1. *For any $t \in [0,1]$,*

$$0 \leq \underline{\dim}_M(C \cap (C+t)) \leq \overline{\dim}_M(C \cap (C+t)) \leq \frac{\log 2}{\log 3}.$$

Proof. Let K be any closed subset of C. The result is trivial if K is empty, so assume $K \neq \varnothing$. Then any cover of C is automatically a cover of K and $1 \leq N_k(K) \leq N_k(C)$ for all $k \in \mathbb{N}$. Thus, $\frac{\log N_k(K)}{\log n^k} \leq \frac{\log N_k(C)}{\log n^k}$ for all k so that

$$\overline{\dim}_M(K) = \limsup_{k\to\infty} \frac{\log\ N_k(K)}{\log n^k} \leq \limsup_{k\to\infty} \frac{\log\ N_k(C)}{\log n^k} = \frac{\log m}{\log n}$$

Furthermore, since $1 \leq N_k(K)$ for all k, then $N_k(K)$ is a nonnegative sequence of integers so that $0 \leq \frac{\log\ N_k(K)}{\log n^k}$ for all k. Thus, $0 \leq \liminf\limits_{k\to\infty} \frac{\log\ N_k(K)}{\log n^k} = \underline{\dim}_M(K)$. It is by definition of the limits superior and inferior that $\liminf\limits_{k\to\infty} a_k \leq \limsup\limits_{k\to\infty} a_k$ for any real sequence (a_k). $\square$

We can completely characterize the dimension for any t which has a finite ternary representation.

Theorem 6.2. *Let C denote the Middle Thirds Cantor set and let $t \in [0,1]$. If there exists $k \in \mathbb{N}$ such that $t = \lfloor t \rfloor_k$ then*

$$\dim_M(C \cap (C+t)) = \begin{cases} \frac{\log 2}{\log 3} & \text{if } \sigma_k(t) = 1 \\ 0 & \text{if } \sigma_k(t) = -1 \end{cases}$$

Proof. Let $t \in [0,1]$ be given such that $t = \lfloor t \rfloor_k$ for some $k \in \mathbb{N}$. Without loss of generality, let k be the minimal element of $\left\{ j \mid \left(t - \lfloor t \rfloor_j \right) = 0 \right\}$. Since $t = \lfloor t \rfloor_k = \lfloor t \rfloor_k + \sum_{j=k+1}^{\infty} \frac{t_j}{3^j}$, then $t_{k+h} = 0$ for all $h > 0$. Therefore, $\sigma_k(t) = \sigma_{k+h}(t)$ and

$$C_{k+h} \cap \left(C_{k+h} + \lfloor t \rfloor_{k+h} \right) = C_{k+h} \cap \left(C_{k+h} + t \right).$$

Let $h > 0$ be arbitrary. We proceed with two cases:

1. Suppose $\sigma_k(t) = 1$. Thus, $C_k \cap (C_k + \lfloor t \rfloor_k)$ is in the interval case and contains $2^{\nu_k(t)}$ ternary intervals. We no longer translate $(C_{k+h} + t)$ For any $h > 0$, so each n-ary interval of $C_{k+h} \cap (C_{k+h} + t)$ is refined into a small Cantor set. Therefore, there are a minimum of $2^{\nu_k(t)+h}$ intervals of length $\frac{1}{3^{k+h}}$ required to cover $C_{k+h} \cap (C_{k+h} + t)$. Also, by Proposition 5.1 each interval contained in $C_{k+h} \cap (C_{k+h} + t)$ contains points of $C \cap (C + t)$ so that $2^{\nu_k(t)+h}$ is the minimal number of intervals of length $\frac{1}{3^{k+h}}$ required to cover $C \cap (C + t)$. That is, for all $h > 0$,

$$N_{k+h}(C \cap (C + t)) \geq 2^{\nu_k(t)+h}.$$

Therefore,

$$\liminf_{l \to \infty} \frac{\log\ N_h(C \cap (C + t))}{\log 3^h} \geq \lim_{h \to \infty} \frac{(\nu_k(t) + h)\log 2}{(k + h)\log 3} = \frac{\log 2}{\log 3}.$$

The upper and lower dimensions are identical by Proposition 6.1, so that

$$\dim_M (C \cap (C + t)) = \frac{\log 2}{\log 3}.$$

2. Suppose $\sigma_k(t) = -1$. Thus, $C_k \cap (C_k + \lfloor t \rfloor_k)$ is in the potential interval and potentially empty cases so that $C_k \cap (C_k + t)$ contains a finite number of points. Let $r = \#C_k \cap (C_k + t)$. Since $C \cap (C + t) \subset (C_k \cap (C_k + t))$, then $C \cap (C + t)$ contains at most r points. Thus,

$$\limsup_{h \to \infty} \frac{\log\ N_h(C \cap (C + t))}{\log 3^h} \leq \lim_{h \to \infty} \frac{\log r}{\log 3^h} = 0.$$

The lower Minkowski dimension must also be 0 according to Proposition 6.1. Thus, the dimension exists and $\dim_M (C \cap (C + t)) = 0$. $\square$

Theorem 6.2 gives the dimension for any $t \in [0,1]$ having finite ternary representation. Thus, for the remainder of this section we expand our basic assumptions:

Let $t \in [0,1]$ such that the ternary expansion of t does not end in repeating 2's or repeating 0's. That is, for any $k \in \mathbb{N}$ there exist $k \leq h_0$ and $k \leq h_2$ such that $t_{h_0} \neq 0$ and $t_{h_2} \neq 2$.

Proposition 6.3. *Let $t \in [0,1]$ be such that the triadic expansion of t does not terminate in repeating 0's or 2's. Then for all k,*

$$N_k\left(C \cap (C+t)\right) = 2^{\nu_k(t)}.$$

Proof. Since t does not terminate in repeating zeroes, then $\left(t - \lfloor t \rfloor_k\right) > 0$ for any k. Thus, $2^{\nu_k(t)}$ corresponds to the number of intervals in $C_k \cap (C_k + t)$ based on our analysis in Section 5. Since potentially empty cases contained in $C_k \cap \left(C_k + \lfloor t \rfloor_k\right)$ always vanish after a finite number of transitions, then $N_k\left(C \cap (C+t)\right) \leq 2^{\nu_k}$.

According to the proof of Proposition 5.1, for any $k \in \mathbb{N}$ we can choose the specific collection of 2^{ν_k} intervals $J \subset C_k$ such that this collection is the minimum required to cover $C \cap (C+t)$. Therefore, $N_k\left(C \cap (C+t)\right) \geq 2^{\nu_k}$. $\square$

In Section 5 we showed that the sets A_t and B_t contain all values of k which increase the total number of intervals or potential intervals in the transition from $C_k \cap \left(C_k + \lfloor t \rfloor_k\right)$ to $C_{k+1} \cap \left(C_{k+1} + \lfloor t \rfloor_{k+1}\right)$. Furthermore, if $k \notin A_t \cup B_t$, then $\nu_{k+1}(t) = \nu_k(t)$. This leads to the following result:

Lemma 6.4. *If $A_t \cup B_t$ is finite, then $C \cap (C+t)$ is finite and* $\dim_M\left(C \cap (C+t)\right) = 0$.

Proof. Suppose $A_t \cup B_t$ is finite for some t. Define $k = \max\left(A \cup B\right) < \infty$ so that $k+l \notin A \cup B$ for all $l > 0$ and $\nu_k(t) = \nu_{k+1}(t) = \cdots = \nu_{k+l}(t) = \cdots$. Thus, $C \cap (C+t)$ contains at most $2^{\nu_k(t)}$ points and

$$\limsup_{l \to \infty} \frac{\log\ N_l\left(C \cap (C+t)\right)}{\log 3^l} \leq \lim_{l \to \infty} \frac{\log 2^{\nu_k(t)}}{\log 3^l} = 0.$$

The lower dimension must also be 0 by Proposition 6.1. Thus the dimension exists and is equal to 0. $\square$

We are able to determine the Minkowski dimension of $C \cap (C+t)$ for a number of cases. Note that Theorem 6.2 and Lemma 6.4 only apply to certain values of t such that the Minkowski dimension is either 0 or $\frac{\log 2}{\log 3}$. Since the Minkowski dimension is bounded by these values, we are interested in constructing values of t such that inequality holds: $0 < \dim_M (C \cap (C+t)) < \frac{\log 2}{\log 3}$.

Theorem 6.5. *Let*

$$\mathcal{F}_\alpha := \left\{ s \in [0,1] \mid \dim_M (C \cap (C+s)) = \alpha \frac{\log 2}{\log 3} \right\}.$$

Then $\mathcal{F}_\alpha$ is dense in the interval $[0,1]$ for any $0 \leq \alpha \leq 1$.

Proof. Let $0 \leq \alpha \leq 1$, $u \in [0,1]$, and $\varepsilon > 0$ be given. Choose $k \in \mathbb{N}$ such that

$$u - \varepsilon < 0.u_1 \ldots u_k \overline{0} < 0.u_1 \ldots u_k \overline{2} < u + \varepsilon.$$

Define $s_j = u_j$ for all $1 \leq j \leq k$ so that $|s - u| < \varepsilon$ for any choice of remaining digits s_j for $j > k$. Furthermore, regardless of the value of $\sigma_k(s)$ we are able to choose s_{k+1} so that $\sigma_{k+1}(s)$ is either -1 or 1 as desired.

Note that if $\alpha = 1$, then choose s_{k+1} so that $\sigma_{k+1}(s) = 1$ and let $s_j = 0$ for $j > k$. Similarly, if $\alpha = 0$, then choose s_{k+1} so that $\sigma_{k+1}(s) = -1$ and let $s_j = 0$ for $j > k$. In either case we obtain the desired result through Theorem 6.2. Thus, we proceed with values $0 < \alpha < 1$ for the remainder of the proof. We also choose s_{k+1} such that $\sigma_{k+1}(s) = 1$ and we begin in the interval case.

Define the sequence $x_j := \lfloor j \cdot \alpha \rfloor$ for all $j > 0$ so that x_j is the unique positive integer such that $x_j \leq j \cdot \alpha < x_j + 1$. Equivalently, $0 \leq \alpha - \frac{x_j}{j} < \frac{1}{j}$ so that the sequence $\left(\frac{x_j}{j}\right)$ converges to α as $j \to \infty$. Since $0 < \alpha < 1$, then $x_j \leq x_{j+1} \leq x_j + 1$ for all j. Thus, for all

$j \geq k+1$ choose the remaining digits of s as follows:

$$s_{j+1} = \begin{cases} 2 & \text{if } x_{j+1} = x_j \\ 0 & \text{if } x_{j+1} = x_j + 1. \end{cases}$$

Therefore, we have chosen values of s_j for $j > k$ so that $\sigma_j(s)$ is always 1. Define $\nu_i(s) = \#\{j < i \mid j \in A_s\}$ and $A_s = \{j \mid \sigma_j(s) = 1 \text{ and } s_{j+1} = 0\}$. Then, by Proposition 6.3, for any $j > k+1$

$$N_j(C \cap (C+s)) = 2^{\nu_j(s)}.$$

Furthermore, $\nu_{j+1}(s) = x_{j+1} - x_j + \nu_j(s)$ for any $j > k$. Thus, $\nu_{l+k+1}(s) = x_l - x_{k+1} + \nu_{k+1}(s)$ for all $l > 0$. Therefore,

$$\frac{\log N_{l+k+1}(C \cap (C+s))}{\log 3^{l+k+1}} = \frac{x_l \log 2 - x_{k+1} \log 2 + \nu_{k+1}(s) \log 2}{l \log 3 + (k+1) \log 3}.$$

Since x_{k+1} and $\nu_{k+1}(s)$ are fixed, finite values, then

$$\lim_{l \to \infty} \frac{x_l \log 2 - x_{k+1} \log 2 + \nu_{k+1}(s) \log 2}{l \log 3 + (k+1) \log 3} = \lim_{l \to \infty} \frac{x_l}{l} \frac{\log 2}{\log 3} = \alpha \frac{\log 2}{\log 3}.$$

Thus, the Minkowski dimension of $C \cap (C+s)$ exists and is equal to $\alpha \frac{\log 2}{\log 3}$ as desired. $\square$

Corollary 6.6. *Define the function $\psi(u) := \dim_M (C \cap (C+u))$ and let $T \subseteq [0,1]$ be the points where the dimension exists. For any $u_0 \in [0,1]$ and any $\delta > 0$ define the open ball $B_\delta(u_0) = (u_0 - \delta, u_0 + \delta)$. Then the image of $T \cap B_\delta(u_0)$ under ψ is the closed interval $\left[0, \frac{\log 2}{\log 3}\right]$. In symbols,*

$$\psi(T \cap B_\delta(u_0)) = \left[0, \frac{\log 2}{\log 3}\right].$$

Corollary 6.7. *The mapping $u \mapsto \underline{\dim}_M (C \cap (C+u))$ defined on $[0,1]$ is everywhere discontinuous.*

The proof of Theorem 6.5 gives a countable, dense set of values s such that the Minkowski dimension of $C \cap (C+s)$ is $\alpha \frac{\log 2}{\log 3}$. This result was also proven using different methods as Result B of [DaHu95], and as Proposition 1 of [NeLi02].

7. A Formula

To reiterate assumptions, let $t \in [0,1]$ such that the ternary expansion of t does not terminate in repeating 0's or repeating 2's. By Lemma 6.4, if $A_t \cup B_t$ is a finite set then $\dim_M C \cap (C+t) = 0$. Proposition 6.3 gives us the formula $N_k (C \cap (C+t)) = 2^{\nu_k(t)}$ for all $k \in \mathbb{N}$. In general,

$$\overline{\dim}_M (C \cap (C+t)) = \limsup_{k\to\infty} \frac{\nu_k(t)}{k} \frac{\log 2}{\log 3}$$
$$\underline{\dim}_M (C \cap (C+t)) = \liminf_{k\to\infty} \frac{\nu_k(t)}{k} \frac{\log 2}{\log 3}.$$

Therefore, if $A_t \cup B_t$ is not finite, then the Minkowski dimension of $C \cap (C+t)$ is determined by the asymptotic density of $A_t \cup B_t$ in $\mathbb{N}$. Thus, the Minkowski dimension is not guaranteed to exist for an arbitrary value t.

Let $t \in C$. By definition, $t_k \neq 1$ for any $k \in \mathbb{N}$ so that $\sigma_k(t) = 1$ for all k. Thus, $B_t = \varnothing$ and

$$\nu_k(t) = \#\{j < k \mid j \in A_t \cup B_t\} = \#\{j \le k \mid t_j = 0\}.$$

If t is irrational we will naturally have an uneven distribution to the digits $t_j = 0$ and the upper and lower dimensions may not be equal.

$$\overline{\dim}_M (C \cap (C+t)) = \limsup_{k\to\infty} \frac{\#\{j \le k \mid t_j = 0\}}{k} \frac{\log 2}{\log 3}$$
$$\underline{\dim}_M (C \cap (C+t)) = \liminf_{k\to\infty} \frac{\#\{j \le k \mid t_j = 0\}}{k} \frac{\log 2}{\log 3}$$

This allows us to construct an uncountable set of values ω such that $C \cap (C+\omega)$ does not have Minkowski dimension: Define $\widehat{C} \subseteq C$ which contains any value $z \in C$ such that $0.z_1 z_2 \cdots$ does not terminate in repeating 2's. This set is nonempty since $0 \in \widehat{C}$. Further, $\widehat{C}$ contains all values of C which are not the right endpoint of any interval J of length $\frac{1}{3^k}$ contained in C_k for any k. Thus, $\widehat{C}$ is an uncountable set. Let $z \in \widehat{C}$ be arbitrary.

For each $k > 0$, let $y_k(z)$ be the (base 3) natural number such that

$$y_k(z) \text{ consists of } 2^{k-1} \text{ twos followed by } 2^{k-1} \text{ zeroes if } z_k = 0$$

$$y_k(z) \text{ consists of } 2^{k-1} \text{ zeroes followed by } 2^{k-1} \text{ twos if } z_k = 2.$$

In essence, $z_3 = 0$ implies $y_3 = 22220000$, and $z_3 = 2$ implies $y_3 = 00002222$. Let $\omega(z)$ be the irrational number defined by

$$\omega(z) = \sum_{k=1}^{\infty} y_k(z) \cdot \left(\frac{1}{3}\right)^{2^{k+1}-2}.$$

For example, $\omega(0) = 0.20220022220000\cdots$. Thus, for any choice of z and any $k \in \mathbb{N}$ there are exactly $2^k - 1$ zeroes in the first $2\left(2^k - 1\right)$ digits. That is,

$$2^k - 1 = \#\left\{j \leq 2\left(2^k - 1\right) \mid \omega_j(z) = 0\right\}.$$

Since $i < 2\left(2^i - 1\right)$ for any $i \in \mathbb{N}$, then

$$\limsup_{k\to\infty} \frac{\#\{j \leq k \mid \omega_j(z) = 0\}}{k} \frac{\log 2}{\log 3} \geq \lim_{k\to\infty} \frac{2^k - 1}{2\left(2^k - 1\right)} \frac{\log 2}{\log 3} = \frac{1}{2}\frac{\log 2}{\log 3}.$$

However, since the ternary representation of z does not terminate in repeating 2's, then for any $k \in \mathbb{N}$ there exists an $h + 1 > k$ such that $z_{h+1} \neq 2 \iff z_{h+1} = 0$. By definition, $y_{h+1}(z)$ consists of 2^h twos followed by an equal number of zeroes. Since there are $2^h - 1$ zeroes in the first $2\left(2^h - 1\right)$ digits, we can proceed by another 2^h digits and only add twos. Therefore, there are only $2^h - 1$ zeroes in the first $2\left(2^h - 1\right) + 2^h = 3\left(2^h - 1\right) + 1$ digits. Because k is arbitrary, then for all k there exists some $h_k + 1 > k$ such that this is true. Thus,

$$\liminf_{k\to\infty} \frac{\#\{j \leq k \mid \omega_j(z) = 0\}}{k} \frac{\log 2}{\log 3} \leq \lim_{k\to\infty} \frac{2^{h_k} - 1}{3\left(2^{h_k} - 1\right) + 1} \frac{\log 2}{\log 3} = \frac{1}{3}\frac{\log 2}{\log 3}.$$

By this construction, $\omega(z) \in C \subset [0, 1]$ is irrational and does not have Minkowski dimension. Furthermore, for any $z, z' \in \widehat{C}$, $\omega(z) = \omega(z')$ is and only if $z = z'$ so that we generate a distinct irrational value $\omega(z)$ for each z. Thus, $\left\{\omega(z) \mid z \in \widehat{C}\right\}$ is an uncountable set.

Note also that for any $0 < \alpha \leq 1$, we can choose an increasing sequence (x_j) such that $\frac{x_j}{j} \to \alpha$ and use the method from Theorem 6.5 to construct a countable dense set of values t which have Minkowski dimension.

Theorem 7.1. *For any rational $t \in [0,1]$, let p be a period of t such that $t_{k+p} = t_k$ for sufficiently large values of k. Then*

$$\dim_M C \cap (C+t) = \frac{\#\{k \leq j < k+2p \mid j \in A_t \cup B_t\}}{2p} \frac{\log 2}{\log 3}.$$

Proof. Let $t \in [0,1]$ be rational so that the ternary representation is periodic after a certain point k. Without loss of generality, assume that k is the smallest value where t becomes periodic. Let $p > 0$ be a period of t so that $t_{h+p} = t_h$ for all $h \geq k$.

Since $\sigma_k(t) = \pm 1$ as determined by the first k digits of t, then $\sigma_{k+p}(t) = \sigma_k(t) \cdot \prod_{i=1}^{i=p} (-1)^{t_{k+i}}$ by definition. However, the repeating sequence $t_k \cdots t_{k+p-1}$ may contain an odd number of digits $t_j = 1$, which implies $\sigma_{k+p}(t) = -\sigma_k(t)$. Thus, we consider a period of $2p$ so that the repeating sequence contains an even number of digits $t_j = 1$. This guarantees that $\sigma_{h+2p}(t) = \sigma_h(t)$ for all $h \geq k$ and $\sigma_h(t)$ has period $2p$. We write $t = 0.t_1 \cdots \overline{t_k \cdots t_{k+2p-1}}$.

Thus, $t_{h+2p} = t_h$ and $\sigma_{h+2p}(t) = \sigma_h(t)$ for all $h \geq k$. Therefore, if $h \in A_t \cup B_t$, then $h + 2p \in A_t \cup B_t$. Define r_h to be the number of $h \leq j < h+2p$ such that $j \in A_t \cup B_t$. Then $r_h = r_k$ for $h \geq k$. Since $r = r_k = \#\{k \leq j < k+2p \mid j \in A_t \cup B_t\}$, then $\nu_{h+2p}(t) = r + \nu_h(t)$ and

$$\lim_{j\to\infty} \frac{\nu_{h+j2p}(t)}{h+j2p} = \lim_{j\to\infty} \frac{\nu_h(t) + jr}{h+j2p} = \frac{r}{2p}.$$

The limit above is identical for any $h \in \{k, k+1, \cdots, k+2p-1\}$ and since t is periodic then $\lim\limits_{j\to\infty} \frac{\nu_j(t)}{j} = \frac{r}{2p}$. Hence,

$$\overline{\dim}_M (C \cap (C+t)) = \limsup_{j\to\infty} \frac{\nu_j}{j} \frac{\log 2}{\log 3} = \frac{r}{2p} \frac{\log 2}{\log 3}$$
$$\underline{\dim}_M (C \cap (C+t)) = \liminf_{j\to\infty} \frac{\nu_j}{j} \frac{\log 2}{\log 3} = \frac{r}{2p} \frac{\log 2}{\log 3}.$$

This implies that the Minkowski dimension exists and is equal to

$$\dim_M C \cap (C+t) = \frac{r}{2p} \frac{\log 2}{\log 3}.$$

$\square$

Part 3. The Deleted Digits Cantor Set

8. Defining A Deleted Digits Set

For any bounded $X \subseteq \mathbb{R}$, define the *translation* of X by r, $(X + r) := \{x + r \mid x \in X\}$. Similarly, we will use the notation $r \cdot X := \{r \cdot x \mid x \in X\}$, $X^{+} := X \cap [0, \infty)$, and $X^{-} := X \cap (-\infty, 0]$. When X is a finite set, denote the cardinality of X by $\#X$.

Let $n \geq 3$. For any arbitrary real number $t \in [0, 1]$, the *base n expansion* is $t = \sum_{k=1}^{\infty} \frac{t_k}{n^k} = 0._n t_1 t_2 \cdots$ where the sequence (t_k) is determined by Equation (2.1). We will assume that the base n expansion of t does not terminate in repeating $n-1$ digits, or equivalently: for any $k \in \mathbb{N}$ there exists $h \geq k$ such that $t_h \neq n-1$. As with the Middle Thirds Cantor set, we forego special notation and assume the base n expansion.

Define the *set of digits* $D \subset \mathbb{N}$ to be a set containing at least two elements,

$$D := \{d_j \mid 0 \leq d_1 < d_2 < \cdots < d_m \leq n-1\}.$$

For each $j \in \{1, \ldots, m\}$, define the continuous contraction mapping $f_j(x) = \frac{(x+d_j)}{n}$ and let $f_{n,D}(K) = \bigcup_{j=1}^{m} f_j(K)$ for any compact $K \subset \mathbb{R}$. Let $C_0 = [0, 1]$ and inductively define $C_{k+1} = f_{n,D}(C_k)$ for all $k \geq 0$ so that for each k, $C_k = \{0.t_1 t_2 \cdots \mid t_j \in D \text{ for } j \leq k\}$. A *deleted digits set* is defined to be the compact set which depends on both n and D,

$$C_{n,D} := \bigcap_{k=0}^{\infty} C_k = \left\{ \sum_{k=1}^{\infty} \frac{t_k}{n^k} \mid t_k \in D \right\}.$$

In general, we will use $C = C_{n,D}$ to denote a deleted digits set with the understanding that C is determined by the choices of n and D. Similarly, we will denote $f = f_{n,D}$. By Theorem 3.1 of [Hut81], C is the unique nonempty, compact subset of $\mathbb{R}$ such that $f(C) = C$.

Note that if $d_1 \neq 0$ then $(D - d_1)$ is also a set of digits which generates $C_{n,D-d_1} = \left\{\sum_{k=0}^{\infty} \frac{x_k}{n^k} \mid x_k \in D - d_1\right\}$. For any $x \in C_{n,D-d_1}$, $x + \frac{d_1}{n-1} = \sum_{k=0}^{\infty} \frac{x_k + d_1}{n^k} = \sum_{k=0}^{\infty} \frac{y_k}{n^k} = y$ for some $y \in C_{n,D}$. By the same equation, for any $y \in C_{n,D}$, $y - \frac{d_1}{n-1} = x$ for some $x \in C_{n,D-d_1}$ so that $C_{n,D-d_1} = C_{n,D} - \frac{d_1}{n-1}$. Since the structure of the original set is unchanged when translating by the real value $\frac{-d_1}{n-1}$, we will assume that $d_1 = 0$ without loss of generality.

The definition of the set of digits includes $D = \{0, 1, \ldots, n-1\}$, which results in $C = [0,1]$ for any $n \geq 3$. In order to guarantee that C is not trivial, we define a *deleted digits Cantor set* to be a deleted digits set C such that $D \subsetneqq \{0, 1, \ldots, n-1\}$. Furthermore, we impose a separation condition on the set of digits so that $d_{j+1} - d_j \geq 2$ for all $1 \leq j < m$. Thus, for the remainder of Part 3 we will use the following conditions on the digits set:

$$D = \{d_j \mid 0 = d_1 < d_2 < \cdots < d_m \leq n-1 \text{ and } d_{j+1} - d_j \geq 2\}.$$

With this definition we can construct the Middle Thirds Cantor set using the set of digits $D = \{0, 2\}$ for $n = 3$. We say that a deleted digits Cantor set is *uniform* if there exists an integer $c > 1$ such that $d_j = (j-1) \cdot c$ for all $d_j \in D$.

9. The Set $\mathcal{F}$

We are interested in the structure of a deleted digits Cantor set C intersected with its translate by a real value t:

$$C \cap (C + t).$$

It is important to note that the largest element of C is $0.d_m d_m \cdots = \sum_{j=1}^{\infty} \frac{d_m}{n^j} = \frac{d_m}{n-1}$ so that $C \subset \left[0, \frac{d_m}{n-1}\right]$. Thus, for any $t > \frac{d_m}{n-1}$ we have $(C+t) \subset \left(\frac{d_m}{n-1}, \infty\right)$ so that $C \cap (C+t) \subset \left[0, \frac{d_m}{n-1}\right] \cap \left(\frac{d_m}{n-1}, \infty\right) = \varnothing$. Similarly, for any $t < -\frac{d_m}{n-1}$, $(C+t) \subset (-\infty, 0)$ so that $C \cap (C+t) = \varnothing$. Therefore, the set $C \cap (C+t)$ is only interesting when $t \in \left[-\frac{d_m}{n-1}, \frac{d_m}{n-1}\right]$. In order to identify the values of t resulting in a non-empty intersection, define the set:

$$\mathcal{F} := \{t \mid C \cap (C+t) \neq \varnothing\}$$

This set will play a key role in our analysis of $C \cap (C+t)$, thus we will investigate some of the properties of $\mathcal{F}$. We make a special note that Section 9 is the generalization of Lemma 4.1 from Part 2.

Lemma 9.1. *For any deleted digits set C, $\mathcal{F} = C - C$.*

Proof. Let $t \in \mathcal{F}$ be arbitrary. Then $C \cap (C+t) \neq \varnothing$ by definition of $\mathcal{F}$ and we can choose $x, y \in C$ such that $x = y + t$. Thus, $t = x - y$ so that $t \in C - C$ and $\mathcal{F} \subseteq C - C$.

Furthermore, $x = y + (x - y)$ for any $x, y \in C$ so that $C \cap (C + (x - y)) \neq \varnothing$. Hence, $(x - y) \in \mathcal{F}$ and $C - C \subseteq \mathcal{F}$. $\square$

It is a direct result of Lemma 9.1 that $0 \in C$ implies $C \subset \mathcal{F}$. Furthermore, since $(x - y) \in C - C$ if and only if $(y - x) \in C - C$, then $\mathcal{F}$ is symmetric around 0 and $\mathcal{F}^- = -\mathcal{F}^+$. Thus, we will investivate nonnegative values of t. Since $t = 0$ generates the set $C \cap (C + 0) = C$ and $t = \frac{d_m}{n-1}$ generates $C \cap \left(C + \frac{d_m}{n-1}\right) = \left\{\frac{d_m}{n-1}\right\}$ we will focus on the cases when $t \in \left(0, \frac{d_m}{n-1}\right)$.

Lemma 9.2. *$\mathcal{F}$ is compact.*

Proof. $\mathcal{F}$ is nonempty according to Lemma 9.1. Since the set $C \cap (C + t)$ is empty for t outside of $\left[-\frac{d_m}{n-1}, \frac{d_m}{n-1}\right]$, then $\mathcal{F} \subseteq \left[-\frac{d_m}{n-1}, \frac{d_m}{n-1}\right]$ and the set is bounded. We only need to show that $\mathcal{F}$ is closed.

From Section 8 we know that C and $C + t$ are both compact. Let $x \in \mathbb{R}$ such that $C \cap (C + x) = \varnothing$ and let $\varepsilon := \inf\{|a - b| \mid a \in C,\, b \in (C + x)\} \geq 0$.

If $\varepsilon = 0$, then we can construct sequences $(a_k) \subset C$ and $(b_k) \subset (C + x)$ such that $|a_k - b_k| < \frac{1}{k}$ for all $k \in \mathbb{N}$. Since C is compact, then (a_k) contains a convergent subsequence $(a_{k_i}) \to \alpha \in C$. However, $|b_{k_i} - \alpha| \leq |a_{k_i} - b_{k_i}| + |a_{k_i} - \alpha| \mapsto 0$ so that the subsequence (b_{k_i}) also converges to α. Since $C + x$ is also compact, then $\alpha \in (C + x)$ which contradicts the assumption that $C \cap (C + x) = \varnothing$. Thus, $\varepsilon > 0$.

Suppose $y \in \mathbb{R}$ such that $|x - y| < \varepsilon$. For any $a \in C$ and $b \in (C + x)$, then $b + (y - x)$ is any element of $(C + y)$ and $|a - b - (y - x)| \leq |a - b| + |x - y|$. Since $|x - y|$ is constant for a given y,

$$\begin{aligned}\inf\{|a - b| \mid a \in C,\, b \in (C + x)\} &\leq \inf\{|a - b| + |x - y| \mid a \in C,\, b + (y - x) \in (C + y)\} \\ &= |x - y| + \inf\{|a - b| \mid a \in C,\, b \in (C + y)\}.\end{aligned}$$

Hence, $\inf\{|a - b| \mid a \in C,\, b \in (C + y)\} \geq \varepsilon - |x - y| > 0$ and $C \cap (C + y) = \varnothing$. Therefore, the complement of $\mathcal{F}$ is open in $\mathbb{R}$. $\square$

Define the *set of differences*

$$\Delta := D - D = \{d_i - d_j \mid d_i, d_j \in D\}.$$

Note that Δ is nonempty since $0 \in D$ implies $\pm D \subset \Delta$. Since D contains only nonnegative values then Δ contains at least $2m - 1$ distinct elements. Let $M = \#\Delta$. We will assume that Δ is an ordered set so that $\delta_i < \delta_j$ for any $1 \leq i < j \leq M$. For each $\delta \in \Delta$, define the continuous contraction mappings which rely on n and D:

$$g_\delta(x) = \frac{x + \delta}{n}. \tag{9.1}$$

Define $g_{n,\Delta}(K) = \bigcup_{\delta \in \Delta} g_\delta(K)$ for any compact $K \subset \mathbb{R}$ and denote $g = g_{n,\Delta}$. Since $D \subset \Delta$, then $g_{d_j}(x) = f_j(x)$ for all $1 \leq j \leq m$. This observation leads the the following result:

Lemma 9.3. *$\mathcal{F}$ is the unique non-empty compact set such that $\mathcal{F} = g(\mathcal{F})$.*

Proof. $\mathcal{F} = C - C$ by Lemma 9.1. Due to the self-similar nature of C,

$$\begin{aligned} C - C &= \bigcup_{d_i \in D} f_i(C) - \bigcup_{d_j \in D} f_j(C) \\ &= \bigcup_{d_i, d_j \in D} f_i(C) - f_j(C) \\ &= \bigcup_{\delta \in \Delta} g_\delta(C - C) \\ &= g(C - C). \end{aligned}$$

Since $f_i(x) - f_j(y) = \frac{x+d_i}{n} - \frac{y+d_j}{n} = \frac{(x-y)+(d_i-d_j)}{n} = g_\delta(x - y)$ for $\delta = d_i - d_j$. $\square$

Thus, $\mathcal{F}$ is a self-similar set [Hut81]. Furthermore, $g_{d_j}(x) = f_j(x)$ implies that the structure of $\mathcal{F}$ is related to the structure of C. In particular, this allows us to classify the cases when $\mathcal{F}$ is an interval:

Proposition 9.4. *$\mathcal{F}$ is an interval iff*

$$2d_m \geq (n-1)(\delta_{j+1} - \delta_j)$$

for all $j \in \{1,2,\ldots,M-1\}$. Furthermore, if $\mathcal{F}$ is an interval then $\mathcal{F} = \left[-\frac{d_m}{n-1}, \frac{d_m}{n-1}\right]$.

Proof. Let $I = \left[-\frac{d_m}{n-1}, \frac{d_m}{n-1}\right]$. We know that the largest element of C is $\frac{d_m}{n-1}$ and $\mathcal{F} \subseteq I$. Since $C \cap \left(C + \frac{d_m}{n-1}\right) = \left\{\frac{d_m}{n-1}\right\} \neq \varnothing$ and $C \cap \left(C - \frac{d_m}{n-1}\right) = \{0\} \neq \varnothing$ then the endpoints $-\frac{d_m}{n-1}$ and $\frac{d_m}{n-1}$ are contained in $\mathcal{F}$. By definition, a closed interval must contain all points $-\frac{d_m}{n-1} \leq t \leq \frac{d_m}{n-1}$, thus $\mathcal{F}$ is an interval if and only if $\mathcal{F} = I$.

Since $\mathcal{F} \subseteq I$ and we can construct $\mathcal{F}$ using the similarity mappings g_{δ_j} for $\delta_j \in \Delta$ according to Lemma 9.3, then $g_{\delta_j}(\mathcal{F}) \subseteq g_{\delta_j}(I)$ for all $1 \leq j \leq M$. In particular, $g_{\delta_j}(I) = \frac{1}{n}\left[\delta_j - \frac{d_m}{n-1}, \delta_j + \frac{d_m}{n-1}\right]$ and since $\pm\frac{d_m}{n-1} \in \mathcal{F}$, then the endpoints $g_{\delta_j}\left(\pm\frac{d_m}{n-1}\right)$ are contained in $g_{\delta_j}(\mathcal{F})$ for all $1 \leq j \leq M$. Therefore, each $g_{\delta_j}(I)$ contains points of $g(\mathcal{F}) = \mathcal{F}$.

Since g_{δ_j} is continuous for all $1 \leq j < M$, then $g_{\delta_j}(I)$ is a compact interval. If $g_{\delta_j}(I) \cap g_{\delta_{j+1}}(I) = \varnothing$ for some j, then there exists an x such that

$$g_{\delta_j}\left(\frac{d_m}{n-1}\right) = \sup\left\{g_{\delta_j}(I)\right\} < x < \inf\left\{g_{\delta_{j+1}}\right\} = g_{\delta_{j+1}}\left(-\frac{d_m}{n-1}\right).$$

Thus, x is not contained in either set. Because Δ is an ordered set, then $x \notin g(I)$ and $x \notin \mathcal{F} = g(\mathcal{F}) \subseteq g(I)$. Therefore, $\mathcal{F}$ is an interval iff for all $1 \leq j < M$,

$$g_{\delta_j}\left(\frac{d_m}{n-1}\right) \geq g_{\delta_{j+1}}\left(-\frac{d_m}{n-1}\right).$$

By direct substitution using Equation (9.1), the inequality above is equivalent to $2d_m \geq (n-1)(\delta_{j+1} - \delta_j)$. $\square$

Corollary 9.5. *If D consists of even integers, then $\mathcal{F}$ is an interval iff*

$$\Delta = \{-n+1, \ldots, -2, 0, 2, \ldots, n-1\}.$$

When this occurs, n is an odd integer and $\mathcal{F} = [-1, 1]$.

Proof. Let D be a finite subset of $2 \cdot \mathbb{N}$. Since $\Delta = D - D \subset 2 \cdot \mathbb{Z}$ then Δ also consists of even integers.

($\Rightarrow$) Suppose $\mathcal{F}$ is an interval. Then by Proposition 9.4, $2d_m \geq (n-1)(\delta_{j+1} - \delta_j)$ for all $1 \leq j < M$. Thus, $0 < d_m \leq n-1$ by definition of D and since δ_j is even, then for all j

$$n - 1 \geq d_m \geq (n-1)\frac{(\delta_{j+1} - \delta_j)}{2}.$$

Therefore, $\delta_{j+1} - \delta_j = 2$ for all j. When this occurs, the equation above requires $d_m = n-1$ so that $\Delta = \{-n+1, \cdots, -2, 0, 2, \ldots, n-1\}$. Furthermore, $\mathcal{F}$ is necessarily the interval $\left[-\frac{d_m}{n-1}, \frac{d_m}{n-1}\right] = [-1, 1]$.

($\Leftarrow$) Suppose $\Delta = \{-n+1, \ldots, -2, 0, 2, \ldots, n-1\}$. Since $\Delta = D - D$, then

$$n - 1 = \max(\Delta) = \max(D) - \min(D) = d_m - d_1.$$

Therefore, $d_m = n-1+d_1$ and $n-1 \geq d_m > d_1 \geq 0$ by definition of D. Thus, $n-1 \geq n-1+d_1$ so that $d_1 = 0$ and $d_m = n-1$. Since $\delta_{j+1} - \delta_j = 2$ for all $1 \leq j < M$, then

$$2 \cdot d_m = 2(n-1) = (n-1)(\delta_{j+1} - \delta_j).$$

Proposition 9.4 shows that $\mathcal{F} = \left[-\frac{d_m}{n-1}, \frac{d_m}{n-1}\right] = [-1, 1]$. $\square$

Note that Corollary 9.5 does not necessarily require $D = \{0, 2, 4, \ldots, n-1\}$. For example, if $D = \{0, 2, 6\}$ and $n = 7$, then $\Delta = \{0, \pm 2, \pm 4, \pm 6\}$ and $\mathcal{F} = [-1, 1]$.

Given any finite set of continuous contraction mappings $\{\psi_j \mid 1 \leq j \leq R\}$, the *open set condition* holds for this collection of functions if there exists a bounded, open set V such that $\psi_i(V) \cap \psi_j(V) = \varnothing$ for $i \neq j$ and

$$\psi(V) = \bigcup_{j=1}^{R} \psi_j(V) \subset V.$$

Let $\overline{V}$ denote the closure of V. By defining $V_0 = V$ and $V_{k+1} = \psi(V_k)$ for $k \in \mathbb{N}$, then $\bigcap_{k=0}^{\infty} \overline{V_k}$ is the unique nonempty compact set invariant under ψ [Fal90].

The proof of Proposition 9.4 also shows when the collection $\{g_\delta \mid \delta \in \Delta\}$ will satisfy the open set condition using an open interval. There exists a bounded, open interval I such that the disjoint union $\bigcup_{\delta \in \Delta} g_\delta(I)$ is a subset of I if and only if for all $1 \leq j < M$ the following equation holds:

$$2d_m \leq (n-1)\left(\delta_{j+1} - \delta_j\right). \tag{9.2}$$

When this occurs we can choose the bounded, open set $I = \left(-\frac{d_m}{n-1}, \frac{d_m}{n-1}\right)$.

Lemma 9.6. *If D is a finite subset of $c \cdot \mathbb{N}$ for some integer $c > 1$, then the collection $\{g_\delta \mid \delta \in \Delta\}$ satisfies the open set condition.*

Proof. Let $c > 1$ and $D \subset c \cdot \mathbb{N}$. By definition $\Delta = D - D \subset c \cdot \mathbb{Z}$ so that for any $1 \leq j < M$, $\delta_{j+1} - \delta_j \geq c$. Since $n - 1 \geq d_m > 0$, then

$$(n-1)\left(\delta_{j+1} - \delta_j\right) \geq c \cdot (n-1) \geq 2(n-1) \geq 2d_m.$$

□

Proposition 9.7. *If D is a finite subset of $c \cdot \mathbb{N}$ for some integer $c > 1$, then $\mathcal{F} = \left[-\frac{d_m}{n-1}, \frac{d_m}{n-1}\right]$ or there is a deleted digits Cantor set B such that $\mathcal{F} = c \cdot B - \frac{d_m}{n-1}$.*

Proof. Let $c > 1$ and $D \subset c \cdot \mathbb{N}$ so that $\frac{1}{c} \cdot D = \left\{\frac{d_1}{c}, \frac{d_2}{c}, \cdots, \frac{d_m}{c}\right\} \subset \mathbb{N}$. Let $t \in \mathcal{F} = C - C$ be arbitrary and choose $x, y \in C$ such that $t = x - y$. Thus,

$$\frac{(x-y)}{c} = \sum_{k=1}^{\infty} \frac{(x_k - y_k)/c}{n^k} = \frac{t}{c}.$$

Define $E := \frac{1}{c}D - \frac{1}{c}D = \{-e_m, \ldots, 0, \ldots, e_m\}$ so that $\{0, e_m, 2e_m\} \subseteq E + e_m \subseteq \{0, 1, \ldots 2e_m\}$ and $2e_m \leq ce_m = d_m \leq n - 1$. Therefore, $E + e_m$ is also a digit set for the same $n \geq 3$. Let B be the deleted digits set determined by n and $E + e_m$.

Since $\frac{(x_k - y_k)}{c} \in E$, then let $z_k = e_m + \frac{(x_k - y_k)}{c} \in E + e_m$ for all k. Then $z = 0.z_1 z_2 \ldots \in B$ such that

$$z = \sum_{k=1}^{\infty} \frac{z_k}{n^k} = \sum_{k=1}^{\infty} \frac{(x_k - y_k)/c}{n^k} + \sum_{k=1}^{\infty} \frac{e_m}{n^k} = \frac{t}{c} + \frac{e_m}{n-1}.$$

Hence, $t = c \cdot z - \frac{d_m}{n-1} \in c \cdot B - \frac{d_m}{n-1}$. Since $t \in \mathcal{F}$ is arbitrary, then $\mathcal{F} \subseteq c \cdot B - \frac{d_m}{n-1}$.

Let $z \in c \cdot B - \frac{d_m}{n-1}$ be arbitrary. Then $\frac{z}{c} + \frac{e_m}{n-1} = \sum_{k=1}^{\infty} \frac{(z_k/c) + e_m}{n^k}$ is an element of B and $\frac{z_k}{c} \in E$ for all k. By definition of E, for each k there exist $\frac{d_{i,k}}{c}$ and $\frac{d_{j,k}}{c}$ in $\frac{1}{c}D$ such that $\frac{z_k}{c} = \frac{d_{i,k}}{c} - \frac{d_{j,k}}{c}$. Equivalently, $z_k = d_{i,k} - d_{j,k}$. Let $x = \sum_{k=1}^{\infty} \frac{d_{i,k}}{n^k}$ and $y = \sum_{k=1}^{\infty} \frac{d_{j,k}}{n^k}$ so that $x, y \in C$ and $z = \sum_{k=1}^{\infty} \frac{z_k}{n^k} = \sum_{k=1}^{\infty} \frac{d_{i,k} - d_{j,k}}{n^k} = x - y \in \mathcal{F}$. Hence, $c \cdot B - \frac{d_m}{n-1} \subseteq \mathcal{F}$ and thus

$$\mathcal{F} = c \cdot B - \frac{d_m}{n-1}.$$

If $E + e_m = \{0, 1, \ldots, e_m, \ldots, 2e_m\}$ then B is trivially the interval $\left[0, \frac{2e_m}{n-1}\right]$ and $\mathcal{F} = \left[-\frac{d_m}{n-1}, \frac{d_m}{n-1}\right]$. Otherwise, if $E + e_m \subsetneq \{0, 1, \ldots, e_m, \ldots, 2e_m\}$ then $\{0, e_m, 2e_m\} \subseteq E + e_m$ implies that B is a deleted digits Cantor set. $\square$

In general, we say that C is *regular* if $D \subset c \cdot \mathbb{N}$ for some $c > 1$ and C is *irregular* if D is not a subset of $c \cdot \mathbb{N}$ for any $c > 1$. By definition, any uniform deleted digits Cantor set is generated by a set of digits $D \subset c \cdot \mathbb{N}$ and is considered regular. Furthermore, we can characterize the structure of $\mathcal{F}$ by applying Lemma 9.6 and Proposition 9.7. A regular deleted digits Cantor set is not necessarily uniform (for example: $D = \{0, 2, 6\}$ when $n \geq 7$), however any regular digits set $D \subset c \cdot \mathbb{N}$ is the subset of some uniform digit set $Y = \{0, c, 2c, \ldots, rc = d_m\}$.

10. An Analysis of Construction

We begin our analysis by examining the construction of C. For any closed interval $I = [a, b]$ we will refer to the *refinement* of I,

$$\bigcup_{j=1}^{m} \left[a + \frac{d_j}{n}(b-a), a + \frac{d_j + 1}{n}(b-a)\right].$$

Let $k = 1$ and $C_0 = [0, 1]$. Then $C_1 = \bigcup_{j=1}^{m} f_j(C_0)$ from Section 8. Since f_j is a linear, continuous contraction mapping for all $1 \leq j \leq m$, then $f_j(C_0)$ is the closed interval $f_j(C_0) = \left[\frac{d_j}{n}, \frac{1+d_j}{n}\right] = \left[\frac{d_j}{n}, \frac{d_j}{n} + \frac{1}{n}\right]$. Hence, $C_1 = \bigcup_{1 \leq j \leq m} \left[\frac{d_j}{n}, \frac{d_j}{n} + \frac{1}{n}\right]$ is the refinement of the interval $C_0 = [0, 1]$ and consists of m disjoint closed intervals of length $\frac{1}{n}$.

Let $k > 1$. Suppose $C_k = \{0.t_1 t_2 \cdots \mid t_j \in D \text{ for } j \leq k\}$ consists of m^k closed intervals of length $\frac{1}{n^k}$. Let $J \subset C_k$ be an arbitrary closed interval of length $\frac{1}{n^k}$. Then $J = \left[\sum_{i=0}^{k} \frac{t_i}{n^i}, \sum_{i=0}^{k} \frac{t_i}{n^i} + 1\right]$ for some $t_1, t_2, \ldots, t_k \in D$.

Choose $I_j \subset C_k$ such that $I_j = \left[\left(\sum_{i=0}^{k-1} \frac{t_{i+1}}{n^i}\right) + \frac{d_j}{n^k}, \sum_{i=0}^{k-1} \frac{t_{i+1}}{n^i} + \frac{d_j}{n^k} + 1\right]$ for each $1 \leq j \leq m$. Since $t_1 = d_h$ for some $1 \leq h \leq m$, then

$$f_h\left(I_j\right) = \left[\left(\sum_{i=0}^{k} \frac{t_i}{n^i}\right) + \frac{d_j}{n^{k+1}}, \left(\sum_{i=0}^{k} \frac{t_i}{n^i}\right) + \frac{d_j}{n^{k+1}} + 1\right].$$

Thus, $\bigcup_{j=1}^{m} f_j\left(I_h\right)$ is the subset of C_{k+1} which is also the refinement of J. Since J was chosen to be arbitrary, then C_{k+1} is generated by refining each closed interval of length $\frac{1}{n^k}$ contained in C_k. Furthermore, each $J \subset C_k$ refines to m distinct subintervals so that C_{k+1} consists of m^{k+1} disjoint closed intervals of length $\frac{1}{n^{k+1}}$.

For an arbitrary deleted digits set, $C = \bigcap\limits_{k=0}^{\infty} C_k$ and similarly $C + t = \left(\bigcap\limits_{k=0}^{\infty} C_k\right) + t = \bigcap\limits_{k=0}^{\infty} \left(C_k + t\right)$. Thus,

$$C \cap (C + t) = \bigcap_{k=0}^{\infty} \left(C_k \cap \left(C_k + t\right)\right) \tag{10.1}$$

In order to examine $C_k \cap \left(C_k + t\right)$ we proceed by investigating the related sets $C_k \cap \left(C_k + \lfloor t \rfloor_k\right)$. Given the construction of C, all intervals of C_k (and its translate $C_k + \lfloor t \rfloor_k$) will have similar properties: intervals are closed with length $\frac{1}{n^k}$ where the left endpoint is $\frac{h}{n^k}$ for some $h \in \mathbb{N}$. In general, for any $k \in \mathbb{N}$ and any $h \in \mathbb{Z}$ we say that a closed interval $I = \left[\frac{h}{n^k}, \frac{h+1}{n^k}\right]$ is an *n-ary* interval. As n-ary intervals are the basis of our analysis, we will often use the terms interval and n-ary interval interchangeably, and mention specifically when a given interval is not n-ary.

We have shown that C_{k+1} is generated by refining each interval contained in C_k. Similarly, $C_{k+1} + \lfloor t \rfloor_{k+1}$ is generated by first refining each interval of $C_k + \lfloor t \rfloor_k$ and then translating the result by an additional $\frac{t_{k+1}}{n^k}$. We say that $C_k \cap \left(C_k + \lfloor t \rfloor_k\right)$ *transitions* to $C_{k+1} \cap \left(C_{k+1} + \lfloor t \rfloor_{k+1}\right)$ by first generating the sets C_{k+1} and $C_{k+1} + \lfloor t \rfloor_{k+1}$ and then taking the intersection.

In this way there are only four types of interactions between any two intervals $J = [\alpha, \beta] \subset C_k$ and $I = [a, b] \subset C_k + \lfloor t \rfloor_k$. We will examine the interactions of any subintervals of I and J after transitioning to $C_{k+1} \cap \left(C_{k+1} + \lfloor t \rfloor_{k+1}\right)$.

10.1. **The empty case $I \cap J = \varnothing$:** Suppose $b < \alpha$ or $a > \beta$. Since I and J are both n-ary intervals, the distance between I and J is $\min\{|\alpha - b|, |a - \beta|\} \geq \frac{1}{n^k}$. Since $\frac{t_{k+1}}{n^{k+1}} < \frac{1}{n^k}$, then regardless of the value of t_{k+1}, the refined subintervals of I are not translated by an amount large enough to intersect subintervals of J (or subintervals of I are translated to be farther from J). In either case, the intersections of subintervals of I and J are also empty in $C_{k+1} \cap \left(C_{k+1} + \lfloor t \rfloor_{k+1}\right)$.

10.2. **The potentially empty case $I - \frac{1}{n^k} = J$:** Figure 10.1 depicts an arbitrary potentially empty case before translating by $\frac{t_{k+1}}{n^k}$. The figure shows the refinement of J on top and the refinement of I on bottom. Retained subintervals are denoted as black bands, while deleted subintervals are shown as a thin black line for visual spacing purposes. All n-ary intervals other than I and J have also been removed for clarity.

If $t_{k+1} = 0$ and $\frac{d_m}{n-1} = 1$, then we merely refine both I and J without further translating I. Since $\frac{d_m}{n-1} = 1$ implies $n - 1 = d_m \in D$, then $\beta \in f_m(J)$. Furthermore, $a \in f_1(I)$ so that we maintain the potentially empty case. If $t_{k+1} \neq 0$, then a is not an element in the refinement and translation of I and the intersection is empty. Similarly, if $\frac{d_m}{n-1} < 1$, then β is not an element in the refinement of J and the intersection is empty.

We say that an n-ary interval $J \subset C_k$ is in the *potentially empty case* if there exists an n-ary interval $I \subset C_k + \lfloor t \rfloor_k$ such that $I - \frac{1}{3^k} = J$.

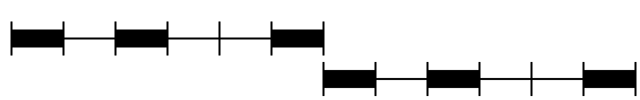

FIGURE 10.1. A potentially empty case for $D = \{0, 2, 5\}$ when $n = 6$.

Remark 10.1. Both the potentially empty and empty cases are said to be *irrecoverable*. That is, once one of these cases occurs we will only have trivial intersection of any subintervals in the transition to step $k+1$. The potentially empty case contains at most a finite number of isolated points at step k which vanish on any subsequent transition such that $t_{k+h} > 0$. The separation condition guarantees that distinct intervals in C_k are a minimum of $\frac{1}{n^k}$ apart and $t - \lfloor t \rfloor_k = \sum_{j=k+1}^{\infty} \frac{t_j}{n^j} < \frac{1}{n^k}$ implies that any further translations of I do not intersect any other intervals in C_k. For these reasons, we are only interested in the remaining three cases.

10.3. **The interval case $I = J$ and the potential interval case $I + \frac{1}{n^k} = J$:** In general, we say that an n-ary interval $J \subset C_k$ of length $\frac{1}{n^k}$ is in the *interval case* if there exists an n-ary interval $I \subset C_k + \lfloor t \rfloor_k$ such that $I = J$. Similarly, we say that J is in the *potential interval case* if $I + \frac{1}{n^k} = J$.

The set $C_k \cap (C_k + \lfloor t \rfloor_k)$ is in the *interval case* if there exists at least one $J \subset C_k$ in the interval case and for any $I \cap J \neq \varnothing$, I and J are in the interval case or potentially empty case. Similarly, $C_k \cap (C_k + \lfloor t \rfloor_k)$ is in the *potential interval case* if there exists at least one $J \subset C_k$ in the potential interval case and for any $I \cap J \neq \varnothing$, I and J are in the potential interval case or potentially empty case.

Figure 10.2 depicts an arbitrary interval case after refining and translating by an additional $\frac{3}{n^{k+1}}$. The refinement of J is on top, while the refinement and translation of I is on bottom.

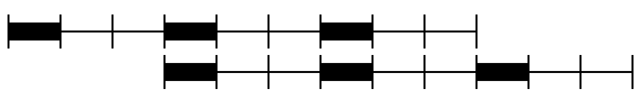

FIGURE 10.2. An interval case translated by $\frac{3}{n^{k+1}}$ when $D = \{0, 3, 6\}$ and $n = 9$.

In Section 9 we defined $\Delta = D - D$. We now show that Δ^+ contains all of the possible digits t_k which preserve the interval case in the transition to step $k + 1$.

Lemma 10.2. *If $C_k \cap (C_k + \lfloor t \rfloor_k)$ contains at least one n-ary interval of length $\frac{1}{n^k}$ and $t_{k+1} \in \Delta^+$, then the transition of this interval to $C_{k+1} \cap \left(C_{k+1} + \lfloor t \rfloor_{k+1}\right)$ will contain a distinct n-ary interval of length $\frac{1}{n^{k+1}}$ for each element of $D \cap (D + t_{k+1})$.*

Proof. Let $t_{k+1} = (d_i - d_j) \in \Delta^+$ be arbitrary for some $1 \leq j \leq i \leq m$. Let $I = J$ be an n-ary interval contained in $C_k \cap (C_k + \lfloor t \rfloor_k)$ where $J \subset C_k$ and $I \subset C_k + \lfloor t \rfloor_k$. By construction of C_k, $I = J = \left[\frac{h}{n^k}, \frac{h}{n^k} + \frac{1}{n^k}\right]$ for some $h \in \mathbb{N}$.

For each $1 \leq p \leq m$, refine I and J into disjoint subintervals

$$I_p = \left[\frac{h}{n^k} + \frac{d_p}{n^{k+1}}, \frac{h}{n^k} + \frac{d_p + 1}{n^{k+1}}\right] \subset (C_{k+1} + \lfloor t \rfloor_k)$$
$$J_p = \left[\frac{h}{n^k} + \frac{d_p}{n^{k+1}}, \frac{h}{n^k} + \frac{d_p + 1}{n^{k+1}}\right] \subset C_{k+1}.$$

Furthermore, $I_j + \frac{t_{k+1}}{n^{k+1}} \subset \left(C_{k+1} + \lfloor t \rfloor_{k+1}\right)$ such that

$$\begin{aligned} I_j + \frac{t_{k+1}}{n^{k+1}} &= \left[\frac{hn + d_j + t_{k+1}}{n^{k+1}}, \frac{hn + d_j + t_{k+1} + 1}{n^{k+1}}\right] \\ &= \left[\frac{hn + d_i}{n^{k+1}}, \frac{hn + d_i + 1}{n^{k+1}}\right] \\ &= J_i. \end{aligned}$$

Thus, $I_j + \frac{t_{k+1}}{n^{k+1}} = J_i$ is an n-ary interval of length $\frac{1}{n^{k+1}}$ contained in $C_{k+1} \cap \left(C_{k+1} + \lfloor t \rfloor_{k+1}\right)$.

Since d_i, d_j are arbitrary, then each i such that $d_i = d_j + t_{k+1}$ generates a unique subinterval satisfying the above equation. More specifically, each interval case $I = J$ transitions to a distinct subinterval case corresponding to each element of $D \cap (D + t_{k+1})$. □

As the set Δ^+ preserves the interval case, the set $\Delta^+ - 1$ contains all possible digits t_k that transition an interval case to a potential interval case.

Lemma 10.3. *If $C_k \cap (C_k + \lfloor t \rfloor_k)$ contains at least one n-ary interval of length $\frac{1}{n^k}$ and $t_{k+1} \in \Delta^+ - 1$, then for each element of $D \cap (D + t_{k+1} + 1)$ the transition of this interval will contain a distinct pair of intervals $J \subset C_{k+1}$ and $I \subset \left(C_{k+1} + \lfloor t \rfloor_{k+1}\right)$ such that $I + \frac{1}{n^{k+1}} = J$.*

Proof. Let $t_{k+1} = (d_i - d_j - 1) \in (\Delta^+ - 1)$ be arbitrary for some $1 \leq j \leq i \leq m$. Let $I = J$ be intervals as defined in the proof of Lemma 10.2. Then $t_{k+1} + 1 \in \Delta^+$ and as shown in the proof of Lemma 10.2:

$$\begin{aligned} I_j + \frac{1 + t_{k+1}}{n^{k+1}} &= \left[\frac{hn + d_j + t_{k+1} + 1}{n^{k+1}}, \frac{hn + d_j + t_{k+1} + 2}{n^{k+1}}\right] \\ &= \left[\frac{hn + d_i}{n^{k+1}}, \frac{hn + d_i + 1}{n^{k+1}}\right] \\ &= J_i. \end{aligned}$$

Therefore, $I_j + \frac{t_{k+1}}{n^{k+1}}$ and J_i are the desired intervals.

Since d_i, d_j are arbitrary, then each i such that $d_i = d_j + t_{k+1} + 1$ generates a unique subinterval pair satisfying the above equation. More specifically, each interval case $I = J$ transitions to a distinct subinterval $I_j + \frac{1+t_{k+1}}{n^{k+1}} = J_i$ corresponding to each element of $D \cap (D + t_{k+1} + 1)$. $\square$

Corollary 10.4. *If $C_k \cap (C_k + \lfloor t \rfloor_k)$ contains at least one n-ary interval of length $\frac{1}{n^k}$ and $t_{k+1} \in \{0, 1, \ldots, n-1\} \setminus (\Delta^+ \cup (\Delta^+ - 1))$, then the transition of this interval to $C_{k+1} \cap \left(C_{k+1} + \lfloor t \rfloor_{k+1}\right)$ is an irrecoverable case.*

Proof. Let $t_{k+1} \in \{0, 1, \ldots, n-1\} \setminus (\Delta^+ \cup (\Delta^+ - 1))$ so that $t_{k+1} \notin \Delta^+$ and $t_{k+1} \notin \Delta^+ - 1$. By the proof of Lemma 10.2,

$$\begin{aligned} I_j + \frac{t_{k+1}}{n^{k+1}} = J_i &\iff hn + d_j + t_{k+1} = hn + d_i \\ &\iff t_{k+1} = d_i - d_j. \end{aligned}$$

Thus, we cannot have an interval case since $t_{k+1} \notin \Delta^+$. Similarly, by the proof of Lemma 10.3,

$$\begin{aligned} I_j + \frac{1 + t_{k+1}}{n^{k+1}} = J_i &\iff hn + d_j + t_{k+1} + 1 = hn + d_i \\ &\iff t_{k+1} = d_i - d_j - 1. \end{aligned}$$

We cannot have a potential interval case since $t_{k+1} \notin \Delta^+ - 1$. Hence, for all $i, j \in \{1, 2, \ldots, m\}$, the intersections $I_j \cap J_i$ are either empty or potentially empty, which are irrecoverable by definition. $\square$

Lemma 10.2 and Lemma 10.3 require at least one pair of intervals $I = J$ contained in $C_k \cap (C_k + \lfloor t \rfloor_k)$. When $C_k \cap (C_k + \lfloor t \rfloor_k)$ is in the potential interval case we examine the set $n - \Delta^+$. Figure 10.3 depicts an arbitrary potential interval case before translating by $\frac{t_{k+1}}{n^k}$.

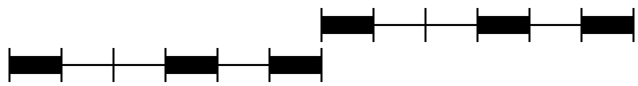

FIGURE 10.3. A potential interval case for $D = \{0, 3, 5\}$ when $n = 6$.

For any element $\delta \in \Delta^+$, $0 \le \delta \le n-1$ by definition of Δ^+ so that $1 \le (n - \delta) \le n$. The set $n - \Delta^+$ contains all of the possible digits t_k which transition a potential interval case to an interval case.

Lemma 10.5. *If $C_k \cap (C_k + \lfloor t \rfloor_k)$ contains at least one pair (I, J) of n-ary potential intervals $I + \frac{1}{n^k} = J$ and $t_{k+1} \in n - \Delta^+$, then the transition of this interval pair to $C_{k+1} \cap \left(C_{k+1} + \lfloor t \rfloor_{k+1}\right)$ will contain a distinct n-ary interval of length $\frac{1}{n^{k+1}}$ for each element of $D \cap (D + n - t_{k+1})$.*

Proof. Let $t_{k+1} = n - (d_i - d_j) \in (n - \Delta^+)$ be arbitrary for some $1 \le j \le i \le m$. Let $J \subset C_k$ and $I \subset C_k + \lfloor t \rfloor_k$ be n-ary intervals such that $I + \frac{1}{n^k} = J$. Thus, for some $h \in \mathbb{N}$, $I = \left[\frac{h-1}{n^k}, \frac{h}{n^k}\right]$ and $J = \left[\frac{h}{n^k}, \frac{h+1}{n^k}\right]$.

For each $1 \le p \le m$, refine I and J into disjoint subintervals

$$I_p = \left[\frac{h-1}{n^k} + \frac{d_p}{n^{k+1}}, \frac{h-1}{n^k} + \frac{d_p + 1}{n^{k+1}}\right] \subset (C_{k+1} + \lfloor t \rfloor_k)$$
$$J_p = \left[\frac{h}{n^k} + \frac{d_p}{n^{k+1}}, \frac{h}{n^k} + \frac{d_p + 1}{n^{k+1}}\right] \subset C_{k+1}.$$

Furthermore, $I_i + \frac{t_{k+1}}{n^{k+1}} \subset \left(C_{k+1} + \lfloor t \rfloor_{k+1}\right)$ such that

$$\begin{aligned} I_i + \frac{t_{k+1}}{n^{k+1}} &= \left[\frac{hn - n + d_i + t_{k+1}}{n^{k+1}}, \frac{hn - n + d_i + t_{k+1} + 1}{n^{k+1}}\right] \\ &= \left[\frac{hn + d_j}{n^{k+1}}, \frac{hn + d_j + 1}{n^{k+1}}\right] \\ &= J_j. \end{aligned}$$

Thus, $I_i + \frac{t_{k+1}}{n^{k+1}} = J_j$ is an interval of length $\frac{1}{n^{k+1}}$ contained in $C_{k+1} \cap \left(C_{k+1} + \lfloor t \rfloor_{k+1}\right)$.

Since d_i, d_j are arbitrary, then each i such that $d_i = d_j + n - t_{k+1}$ generates a unique subinterval satisfying the above equation. More specifically, each potential interval case

$I + \frac{1}{n^k} = J$ transitions to a distinct subinterval $I_i + \frac{t_{k+1}}{n^{k+1}} = J_j$ corresponding to each element of $D \cap (D + n - t_{k+1})$. $\square$

Finally, the set $n - \Delta^+ - 1$ contains all of the possible digits t_k which preserve the potential interval case in the transition.

Lemma 10.6. *If $C_k \cap (C_k + \lfloor t \rfloor_k)$ contains at least one pair (I, J) of n-ary potential intervals of length $\frac{1}{n^k}$ where $I + \frac{1}{n^k} = J$ and $t_{k+1} \in n - \Delta^+ - 1$ then for each element of $D \cap (D + n - (t_{k+1} + 1))$ the transition of this interval pair to $C_{k+1} \cap \left(C_{k+1} + \lfloor t \rfloor_{k+1}\right)$ will contain a distinct interval pair (I_i, J_j) of length $\frac{1}{n^{k+1}}$ with $I_i + \frac{1}{n^{k+1}} = J_j$.*

Proof. Let $t_{k+1} = n - (d_i - d_j) - 1 \in n - \Delta^+ - 1$ be arbitrary for some $1 \le i \le j \le m$. Let $I + \frac{1}{n^k} = J$ be intervals as defined in the proof of Lemma 10.5. Then $t_{k+1} + 1 \in n - \Delta^+$ and as shown in Lemma 10.5,

$$\begin{aligned} I_i + \frac{1 + t_{k+1}}{n^{k+1}} &= \left[\frac{hn + d_j}{n^{k+1}}, \frac{hn + d_j + 1}{n^{k+1}}\right] \\ &= J_j. \end{aligned}$$

Therefore, $I_i = \left[\frac{hn+d_j-1}{n^{k+1}}, \frac{hn+d_j}{n^{k+1}}\right]$ so that I_i and J_j are the desired intervals.

Since d_i, d_j are arbitrary, then each i such that $d_i = d_j + n - (t_{k+1} + 1)$ generates a unique subinterval satisfying the equation above. More specifically, each potential interval case $I + \frac{1}{n^k} = J$ transitions to a distinct potential subinterval $I_i + \frac{1+t_{k+1}}{n^{k+1}} = J_j$ corresponding to each element of $D \cap (D + n - (t_{k+1} + 1))$. $\square$

Corollary 10.7. *If $C_k \cap (C_k + \lfloor t \rfloor_k)$ contains a pair of n-ary potential intervals (I, J) of length $\frac{1}{n^k}$ such that $I + \frac{1}{n^k} = J$ and $t_{k+1} \in \{0, 1, \ldots, n-1\} \setminus ((n - \Delta^+) \cup (n - \Delta^+ - 1))$, then the transition of this pair of intervals to $C_{k+1} \cap \left(C_{k+1} + \lfloor t \rfloor_{k+1}\right)$ is an irrecoverable case.*

Proof. Let $t_{k+1} \in \{0, 1, \ldots, n-1\} \setminus ((n - \Delta^+) \cup (n - 1 - \Delta^+))$ so that $t_{k+1} \notin n - \Delta^+$ and $t_{k+1} \notin n - \Delta^+ - 1$. By the proof of Lemma 10.5,

$$\begin{aligned} I_i + \frac{t_{k+1}}{n^{k+1}} = J_j &\iff hn - n + d_i + t_{k+1} = hn + d_j \\ &\iff t_{k+1} = n - (d_i - d_j)\,. \end{aligned}$$

Thus we cannot have an interval case since $t_{k+1} \notin n - \Delta^+$. Similarly, by the proof of Lemma 10.6,

$$\begin{aligned} I_i + \frac{1 + t_{k+1}}{n^{k+1}} = J_j &\iff hn - n + d_i + t_{k+1} + 1 = hn + d_j \\ &\iff t_{k+1} = n - (d_i - d_j) - 1. \end{aligned}$$

We cannot have a potential interval case since $t_{k+1} \notin n - 1 - \Delta^+$. Hence for all $i, j \in \{1, 2, \ldots, m\}$ the intersections $I_i \cap J_j$ are either empty or potentially empty, which are irrecoverable by definition. $\square$

10.4. **The simultaneous case:** We say that $C_k \cap (C_k + \lfloor t \rfloor_k)$ is in the *simultaneous case* if there exist distinct n-ary intervals $J_I, J_P \subset C_k$ such that J_I is in the interval case and J_P is in the potential interval case. Figure 10.4 depicts an arbitrary interval case transitioning to the simultaneous case. The refinement of J is on top and refinement of I is translated by $\frac{t_{k+1}}{6^{k+1}} = \frac{2}{6^{k+1}}$ below. For clarity, the subinterval J_I is depicted in red while the subinterval J_P is depicted in yellow.

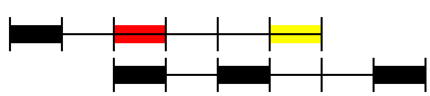

FIGURE 10.4. An example of the simultaneous case for $D = \{0, 2, 5\}$ when $n = 6$.

Remark 10.8. The simultaneous case can only describe the set $C_k \cap (C_k + \lfloor t \rfloor_k)$ for $k \in \mathbb{N}$. If an n-ary interval $J \subset C_k$ is in the simultaneous case, then there necessarily exist intervals I and K in $C_k + \lfloor t \rfloor_k$ such that $I + \frac{1}{n^k} = J$ and $K = J$. However, this implies that $I + \frac{1}{n^k} = K$

and thus $d_j + 1 = d_{j+1}$ for some $1 \leq j \leq m$, which is not possible due to the separation condition on D.

Since the deleted digits Cantor set is arbitrary, then for any intersection $C_k \cap (C_k + \lfloor t \rfloor_k)$ we have the following possibilities: the interval case, the potential interval case, the irrecoverable case, or the simultaneous case. The previous Lemmas of this section completely determine the transitions of interval and potential interval cases. Furthermore, irrecoverable cases will transition to irrecoverable cases by Remark 10.1, so it remains to discuss the simultaneous case.

By definition, $C_k \cap (C_k + \lfloor t \rfloor_k)$ is in the simultaneous case if the set contains both interval and potential interval cases. Since Lemma 10.2 and Lemma 10.5 do not impose conditions on D or n, it is possible that $\Delta^+ \cap (\Delta^+ - 1)$ is nonempty. Thus, if $C_k \cap (C_k + \lfloor t \rfloor_k)$ is in the interval case and $t_{k+1} \in \Delta^+ \cap (\Delta^+ - 1)$, then the transition to step $k+1$ contains both intervals and potential intervals. Hence, we can combine the results of these four Lemmas to investigate the simultaneous case.

11. Counting Intervals

We know that intervals contained in $C_k \cap (C_k + \lfloor t \rfloor_k)$ have length $\frac{1}{n^k}$. Since we assume that the base n expansion of t does not terminate in repeating $n-1$ digits, then for any $k \in \mathbb{N}$

$$0 \leq t - \lfloor t \rfloor_k < \frac{1}{n^k}.$$

Let $J \subset C_k$ and $I \subset C_k + \lfloor t \rfloor_k$ be n-ary intervals of length $\frac{1}{n^k}$. If $I = J$ is an interval of $C_k \cap (C_k + \lfloor t \rfloor_k)$, then $I + (t - \lfloor t \rfloor_k)$ is a subset of $C_k + t$ such that $(I + (t - \lfloor t \rfloor_k)) \cap J$ is an interval of length $\frac{1}{n^k} - (t - \lfloor t \rfloor_k) > 0$. Thus, further translation by the factor $t - \lfloor t \rfloor_k$ may reduce the length of existing intervals, but will not affect the total number of intervals. This means that any n-ary interval of $C_k \cap (C_k + \lfloor t \rfloor_k)$ will correspond to an interval in $C_k \cap (C_k + t)$.

By a similar argument, if $I + \frac{1}{n^k} = J$ is a potential interval of $C_k \cap (C_k + \lfloor t \rfloor_k)$, then $(I + (t - \lfloor t \rfloor_k)) \cap J$ is a closed, nonempty subset of $C_k \cap (C_k + t)$. If $t = \lfloor t \rfloor_k$ then there is no further translation and $(I + (t - \lfloor t \rfloor_k)) \cap J = \{\alpha\}$ is an isolated point. However, if there exists some $h > k$ where $t_h > 0$ then potential intervals of $C_k \cap (C_k + \lfloor t \rfloor_k)$ will be expanded into closed intervals of length $t - \lfloor t \rfloor_k > 0$ in $C_k \cap (C_k + t)$. This means that potential intervals in $C_k \cap (C_k + \lfloor t \rfloor_k)$ will also correspond to intervals in $C_k \cap (C_k + t)$.

Therefore, we can count the number of closed intervals contained in $C_k \cap (C_k + t)$ for any t by analyzing the related sets $C_k \cap (C_k + \lfloor t \rfloor_k)$. We will show in Section 12 that potentially empty cases to not affect the dimension of $C \cap (C + t)$, and so we are only interested in counting the recoverable cases of $C_k \cap (C_k + \lfloor t \rfloor_k)$.

Section 10 gives a complete description of the transition from $C_k \cap (C_k + \lfloor t \rfloor_k)$ to $C_{k+1} \cap \left(C_{k+1} + \lfloor t \rfloor_{k+1}\right)$. Since $C_0 = [0, 1]$, then we begin in the interval case and can examine the transitions using the sequence (t_k). For clarity of notation, if R and S are arbitrary sets, we will say $h \in R$ *only* whenever $h \in R \setminus S$ (where $\setminus$ denotes set subtraction). Let $i = \sqrt{-1}$ and define the function $\xi : \{0, \pm 1, i\} \times \{0, 1, ..., n-1\} \to \{0, \pm 1, \pm i\}$ based on the results of the key Lemmas in Section 10:[1]

[1] Strictly speaking, $-1 \in \left(\Delta^+ - 1\right)$ and $n \in \left(n - \Delta^+\right)$ are outside the domain of ξ. We ignore these possibilities since $0 \leq t_k \leq n-1$ by Lemma 2.1.

$$\xi(0,h) := 0$$

$$\xi(1,h) := \begin{cases} 1 & \text{for } h \in \Delta^{+} \text{ only} \\ -1 & \text{for } h \in (\Delta^{+}-1) \text{ only} \\ i & \text{for } h \in \Delta^{+} \cap (\Delta^{+}-1) \\ 0 & \textit{otherwise} \end{cases}$$

$$\xi(-1,h) := \begin{cases} -1 & \text{for } h \in (n-\Delta^{+}) \text{ only} \\ 1 & \text{for } h \in (n-\Delta^{+}-1) \text{ only} \\ -i & \text{for } h \in (n-\Delta^{+}) \cap (n-\Delta^{+}-1) \\ 0 & \textit{otherwise} \end{cases}$$

$$\xi(i,h) := \begin{cases} -i & \text{for } h \in [\Delta^{+} \cup (n-\Delta^{+})] \text{ only} \\ i & \text{for } h \in [(\Delta^{+}-1) \cup (n-\Delta^{+}-1)] \text{ only} \\ 1 & \text{for } h \in [\Delta^{+} \cup (n-\Delta^{+})] \cap [(\Delta^{+}-1) \cup (n-\Delta^{+}-1)] \\ 0 & \textit{otherwise.} \end{cases}$$

Since $\xi(z,h)$ does not depend on t, then we can determine the function completely based on D and n. However, as shown above, the transitions are determined based on values of t_{k+1}. For any $t = 0.t_1 t_2 \cdots \in [0,1]$, define $\sigma_k : \mathbb{R} \to \{0, \pm 1, i\}$ such that for all $k \in \mathbb{Z}^{+}$:

$$\sigma_0(t) := 1$$

$$\sigma_{k+1}(t) := \xi(\sigma_k(t), t_{k+1}) \cdot \sigma_k(t).$$

From these definitions it is possible for $\xi(i,h) = -i$, however the product has been carefully chosen so that $\sigma_k(t) \in \{0, \pm 1, i\}$ for all k. Additionally, the function ξ has been selected according to the Lemmas of Section 10 so that $\sigma_k(t)$ holds the following property:

Proposition 11.1. *The value $\sigma_k(t) = 1$ indicates that $C_k \cap (C_k + \lfloor t \rfloor_k)$ is in the interval case. Similarly, $\sigma_k(t) = -1$ indicates the potential interval case, $\sigma_k(t) = i$ indicates the simultaneous case, and $\sigma_k(t) = 0$ indicates the irrecoverable case.*

Proof. For $k = 0$, $\sigma_0(t) = 1$ and $C_0 \cap (C + \lfloor t \rfloor_0) = C_0 = [0, 1]$ so that the claim is true. Suppose the claim is true for some k.

Suppose $\sigma_k(t) = 1$. By Lemma 10.2 $t_{k+1} \in \Delta^+$ results in the interval case and $t_{k+1} \in (\Delta^+ - 1)$ results in the potential interval case by Lemma 10.3. Thus, $t_{k+1} \in \Delta^+ \cap (\Delta^+ - 1)$ results in both interval and potential interval cases. By Remark 10.1 any remaining values of t_{k+1} will result in an irrecoverable case.

Suppose $\sigma_k(t) = -1$. By Lemma 10.5 $t_{k+1} \in (n - \Delta^+)$ results in the interval case and $t_{k+1} \in (n - \Delta^+ - 1)$ results in the potential interval case by Lemma 10.6. Thus, $t_{k+1} \in (n - \Delta^+) \cap (n - \Delta^+ - 1)$ results in both interval and potential interval cases. By Remark 10.1 any remaining values of t_{k+1} will result in an irrecoverable case.

Suppose $\sigma_k(t) = i$. As this is a combination of the previous two cases, then by the cited Lemmas we will have the interval case for $t_{k+1} \in \Delta^+ \cup (n - \Delta^+)$ and the potential interval case for $t_{k+1} \in (\Delta^+ - 1) \cup (n - \Delta^+ - 1)$. Similarly, we can maintain the simultaneous case if t_{k+1} is contained in both sets.

Suppose $\sigma_k(t) = 0$. By Remark 10.1, once $C_k \cap (C_k + \lfloor t \rfloor_k)$ is in the irrecoverable case for some k, the remaining choices t_j for $j > k$ will always result in a finite or empty set $C_j \cap \left(C_j + \lfloor t \rfloor_j\right)$. Thus, our choice $\xi(0, t_j) = 0$ always results in the irrecoverable case for all $j > k$.

Thus the claim is true for $k + 1$. By induction, the claim is true for all $k \in \mathbb{N}$. $\square$

The set $\Delta^+ \cap (\Delta^+ - 1)$ is empty if and only if $(n - \Delta^+) \cap (n - \Delta^+ - 1)$ is empty. Thus, if D and n are such that $\Delta^+ \cap (\Delta^+ - 1) = \varnothing$, then the simultaneous case $\sigma_k(t) = i$ cannot occur for any k.

For example, suppose C is regular. Then D is a subset of $Y = \{0, c, 2c, \ldots, d_m\}$ for some integer $c > 1$. Thus, $\Delta \subseteq Y - Y = (-Y) \cup Y$ so that $\Delta^+ \subseteq Y$ and $\Delta^+ - 1 \subseteq$

$\{-1, c-1, \ldots, d_m - 1\}$. Since $c(j+1) - cj = c > 1$ for all $cj \in Y$, then $D \subseteq Y$ implies $d_j - 1 \notin D$ for all $1 \leq j \leq m$.

Whenever $\Delta^+ \cap (\Delta^+ - 1) = \varnothing$, then the definition of ξ simplifies to:

$$\xi(0, t_{k+1}) = 0$$

$$\xi(1, t_{k+1}) = \begin{cases} 1 & \text{for } t_{k+1} \in \Delta^+ \text{ only} \\ -1 & \text{for } t_{k+1} \in (\Delta^+ - 1) \text{ only} \\ 0 & \textit{otherwise} \end{cases}$$

$$\xi(-1, t_{k+1}) = \begin{cases} 1 & \text{for } t_{k+1} \in (n - \Delta^+ - 1) \text{ only} \\ -1 & \text{for } t_{k+1} \in (n - \Delta^+) \text{ only} \\ 0 & \textit{otherwise.} \end{cases}$$

Furthermore, when C is uniform, $c = 2$, and $d_m = c(m-1) = n-1$, then $D = \Delta^+ = \{0, 2, \ldots, n-1\}$ and $\Delta^+ - 1 = \{-1, 1, 3, \ldots, n-2\}$ form a partition of $\{0, 1, \ldots, n-1\}$. Since C is also regular, then we eliminate both the simultaneous case $\sigma_k(t) = i$ and the irrecoverable case $\sigma_k(t) = 0$ so that the calculation simplifies to the σ_k definition given for the Middle Thirds Cantor set in Part 2.

Based on the analysis of Section 10 the set Δ^+ transitions interval cases to subinterval cases (denoted II) and the set $\Delta^+ - 1$ transitions interval cases to potential subinterval cases (denoted IP). In the same way, $n - \Delta^+$ transitions potential interval cases to subinterval cases (denoted PI) and $n - \Delta^+ - 1$ transitions potential interval cases to potential subinterval cases (denoted PP). Using this notation, define the following counting functions $\mathbb{Z} \to \mathbb{N}$:

$$\mu_{II}(h) = \#D \cap (D + h)$$

$$\mu_{IP}(h) = \#D \cap (D + h + 1)$$

$$\mu_{PI}(h) = \#D \cap (D + n - h)$$

$$\mu_{PP}(h) = \#D \cap (D + n - h - 1)\,. \tag{11.1}$$

According to Lemma 10.2, each interval at step k transitions to $\mu_{II}(t_{k+1})$ corresponding subintervals and by Lemma 10.5, each potential interval at step k transitions to $\mu_{PI}(t_{k+1})$ corresponding subintervals. According to Corollary 10.4, these are the only possibilities for intervals in $C_{k+1} \cap \left(C_{k+1} + \lfloor t \rfloor_{k+1}\right)$. Define I_k to be the number of intervals at step k, and P_k to be the number of potential intervals at step k. Then

$$I_{k+1} = I_k \cdot \mu_{II}(t_{k+1}) + P_k \cdot \mu_{PI}(t_{k+1}). \tag{11.2}$$

Similarly, according to Lemma 10.3 and Lemma 10.6, each interval and potential interval at step k transitions to $\mu_{IP}(t_{k+1})$ and $\mu_{PP}(t_{k+1})$ corresponding potential subintervals at step $k+1$, respectively. By Corollary 10.7, these are the only possibilities for potential intervals in $C_{k+1} \cap \left(C_{k+1} + \lfloor t \rfloor_{k+1}\right)$. Thus,

$$P_{k+1} = I_k \cdot \mu_{IP}(t_{k+1}) + P_k \cdot \mu_{PP}(t_{k+1}). \tag{11.3}$$

Define the column vector $\overrightarrow{\nu_k}(t) := \begin{bmatrix} I_k & P_k \end{bmatrix}^T$ for all $k \in \mathbb{N}$. Then we can combine Equation (11.2) with Equation (11.3) so that

$$\begin{gathered} \overrightarrow{\nu_{k+1}}(t) = \eta(t_{k+1}) \cdot \overrightarrow{\nu_k}(t) \\ \text{where } \eta(t_{k+1}) := \begin{bmatrix} \mu_{II}(t_{k+1}) & \mu_{PI}(t_{k+1}) \\ \mu_{IP}(t_{k+1}) & \mu_{PP}(t_{k+1}) \end{bmatrix}. \end{gathered} \tag{11.4}$$

It is important to note that the separation condition guarantees $(D+h) \cap (D+h+1) = \varnothing$ since $d_i - d_j \geq 2$. For the same reason, $(D+n-h) \cap (D+n-h-1) = \varnothing$. Hence, $D \cap (D+h)$ and $D \cap (D+h+1)$ are disjoint subsets of D so that for any h,

$$\mu_{II}(h) + \mu_{IP}(h) = \#D \cap (D+h) + \#D \cap (D+h+1) \leq \#D = m.$$

This implies that any subinterval in the transition will belong to at most one recoverable case. That is, a given n-ary interval J cannot be in both the interval and potential interval cases simultaneously due to the spacing of the interval digits. This is significant so that the functions μ_{II} and μ_{IP} cannot "double count" any subintervals in the transition to step $k+1$.

Similarly, $\mu_{PI}(h) + \mu_{PP}(h) = \#D \cap (D+n-h) + \#D \cap (D+n-h-1) \leq \#D = m$. As the original n-ary intervals in step k also cannot be in the simultaneous case, then μ_{II} and μ_{PI} cannot "double count" any subintervals in the transition to step $k+1$. Likewise, this is true for any combination of the four counting functions in Equation (11.4).

The matrices $\eta(h)$ are completely determined by the choice of D and n, hence all n possible matrices are determined independently of $t \in \mathbb{R}$. Furthermore, since $C_0 = [0,1]$, then $\overrightarrow{\nu_0}(t) = \overrightarrow{\nu_0} = \begin{bmatrix} 1 & 0 \end{bmatrix}^T$ for all t. By induction, we can expand the product in Equation (11.4) so that for any k,

$$\overrightarrow{\nu_k}(t) = \eta(t_k) \cdot \eta(t_{k-1}) \cdot \cdots \cdot \eta(t_1) \cdot \overrightarrow{\nu_0}. \tag{11.5}$$

Since $\overrightarrow{\nu_k}(t)$ counts both intervals and potential intervals exactly, then $\begin{bmatrix} 1 & 1 \end{bmatrix} \cdot \overrightarrow{\nu_k}(t) = I_k + P_k$ represents the total number of recoverable cases at step k. Define M_k to be the finite number of potentially empty cases at step k. Then $I_k + P_k + M_k$ is an upper bound for the number of n-ary intervals of length $\frac{1}{n^k}$ required to cover $C_k \cap (C_k + \lfloor t \rfloor_k)$. Based on the observation at the beginning of this section, $I_k + P_k + M_k$ is also an upper bound for the number of n-ary intervals of length $\frac{1}{n^k}$ required to cover $C_k \cap (C_k + t)$, but may not be the least upper bound.

Theorem 11.2. *Let $\mu = \mu_{II}$. Then the functions μ_{II}, μ_{IP}, μ_{PI}, and μ_{PP} can all be expressed in terms of μ. Futhermore, the collection of matrices $\mathcal{M} := \{\eta(h) \mid 0 \leq h \leq n-1\}$ holds a symmetric property such that*

$$\eta(h) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \Longleftrightarrow \qquad \eta((n-1)-h) = \begin{bmatrix} d & c \\ b & a \end{bmatrix}$$

$$\eta(h) = \begin{bmatrix} * & * \\ c & d \end{bmatrix} \qquad \Longleftrightarrow \qquad \eta(h+1) = \begin{bmatrix} c & d \\ * & * \end{bmatrix}.$$

Proof. Let $h \in \mathbb{Z}$ and $\mu = \mu_{II}$. Then by Equation (11.1):

$$\mu_{II}(h) = \mu(h)$$

$$\mu_{IP}(h) = \mu(h+1)$$

$$\mu_{PI}(h) = \mu(n-h)$$

$$\mu_{PP}(h) = \mu(n-(h+1)).$$

Thus, the matrix $\eta(h)$ can be determined by

$$\eta(h) = \begin{bmatrix} \mu(h) & \mu(n-h) \\ \mu(h+1) & \mu(n-(h+1)) \end{bmatrix}.$$

For any $0 \leq h \leq n-1$, this observation reveals that

$$\eta((n-1)-h) = \begin{bmatrix} \mu(n-(h+1)) & \mu(h+1) \\ \mu(n-h) & \mu(h) \end{bmatrix}$$

$$\text{and } \eta(h+1) = \begin{bmatrix} \mu(h+1) & \mu(n-(h+1)) \\ \mu(h+2) & \mu(n-(h+2)) \end{bmatrix}.$$

□

Corollary 11.3. *The counting matrices $\eta(h)$ are independent of the choice of t. Due to the symmetric property of $\mathcal{M}$, all n matrices are completely determined if the first (or last) $\left\lfloor \frac{n+1}{2} \right\rfloor$ matrices are known. That is, $\eta((n-1)-h)$ is uniquely determined whenever $\eta(h)$ is known and vice-versa.*

We will show in Section 12 that we are interested in cases where the sequence $I_k + P_k$ increases to ∞. For this reason, we are particularly interested in transitions which strictly increase the number of recoverable cases. Specifically, we want to show that $I_{k+1} + P_{k+1} > I_k + P_k$. It is important to identify the values of t_{k+1} which can result in an increase for an arbitrary deleted digits Cantor set C.

Proposition 11.4. *For any $n \geq 3$ and digit set D satisfying the separation condition,*

$$\eta(0) = \begin{bmatrix} m & 0 \\ 0 & \left\lfloor \frac{d_m}{n-1} \right\rfloor \end{bmatrix} \qquad \eta(n-1) = \begin{bmatrix} \left\lfloor \frac{d_m}{n-1} \right\rfloor & 0 \\ 0 & m \end{bmatrix}$$

$$\eta\left(d_m\right)=\begin{bmatrix} 1 & * \\ 0 & * \end{bmatrix} \qquad\qquad \eta\left(n-d_m\right)=\begin{bmatrix} * & 1 \\ * & 0 \end{bmatrix}.$$

Proof. Let D and $n \geq 3$ be given. By Theorem 11.2,

$$\eta\left(0\right)=\begin{bmatrix} \mu\left(0\right) & \mu\left(n\right) \\ \mu\left(1\right) & \mu\left(n-1\right) \end{bmatrix}.$$

We are able to calculate the matrix entries of $\eta\left(0\right)$ as follows: $\mu\left(0\right)=\#D\cap\left(D+0\right)=\#D=m$ and since $d_i<n$ for any $d_i\in D$, then $\mu\left(n\right)=\#D\cap\left(D+n\right)=\#\varnothing=0$. By the separation condition on D, $\mu\left(1\right)=\#D\cap\left(D+1\right)=\#\varnothing=0$.

Suppose $\frac{d_m}{n-1}=1$ so that $d_m=n-1$ is the largest element of D. Thus, $\mu\left(n-1\right)=\#D\cap\left(D+\left(n-1\right)\right)=\#\left\{n-1\right\}=1$. Otherwise, $\frac{d_m}{n-1}<1$ so that $d_i<n-1$ for all $d_i\in D$. Therefore, $\mu\left(n-1\right)=\#D\cap\left(D+\left(n-1\right)\right)=\#\varnothing=0$. In either case $\mu\left(n-1\right)=\left\lfloor\frac{d_m}{n-1}\right\rfloor$. Thus,

$$\eta\left(0\right)=\begin{bmatrix} m & 0 \\ 0 & \left\lfloor\frac{d_m}{n-1}\right\rfloor \end{bmatrix}.$$

By Theorem 11.2, $\eta\left(n-1\right)$ is determined by the symmetry of $\mathcal{M}$.

Similarly, we are able to calculate the matrix entries of $\eta\left(d_m\right)$ as follows: $\mu\left(d_m\right)=\#\left\{d_m\right\}=1$ and $\mu\left(d_m+1\right)=\#\varnothing=0$. Since $\mu\left(n-d_m\right)$ and $\mu\left(n-\left(d_m+1\right)\right)$ depend on both D and n, then

$$\eta\left(d_m\right)=\begin{bmatrix} 1 & * \\ 0 & * \end{bmatrix}.$$

The separation condition on D guarantees that $d_m-1\notin D$ so that $\mu\left(d_m-1\right)=\#\varnothing=0$. Thus, $\eta\left(d_m-1\right)=\begin{bmatrix} 0 & * \\ 1 & * \end{bmatrix}$ according to Theorem 11.2, and by the symmetry of $\mathcal{M}$

$$\eta\left(\left(n-1\right)-\left(d_m-1\right)\right)=\eta\left(n-d_m\right)=\begin{bmatrix} * & 1 \\ * & 0 \end{bmatrix}.$$

$\square$

While the matrices $\eta(0)$ and $\eta(n-1)$ can be determined in the general case, the remaining matrices $\eta(h)$ depend on the chosen set of digits D.

Since $m \geq 2$ then if $\sigma_k(t) = 1$ and $t_{k+1} = 0$ then $\overrightarrow{\nu_{k+1}}(t) = \overrightarrow{\nu_k}(t) \cdot \eta(0)$ and $I_{k+1} = m \cdot I_k$. Because no potential intervals exist at step k, then $P_{k+1} = P_k = 0$. Thus, $I_{k+1} + P_{k+1} = m \cdot (I_k + P_k)$ and we have an increase in the number of recoverable cases. Similarly, if $\sigma_k(t) = -1$ and $t_{k+1} = n-1$ then $\overrightarrow{\nu_{k+1}}(t) = \eta(n-1) \cdot \overrightarrow{\nu_k}(t)$. Since no interval cases exist at step k then $P_{k+1} = m \cdot P_k$ and we have an increase in the number of recoverable cases in the transition. Thus, Proposition 11.4 shows that for any D and n there are at least two digits which can result in $I_{k+1} + P_{k+1} > I_k + P_k$.

The remaining matrices $\eta(h)$ for $0 < h < n-1$ are dependent on D and n, and it is possible that 0 and $n-1$ are the only digits which can result in $I_{k+1} + P_{k+1} > I_k + P_k$. We say that a deleted digits Cantor set C is *sparse* if 0 and $n-1$ are the only digits which can result in an increase in the number of recoverable cases in the transition. When this occurs, $I_k + P_k \geq I_{k+1} + P_{k+1}$ for any remaining digits and we have the equivalent definition: C is sparse if for all $t \in [0,1]$ and any $0 < h < n-1$,

$$\begin{bmatrix} 1 & 1 \end{bmatrix} \cdot \overrightarrow{\nu_k}(t) \geq \begin{bmatrix} 1 & 1 \end{bmatrix} \cdot \eta(h) \cdot \overrightarrow{\nu_k}(t). \tag{11.6}$$

When covering $C_k \cap (C_k + t)$ we are only interested in intervals of $C_k \cap (C_k + \lfloor t \rfloor_k)$ which contain points in $C_k \cap (C_k + t)$, so it is important to identify these cases.

Proposition 11.5. *Let C be a deleted digits Cantor set such that $\mathcal{F} = [-1,1]$ and let $t \in \mathcal{F}^+$ be fixed. For any $J = \left[\frac{h}{n^k}, \frac{h+1}{n^k}\right] \subset C_k$, if there exists $I \subset C_k + \lfloor t \rfloor_k$ such that either $I = J$ or $I + \frac{1}{n^k} = J$ then the corresponding closed set $J \cap (I + (t - \lfloor t \rfloor_k)) \subset C_k \cap (C_k + t)$ contains at least one point of $C \cap (C + t)$.*

Proof. Fix $t \in \mathcal{F}^+ = [0,1]$ and let $J = \left[\frac{h}{n^k}, \frac{h+1}{n^k}\right] \subset C_k$ be arbitrary. Let $I \subset (C_k + \lfloor t \rfloor_k)$ be an interval such that either $I = J$ or $I + \frac{1}{n^k} = J$. Note that I is unique for any J in the interval case since n-ary intervals are at least $\frac{1}{n^k}$ apart. The proof is divided into two cases:

1. Suppose $I = J$. By definition, $0 \leq (t - \lfloor t \rfloor_k) \leq \frac{1}{n^k}$ and $(I + (t - \lfloor t \rfloor_k)) \cap J$ is nonempty from the analysis in Section 10.

Since $J \subset C_k$ is n-ary, then J refines to the small Cantor set $\frac{1}{n^k}C + h$ based on the construction of C. Let $J_0 = J$ and let J_{i+1} be the refinement of J_i for all $i \in \mathbb{N}$. Since $J_0 \subset C_k$ and C_{k+1} is the refinement of all intervals of C_k, then $J_i \subset C_{k+i}$ for all i. Thus, $n^k \cdot J_0 - h = [0,1] = C_0$ and since $n^k \cdot J_{i+1} - h$ is the refinement of $n^k \cdot J_i - h = C_i$, then $n^k \cdot J_i - h = C_i$ for all i. Therefore,

$$C = \bigcap_{i=0}^{\infty} \left(n^k \cdot J_i - h\right) = n^k \cdot \bigcap_{i=0}^{\infty} (J_i) - h.$$

Define $\widetilde{t} := n^k \left(t - \lfloor t \rfloor_k\right)$. Since $0 \leq \widetilde{t} \leq 1$ and $I = J$, then for refinements $I_i = J_i$,

$$C + \widetilde{t} = n^k \cdot \bigcap_{i=0}^{\infty} \left(I_i + \left(t - \lfloor t \rfloor_k\right)\right) - h.$$

Furthermore, because $\widetilde{t} \in \mathcal{F}^+ = [0,1]$, then $C \cap \left(C + n^k \left(t - \lfloor t \rfloor_k\right)\right) \neq \varnothing$ by definition of $\mathcal{F}$. Let $r \in C \cap \left(C + \widetilde{t}\right)$. We need to show that $\frac{r+h}{n^k} \in C \cap (C+t)$.

Then $r \in n^k \cdot \bigcap_{i=0}^{\infty} (J_i) - h$ such that $\frac{r+h}{n^k} \in \bigcap_{i=0}^{\infty} (J_i) \subset J$. Since $\bigcap_{i=0}^{\infty} (J_i)$ represents the refinement of J in the construction of C, then $\bigcap_{i=0}^{\infty} (J_i) \subset \bigcap_{i=0}^{\infty} (C_{k+i}) = C$ so that $\frac{r+h}{n^k} \in C$.

Similarly, $r \in n^k \cdot \bigcap_{i=0}^{\infty} \left(I_i + \left(t - \lfloor t \rfloor_k\right)\right) - h$ such that $\frac{r+h}{n^k} \in \bigcap_{i=0}^{\infty} \left(I_i + \left(t - \lfloor t \rfloor_k\right)\right)$. Since $\left(I_i + \left(t - \lfloor t \rfloor_k\right)\right)$ are refinements of $\left(I + \left(t - \lfloor t \rfloor_k\right)\right) \subset (C_k + t)$, then $\bigcap_{i=0}^{\infty} \left(I_i + \left(t - \lfloor t \rfloor_k\right)\right) \subset \bigcap_{i=0}^{\infty} (C_{k+i} + t) = (C+t)$ and $\frac{r+h}{n^k} \in (C+t)$.

Therefore, $\frac{r+h}{n^k} \in C \cap (C+t)$ such that $\frac{r+h}{n^k} \in J \cap \left(I + \left(t - \lfloor t \rfloor_k\right)\right)$. Hence, $J \cap \left(I + \left(t - \lfloor t \rfloor_k\right)\right)$ contains at least one point of $C \cap (C+t)$.

2. Suppose $I + \frac{1}{n^k} = J$. Then $-\frac{1}{n^k} \leq \left(t - \lfloor t \rfloor_k\right) - \frac{1}{n^k} \leq 0$ such that

$$I + \left(t - \lfloor t \rfloor_k\right) = I + \frac{1}{n^k} + \left[\left(t - \lfloor t \rfloor_k\right) - \frac{1}{n^k}\right].$$

Define $\widetilde{t} := n^k \cdot \left(t - \lfloor t \rfloor_k\right) - 1 \in \mathcal{F}^- = [-1,0]$. Thus, we can use the same argument above so that $J \cap \left(I + \left(t - \lfloor t \rfloor_k\right)\right)$ contains at least one point of $C \cap (C+t)$ by definition of $\mathcal{F}$. $\square$

12. Minkowski-Bouligand Dimension

Given a compact set K and $n, k \in \mathbb{N}$ let $N_k(K)$ denote the smallest number of closed intervals of length $\frac{1}{n^k}$ needed to cover K. The *upper Minkowski dimension* of K is

$$\overline{\dim}_M(K) = \limsup_{k\to\infty} \frac{\log\, N_k(K)}{\log n^k}.$$

Similarly, the *lower Minkowski dimension* is $\underline{\dim}_M(K) = \liminf_{k\to\infty} \frac{\log\, N_k(K)}{\log n^k}$. When the upper and lower Minkowski dimensions are equal then $\dim_M(K) = \overline{\dim}_M(K) = \underline{\dim}_M(K)$ is the *Minkowski dimension* of K. The Minkowski dimension is also referred to as the Minkowski-Bouligand dimension or box-counting dimension of K.

By construction of C, $N_0(C) = 1$ and $N_{k+1}(C) = m \cdot N_k(C)$ for all $k \in \mathbb{N}$. Thus, $N_k(C) = m^k$ for all k and

$$\dim_M(C) = \lim_{k\to\infty} \frac{\log\, N_k(C)}{\log\, n^k} = \lim_{k\to\infty} \frac{k \cdot \log m}{k \cdot \log n} = \frac{\log m}{\log n}.$$

Lemma 12.1. *For any* $t \in \mathbb{R}$,

$$0 \le \underline{\dim}_M(C \cap (C+t)) \le \overline{\dim}_M(C \cap (C+t)) \le \frac{\log m}{\log n}.$$

Proof. Let K be any closed subset of C. The result is trivial if K is empty, so assume $K \neq \varnothing$. Then any cover of C is automatically a cover of K and $1 \le N_k(K) \le N_k(C)$ for all $k \in \mathbb{N}$. Thus, $\frac{\log N_k(K)}{\log n^k} \le \frac{\log N_k(C)}{\log n^k}$ for all k so that

$$\overline{\dim}_M(K) = \limsup_{k\to\infty} \frac{\log\, N_k(K)}{\log n^k} \le \limsup_{k\to\infty} \frac{\log\, N_k(C)}{\log n^k} = \frac{\log m}{\log n}$$

Furthermore, since $1 \le N_k(K)$ for all k, then $N_k(K)$ is a nonnegative sequence of integers so that $0 \le \frac{\log\, N_k(K)}{\log n^k}$ for all k. Thus, $0 \le \liminf_{k\to\infty} \frac{\log\, N_k(K)}{\log n^k} = \underline{\dim}_M(K)$. It is by definition of the limits superior and inferior that $\liminf_{k\to\infty} a_k \le \limsup_{k\to\infty} a_k$ for any real sequence (a_k). $\square$

Lemma 12.2. *Let K be a compact subset of $\mathbb{R}$ and $A \subset \mathbb{R}$ a finite set. Then*

$$\overline{\dim}_M (K \cup A) = \overline{\dim}_M (K)$$

$$\underline{\dim}_M (K \cup A) = \underline{\dim}_M (K) .$$

Proof. The result is trivial if $A \subseteq K$. Without loss of generality, assume $A \cap K = \varnothing$ (otherwise $A \setminus (A \cap K)$ is our finite set of interest). We have two cases:

1. If K is also finite then $K \cup A$ is a finite set containing $r < \infty$ elements. Define

$$\varepsilon := \min \{|a-b| \mid a, b \in K \cup A, a \neq b\} > 0.$$

Choose $k \in \mathbb{N}$ such that $\frac{1}{n^k} < \varepsilon$. Then for all $h \geq k$, any closed interval J of length $\frac{1}{n^h} \leq \frac{1}{n^k} < \varepsilon$ is sufficiently small so that J contains at most one point of $K \cup A$. Hence, $N_h (K \cup A) = N_k (K \cup A) = r$ for any $h \geq k$. Thus,

$$0 \leq \limsup_{h\to\infty} \frac{\log N_h (K)}{\log n^h} \leq \limsup_{h\to\infty} \frac{\log N_h (K \cup A)}{\log n^h} = \limsup_{h\to\infty} \frac{\log r}{\log n^h} = 0.$$

Thus, $\overline{\dim}_M (K \cup A) = \overline{\dim}_M (K) = 0$. By Lemma 12.1,

$$0 \leq \liminf_{h\to\infty} \frac{\log N_h (K \cup A)}{\log n^h} \leq \overline{\dim}_M (K) = 0.$$

Hence, $\underline{\dim}_M (K \cup A) = \underline{\dim}_M (K) = 0$.

2. If K contains an infinite number of points, then $N_k (K) \nearrow \infty$. Since A is finite, define

$$V := \min \{|a-b| \mid a, b \in A, a \neq b\}$$

$$W := \inf \{|r-a| \mid r \in K, a \in A\} .$$

Choose $k \in \mathbb{N}$ such that $\frac{1}{n^k} \leq \min \{V, W\}$. Then for all $h \geq k$, any closed interval J of length $\frac{1}{n^h} \leq \frac{1}{n^k} < \varepsilon$ is sufficiently small so that J contains at most one point of A. Furthermore, any J containing a point of A is also sufficiently small so that $J \cap K = \varnothing$. Therefore,

$N_h(K\cup A) = N_h(K) + \#A$ so that

$$\begin{aligned}
\limsup_{h\to\infty}\frac{\log N_h(K\cup A)}{\log n^h} &= \limsup_{h\to\infty}\frac{\log\left(N_h(K)+\#A\right)}{\log n^h}\\
&= \limsup_{h\to\infty}\frac{\log N_h(K)+\log\left(1+\frac{\#A}{N_k(K)}\right)}{\log n^h}\\
&= \limsup_{h\to\infty}\frac{\log N_h(K)}{\log n^h}.
\end{aligned}$$

Thus, $\overline{\dim}_M(K\cup A) = \overline{\dim}_M(K)$. Similarly, $\liminf_{h\to\infty}\frac{\log N_h(K\cup A)}{\log n^h} = \liminf_{h\to\infty}\frac{\log N_h(K)}{\log n^h}$ so that $\underline{\dim}_M(K\cup A) = \underline{\dim}_M(K)$. ☐

In Section 10 we described the potentially empty case which may exist in $C_k\cap(C_k+\lfloor t\rfloor_k)$, but is not counted by the algorithm of Section 11. For any $k\in\mathbb{N}$, the total number of potentially empty cases M_k is at most finite. In the transition to step $k+1$ there are three possibilities:

(1) If $t=\lfloor t\rfloor_k$ and $d_m=n-1$, then $t_{k+h}=0$ for all $h\geq 0$ and the set $C_k\cap(C_k+t)$ contains M_k potentially empty points. Since $d_m=n-1$, then $C_{k+h}\cap(C_{k+h}+t)$ contains the same set of M_k potentially empty points according to Subsection 10.2. Because these points are a minimum distance of $\frac{1}{n^k}$ from other points or intervals, then by Lemma 12.2 these points do not affect the calculation of the upper and lower dimensions of $C\cap(C+t)$.

(2) If $t=\lfloor t\rfloor_k$ and $d_m<n-1$, then the finite set of potentially empty points at step k vanishes according to Subsection 10.2. Since $C_k+\lfloor t\rfloor_k=C_k+t$ does not translate further, then no new potentially empty points can appear in further transitions $h>k$.

(3) If $t-\lfloor t\rfloor_k>0$, then there exists $h\geq 0$ such that $t_{k+h}>0$. Thus, the finite set of M_k potentially empty points vanishes and is (possibly) replaced by a new set of M_{k+h} potentially empty points. By construction of $C_k\cap(C_k+\lfloor t\rfloor_k)$, potentially empty points at step $k+h$ cannot be generated from irrecoverable cases. Hence, the M_{k+h} isolated points arise from a potential interval or interval case in step k.

Since we are translating by $0 < t - \lfloor t \rfloor_k < \frac{1}{n^k}$, the new potentially empty points at step $k+h$ are already covered by the natural covering of $C_k \cap (C_k + \lfloor t \rfloor_k)$.

In all of the above cases, the isolated potentially empty points do not affect the Minkowski dimension of the set. Specifically, whenever $t = \lfloor t \rfloor_k$ for some k, we are able to completely determine the Minkowski dimension of the set $C \cap (C + t)$.

Theorem 12.3. *Let D and n be given such that D satisfies the separation condition. If there exists $k \in \mathbb{N}$ such that $t = \lfloor t \rfloor_k$, then*

$$\dim_M (C \cap (C+t)) = \begin{cases} \frac{\log(m)}{\log(n)} & \text{if } \sigma_k(t) \in \{1, i\} \\ 0 & \text{if } \sigma_k(t) \in \{-1, 0\}. \end{cases}$$

Proof. Let $t \in [0,1]$ be given such that $t = \lfloor t \rfloor_k$ for some k. Without loss of generality, let k be the minimal element of $\{h \mid (t - \lfloor t \rfloor_h) = 0\}$. Thus, $t - \lfloor t \rfloor_k = \sum_{h=k+1}^{\infty} \frac{t_h}{n^h} = 0$ so that $t_h = 0$ for all $h \geq k$. Therefore,

$$\eta(t_h) = \begin{bmatrix} m & 0 \\ 0 & \left\lfloor \frac{d_m}{n-1} \right\rfloor \end{bmatrix} \; \forall h \geq k$$

$$\text{and } \overrightarrow{\nu_h}(t) = \eta(h) \cdots \eta(k+1) \cdot \overrightarrow{\nu_k}(t)$$

$$= \begin{bmatrix} m^{h-k} \cdot I_k \\ \left\lfloor \frac{d_m}{n-1} \right\rfloor^{h-k} \cdot P_k \end{bmatrix}.$$

Note that $C_h \cap (C_h + t) = C_h \cap (C_h + \lfloor t \rfloor_h)$ for all $h \geq k$. Since potentially empty cases will either remain isolated points or vanish completely, then $M_{k+1} = M_h = M$ for all $h > k$. Similarly, $\left\lfloor \frac{d_m}{n-1} \right\rfloor \in \{0,1\}$ so that $P_h = P$ for all $h > k$. Define

$$A := \{x \mid x \text{ is an isolated point of } C_h \cap (C_h + t) \text{ for all } h \geq k\}.$$

Then $\#A = P + M$.

Let $h > k$ be arbitrary. Since $C_h \cap (C_h + t) = C_h \cap (C_h + \lfloor t \rfloor_h)$, then $C_h \cap (C_h + t)$ contains n-ary intervals of length $\frac{1}{n^h}$. Define the collection

$$K_h := \{J \mid J \text{ an } n\text{-ary interval of } C_h \cap (C_h + t)\}.$$

Thus, K_h contains I_h distinct intervals J of length $\frac{1}{n^h}$ and is trivially covered by these intervals.

Furthermore, isolated points of A can be covered by a finite number of additional intervals $r < \infty$. Since potential intervals and potentially empty points are at least $\frac{1}{n^h}$ apart, then $0 \leq r \leq \#A$. Since $C_h \cap (C_h + t)$ is composed only of interval cases, potential interval cases, and potentially empty cases, then $C_h \cap (C_h + t) = K_h \cup A$. By Lemma 12.2,

$$\overline{\dim}_M (C \cap (C + t)) = \limsup_{h \to \infty} \frac{\log\ N_h (K_h)}{\log n^h}$$
$$\underline{\dim}_M (C \cap (C + t)) = \liminf_{h \to \infty} \frac{\log\ N_h (K_h)}{\log n^h}.$$

Note that $N_h (K_h)$ represents the smallest number of intervals of length $\frac{1}{n^h}$ required to cover K_h. Since K_h consists of I_h distinct n-ary intervals, then $N_h (K_h) = I_h = m^{h-k} \cdot I_k$.

Suppose $\sigma_k (t) \in \{1, i\}$. Then $C_k \cap (C_k + t)$ contains at least one interval and $I_k \geq 1$. Since $I_h = m^{h-k} \cdot I_k \geq I_k \geq 1$, then $\sigma_h (t) \in \{1, i\}$ for all $h > k$. Since $\log I_k - \log m^k$ is a fixed, finite value, then

$$\begin{aligned} \underline{\dim}_M (C \cap (C + t)) &= \liminf_{h \to \infty} \frac{\log m^h - \log m^k + \log I_k}{\log n^h} \\ &\geq \lim_{h \to \infty} \frac{\log m^h}{\log n^h} \\ &= \frac{\log m}{\log n}. \end{aligned}$$

The upper Minkowski dimension will also be $\frac{\log m}{\log n}$ according to Lemma 12.1, so that $\dim_M (C \cap (C + t)) = \frac{\log m}{\log n}$. To be exact, since each interval of length $\frac{1}{n^k}$ in $C_k \cap (C_k + t)$ is no longer translated, then $C \cap (C + t)$ contains I_k distinct copies of $\frac{1}{n^k} C$.

Suppose $\sigma_k(t) \in \{-1, 0\}$. Then $C_k \cap (C_k + t)$ contains no intervals and $I_k = 0$. Specifically, $C \cap (C + t) \subset A$ and $\overline{\dim}_M (C \cap (C + t)) = \limsup_{h \to \infty} \frac{\log r}{\log n^h} = 0$. The lower Minkowski dimension is necessarily zero according to Lemma 12.1, so that $\dim_M (C \cap (C + t)) = 0$. □

Theorem 12.3 allows us to completely characterize the Minkowski dimension for any t having a finite base n expansion. Therefore, for the remainder of Part 3 we expand our basic assumptions on t:

Let $t \in (0, 1)$ such that the base n expansion of t does not terminate in repeating 0 or $n - 1$ digits. That is, for any $k \in \mathbb{N}$ there exist $k \leq h_0$ and $k \leq h_{n-1}$ such that $t_{h_0} \neq 0$ and $t_{h_{n-1}} \neq n - 1$.

Proposition 12.4. *For any $k \in \mathbb{N}$, if $t - \lfloor t \rfloor_k > 0$ then $\begin{bmatrix} 1 & 1 \end{bmatrix} \cdot \overrightarrow{\nu_k}(t) = I_k + P_k$ is the natural number representing the smallest number of intervals of length $\frac{1}{n^k}$ required to cover $C_k \cap (C_k + t)$.*

Proof. Let $t \in [0, 1]$ be given such that the base n representation of t does not terminate in repeating zeroes or $n - 1$ digits. Let $k \in \mathbb{N}$ be arbitrary. Choose $J \subset C_k$ and $K \subset C_k + \lfloor t \rfloor_k$ to be arbitrary n-ary intervals of length $\frac{1}{n^k}$ such that $J \cap K \neq \varnothing$. That is, these intervals produce either a potential interval, interval, or potentially empty case. In general $0 \leq t - \lfloor t \rfloor_k < \frac{1}{n^k}$. We consider each case:

1. Suppose $J = K$ (the interval case). Then $K + (t - \lfloor t \rfloor_k) \subset (C_k + t)$ and $J \cap (K + (t - \lfloor t \rfloor_k)) \neq \varnothing$. Specifically, $J \cap (K + (t - \lfloor t \rfloor_k))$ is a closed interval of length $\frac{1}{n^k} - (t - \lfloor t \rfloor_k) > 0$ contained in $C_k \cap (C_k + t)$. Therefore J is an n-ary interval of length $\frac{1}{n^k}$ which covers the corresponding points in $C_k \cap (C_k + t)$. By construction, K is at least a distance of $\frac{1}{n^k}$ from other intervals in $(C_k + \lfloor t \rfloor_k)$, so that J does not intersect any other intervals in $C_k + t$.

Since $J = K$ is arbitrary, then each interval case at step k has a corresponding interval $J \subset C_k$ which covers a portion of $C_k \cap (C_k + t)$. Based on the analysis of Section 10, there are I_k such cases. Note that by construction of C, these intervals may contain points of $C \cap (C + t)$.

2. Suppose $J = K + \frac{1}{n^k}$ (the potential interval case). Then, $K + (t - \lfloor t \rfloor_k) \subset (C_k + t)$ and $J \cap (K + (t - \lfloor t \rfloor_k)) \neq \varnothing$ is a closed interval of length $(t - \lfloor t \rfloor_k) > 0$ contained in $C_k \cap (C_k + t)$. Thus, J is an n-ary interval of length $\frac{1}{n^k}$ which covers the corresponding points in $C_k \cap (C_k + t)$. By construction, K is at least a distance $\frac{1}{n^k}$ from other intervals in $(C_k + \lfloor t \rfloor_k)$, so that J will not intersect any other intervals in $C_k + t$.

Since $J = K + \frac{1}{n^k}$ is arbitrary, then each potential interval case at step k has a corresponding interval $J \subset C_k$ which covers a portion of $C_k \cap (C_k + t)$. As J cannot be in both interval case and potential interval case simultaneously, then the collection of potential intervals J is disjoint from the interval cases above. Furthermore, there are P_k such cases and the chosen intervals J may contain points in $C \cap (C + t)$.

3. Suppose $J + \frac{1}{n^k} = K$ (the potentially empty case). Since $t - \lfloor t \rfloor_k > 0$, then $J \cap (K + (t - \lfloor t \rfloor_k)) = \varnothing$ is a subset of $C_k \cap (C_k + t)$. Hence, potentially empty cases do not affect the minimal number of intervals required to cover $C_k \cap (C_k + t)$. Trivially, $J \cap (K + (t - \lfloor t \rfloor_k))$ does not contain points of $C \cap (C + t)$. Similarly, if $J \cap K = \varnothing$ then $J \cap (K + (t - \lfloor t \rfloor_k))$ is an empty subset of $C_k \cap (C_k + t)$.

Since $C \cap (C + t) \subset C_k \cap (C_k + t)$, then the intervals J chosen from the interval and potential interval cases are a cover of $C_k \cap (C_k + t)$. Furthermore, there are precisely $I_k + P_k$ such intervals. By construction of C, any distinct intervals from this collection are a minimum distance of $\frac{1}{n^k}$ apart, so there is no more efficient way to cover the set. Therefore, $I_k + P_k$ is the minimum number of n-ary intervals J required to cover $C_k \cap (C_k + t)$. $\square$

Corollary 12.5. *For any $t \in [0, 1]$ and an arbitrary $k \in \mathbb{N}$, $0 \leq I_k + P_k \leq m^k$.*

Proof. Choose the collection of n-ary intervals $J \subset C_k$ corresponding to the interval and potential interval cases according to Proposition 12.4. Since t may terminate in repeating zeroes, we include the intervals $J \subset C_k$ corresponding to the potentially empty cases. Because potentially empty cases are disjoint from the interval and potential interval cases, then $I_k + P_k + M_k$ is at most the total number of n-ary intervals contained in C_k. Thus, $I_k + P_k + M_k \leq m^k$ by construction of C. Since $M_k \geq 0$, then

$$I_k + P_k \leq m^k.$$

$\square$

Since $C \cap (C+t) = \bigcap_{k=0}^{\infty} [C_k \cap (C_k + t)]$, then the collection of intervals J chosen via the proof of Proposition 12.4 will also be a cover of $C \cap (C+t)$, but may not be the minimal cover. Thus, for any $t \in [0,1]$ which does not terminate in repeating zeroes, $N_k (C_k \cap (C_k + t)) \leq I_k + P_k$ for all $k \in \mathbb{N}$. Since $k \in \mathbb{N}$ is arbitrary, then

$$\overline{\dim}_M (C \cap (C+t)) \leq \limsup_{k \to \infty} \frac{\log\ (I_k + P_k)}{\log n^k}.$$

Furthermore, if either $I_k = 0$ or $P_k = 0$ for any $k \in \mathbb{N}$ we obtain the following result:

Proposition 12.6. *Let $t \in \mathcal{F}^+$ such that the base n expansion of t does not terminate in repeating 0 or $n-1$ digits. If there exists $k \in \mathbb{N}$ such that $\sigma_k (t) = \pm 1$, then*

$$N_k (C \cap (C+t)) = I_k + P_k.$$

Proof. Let $t \in \mathcal{F}^+$ be given such that t does not terminate in repeating 0 or $n-1$ digits.

Proposition 12.4 gives us that $N_k (C_k \cap (C_k + t)) \leq I_k + P_k$. Since $C \cap (C+t) \subseteq C_k \cap (C_k + t)$, then regardless of the value of $\sigma_k (t)$, for all k

$$N_k (C \cap (C+t)) \leq I_k + P_k.$$

We need to show that the reverse inequality is true. Suppose there exists $k \in \mathbb{N}$ such that $\sigma_k (t) = \pm 1$. That is, $C_k \cap (C_k + \lfloor t \rfloor_k)$ is in either the interval case or the potential interval case.

1. If $C_k \cap (C_k + \lfloor t \rfloor_k)$ is in the interval case, then choose V_k to be the collection of n-ary intervals J in the interval case as in the proof of Proposition 12.4. Thus, V_k contains I_k distinct intervals $J \subset C_k$.

For each $J \in V_k$, $J \cap (K + (t - \lfloor t \rfloor_k))$ is an interval of length $\frac{1}{n^k} - (t - \lfloor t \rfloor_k) > 0$ in $C_k \cap (C_k + t)$. There are no potential interval cases at step k since $\sigma_k (t) = 1$. Furthermore, potentially empty cases do not contain points in $C_k \cap (C_k + t)$ since $(t - \lfloor t \rfloor_k) > 0$. Therefore, $C_k \cap (C_k + t)$ consists only of the I_k intervals $J \cap (K + (t - \lfloor t \rfloor_k))$. Specifically, $C_k \cap (C_k + t)$

does not contain any isolated points. Thus,

$$C \cap (C+t) \subseteq C_k \cap (C_k+t) \subset \bigcup_{J \in V_k} J.$$

By self-similarity, either all intervals $J \cap (K + (t - \lfloor t \rfloor_k))$ in $C_k \cap (C_k+t)$ contain at least one point of $C \cap (C+t)$, or none do. However, if no intervals $J \in V_k$ contain points in $C \cap (C+t)$, then $C \cap (C+t)$ is necessarily empty, which contradicts $t \in \mathcal{F}^+$. Therefore, all intervals $J \in V_k$ contain at least one point of $C \cap (C+t)$ and the collection V_k is the minimum required to cover $C \cap (C+t)$. Since $C_k \cap (C_k + \lfloor t \rfloor_k)$ is in the interval case, there are I_k such intervals and $P_k = 0$. Therefore, $N_k (C \cap (C+t)) \geq I_k + P_k$.

2. If $C_k \cap (C_k + \lfloor t \rfloor_k)$ is in the potential interval case, the argument above is valid with minor modifications based on Proposition 12.4. Choose V_k to be the collection of n-ary intervals $J \subset C_k$ in the potential interval case. Then $C_k \cap (C_k+t)$ consists only of the P_k intervals $J \cap (I + (t - \lfloor t \rfloor_k))$ of length $(t - \lfloor t \rfloor_k) > 0$. Since $t \in \mathcal{F}^+$ implies $C \cap (C+t) \neq \varnothing$, then all $J \in V_k$ contain at least one point of $C \cap (C+t)$ and thus form a minimum cover.

Hence, $N_k (C \cap (C+t)) \geq I_k + P_k$. □

We would like to show in general that $t \in \mathcal{F}^+$ implies $N_k (C \cap (C+t)) \geq I_k + P_k$. However, for specific choices of D, n, and t, simultaneous cases may contain interval or potential interval cases which are not required to cover $C \cap (C+t)$.

Example 12.7. Let $D = \{0, 2, 6, 9\}$ for $n = 16$. It is sufficient to consider only the digits 6 and 4 so that

$$\eta(6) = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \qquad \eta(4) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Choose $t = 0.64t_3\ldots$ for some remaining digits $t_3, t_4, \ldots$. At step $k=1$, $C_1 \cap (C_1 + 0.6)$ is in the simultaneous case and $I_1 + P_1 = 2$. However, the potential interval case transitions to an irrecoverable case at step $k = 2$. Our method counts this potential interval case, yet these cases cannot contain points in $C \cap (C+t)$. By this construction, we have the strict inequality $N_1 (C \cap (C+t)) = 1 < I_1 + P_1 = 2$. Furthermore, by choosing $t = 0.66\ldots64t_j\ldots$

for some remaining digits $t_j, t_{j+1}, \ldots$, we can generate a finite number of potential interval cases which are not required to cover the set $C \cap (C + t)$.

Unfortunately, for an arbitrary non-terminating value t, it may not be feasible to calculate the infinite matrix product that arises from Equation (11.5). However, with appropriate assumptions on D and n we are able to construct a countable set of values s such that $C \cap (C + s)$ has a specific Minkowski dimension.

Lemma 12.8. *Let D and n be given such that $d_m < n - 1$ and the separation condition is satisfied. For any $t \in \mathcal{F}^+$, there exists an element $s \in \mathcal{F}^+$ such that $\lfloor s \rfloor_k = \lfloor t \rfloor_k$ and $\sigma_{k+1}(s) = 1$.*

Proof. Let $t \in \mathcal{F}^+$ be arbitrary. Let $s_i = t_i$ for all $1 \leq i \leq k$ so that $\lfloor s \rfloor_k = \lfloor t \rfloor_k$. We will show that there exists a digit s_{k+1} such that $\sigma_{k+1}(s) = 1$.

Suppose $\sigma_k(t) = 0$. Then $C_k \cap (C_k + \lfloor t \rfloor_k)$ consists of a finite number of potentially empty cases. However, by the argument in Section 12, since $d_m < n - 1$ then $C_{k+1} \cap (C_{k+1} + t)$ is empty for any value of t_{k+1} and $t - \lfloor t \rfloor_k$. Therefore,

$$C \cap (C + t) \subseteq C_{k+1} \cap (C_{k+1} + t) = \varnothing.$$

This contradicts that $t \in \mathcal{F}^+$. Hence, $\sigma_k(t) \in \{\pm 1, i\}$.

Suppose $\sigma_k(t) \in \{1, i\}$. Then $C_k \cap (C_k + \lfloor s \rfloor_k)$ contains at least one interval case so that $I_k \geq 1$ and $P_k \geq 0$. Since $\left\lfloor \frac{d_m}{n-1} \right\rfloor = 0$, then according to Proposition 11.4,

$$\eta(0) = \begin{bmatrix} m & 0 \\ 0 & 0 \end{bmatrix}.$$

Thus, we can choose $s_{k+1} = 0$ so that $I_{k+1} = m \cdot I_k \geq 1$ and $P_{k+1} = 0$. Hence, $C_{k+1} \cap \left(C_{k+1} + \lfloor s \rfloor_{k+1}\right)$ is in the interval case and $\sigma_{k+1}(s) = 1$.

Suppose $\sigma_k(t) = -1$. Then $C_k \cap (C_k + \lfloor s \rfloor_k)$ contains at least one potential interval case so that $I_k = 0$ and $P_k \geq 1$. By Proposition 11.4 there exist $0 \leq p, q \leq m$ such that

$$\eta(n - d_m) = \begin{bmatrix} p & 1 \\ q & 0 \end{bmatrix}.$$

Thus, we can choose $s_{k+1} = n - d_m$ so that $I_{k+1} = p \cdot I_k + P_k = P_k \geq 1$ and $P_{k+1} = q \cdot I_k = 0$. Hence, $C_{k+1} \cap \left(C_{k+1} + \lfloor s \rfloor_{k+1}\right)$ is in the interval case and $\sigma_{k+1}(s) = 1$.

Choose remaining digits $s_j = 0$ for all $j > k+1$. Thus, $\dim_M(C \cap (C+s)) = \frac{\log m}{\log n}$ by Theorem 12.3 so that $s \in \mathcal{F}^+$. $\square$

Theorem 12.9. *Let D and n be given such that $d_m < n-1$ and the separation condition is satisfied. Let*

$$\mathcal{F}_\alpha := \left\{ s \in \mathcal{F} \mid \dim_M(C \cap (C+s)) = \alpha \frac{\log m}{\log n} \right\}.$$

Then $\mathcal{F}_\alpha$ is dense in $\mathcal{F}$ for any $0 \leq \alpha \leq 1$.

Proof. Since $\mathcal{F}^- = -\mathcal{F}^+$ it is sufficient to show that $\mathcal{F}_\alpha^+$ is dense in $\mathcal{F}^+$.

Let $t \in \mathcal{F}^+$ and $\varepsilon > 0$ be arbitrary. Choose $k \in \mathbb{N}$ such that $0 < \frac{1}{n^k} < \varepsilon$. Let $s_i = t_i$ for all $1 \leq i \leq k$ so that $|s - t| < \varepsilon$ regardless of the choice of remaining digits s_h for $h > k$. Let $0 \leq \alpha \leq 1$ be given.

Consider the value of $\sigma_k(s) = \sigma_k(t)$. If $\sigma_k(t) = 0$, then $C \cap (C+t) \subseteq \varnothing$ by the proof of Lemma 12.8, which contradicts that $t \in \mathcal{F}^+$.

Thus, $\sigma_k(s) = \sigma_k(t) \in \{\pm 1, i\}$. Choose s_{k+1} according to the proof of Lemma 12.8 so that $\sigma_{k+1}(s) = 1$ and we begin in the interval case.

For all $j > 0$ define $x_j := \lfloor j \cdot \alpha \rfloor$. Note that x_j is the unique positive integer such that $x_j \leq j \cdot \alpha \leq x_j + 1$. Thus, $0 \leq \alpha - \frac{x_j}{j} \leq \frac{1}{j}$ for all $j > 0$ so that the sequence $\left(\frac{x_j}{j}\right)$ converges to α as $j \to \infty$. Because $0 \leq \alpha \leq 1$, then $x_j \leq x_{j+1} \leq x_j + 1$ for all j. Therefore, for any $j \geq k+1$ we can choose the remaining digits of s as follows:

$$s_{j+1} = \begin{cases} d_m & \text{if } x_{j+1} = x_j \\ 0 & \text{if } x_{j+1} = x_j + 1. \end{cases}$$

Furthermore, according to Theorem 11.2 there exist $0 \leq p, q \leq m$ such that

$$\eta\left(d_m\right) = \begin{bmatrix} 1 & p \\ 0 & q \end{bmatrix}.$$

In this way, $\sigma_j(s) = 1$ for all $j \geq k+1$ and by Proposition 12.6, $N_j\left(C \cap (C+s)\right) = I_j + P_j = I_j$. More specifically, $I_{j+1} = I_j$ when $s_{j+1} = d_m$ and $I_{j+1} = m \cdot I_j$ when $s_{j+1} = 0$. Let $w_j = \#\left\{k+1 \leq i < j \mid s_i = 0\right\}$ then for any $j \geq k+1$:

$$N_j\left(C \cap (C+s)\right) = I_j = m^{w_j} \cdot I_{k+1}.$$

By definition, $w_{j+1} = x_{j+1} - x_j + w_j$ for all $j \geq k+1$. Hence, for all $h > 0$, $w_{h+k+1} = x_h - x_{k+1}$ and

$$\frac{\log N_{h+k+1}\left(C \cap (C+s)\right)}{\log n^{h+k+1}} = \frac{x_h \log m - x_{k+1} \log m + \log I_{k+1}}{h \log n + (k+1) \log n}.$$

Yet, x_{k+1} and $I_{k+1} \geq 1$ are both fixed, finite values so that

$$\lim_{h\to\infty} \frac{x_h \log m - x_{k+1} \log m + \log I_{k+1}}{h \log n + (k+1) \log n} = \lim_{h\to\infty} \frac{x_h}{h} \frac{\log m}{\log n} = \alpha \frac{\log m}{\log n}.$$

Therefore, the Minkowski dimension of $C \cap (C+s)$ exists and is equal to $\alpha \frac{\log m}{\log n}$ as desired. □

Corollary 12.10. *The mapping* $u \mapsto \underline{\dim}_M\left(C \cap (C+t)\right)$ *defined on* $\mathcal{F}^+$ *is everywhere discontinuous.*

We now show that our restrictions on the set of digits D are required for Theorem 12.9. In particular, Example 12.12 demonstrates when $d_m = n-1$ and Example 12.13 violates the separation condition. Example 12.11 and Example 12.14 meet neither condition.

Example 12.11. If there is no separation condition on D and $d_m = n-1$ then choose $D = \{0, 1, 2, \ldots, n-1\}$ so that $C = \mathcal{F}^+ = [0,1]$. In this case, if $t \in (-1,1)$ then $C \cap (C+t)$ is a closed interval with dimension $\frac{\log n}{\log n} = 1$. Otherwise, if $t = \pm 1$ then $C \cap (C+t)$ contains only a single point (0 or 1 respectively) and trivially has dimension 0.

Example 12.12. Let $D = \{0, 2, 4, 7\}$ and $n = 8$. Note that $d_m = n - 1$. Then $\mathcal{F} = [-1, 1]$ by Proposition 9.4 and we have the following matrices:

$$\eta(0) = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix} \qquad \eta(7) = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$$

$$\eta(1) = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix} \qquad \eta(6) = \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}$$

$$\eta(2) = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix} \qquad \eta(5) = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$$

$$\eta(3) = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \qquad \eta(4) = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

Choose $t = 0.4\overline{0}$ so that $\sigma_1(t) = i$ and $\overrightarrow{\nu_1}(t) = \begin{bmatrix} I_1 & P_1 \end{bmatrix}^T = \begin{bmatrix} 1 & 1 \end{bmatrix}^T$. By Theorem 12.3, $\dim_M(C \cap (C + t)) = \frac{\log m}{\log n}$. Let $\varepsilon = \frac{1}{8}$ and $s = 0.4s_2\ldots$ where, by construction, $|s - t| < \varepsilon$. We will show that it is not possible to construct remaining digits such that $\underline{\dim}_M(C \cap (C + s))$ is arbitrarily close to 0.

We first show for any $u \in \mathcal{F}^+$, if $I_k > 0$ and $P_k > 0$ for some k then $I_{k+1} > 0$ and $P_{k+1} > 0$. Suppose $I_k > 0$ and $P_k > 0$ for some k. If $u_{k+1} \in \{0, 2, 3, 4, 5, 7\}$, then $I_{k+1} \geq \mu_{II}(u_{k+1}) \cdot I_k$. Since $\mu_{II}(u_{k+1}) \geq 1$ for this set of digits, then $I_{k+1} \geq I_k > 0$. Similarly, if $u_{k+1} \in \{1, 6\}$, then $I_{k+1} \geq \mu_{PI}(u_{k+1}) \cdot P_k$ for some $\mu_{PI}(u_{k+1}) \geq 1$. Therefore, $I_{k+1} > 0$ for any choice of $0 \leq u_{k+1} \leq 7$. By a similar argument, either $P_{k+1} \geq \mu_{PP}(u_{k+1}) \cdot P_k > 0$ or $P_{k+1} \geq \mu_{IP}(u_{k+1}) \cdot I_k > 0$ for all $0 \leq u_{k+1} \leq 7$. Thus our claim is true for $k + 1$. By induction, if $I_k > 0$ and $P_k > 0$ for some k, then $I_h > 0$ and $P_h > 0$ for all $h \geq k$.

Let $k \in \mathbb{N}$ such that $I_k > 0$ and $P_k > 0$. We are interested in $\overrightarrow{\nu_{k+1}}(s) = \begin{bmatrix} I_{k+1} & P_{k+1} \end{bmatrix}^T = \eta(s_{k+1}) \cdot \begin{bmatrix} I_k & P_k \end{bmatrix}^T$. Thus,

$$\overrightarrow{\nu_{k+1}}(s) = \begin{cases} \begin{bmatrix} (4 \cdot I_k) & (P_k) \end{bmatrix}^T & \text{if } s_{k+1} = 0 \\ \begin{bmatrix} (P_k) & (2 \cdot I_k) \end{bmatrix}^T & \text{if } s_{k+1} = 1 \\ \begin{bmatrix} (2 \cdot I_k) & (I_k + P_k) \end{bmatrix}^T & \text{if } s_{k+1} = 2 \\ \begin{bmatrix} (I_k + P_k) & (I_k + P_k) \end{bmatrix}^T & \text{if } s_{k+1} = 3 \\ \begin{bmatrix} (I_k + P_k) & (I_k + P_k) \end{bmatrix}^T & \text{if } s_{k+1} = 4 \\ \begin{bmatrix} (I_k + P_k) & (2 \cdot P_k) \end{bmatrix}^T & \text{if } s_{k+1} = 5 \\ \begin{bmatrix} (2 \cdot P_k) & (I_k) \end{bmatrix}^T & \text{if } s_{k+1} = 6 \\ \begin{bmatrix} (I_k) & (4 \cdot P_k) \end{bmatrix}^T & \text{if } s_{k+1} = 7 \end{cases}$$

Note that we at least double either I_k or P_k for all digits except 3 and 4. Suppose $s_{k+1} \in \{3, 4\}$. If $I_k \leq P_k$, then $I_{k+1} = I_k + P_k \geq 2 \cdot I_k$ and $P_{k+1} = I_k + P_k \geq P_k$. Similarly, if $I_k > P_k$, then $I_{k+1} = I_k + P_k \geq I_k$ and $P_{k+1} = I_k + P_k \geq 2 \cdot P_k$. Therefore, one of the following pairs of inequalities holds for each integer $0 \leq h \leq 7$:

$I_{k+1} \geq 2 \cdot I_k$	and	$P_{k+1} \geq P_k$
$I_{k+1} \geq 2 \cdot P_k$	and	$P_{k+1} \geq I_k$
$I_{k+1} \geq I_k$	and	$P_{k+1} \geq 2 \cdot P_k$
$I_{k+1} \geq P_k$	and	$P_{k+1} \geq 2 \cdot I_k$

Thus, for any digit $0 \leq s_{k+1} \leq 7$ we multiply one of I_k and P_k by at least 2 and the other by at least 1. Let $h > 0$. Then for some integer sequence (a_h) such that $a_h \leq a_{h+1} \leq a_h + 1$ for all h,

$$I_{k+h} \geq 2^{a_h} \qquad \text{and} \qquad P_{k+h} \geq 2^{h - a_h - 1}.$$

Hence, either $a_h \geq \frac{(h-1)}{2}$ or $h - a_h - 1 \geq \frac{(h-1)}{2}$ for all $h > 0$. Therefore, $I_{k+h} + P_{k+h} \geq \max\{I_{k+h}, P_{k+h}\} \geq 2^{(h-1)/2}$ regardless of the choice of remaining digits s_{k+h}. Furthermore, since each interval and potential interval contains points in $C \cap (C+s)$ by Proposition 11.5, then $N_{k+h}(C \cap (C+s)) \geq 2^{(h-1)/2}$.

For this example $k = 1$ and $I_1 = P_1 = 1$ so that $I_h + P_h \geq 2^{(h-1)/2}$. Thus,

$$\underline{\dim}_M (C \cap (C+s)) \geq \liminf_{h\to\infty} \frac{\log 2^{(h-1)/2}}{\log n^h} = \frac{\log \sqrt{2}}{\log 8} = \frac{1}{6}$$

The argument above only includes $s \in [t, t+\varepsilon)$. However, since $\eta(3) = \eta(4)$, then this equation also holds for any choice of $s = 0.3s_2 \ldots$. Thus, $\underline{\dim}_M (C \cap (C+s)) \geq \frac{1}{6}$ for any $s \in (t-\varepsilon, t+\varepsilon)$ and it is impossible to construct a set of smaller dimension. Specifically, $\mathcal{F}_\alpha \cap (0.3, 0.5)$ is empty for $0 \leq \alpha < \frac{1}{6}$. Furthermore, this is true for any $t \in [0,1]$ such that $\sigma_k(t) = i$ for some $k \in \mathbb{N}$.

Let $s = 0.4\overline{1}$ and $h > 0$. Then $k = 1$ so that $I_{1+2h} = P_{1+2h} = 2^h$ for the subsequence of even values $2h$. Therefore,

$$\underline{\dim}_M (C \cap (C+s)) \leq \liminf_{h\to\infty} \frac{\log(I_{1+2h} + P_{1+2h})}{\log n^{1+2h}} = \frac{h \log 2}{(1+2h)\log 8} = \frac{\log \sqrt{2}}{\log 8}.$$

It follows from the argument above that $\underline{\dim}_M (C \cap (C+s)) = \frac{1}{6}$.

Example 12.13. Without the separation condition on D it is possible that $d_{i+1} = d_i + 1$ for some i. Let $n = 5$ and $D = \{0, 1, 2, 3\}$. Thus, $\mathcal{F} = \left[-\frac{3}{4}, \frac{3}{4}\right]$ by Proposition 9.4.

An arbitrary quinary interval $J \subset C_k$ can be in both the interval and potential interval case simultaneously. We will denote S_k to be the number of intervals $J \subset (C_k + \lfloor t \rfloor_k)$ in the simultaneous case at step k, I_k to be the number of intervals J which are in only the interval case (and not the potential interval case), and P_k to be the number of intervals J which are in only the potential interval case. Figure 12.1 shows an example of the simultaneous case with other intervals of C_k and $C_k + \lfloor t \rfloor_k$ removed for clarity. The top row shows the refinement of $J \subset C_k$ and the bottom row is the refinement of intervals $K, K - \frac{1}{5^k} \subset C_k + \lfloor t \rfloor_k$ such that $K = J$. The figure shows retained subintervals in black and removed subintervals as a thin line to allow better visual spacing.

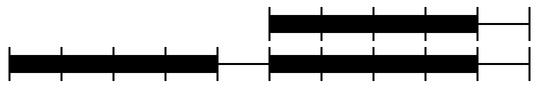

FIGURE 12.1. A simultaneous case for $D = \{0, 1, 2, 3\}$ when $n = 5$.

Thus, we can expand the matrix definitions from Section 11 to include the simultaneous case. Let $\overrightarrow{\nu_k} = \begin{bmatrix} I_k & P_k & S_k \end{bmatrix}^T$. Then $\overrightarrow{\nu_0} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T$ and we can define the following matrix extension (where $\mu_{QR}(h)$ represents the number of R cases in step $k+1$ resulting from each Q case at step k):

$$\eta(h) = \begin{bmatrix} \mu_{II}(h) & \mu_{PI}(h) & \mu_{SI}(h) \\ \mu_{IP}(h) & \mu_{PP}(h) & \mu_{SP}(h) \\ \mu_{IS}(h) & \mu_{PS}(h) & \mu_{SS}(h) \end{bmatrix}.$$

We will generate all five 3×3 matrices by examination of the intersections. Let $J \subset C_k$ and $K, \left(K - \frac{1}{5^k}\right) \subset C_k + \lfloor t \rfloor_k$ be quinary intervals such that $K = J$. By counting interval and potential interval cases as in Section 11, if $t_{k+1} = 2$ then each interval case at step k results in one subinterval case, no potential subinterval cases, and one simultaneous case. Additionally, each potential interval case at step k results in one potential subinterval case, one simultaneous case, and no subinterval cases. Figure 12.2 shows intervals J, K, and $K - \frac{1}{5^k}$ as in Figure 12.1, but with K translated to the right by $\frac{t_{k+1}}{5^{k+1}} = \frac{2}{5^{k+1}}$. Thus, the transition contains one empty intersection, one interval case I_k (denoted in red), one potential interval case P_k (denoted in yellow), and two more simultaneous cases S_k (again denoted in black).

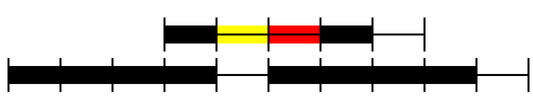

FIGURE 12.2. A simultaneous case for $D = \{0, 1, 2, 3\}$ when $n = 5$ (translated by $t_{k+1} = 2$).

Thus, we obtain the values for the column vectors of the $\eta(2)$ matrix and can generate all 5 matrices using this method:

$$\eta(0) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 3 & 0 & 3 \end{bmatrix} \qquad \eta(4) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 3 & 3 \end{bmatrix}$$

$$\eta(1) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 2 & 0 & 2 \end{bmatrix} \qquad \eta(3) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 2 & 2 \end{bmatrix}$$

$$\eta(2) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix}.$$

Consider any $t \in \left[0, \frac{3}{5}\right) \subset \mathcal{F}$ such that the quinary representation does not terminate in repeating 4's. Thus, $t_1 \in \{0, 1, 2\}$ and since $S_1 \geq I_0 > 0$, then $C_1 \cap (C_1 + \lfloor t \rfloor_1)$ contains at least one simultaneous case. However, since $\mu_{SS}(h) \geq 2$ for all $0 \leq h \leq 4$, then $S_{k+1} \geq 2S_k > 0$ for all k. Therefore, for any choice of remaining digits $t_2, t_3, \ldots$, since $S_1 > 0$, then $S_k \geq 2^{k-1}$ for all k.

By this analysis, once a simultaneous case occurs at step k then every step $h \geq k$ will contain a sub-simultaneous case. Additionally, since $t - \lfloor t \rfloor_h < \frac{1}{5^h}$ then any simultaneous case in $C_h \cap (C_h + \lfloor t \rfloor_h)$ will correspond to a closed interval J of length $\frac{1}{5^h} - (t - \lfloor t \rfloor_h)$ in $C_k \cap (C_k + t)$. By induction, choose simultaneous case quinary intervals $J_h \subset C_h \cap (C_h + t)$ for all $h \geq k$ such that $\cdots \subset J_h \subset \cdots \subset J_{k+1} \subset J_k$. Since $\{J_h\}$ is a sequence of nested, closed intervals, then $\bigcap_{h=k}^{\infty} J_h$ is nonempty. Since $\bigcap_{h=k}^{\infty} J_h \subset \bigcap_{h=k}^{\infty} C_h \cap (C_h + t) = C \cap (C + t)$, then every simultaneous case at step k contains at least one point of $C \cap (C + t)$.

Therefore, the collection of S_k intervals at step k are required at minimum to cover $C \cap (C + t)$. Thus, $N_k(C \cap (C + t)) \geq S_k \geq 2^{k-1} S_1$ and

$$\underline{\dim}_M (C \cap (C + t)) \geq \lim_{k \to \infty} \frac{(k-1)\log 2 + \log S_1}{k \log 5} = \frac{\log 2}{\log 5}$$

Hence, for any $t \in \left[0, \frac{3}{5}\right)$ the lower Minkowski dimension cannot be smaller than $\frac{\log 2}{\log 5}$.

Note that when $D = \{0, 1, \dots, n-2\}$ and $n \geq 5$ then $\mu_{IS}(h) = n - 2 - h \geq 1$ for $0 \leq h \leq n-3$. Furthermore, for any $0 < h < n-1$, we can refer to Figure 12.2 so that the transition to step $k+1$ contains one empty subinterval, one subinterval case, one potential subinterval case, and the remaining $n-3$ intersections are in the simultaneous case. Therefore, $\mu_{SS}(h) \geq n-3$ for $0 < h < n-1$. If $h = 0$, then the transition contains one empty subinterval, one subinterval case, and $n-2$ simultaneous cases. Similarly, if $h = n-1$, then the transition contains one empty subinterval, one potential subinterval case, and $n-2$ simultaneous cases. Therefore, $\mu_{SS}(0) = \mu_{SS}(n-1) = n-2$. Hence, for any $t \in \left[0, \frac{n-2}{n}\right)$ which does not terminate in repeated $n-1$ digits, $S_1 \geq 1$ and $S_{k+1} \geq (n-3) S_k$ for all k.

By the argument above, $\underline{\dim}_M (C \cap (C+t)) \geq \lim_{k\to\infty} \frac{(k-1)\log(n-3)+\log S_1}{k \log n} = \frac{\log(n-3)}{\log n}$.

It is interesting to note that this method does not apply to the case $n = 3$. If $D = \{0, 1\}$ and $n = 3$, then the deleted digits Cantor set $C = \frac{1}{2}\widehat{C}$, where $\widehat{C}$ denotes the traditional Middle Thirds Cantor set.

Example 12.14. In Example 12.11, C is a deleted digits set, but not a deleted digits Cantor set. Let $n \geq 5$ and choose $D = \{0, 1, 2, \dots n-1\} \setminus \{i\}$ for some $0 < i < n-1$. Thus, we can use a variation of the argument used in Example 12.13 above so that $\underline{\dim}_M (C \cap (C+t)) \geq \frac{\log(n-3)}{\log n}$ for any $t \in \left[0, \frac{1}{n}\right)$.

13. The Asymptotic Density Formula

We introduce an alternate method for calculating the box dimension based on Proposition 12.6. Since Theorem 12.3 completely determines the dimension for any t terminating in repeating zeroes, we will assume that t does not terminate in repeating zeroes or repeating $n-1$ digits. Equivalently, for any $k \in \mathbb{N}$ there exist $k \leq h_0$ and $k \leq h_{n-1}$ such that $t_{h_0} \neq 0$ and $t_{h_{n-1}} \neq n-1$.

All n matrices $\eta(h)$ are determined based on the set of digits D and n by Corollary 11.3. Define

$$H_I := \{h \mid \text{ exactly one of } \mu_{II}(h), \mu_{IP}(h) \text{ is zero}\}$$

$$H_P := \{h \mid \text{ exactly one of } \mu_{PI}(h), \mu_{PP}(h) \text{ is zero}\}.$$

Thus, we construct the sets H_I and H_P by examining the column vectors of $\eta(h)$ for $0 \leq h \leq n-1$. Any matrix $\eta(h)$ containing exactly one zero in the first column implies $h \in H_I$. Similarly, any matrix $\eta(h)$ containing exactly one zero in the second column implies $h \in H_P$. By Example 12.12, $5 \in H_I$ when $D = \{0, 2, 4, 7\}$ and $n = 8$.

By this construction, if $C_k \cap (C_k + \lfloor t \rfloor_k)$ is in the interval case, then H_I contains all values $t_{k+1} = h$ which do not generate a simultaneous case or an irrecoverable case. Similarly, if $C_k \cap (C_k + \lfloor t \rfloor_k)$ is in the potential interval case, then H_P contains all values $t_{k+1} = h$ which do not generate a simultaneous case or an irrecoverable case.

Proposition 11.4 shows that $0, d_m \in H_I$ and $n-1, n-1-d_m \in H_P$ for any D and n. Thus, H_I and H_P are non-empty. It is a direct result of Theorem 11.2 that $H_P = n-1-H_I$.

We focus on specific types of matrices for our analysis. For any $0 \leq h \leq n-1$, we say that a matrix $\eta(h)$ is *partial* if one of the following equations holds for some $a, b \in \mathbb{N}$:

$$\eta(h) = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \qquad \eta(h) = \begin{bmatrix} 0 & b \\ a & 0 \end{bmatrix}$$

$$\eta(h) = \begin{bmatrix} 0 & 0 \\ a & b \end{bmatrix} \qquad \eta(h) = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}.$$

If $a = b = 0$, then $\eta(h)$ is the zero matrix, however any nonzero partial matrix $\eta(h)$ is contained in at least one of H_I or H_P. We say that the collection $\mathcal{M} = \{\eta(h) \mid 0 \leq h \leq n-1\}$ is *partial* if $\eta(h)$ is partial for all $0 \leq h \leq n-1$.

Let $\mathcal{E}$ be the set containing any value $t \in \mathcal{F}^+$ such that $t_1 \in H_I$ and

$$t_{k+1} \in \begin{cases} H_I & \text{if } \sigma_k(t) = 1 \\ H_P & \text{if } \sigma_k(t) = -1. \end{cases}$$

Thus, $0 \in \mathcal{E}$ so the set is nonempty. Furthermore, $\mathcal{E} = \{t \in \mathcal{F}^+ \mid \sigma_k(t) = \pm 1 \text{ for all } k\}$ and we are able to classify when $\mathcal{E} = \mathcal{F}^+$.

Theorem 13.1. *The following are equivalent:*

(1) $\Delta^+ \cap (\Delta^+ - 1)$ *is empty.*

(2) $\mathcal{M}$ *is partial.*

(3) $\mathcal{E} = \mathcal{F}^+$.

Proof. Let D and $n \geq 3$ be given.

($1 \Rightarrow 2$) Suppose $\Delta^+ \cap (\Delta^+ - 1) = \varnothing$. Suppose there exists some $0 \leq j \leq n-1$ such that $\mu(j) > 0$ and $\mu(j+1) > 0$. Then by Equation (11.1),

$$\mu(j) = \#D \cap (D + j) > 0$$

$$\mu(j+1) = \#D \cap (D + j + 1) > 0.$$

Thus, there exist integers $1 \leq r, s, u, v \leq m$ such that $d_r = d_s + j$ and $d_u = d_v + j + 1$. Since $(d_r - d_u), (d_s - d_v) \in \Delta$, then $(d_r - d_u) = (d_s - d_v) - 1 \in \Delta$.

If $(d_r - d_u) \in \Delta^+$ then $(d_r - d_u) \in \Delta^+ \cap (\Delta^+ - 1)$ which is a contradiction. However, if $(d_r - d_u) \in \Delta^-$, then $(d_u - d_r) \in \Delta^+$ such that $(d_u - d_r) - 1 = (d_v - d_s) \in \Delta^+$. Hence, $(d_v - d_s) \in \Delta^+ \cap (\Delta^+ - 1)$ which contradicts that $\Delta^+ \cap (\Delta^+ - 1)$ is empty.

Therefore, for all $0 \leq h \leq n-1$ at most one of $\mu(h)$ and $\mu(h+1)$ is nonzero. Similarly, at most one of $\mu(n-h)$ and $\mu(n-h-1)$ is nonzero by symmetry of $\mathcal{M}$. Hence, each $\eta(h)$ matrix contains at least one zero in each column so that $\eta(h)$ is partial for all $0 \leq h \leq n-1$. Therefore, $\mathcal{M}$ is partial by definition.

($2 \Rightarrow 1$) Suppose $\eta(h)$ is partial for all $0 \leq h \leq n-1$ and $\Delta^+ \cap (\Delta^+ - 1) \neq \varnothing$.

Let $j \in \Delta^+ \cap (\Delta^+ - 1)$ be arbitrary. Therefore, $j, (j+1) \in \Delta^+$ and since $\Delta = D - D$ then $D \cap (D + j) \neq \varnothing$ and $D \cap (D + j + 1) \neq \varnothing$. However, according to Theorem 11.2,

$\mu_{II}(j) = \mu(j) \geq 1$ and $\mu_{IP}(j) = \mu(j+1) \geq 1$ so that

$$\eta(j) = \begin{bmatrix} \mu(j) & * \\ \mu(j+1) & * \end{bmatrix}.$$

This contradicts our assumption that $\eta(j)$ is partial. Since j is arbitrary then $\Delta^+ \cap (\Delta^+ - 1) = \varnothing$.

$(1 \Rightarrow 3)$ Suppose $\Delta^+ \cap (\Delta^+ - 1) = \varnothing$. Equivalently, $\sigma_k(t) \in \{0, \pm 1\}$ for all k and all $t \in \mathcal{F}^+$ by definition of ξ and σ. Note that if $\sigma_k(t) = \pm 1$ for all k and all $t \in \mathcal{F}^+$, then $\mathcal{F}^+ \subseteq \mathcal{E} \subseteq \mathcal{F}^+$ and we are finished.

Suppose $t \in \mathcal{F}^+$ such that $\sigma_k(t) = 0$ for some k. Then $C_k \cap (C_k + \lfloor t \rfloor_k)$ consists of a finite number of potentially empty points. To reiterate the argument of Section 12, if $t - \lfloor t \rfloor_k > 0$, then $C \cap (C + t) \subset C_{k+1} \cap (C_{k+1} + t) = \varnothing$ which contradicts that $t \in \mathcal{F}$. Similarly, if $d_m < n - 1$, then $C \cap (C + t) \subset C_{k+1} \cap (C_{k+1} + t) = \varnothing$. Therefore, any $t \in \mathcal{F}^+$ such that $\sigma_k(t) = 0$ requires $d_m = n - 1$ and $t = \lfloor t \rfloor_k$.

Without loss of generality, assume k is the minimal element of $\{l \mid t - \lfloor t \rfloor_l = 0\}$. By definition of the potentially empty case, there exists at least one pair of n-ary intervals (I, J) of length $\frac{1}{n^k}$ such that $J = I - \frac{1}{n^k}$. Thus, $C_k \cap \left(C_k + \lfloor t \rfloor_k - \frac{1}{n^k}\right)$ contains at least one interval.

Let $\tau = t$ as defined by Corollary 2.5. Since simultaneous cases cannot occur, then $\sigma_k\left(t - \frac{1}{n^k}\right) = \sigma_k(\tau) = 1$. Because k is minimal, then $\sigma_l(\tau) = \pm 1$ for $1 \leq l \leq k$. Furthermore, $d_m = n - 1 \in D$ implies that $\sigma_{k+h}(\tau) = 1$ for all $h > 0$. Therefore, $\tau = t \in \mathcal{E}$. Since $t \in \mathcal{F}^+$ is arbitrary, then $\mathcal{F}^+ \subseteq \mathcal{E}$.

$(1 \Leftarrow 3)$ Suppose $\Delta^+ \cap (\Delta^+ - 1) \neq \varnothing$. Let $j \in \Delta^+ \cap (\Delta^+ - 1)$ so that $\mu(j) \geq 1$ and $\mu(j+1) \geq 1$. Thus,

$$\eta(j) = \begin{bmatrix} \mu(j) & * \\ \mu(j+1) & * \end{bmatrix}.$$

Define $u = \frac{j}{n} \in [0, 1]$ so that $\sigma_1(u) = i$ and $u \notin \mathcal{E}$. However, according to Theorem 12.3, $\dim_M(C \cap (C + u)) = \frac{\log m}{\log n} > 0$ so that $C \cap (C + u)$ cannot be empty. Hence, $u \in \mathcal{F}^+$ and $\mathcal{E} \subsetneq \mathcal{F}^+$. $\square$

Example 13.2. Let $n = 7$ and $D = \{0,2,6\}$. Then $\Delta^{+} = \{0,2,4,6\}$ and $\Delta^{+} - 1 = \{-1,1,3,5\}$ so that $\Delta^{+} \cap (\Delta^{+} - 1) = \varnothing$. By Theorem 9.4, $\mathcal{E} = \mathcal{F}^{+} = [0,1]$.

By Theorem 11.2 the matrices $\eta(j)$ only contain values $\mu(h)$ for $0 \leq h \leq n$. We are interested in the set of non-trivial values

$$X := \{\mu(h) \mid 0 \leq h \leq n \text{ and } \mu(h) \neq 0\}.$$

According to Proposition 11.4, $1, m \in X$ so the set is nonempty. In fact, since $D \cap (D+h) \subseteq D$ for all h, then $0 \leq \mu(h) \leq m$. By definition, for any $x \in X$, there exists an h such that $\mu(h) = x$. Thus, if $C_k \cap (C_k + \lfloor t \rfloor_k)$ is in the interval case and $\mu_{II}(t_{k+1}) = x$ then $I_{k+1} = x \cdot I_k$. Similarly, if $C_k \cap (C_k + \lfloor t \rfloor_k)$ is in the potential interval case and $\mu_{PP}(t_{k+1}) = x$ then $P_{k+1} = x \cdot P_k$.

In Section 10 we were interested in transitions resulting in a non-trivial increase in the number of intervals or potential intervals. Since the set X contains all possible increases in the total number of intervals, then for each $x \in X$ and $t \in \mathcal{E}$, define

$$A_t(x) = \{j \mid \sigma_j(t) = 1 \text{ and } \mu_{II}(t_{j+1}) = x \text{ or } \mu_{IP}(t_{j+1}) = x\}$$

$$B_t(x) = \{j \mid \sigma_j(t) = -1 \text{ and } \mu_{PI}(t_{j+1}) = x \text{ or } \mu_{PP}(t_{j+1}) = x\}$$

If $j \in A_t(x)$, then $\sigma_j(t) = 1$ implies that there is at least one interval contained in $C_j \cap \left(C_j + \lfloor t \rfloor_j\right)$ and the transition to step $j+1$ contains a total of $x \cdot I_j$ subintervals. Since $x \geq 1$ by definition, then there is a non-trivial increase in the number of intervals such that $I_{j+1} = x \cdot I_j \geq I_j$. Similarly, if $j \in B_t(x)$, then $\sigma_j(t) = -1$ implies that there is at least one potential interval contained in $C_j \cap \left(C_j + \lfloor t \rfloor_j\right)$ and the transition to step $j+1$ contains $x \cdot P_j$ subintervals. Since either I_j or P_j is always equal to 0, then $I_{j+1} + P_{j+1} = x_{j+1} \cdot (I_j + P_j)$ for some sequence $(x_j) \subset X$. By induction,

$$I_k + P_k = \left(\prod_{j=1}^{k} x_j\right) \cdot (I_0 + P_0). \tag{13.1}$$

By definition of $\mathcal{E}$, each $j \in \mathbb{N}$ is contained in exactly one $A_t(x) \cup B_t(x)$ so that these sets form a partition of $\mathbb{N}$. For each $x \in X$, define

$$R_k(x) := \#\{j \leq k \mid j \in A_t(x) \cup B_t(x)\}.$$

Since $I_0 + P_0 = 1$, then Equation (13.1) simplifies to:

$$I_k + P_k = \left(\prod_{x \in X} x^{R_k(x)}\right). \tag{13.2}$$

For each $x \in X$, define the *upper asymptotic density:*

$$\overline{\gamma}(x) := \limsup_{k \to \infty} \frac{R_k(x)}{k}.$$

Similarly, the *lower asymptotic density* is $\underline{\gamma}(x) := \liminf_{k \to \infty} \frac{R_k(x)}{k}$. If $\underline{\gamma}(x) = \overline{\gamma}(x)$, then $\gamma(x) = \lim_{k \to \infty} \frac{R_k(x)}{k}$ is the *asymptotic density* of x.

Theorem 13.3. *Let $t \in \mathcal{E}$. If the limit $\gamma(x) = \lim_{k \to \infty} \frac{R_k(x)}{k}$ exists for all $x \in X$, then*

$$\dim_M (C \cap (C + t)) = \sum_{x \in X} \gamma(x) \cdot \frac{\log x}{\log n}.$$

Proof. Let t be given. The proof is divided into two parts:

1. Suppose $t = \lfloor t \rfloor_k$ for some k so that $t_{k+h} = 0$ for all $h > 0$. Without loss of generality, choose k to be the minimal element of $\{l \mid t = \lfloor t \rfloor_l\}$.

Suppose $\sigma_k(t) = 1$ and let $h > 0$. Then by Proposition 12.6, $N_{k+h}(C \cap (C + t)) = I_{k+h}$ and $\mu_{II}(t_{k+h}) = m$. Thus, $k + h \in A_t(m) \cup B_t(m)$ so that for any $x \in X$ such that $x \neq m$, $A(x) \cup B(x)$ is finite. Therefore,

$$\begin{aligned}
\gamma(m) &= \lim_{h \to \infty} \frac{R_h(m)}{h} \\
&= \lim_{h \to \infty} \frac{R_k(m) + h - k}{h} = 1 \\
\gamma(x) &= \lim_{h \to \infty} \frac{R_h(x)}{h} \\
&= \lim_{h \to \infty} \frac{R_k(x)}{h} = 0.
\end{aligned}$$

Hence,

$$\sum_{x\in X}\gamma(x)\cdot\frac{\log x}{\log n}=\gamma(m)\cdot\frac{\log m}{\log n}+\sum_{x\in X\setminus\{m\}}\gamma(x)\cdot\frac{\log x}{\log n}=\frac{\log m}{\log n}.$$

This result is consistent with Theorem 12.3 as the Minkowski dimension of $C\cap(C+t)$.

Suppose $\sigma_k(t)=-1$. Then $N_{k+h}(C\cap(C+t))=P_{k+h}$ and $\mu_{PP}(t_{k+h})=\left\lfloor\frac{d_m}{n-1}\right\rfloor$. If $d_m\neq n-1$, then $\mu_{PP}=0$ so that $\sigma_{k+1}(t)=0$ which contradicts that $t\in\mathcal{E}$. Thus, $d_m=n-1$.

Since $\mu_{PP}(t_{k+h})=1$, then $k+h\in A_t(1)\cup B_t(1)$ so that $A_t(x)\cup B_t(x)$ is finite for $x\neq 1$. Thus, $\gamma(1)=1$ and $\gamma(x)=0$ for $x\neq 1$ so that

$$\sum_{x\in X}\gamma(x)\cdot\frac{\log x}{\log n}=\gamma(1)\cdot\frac{\log 1}{\log n}+\sum_{x\in X\setminus\{1\}}\gamma(x)\cdot\frac{\log x}{\log n}=\frac{\log 1}{\log n}=0$$

Hence, if $t=\lfloor t\rfloor_k$, then $\dim_M(C\cap(C+t))=\sum_{x\in X}\gamma(x)\cdot\frac{\log x}{\log n}$ and our result is consistent with Theorem 12.3.

2. If $t-\lfloor t\rfloor_k>0$ for all $k\in\mathbb{N}$, then Proposition 12.6 and Equation (13.2) give the following equalities:

$$\begin{aligned}\dim_M(C\cap(C+t))&=\lim_{k\to\infty}\frac{\log N_k(C\cap(C+t))}{\log n^k}\\&=\lim_{k\to\infty}\frac{\log(I_k+P_k)}{k\log n}\\&=\lim_{k\to\infty}\frac{\log\left(\prod_{x\in X}x^{R_k(x)}\right)}{k\log n}\\&=\lim_{k\to\infty}\frac{\sum_{x\in X}\log\left(x^{R_k(x)}\right)}{k\log n}\\&=\lim_{k\to\infty}\frac{\sum_{x\in X}(R_k(x)\cdot\log x)}{k\log n}\\&=\lim_{k\to\infty}\sum_{x\in X}\frac{R_k(x)}{k}\cdot\frac{\log x}{\log n}\\&=\sum_{x\in X}\gamma(x)\cdot\frac{\log x}{\log n}.\end{aligned}$$

$\square$

Corollary 13.4. *For any rational $t \in \mathcal{E}$, let p be a period of t after a certain point k. Then the Minkowski dimension of $C \cap (C+t)$ exists and is equal to*

$$\dim_M (C \cap (C+t)) = \sum_{x \in X} \frac{\# \{k \leq j < k+2p \mid j \in A_t(x) \cup B_t(x)\}}{2p} \cdot \frac{\log x}{\log n}.$$

Proof. Let $t \in \mathcal{E}$ be rational. Then the ternary representation must be periodic after a certain point k. Without loss of generality, assume that k is the smallest value where t becomes periodic. Let $p > 0$ be a period of t such that $t_{h+p} = t_h$ for all $h \geq k$.

Since $\sigma_k(t) = \pm 1$ is determined by the first k digits of t by definition of σ, then

$$\sigma_{k+p}(t) = \sigma_k(t) \cdot \prod_{i=k}^{p} \xi(\sigma_i(t), t_{i+1}) = \pm \sigma_k(t).$$

If $\prod_{i=k}^{p} \xi(\sigma_i(t), t_{i+1}) = -1$, then $\sigma_{k+p}(t) = -\sigma_k(t)$. Thus, we consider a period of $2p$ so that $\prod_{i=k}^{2p} \xi(\sigma_i(t), t_{i+1}) = \left[\prod_{i=k}^{p} \xi(\sigma_i(t), t_{i+1})\right]^2 = 1$ and $\sigma_{h+2p}(t) = \sigma_h(t)$ for all $h \geq k$. Thus, $\sigma_h(t)$ has period $2p$ and denote $t = 0.t_1 \cdots \overline{t_k \cdots t_{k+2p-1}}$.

Let $x \in X$ be arbitrary. If $j \notin A_t(x) \cup B_t(x)$ for all $k \leq j < k+2p$, then $\overline{\gamma}(x) = \limsup\limits_{h \to \infty} \frac{R_h(x)}{h} = \limsup\limits_{h \to \infty} \frac{R_k(x)}{h} = 0$. Since the limit inferior is necessarily 0, then the limit $\gamma(x)$ exists and is zero.

Suppose $j \in A_t(x) \cup B_t(x)$ for some $k \leq j < k+2p$. Then $t_{h+2p} = t_h$ and $\sigma_{h+2p}(t) = \sigma_h(t)$ for all $h \geq k$. Therefore, $h \in A_t(x) \cup B_t(x)$ implies $h + 2p \in A_t(x) \cup B_t(x)$. Define

$$r_h(x) := \# \{h \leq j < h+2p \mid j \in A_t(x) \cup B_t(x)\} = R_{h+2p-1}(x) - R_h(x).$$

Then $r_h(x) = r_k(x)$ is constant for $h \geq k$. Let $r(x) = r_k(x)$. Then $I_{h+2p} + P_{h+2p} = (I_h + P_h) \cdot \prod\limits_{x \in X} x^{r(x)}$ so that

$$\begin{aligned} I_{h+2jp} + P_{h+2jp} &= (I_h + P_h) \left(\prod_{x \in X} x^{r(x)} \right)^j \\ &= (I_h + P_h) \left(\prod_{x \in X} x^{j \cdot r(x)} \right). \end{aligned}$$

Furthermore, the following equation holds for each $x \in X$:

$$\begin{aligned}
\lim_{i\to\infty} \frac{R_i(x)}{i} &= \lim_{i\to\infty} \frac{R_k(x) + \#\{k < j \leq k + 2ip \mid j \in A_t(x) \cup B_t(x)\}}{i} \\
&= \lim_{i\to\infty} \frac{R_k(x) + i \cdot r_{k+1}(x)}{i} \\
&= \lim_{i\to\infty} \frac{i \cdot r(x)}{i} \\
&= r(x)
\end{aligned}$$

Therefore, the limit exists and $\gamma(x) = r(x) = \#\{k \leq j < k + 2p \mid j \in A_t(x) \cup B_t(x)\}$. $\square$

Example 13.5. Let $D = \{0, 2, 4, 7, 9\}$ for $n = 10$. Then values $\mu(h) = \#D \cap (D + h)$ are such that $X = \{1, 2, 3, 5\}$. Furthermore, $H_I = \{0, 5, 7, 9\}$ so that

$$\mu(0) = 5$$

$$\mu(5) = \mu(7) = 2$$

$$\text{and } \mu(9) = 1.$$

Let $t = 0.\overline{5079}$. Then $2p = 8$, $\gamma(5) = \gamma(1) = 2$, and $\gamma(2) = 4$ so that

$$\dim_M(C \cap (C + t)) = \frac{1}{4} \cdot \frac{\log 1}{\log 10} + \frac{1}{2} \cdot \frac{\log 2}{\log 10} + 0 \cdot \frac{\log 3}{\log 10} + \frac{1}{4} \cdot \frac{\log 5}{\log 10}.$$

Example 13.6. Let $D = \{0, 2, 4, 6, 9, 12, 15\}$ and $n = 16$. Then $X = \{1, 3, 7\}$ and $H_I = \{0, 1, 2, 13, 14, 15\}$. Let $c_3 = \frac{a_3}{b} = \frac{3}{6}$ and $c_7 = \frac{a_7}{b} = \frac{2}{6}$. Choose the first $2b = 12$ digits of t as follows:

Since $\mu(0) = 7 = m$, then choose the first $a_7 = 2$ digits $t_1 = t_2 = 0$ so that $\sigma_2(t) = 1$. Because $\mu_{IP}(1) = 3$, then choose $t_3 = 1$ which transitions to a potential interval case at step $k = 3$. By the symmetry of $\mathcal{M}$, choose $t_4 = (n - 1) - 1 = 14$ such that $\mu_{PI}(14) = 3$ which transitions back to an interval case. Next choose $t_5 = 1$ which transitions to the potential interval case. Thus we have chosen the next $a_3 = 3$ digits of t needed. In order to define the first $b = 6$ digits, let $t_6 = 0$ so that $\sigma_6(t) = -1$.

Choose the remaining 6 digits such that $t_{h+6} = n - 1 - t_h$. Thus, for $E = 14$ and $F = 15$ in the hexadecimal notation, define remaining digits of t such that

$$t = 0.\overline{001E10FFE1EF}.$$

By Corollary 13.4 we have constructed t such that $R_{12}(7) = 2 \cdot a_7$, $R_{12}(3) = 2 \cdot a_3$, and $R_{12}(1) = 2$. Thus,

$$\dim_M (C \cap (C + t)) = \frac{1}{6} \cdot \frac{\log 1}{\log 16} + \frac{1}{2} \cdot \frac{\log 3}{\log 16} + \frac{1}{3} \cdot \frac{\log 7}{\log 16}.$$

Therefore, for any rational coefficients $0 \leq c_x$ such that $\sum_{x \in X} c_x \leq 1$, we can construct $t \in \mathcal{E}$ such that $\dim_M (C \cap (C + t)) = \sum_{x \in X} c_x \frac{\log x}{\log n}$.

Let D and n be given. For each $x \in X$ choose $c_x \in \mathbb{Q}^+$ such that $\sum_{x \in X} c_x \leq 1$. Denote $c_x = \frac{a_x}{b}$ for $a_x, b \in \mathbb{Z}^+$ and since X is finite, then $\sum_{x \in X} a_x \leq b$. Construct the rational value $t \in \mathcal{E}$ with period $2b$ as follows:

Begin with $m \in X$, and choose $t_1 = t_2 = \cdots = t_{a_m} = 0$. Thus, $\sigma_{a_m}(t) = 1$ and $C_{a_m} \cap \left(C_{a_m} + \lfloor t \rfloor_{a_m}\right)$ is in the interval case. We proceed by inductively choosing the largest value $x \in X$ not already chosen. By definition of X, there exists some $h \in H_I$ such that either $\mu_{II}(h) = x$ or $\mu_{IP}(h) = x$. We need to choose the next a_x digits such that $I_{k+a_x} + P_{k+a_x} = x^{a_x}(I_k + P_k)$. If $\sigma_k(t) = 1$ and $\mu_{II}(h) = x$, then choose the next a_x digits equal to h. If $\sigma_k(t) = -1$ and $\mu_{II}(h) = x$, then choose with a_x digits equal to $(n-1) - h$. However, if $\mu_{IP}(h) = x$ then $\mu_{IP}(h) = \mu_{PI}((n-1) - h)$ and we choose the required the remaining a_x digits by alternating digits h and $(n-1) - h$.

We proceed in this manner until the first $\sum_{x \in X} a_x$ digits have been chosen. Since we require the first b digits to be specifically selected, then choose the remaining $b - \sum_{x \in X} a_x$ digits such that

$$\begin{cases} t_{k+1} = d_m & \text{if } \sigma_k(t) = 1 \\ t_{k+1} = (n-1) - d_m & \text{if } \sigma_k(t) = -1. \end{cases}$$

This choice of remaining digits may result in $R_b(1) > a_1$. However, this will not affect the dimension calculation since $\log 1 = 0$.

Once the first b digits are chosen, define the next b digits based on the value of $\sigma_b(t)$. If $\sigma_b(t) = 1$ then choose $t_{h+b} = t_h$ for $1 \le h \le b$. However, if $\sigma_b(t) = -1$, then choose $t_{h+b} = n - 1 - t_h$ for all $1 \le h \le b$. Thus, we have specifically chosen the first $2b$ digits of t.

Define all remaining digits such that $t = 0.\overline{t_1 \ldots t_{2b}}$. Thus, we have chosen a rational value t having a period $p = 2b$. Thus, for all $x \ne 1$ in X,

$$2(2a_x) = \#\{h < j \le h + 2(2b) \mid j \in A(x) \cup B(x)\}$$

By Corollary 13.4,

$$\dim_M (C \cap (C + t)) = \frac{R_{4b}(1)}{4b} \cdot \frac{\log 1}{\log n} + \sum_{x \in X \setminus \{1\}} \frac{4a_x}{4b} \cdot \frac{\log x}{\log n} = \sum_{x \in X} c_x \cdot \frac{\log x}{\log n}.$$

Lemma 13.7. *Let D and n be given such that $d_m = n - 1$ and the separation condition is satisfied. Let $k \in \mathbb{N}$ be given. For any $t \in \mathcal{E}$ there exists $s \in \mathcal{E}$ such that $|s - t| \le \frac{1}{n^k}$ and $\sigma_k(s) = 1$.*

Proof. Let D and n be given such that $d_m = n - 1$. Let $k \in \mathbb{N}$ and $t \in \mathcal{E}$ be arbitrary. Consider the value of $\sigma_k(t)$. Since $t \in \mathcal{E}$, then $\sigma_k(t) \ne i$ by definition and we only need to show the result for $\sigma_k(t) \in \{0, \pm 1\}$.

If $\sigma_k(t) = \pm 1$ then let $s_i = t_i$ for all $1 \le i \le k$ so that $|s - t| < \frac{1}{n^k}$. If $\sigma_k(s) = 1$, then choose $s_{k+1} = 0$ by Lemma 12.8. Similarly, if $\sigma_k(s) = -1$, then choose $s_{k+1} = n - d_m = 1$. In both cases $\sigma_{k+1}(s) = 1$.

If $\sigma_k(t) = 0$ then $t \in \mathcal{F}$ and $d_m = n - 1$ imply $t = \lfloor t \rfloor_k$ according to the proof of Lemma 12.8. Let (τ_i) be the sequence defined in Corollary 2.5 and define $s_i = \tau_i$ for $1 \le i \le k$. By proof of Theorem 13.1, $\sigma_i(s) = \sigma_i(\tau) = \pm 1$ for all $0 \le i \le k$. Since $\lfloor s \rfloor_k = \lfloor \tau \rfloor_k$, then for any choice of remaining digits s_j,

$$\lfloor s \rfloor_k \le s \le \lfloor s \rfloor_k + \sum_{j=k+1}^{\infty} \frac{n-1}{n^j} = t.$$

Hence $\frac{1}{n^k} = |t - \lfloor s \rfloor_k| \ge |s - t|$. Since $\sigma_k(\tau) = \pm 1$, then choose s_{k+1} according to Lemma 12.8 such that $\sigma_{k+1}(s) = 1$.

It remains to show that $s \in \mathcal{E}$. By construction of s, $\sigma_i(s) = \pm 1$ for all $0 \leq i \leq k+1$. Choose remaining digits $s_j = 0$ for all $j > k+1$ so that $\sigma_j(s) = 1$. Hence, $\sigma_i(s) = \pm 1$ for all i so that $s \in \mathcal{E}$ by definition. □

Theorem 13.8. *Let D and $n \geq 3$ be given such that D satisfies the separation condition. Let*

$$\mathcal{E}_\alpha := \left\{ s \in \mathcal{E} \mid \dim_M (C \cap (C+s)) = \alpha \frac{\log m}{\log n} \right\}.$$

Then for any $0 \leq \alpha \leq 1$, the set $\mathcal{E}_\alpha$ is dense in $\mathcal{E}$.

Proof. Let $t \in \mathcal{E}$ be arbitrary and let $\varepsilon > 0$ be given. Let $0 \leq \alpha \leq 1$ be given. Choose $k \in \mathbb{N}$ such that $0 < \frac{1}{n^k} < \varepsilon$.

If $d_m < n-1$ then choose $s_i = t_i$ for all $1 \leq i \leq k$ so that $|s-t| < \varepsilon$ regardless of any choice of remaining digits s_j for $j \geq k+1$. Then $\sigma_i(s) = \sigma_i(t) = \pm 1$ for all $1 \leq i \leq k$ by definition of $\mathcal{E}$. Choose s_{k+1} according to Lemma 12.8 so that $\sigma_{k+1}(s) = 1$.

If $d_m = n-1$ then choose the first $k+1$ digits of s according to Lemma 13.7. Thus, $\varepsilon > \frac{1}{n^k} \geq |s-t|$ and $\sigma_{k+1}(s) = 1$.

According to the proof of Theorem 12.9, we can choose remaining digits of s such that $s \in \mathcal{F}^+$ and $\dim_M (C \cap (C+t)) = \alpha \frac{\log m}{\log n}$ as desired. Futhermore, the proof of Theorem 12.9 constructs digits such that $\sigma_j(s) = \pm 1$ for all $j > k+1$. Since the first $k+1$ digits of s were chosen such that $\sigma_j(s) = \pm 1$ for all $0 \leq j \leq k+1$, then $s \in \mathcal{E}$ by definition. Hence, $s \in \mathcal{E}_\alpha$. □

Note that Theorem 13.8 includes cases when $d_m = n-1$. Moreover, since $\mathcal{E}$ is nonempty for any D and $n \geq 3$, this result applies to any deleted digits Cantor set. Since Theorem 13.1 gives a characterization for when $\mathcal{E} = \mathcal{F}^+$, then we have the following result:

Corollary 13.9. *Let D and n be given such that $\mathcal{M}$ is partial. Then $\mathcal{F}_\alpha$ is dense in $\mathcal{F}$.*

14. Examples

As deleted digits Cantor sets are defined by the choices of D and n, we calculate some examples using the methods described above.

14.1. **The Middle Thirds Cantor Set.** Let $D = \{0, 2\}$ and $n = 3$. Generate the following sets:

$$\Delta = \{-2, 0, 2\}$$

$$\Delta^+ = \{0, 2\}$$

$$\Delta^+ - 1 = \{-1, 1\}$$

$$n - \Delta^+ = \{1, 3\}$$

$$n - \Delta^+ - 1 = \{0, 2\}.$$

Since $\Delta = \{-n+1, \ldots, 0, \ldots n-1\}$ contains all even digits, then $\mathcal{F} = [-1, 1]$ by Corollary 9.5. Futhermore, we calculate the values of $\mu(h)$ for $0 \leq h \leq n$ such that $\mu(0) = \#\{0, 2\} = 2$, $\mu(1) = \#\varnothing = 0$, $\mu(2) = \#\{2\} = 1$, and $\mu(3) = \#\varnothing = 0$. Hence, $\mathcal{M}$ is the collection of matrices

$$\left\{ \eta(0) = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \eta(1) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \eta(2) = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \right\}.$$

The matrices $\eta(0)$ and $\eta(2)$ are determined by Proposition 11.4 since $m = 2$ and $\frac{d_m}{n-1} = 1$. An analysis of $\mathcal{M}$ shows that $t_{k+1} = 0$ will double the number of intervals, $t_{k+1} = 2$ will double the number of potential intervals, and $t_{k+1} = 1$ will flip intervals and potential intervals. This analysis is consistent with the counting method applied in Part 2.

Furthermore, all matrices are diagonal or anti-diagonal so that $\mathcal{M}$ is partial by definition. Hence, $\mathcal{E} = \mathcal{F}^+ = [0, 1]$ and every interval or potential interval case at step k leads to points in $C \cap (C + t)$ by Proposition 11.5. Additionally, Corollary 13.9 shows that we can construct a set of values s dense in $[-1, 1]$ such that $\dim_M (C \cap (C + s)) = \alpha \frac{\log 2}{\log 3}$ for any $0 \leq \alpha \leq 1$.

14.2. **A Uniform Cantor Set.** Let c be an integer greater than 1 and let $n \geq 3$ be given. Let

$$D = \{0, c, 2c, \ldots, (m-1)c \leq n-1\}.$$

Then $d_j - d_i = (j-1)\,c - (i-1)\,c = (j-i)\,c = d_{j-i+1}$ for any $1 \le i \le j \le m$. Since $d_j - d_i$ can be any element of Δ^+, then $\Delta^+ = D$. Hence,

$$\begin{aligned} \Delta^+ &= D \\ \Delta^+ - 1 &= \{-1, c-1, \ldots, (m-1)\,c - 1\} \\ n - \Delta^+ &= \{n - (m-1)\,c, \ldots, n - c, n\} \\ n - \Delta^+ - 1 &= \{n - (m-1)\,c - 1, \ldots, n - c - 1, n - 1\} \end{aligned}$$

Therefore, $\mu\left((j-1)\,c\right) = \#D \cap (D + (j-1)\,c) = m - j + 1$ for $1 \le j \le m$ and $\mu(h) = \#\varnothing = 0$ for remaining values of h. Thus, all n matrices are uniquely determined by the value of n. Since $D \subset c \cdot \mathbb{N}$, then the structure of $\mathcal{F}$ is determined by Proposition 9.7.

Example 14.1. If $d_m = n - 1$, then $n - 1 - d_j = (m-1)\,c - (j-1)\,c = (m-j)\,c = d_{m-j+1} \in D$. Thus,

$$\eta\left((j-1)\,c\right) = \begin{bmatrix} m - (j-1) & 0 \\ 0 & j \end{bmatrix}.$$

The only remaining non-zero matrices $\eta\left((j-1)\,c \pm 1\right)$ are determined by Theorem 11.2. Thus, $\mathcal{M}$ is partial so that $\mathcal{E} = \mathcal{F}^+$ and $\mathcal{F}_\alpha$ is dense in $\mathcal{F}$ by Corollary 13.9.

14.3. **A Regular Cantor Set.** Let c be an integer greater than 1 and let $n \ge 3$ be given. Let $D = \{0, d_2, d_3, \ldots, d_m \le n-1\} \subset c \cdot \mathbb{N}$.

Define the set of digits $Y = \{0, c, 2c, \ldots, d_m\}$ so that $D \subseteq Y$. Furthermore, Y and n determine a uniform deleted digits Cantor set and according to Section 8,

$$C_{n,D} \subseteq C_{n,Y}.$$

If $d_m < n - 1$, then $\mathcal{F}_\alpha$ is dense in $\mathcal{F}$ by Theorem 12.9.

If $d_m = n - 1$, then Y is defined as in Example 14.1. Since $D \cap (D + h) \subseteq Y \cap (Y + h)$ for all $0 \le h \le n$, then $\mu_D(h) = \#D \cap (D + h) \le \#Y \cap (Y + h) = \mu_Y(h)$. Thus, if $\eta_Y(h)$

is partial then $\eta_D(h)$ is necessarily partial. Hence, $\mathcal{M}$ is partial for both D and Y, so that $\mathcal{E} = \mathcal{F}^+$ by Theorem 13.1. This leads to the following Corollary to Theorem 13.8:

Corollary 14.2. *Let $c > 1$ be an integer and $n \in \mathbb{N}$. If $D \subset c \cdot \mathbb{N}$ is a regular set of digits then $\mathcal{M}$ is partial and $\mathcal{F}_\alpha$ is dense in $\mathcal{F}$.*

14.4. **Sparse Cantor Sets.** A deleted digits Cantor set is *sparse* if 0 and $n-1$ are the only digits which result in a non-trivial increase in the cover of $C_k \cap (C_k + t)$. This includes the subfamily of "Middle" Cantor sets such that $D = \{0, n-1\}$ for some $n \in \mathbb{N}$, and also includes some irregular sets.

Lemma 14.3. *If C is sparse then $\mathcal{E} = \mathcal{F}^+$.*

Proof. We only need to show that $\mathcal{M}$ is partial so that the desired result is obtained from Theorem 13.1. Theorem 11.2 shows that $\eta(0)$ and $\eta(n-1)$ are partial, so it remains to consider $0 < h < n-1$. Since a sparse set satisfies Equation (11.6), then for all $t \in [0,1]$ and all $0 < h < n-1$,

$$\begin{bmatrix} 1 & 1 \end{bmatrix} \cdot \overrightarrow{\nu_k}(t) \geq \begin{bmatrix} 1 & 1 \end{bmatrix} \cdot \eta(h) \cdot \overrightarrow{\nu_k}(t).$$

Consider step $k = 0$. Since $\overrightarrow{\nu_0}(t) = \overrightarrow{\nu_0} = \begin{bmatrix} 1 & 0 \end{bmatrix}^T$, then

$$\begin{aligned} 1 &= \begin{bmatrix} 1 & 1 \end{bmatrix} \cdot \overrightarrow{\nu_0}(t) \\ &\geq \begin{bmatrix} 1 & 1 \end{bmatrix} \cdot \eta(h) \cdot \overrightarrow{\nu_k}(t) \\ &= \mu(h) + \mu(h+1). \end{aligned}$$

Since $\mu(h)$ and $\mu(h+1)$ are natural numbers, then for any h there exist $a, b \in \mathbb{N}$ such that

$$\eta(h) \in \left\{ \begin{bmatrix} 1 & a \\ 0 & b \end{bmatrix}, \begin{bmatrix} 0 & a \\ 0 & b \end{bmatrix}, \begin{bmatrix} 0 & a \\ 1 & b \end{bmatrix} \right\}.$$

Furthermore, by Theorem 11.2,

$$\eta(n-1-h) = \begin{bmatrix} b & * \\ a & * \end{bmatrix}.$$

Thus, $b = \mu(n-1-h)$ and $a = \mu(n-h)$ so that $1 \geq b + a$ by the same argument. By definition, $\eta(h)$ is a partial matrix for all h. $\square$

Proposition 14.4. *Let D and n be given. If $d_j \geq 2(d_{j-1}+1)$ for all $1 < j \leq m$ then C is sparse.*

Proof. We know by Proposition 11.4 that $h \in \{0, n-1\}$ are non-trivial. Suppose $0 < h < n-1$.

Let $1 < j \leq m$ arbitrary. If $1 \leq i < j \leq m$, then $d_i \geq 0$ so that $d_j - d_i \leq d_j$. Furthermore, $d_{j+1} \geq 2(d_j + 1)$ so that

$$\begin{aligned} d_j &\geq 2(d_{j-1}+1) \\ &> 2d_{j-1} \\ &\geq d_{j-1} + d_i. \end{aligned}$$

Equivalently, $d_{j-1} < d_j - d_i \leq d_j$. Since j is arbitrary, then $d_{j-1} < d_j - d_i \leq d_j$ for any $i < j$.

We begin by showing that the condition $d_j \geq 2(d_{j-1}+1)$ implies $\delta_l - \delta_{l-1} \geq 2$ for all $1 < l \leq M$ (where $\delta_l \in \Delta$). Let $\delta \neq \varepsilon$ be arbitrary elements of Δ^+ such that $\delta = d_j - d_i$ and $\varepsilon = d_r - d_s$ for some $1 \leq i < j \leq m$ and $1 \leq s < r \leq m$.

Without loss of generality, suppose $\delta > \varepsilon$. If $d_r > d_j$ then $d_r \geq \varepsilon \geq d_{r-1} \geq d_j \geq \delta$ which is a contradiction. Thus, $d_j \geq d_r$.

If $d_j > d_r$, then

$$\begin{aligned}\delta - \varepsilon &\geq d_j - d_{j-1} - d_r + d_s \\ &\geq d_j - d_{j-1} - d_r \\ &\geq d_j - 2d_{j-1} \\ &\geq 2.\end{aligned}$$

Furthermore, if $d_j = d_r$, then $\delta - \varepsilon = d_s - d_j \geq d_s - d_{s-1} \geq 2$. Hence, $\delta - \varepsilon \geq 2$ for all $\delta > \varepsilon$ in Δ^+. Equivalently, $\delta_l - \delta_{l-1} \geq 2$ for all $1 < l \leq M$.

Since $d_j - d_i = \delta$ is an arbitrary element of Δ^+, then $\Delta^+ \cap (\Delta^+ - 1) = \varnothing$. Thus, $\mathcal{M}$ is partial by Theorem 13.1 and, based on the proof of Lemma 14.3, we only need show that $\mu(h) \leq 1$ for all $0 < h < n-1$. Since $\mu(h) = \#D \cap (D+h) > 0$ if and only if $h \in D - D = \Delta$, then it is sufficient to show that $\mu(\delta) = 1$ for any nonzero $\delta \in \Delta^+$.

Let $\delta = d_j - d_i$ and $\varepsilon = d_r - d_s$ be arbitrary elements of Δ^+. Suppose $\delta = \varepsilon$. Since $d_{r-1} < \delta \leq d_r$ and $d_{j-1} < \varepsilon \leq d_j$, then necessarily $d_j = d_r$. Similarly, $\delta = d_j - d_i = d_j - d_s = \varepsilon$ implies $d_i = d_s$.

Therefore, every nonzero $\delta \in \Delta^+$ is uniquely generated by a pair of digits $d_i, d_j \in D$ where $1 \leq i < j \leq m$. Hence, for all $\delta > 0$, $\mu(\delta) = 1$. $\square$

Example 14.5. Let $D = \{0, 2, 6\}$ for $n = 7$. Since $d_j = 2(d_{j-1} + 1)$ for $1 < j \leq 3$, then C is sparse. Furthermore, $\mathcal{F}^+ = [0, 1]$ by Corollary 9.5 and $\mathcal{E} = \mathcal{F}^+$ by Lemma 14.3.

We can apply Theorem 13.8 so that $\mathcal{F}_\alpha$ is dense in $\mathcal{F}$ for all $0 \leq \alpha \leq 1$. In this example, C is also a regular deleted digits Cantor set and the same results apply by arguments in Subsection 14.3. Since many of our results are obtained by analyzing the matrices $\eta(h)$ for $0 \leq h \leq n-1$, we list all 7 partial matrices of $\mathcal{M}$:

$$\eta(0) = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \qquad \eta(6) = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$$

$$\eta(1) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \eta(5) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\eta(2) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \eta(4) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\eta(3) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Example 14.6. Let $D = \{0, 2, 7, 17\}$ for $n = 18$. Since $d_j \geq 2(d_{j-1} + 1)$ for all $1 < j \leq 4$, then C is a sparse deleted digits Cantor set which is also irregular. Thus, $\mathcal{E} = \mathcal{F}^+$ by Theorem 13.1 where $\mathcal{F}^+$ is a self-similar, proper subset of the interval $[-1, 1]$ by Proposition 9.4. In particular,

$$\mathcal{F} = (-\mathcal{E}) \cup \mathcal{E} = \bigcup_{\delta \in \Delta} g_\delta(\mathcal{F}).$$

Note that the set of nonnegative differences $\Delta^+ = \{0, 2, 5, 7, 10, 15, 17\}$ is consistent with the proof of Proposition 14.4. Furthermore, since $d_m = n - 1$, then we can apply the results of Theorem 13.8. We list all 18 partial matrices of $\mathcal{M}$:

$$\eta(0) = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix} \qquad \eta(17) = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} \qquad \eta(5) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \qquad \eta(12) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\eta(1) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \eta(16) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \eta(6) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \qquad \eta(11) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\eta(2) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \eta(15) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \eta(7) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \eta(10) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\eta(3) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad \eta(14) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \qquad \eta(8) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad \eta(9) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

$$\eta(4) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \qquad \eta(13) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

15. Ongoing Work

There are as yet some unanswered questions based on the results above. Example 12.7 shows that it is possible to generate interval or potential interval cases which do not contain points in $C \cap (C + t)$, however it is not readily apparent how we can identify these "forbidden" combinations of digits.

Additionally, Example 12.13 and Example 12.11 seem to indicate that for any deleted digits Cantor set C there exists a value $0 \leq a < 1$ such that $a \leq \alpha \leq 1$ implies $\mathcal{F}_\alpha$ is dense in $\mathcal{F}$. By Theorem 12.3, $\mathcal{F}_1$ is always a countable dense subset $\mathcal{F}$, however the infimum a is not immediately calculable.

Finally, all deleted digits Cantor sets are calculated using specific n-ary intervals, however it may be possible to expand the results to more general Cantor sets. The Middle Thirds Cantor set is a thoroughly investigated case and has been expanded by removing the open "middle" $1 - 2\beta$ for some real value $0 < \beta < \frac{1}{2}$. It may be possible to expand the definition of deleted digits sets by choosing $D = \{0 \leq r_1 < r_2 < \cdots < r_{2m} \leq 1\}$ such that $r_i \in \mathbb{R}$ and define the contraction mappings $f_i(x) = r_{2i-1} + (r_{2i} - r_{2i-1})\,x$.

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References

[Bar93] Barnsley, Michael F., *Fractals Everywhere*, Morgan Kaufmann, San Francisco (1993).

[DaTi08] Dai, Meifeng and Tian, Lixin, On the intersection of an m-part uniform Cantor Set with its rational translations, *Chaos, Solitons & Fractals* **38** (2008), 962-969.

[DaHu95] Davis, GJ and Hu, T-Y, On the structure of the intersection of two middle thirds Cantor sets, *Publ Math Darc* **39** (1995), 43-60.

[Fal90] Falconer, K.J., *The Geometry of Fractal Sets*, Cambridge University Press, New York (1990).

[Hut81] Hutchinson, John E., Fractals and Self-Similarity, *Indiana University Mathematics Journal* **30** (1981), 713-747.

[LiNe04] Li, Jun and Nekka, Fahima, Intersection of triadic Cantor sets with their translates - II. Hausdorff measure spectrum function and its introduction for the classification of Cantor sets, *Chaos, Solitons & Fractals* **19** (2004), 35-46.

[NeLi02] Nekka, Fahima and Li, Jun, Intersections of triadic Cantor sets with their rational translates - I, *Chaos, Solitons & Fractals* **13** (2002), 1807-1817.

[PePh11] Pedersen, Steen and Phillips, Jason, Intersections of Certain Deleted Digits Sets, *Fractals* to appear.