2011

Analytical and Experimental Vibration Analysis of Variable Update Rate Waveform Generation

Joshua F. Mark
Wright State University

Follow this and additional works at: https://corescholar.libraries.wright.edu/etd_all

Part of the Mechanical Engineering Commons

Repository Citation
https://corescholar.libraries.wright.edu/etd_all/521

This Thesis is brought to you for free and open access by the Theses and Dissertations at CORE Scholar. It has been accepted for inclusion in Browse all Theses and Dissertations by an authorized administrator of CORE Scholar. For more information, please contact corescholar@www.libraries.wright.edu, library-corescholar@wright.edu.
Analytical and Experimental Vibration Analysis of Variable Update Rate Waveform Generation

A thesis submitted in partial fulfillment of the requirements for the degree of Master of Science in Engineering

by

Joshua F. Mark
B.S.M.E., Cedarville University, 2004

2011
Wright State University
I HEREBY RECOMMEND THAT THE THESIS PREPARED UNDER MY SUPERVISION BY Joshua F. Mark ENTITLED Analytical and Experimental Vibration Analysis of Variable Update Rate Waveform Generation BE ACCEPTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF Master of Science in Engineering.

______________________________
Joseph C. Slater, Ph.D., P.E.
Thesis Director

______________________________
George P. G. Huang, Ph.D., P.E.
Chair, Department of Mechanical and Materials Engineering

Committee on Final Examination

______________________________
Joseph C. Slater, Ph.D., P.E.

______________________________
Fred D. Garber, Ph.D.

______________________________
Tommy J. George, Ph.D.

______________________________
Andrew Hsu, Ph.D.
Dean, School of Graduate Studies
ABSTRACT

Mark, Joshua. M.S.Egr., Department of Mechanical and Materials Engineering, Wright State University, 2011. Analytical and Experimental Vibration Analysis of Variable Update Rate Waveform Generation.

Typical vibration analysis of turbine engine components incorporates the use of a function generator to produce a signal that is routed to an excitation source as the forcing function of the test specimen. These waveforms are constructed by varying the amplitude of the signal over time with a fixed update rate (time increment between samples). This research investigates generating chirp waveforms by storing only one sinusoid of points and using an external timing signal to repetitively send out these stored data points. This results in varying the amplitude of the signal over time as well as the update rate. The update rate varies linearly as the frequency varies throughout the chirp. The number of samples stored is fixed as only one sinusoid is stored in memory. This results in a degraded waveform with step changes in the voltages of the analog output signal. This research incorporates a high speed analog output device from National Instruments used to generate the waveform built from user inputs for sweep range, time, and desired samples/cycle. For this research, both single and multiple degree of freedom systems were used to analytically predict the response of the system to the degraded waveforms. A series of experimental tests was conducted using a cantilevered beam to validate the analytical predictions. The response of the test article was captured using a scanning laser vibrometer from which the frequency response function (FRF) was calculated, and in turn, the natural frequencies, mode shapes, and damping characteristics were determined. The differences in the responses of the test article were quantified to determine the effect of the degraded waveform and the minimum number of samples/cycle in a waveform necessary to generate a signal sufficient for accurate modal analysis. A simulated bladed disk was modeled in state space to quantify the accuracy of modal analysis implementing the variable update rate waveforms with traveling wave excitation.
# Contents

1 Introduction  
   1.1 Motivation ................................................. 1  
   1.2 Literature Review ........................................ 2  
   1.3 Overview and Contributions  
      1.3.1 Overview ........................................... 4  
      1.3.2 Contributions ..................................... 4  

2 Background  
   2.1 Engine Order Excitation .................................. 6  
   2.2 Traveling Wave Excitation (TWE)  
      2.2.1 Methodology ....................................... 9  
      2.2.2 Future Design .................................... 12  
   2.3 Waveform Fidelity  
      2.3.1 Vertical (Amplitude) Resolution ................... 13  
      2.3.2 Horizontal (Timing) Resolution ................. 15  
      2.3.3 Nyquist Sampling Theorem ....................... 15  
   2.4 Modal Analysis  
      2.4.1 Transfer Functions ................................ 16  
      2.4.2 Modal Assurance Criteria (MAC) ................. 17  
   2.5 Waveform Generation Theory  
      2.5.1 Linear Chirp Waveform ............................ 18  

3 Analytical Research  
   3.1 Single Degree of Freedom Model ....................... 29  
   3.2 Multiple Degree of Freedom Model ..................... 30  
   3.3 Beam Mesh Convergence .................................. 30  
   3.4 Beam Model Reduction .................................. 32  
   3.5 Disk Mesh Convergence .................................. 37  
   3.6 Disk Model Reduction .................................. 38  
   3.7 Waveform Generation .................................... 43  
   3.8 Linear Time-Invariant Simulation ...................... 46  
      3.8.1 Single Degree of Freedom Simulation .......... 46
List of Figures

1.1 Block diagram of an arbitrary waveform generator. . . . . . . . . . . . . . 3
2.1 Example compressor layout. . . . . . . . . . . . . . . . . . . . . . . . . . 7
2.2 Example Campbell diagram. . . . . . . . . . . . . . . . . . . . . . . . . . 8
2.3 Traveling wave and signals for 12EO with 5 blades. . . . . . . . . . . . . . 10
2.4 Traveling Wave Excitation (TWE) System diagram. . . . . . . . . . . . . . 11
2.5 Multiple Engine Order TWE System diagram. . . . . . . . . . . . . . . . . 13
2.6 Single degree of freedom system. . . . . . . . . . . . . . . . . . . . . . . . 16
2.7 First 3 cycles of a linear chirp waveform capture. . . . . . . . . . . . . . 19
2.8 Linear chirp waveform example with a constant update rate. . . . . . . . 20
2.9 One cycle illustrating 10 samples/cycle. . . . . . . . . . . . . . . . . . . . 23
2.10 Example of multiplied waveform (10 samples/cycle). . . . . . . . . . . . . 25
2.11 Example of TTL signal of multiplied waveform (10 samples/cycle). . . . . 25
2.12 Example of ideal chirp and TTL signal (10 samples/cycle). . . . . . . . . 26
2.13 Example of SPC waveform and TTL signal (10 samples/cycle). . . . . . . . 26
2.14 Example of SPC waveform and TTL signal (10 samples/cycle). . . . . . . . 27
2.15 All waveforms for 1-11 Hz generation (10 samples/cycle). . . . . . . . . . 27
3.1 ANSYS models for multiple degree of freedom simulations. . . . . . . . . . 31
3.2 ANSYS meshes for beam frequency convergence study. . . . . . . . . . . . 31
3.3 Convergence study of elements through length and width. . . . . . . . . . . 32
3.4 Convergence study of elements through thickness. . . . . . . . . . . . . . 33
3.5 Comparison of compliance FRF between SDOF and MDOF model. . . . . . 35
3.6 Comparison of normalized mode shapes for full and reduced beam model. . 36
3.7 Comparison of mode shapes for beam and single blade models. . . . . . . . 37
3.8 ANSYS meshes for beam convergence study. . . . . . . . . . . . . . . . . 38
3.9 Convergence study of disk elements across the arc and radially. . . . . . . 39
3.10 Nodes for reduced model validation for 30 kept degrees of freedom. . . . . 40
3.11 Nodes for reduced model validation for 10 kept degrees of freedom. . . . . 41
3.12 Modeshape comparison between 30 and 10 active degrees of freedom. . . 42
3.13 MAC between 30 and 10 active degrees of freedom reduced models. . . . . 42
3.14 Example of good and poor sampling and resulting FFTs . . . . . . . . . . . 44
3.15 Time data of ideal chirp and SPC waveforms. . . . . . . . . . . . . . . . . 46
3.16 Time data for forcing function and SDOF system response. 47
3.17 FFT of forcing function and SDOF system response. 50
3.18 Resulting Frequency Response Function of SDOF system. 51

4.1 Test equipment setup in TEFF laser vibrometry laboratory. 56
4.2 Polytec scanning laser vibrometer. 57
4.3 Rendering of thick root beam used in testing. 59
4.4 First and third bending mode shapes using beam theory. 60

5.1 SDOF Mode 1 damping results using ideal chirp waveforms. 66
5.2 SDOF Mode 1 frequency results using ideal chirp waveforms. 66
5.3 SDOF Mode 1 results using ideal chirp waveforms. 67
5.4 SDOF Mode 3 results using ideal chirp waveforms. 68
5.5 Damping results for SDOF Mode 1, 100Hz bandwidth. 70
5.6 Modal frequency results for SDOF Mode 1, 100 Hz bandwidth. 71
5.7 Damping results for SDOF Mode 3 results using SPC waveforms. 72
5.8 Modal frequency results for SDOF Mode 3 results using SPC waveforms. 73
5.9 Experimental damping results for Mode 3 using SPC waveforms. 74
5.10 Experimental frequency results for Mode 3 using SPC waveforms. 75
5.11 Damping results for finger disk using ideal chirp waveforms. 76
5.12 Modal frequency results for finger disk using ideal chirp waveforms. 77
5.13 Damping results for finger disk using SPC waveforms. 78
5.14 Modal frequency results for finger disk using SPC waveforms. 79
5.15 MAC results for finger disk using SPC waveforms. 80

B.1 Example waveform for 1-11 Hz sweep using 3 samples/cycle. 89
B.2 Example waveform for 1-11 Hz sweep using 4 samples/cycle. 89
B.3 Example waveform for 1-11 Hz sweep using 5 samples/cycle. 90
B.4 Example waveform for 1-11 Hz sweep using 6 samples/cycle. 90
B.5 Example waveform for 1-11 Hz sweep using 7 samples/cycle. 90
B.6 Example waveform for 1-11 Hz sweep using 8 samples/cycle. 91
B.7 Example waveform for 1-11 Hz sweep using 9 samples/cycle. 91
B.8 Example waveform for 1-11 Hz sweep using 10 samples/cycle. 91
B.9 Example waveform for 1-11 Hz sweep using 20 samples/cycle. 92
B.10 Example waveform for 1-11 Hz sweep using 50 samples/cycle. 92
B.11 Example waveform for 1-11 Hz sweep using 100 samples/cycle. 92
B.12 Example waveform for 1-11 Hz sweep using 500 samples/cycle. 93

C.1 Example TWE waveforms for 1-11 Hz sweep using 3 samples/cycle. 94
C.2 Example TWE waveforms for 1-11 Hz sweep using 4 samples/cycle. 95
C.3 Example TWE waveforms for 1-11 Hz sweep using 5 samples/cycle. 95
C.4 Example TWE waveforms for 1-11 Hz sweep using 6 samples/cycle. 96
C.5 Example TWE waveforms for 1-11 Hz sweep using 7 samples/cycle. 96
C.6 Example TWE waveforms for 1-11 Hz sweep using 8 samples/cycle. 97
C.7 Example TWE waveforms for 1-11 Hz sweep using 9 samples/cycle. 97
C.8 Example TWE waveforms for 1-11 Hz sweep using 10 samples/cycle. 98
List of Tables

2.1 Parameters for the waveform of Figure 2.8 ............................................. 22
2.2 Parameters for the waveform of Figure 2.15 ........................................... 28

3.1 SDOF properties to match beam modal frequencies .............................. 30
3.2 Modal frequencies of full and reduced beam model ................................. 36
3.3 Modal frequencies of cantilevered beam and single blade ......................... 37
3.4 Minimum sampling frequencies for SDOF models .................................. 45
3.5 Test matrix of waveforms for analytical SDOF model (Mode 1) ................ 53
3.6 Test matrix of waveforms for analytical SDOF model (Mode 1) continued .... 54
3.7 Complete test matrix of waveforms for analytical SDOF model (Mode 3) ... 55
3.8 Complete test matrix of TWE waveforms for analytical MDOF model ....... 55

4.1 Specifications for analog output devices ................................................. 58
4.2 Beam geometry ..................................................................................... 59
4.3 Beam material properties ...................................................................... 60
4.4 Beam Theory Modal Frequencies ......................................................... 60
4.5 Complete test matrix of waveforms for experimental bench test ............... 64
Acknowledgment

First and foremost, I would like to thank my LORD and Savior Jesus Christ for sustaining me and my family through this time. I praise Him for gifting me with the ability and opportunity to complete this thesis. Without His saving grace and endless mercy, I would be nothing, for I can do nothing apart from Him. I would particularly like to thank my wife and best friend, Sarah Mark, for encouraging me through all of the long hours spent dedicated to finishing this. She is the love of my life! I would also like to thank my grandma Lonie for taking her time to edit my document. I appreciate everything she does for me and my family, and I love her dearly.

I have had a multitude of colleagues support my research efforts to whom I am greatly indebted. I would like to thank all of my past co-workers in the Turbine Engine Fatigue Facility for allowing me to perform my research in their laboratory. I would like to extend my appreciation to Dr. Tommy George for his support and oversight of my research, to John Justice for catching all of the laughable mistakes in my codes, and to Joe Beck for sharing his knowledge of ANSYS modeling and SEREP. I am especially appreciative of all the discussion generated over coffee breaks with my friend and co-worker Brian Runyon. Much of my research was greatly influenced by our conversations and to him I owe a plethora of thanks.

I would like to extend my thanks as well to Dr. Fred Garber for his willingness to be on my committee. His guidance provided me with the opportunity to improve my technical writing abilities which made this thesis stronger. I would specifically like to thank my advisor, Dr. Joseph Slater, for his insight and patience. His instruction was critical to the completion of my research and I am grateful for his assistance in editing my seemingly endless document revisions.

Again, I thank God, to whom I owe all praise.
Dedicated to

My family... I love you all
Introduction

1.1 Motivation

The traveling wave excitation (TWE) system is designed to simulate engine order excitation in a stationary bladed disk. The traveling wave is used to excite the article to enable identification of the forced response localization and amplification due to mistuning as written in 2002 by Jones and Cross[15]. Gas turbine engines are becoming more complex, and to adequately and accurately characterize the response of a bladed disk, the excitation sources must replicate that of what is seen in the engine. The first generation TWE system uses a function generator, or an arbitrary waveform generator, with a fixed clock rate based on its horizontal resolution capability. In order to meet the expanding technology needs, the system needs to be able to adapt for multiple kinds of excitation. In order to expand the system for future needs, and remain flexible to excitation parameters, an alternative approach to signal generation will be explored.

The focus of this thesis is to analytically predict and experimentally validate the response of a cantilever beam to a variable update rate forcing function and to quantify an adequate number of samples per cycle for accurate vibration testing. Then, to analytically predict the response of a simulated bladed disk to traveling wave excitation implementing the variable update rate waveforms generated using the Single Cycle algorithm developed for this research.
1.2 Literature Review

Function generators have come a long way since 1951 when the Hewlett-Packard Company released its 202A that produced sine, square, and triangle waves at frequencies ranging from 0.008 to 1200 cycles per second (the Hertz had not been invented yet) in five bands. The 202A used vacuum tubes and weighed in at 38 pounds [24].

Since then, function generators have improved and continue to be used to generate signals to use as stimulus for electronic measurements. Depending on the requirements, there are various types of generators to meet the need. There are two main groups, mixed signal generators, comprised of arbitrary waveform and function generators, and logic sources, which generate pure digital signals, such as pulse or pattern generators [12].

Mixed signal generators are designed to produce waveforms with analog characteristics including sine, triangle waves, and square waves. The signals can be controlled by amplitude, frequency, phase, and DC offset. Arbitrary waveform generators (AWG) use digital sampling techniques to build waveforms based on desired shape. The AWG generates a periodic waveform for which the user defines the shape of one period and is defined by a mathematical expression in the form of a set of sample points [5].

Arbitrary function generators (AFG) use a similar sampling technique to build the desired waveform based on function of the signal, storing just the sampled portion of the entire waveform. The AFG is useful in applications where rapid variations in the waveform are required. Arbitrary waveform generators and function generators produce their analog signals in reference to their sample or clock rate. This is specified for each device based on its hardware capability. These clock rates are fixed throughout the waveform and affect the frequency and fidelity of the generated signal [12]. The lower the clock rate, the lower the fidelity becomes. To produce the same signal with fewer points stored, the sample rate is reduced, saving buffer memory, but lowering the fidelity of the waveform because there are fewer samples per cycle. The sampled data stored in the memory is read through the shift register and fed to the digital-to-analog converter (DAC). The frequency and amplitude
of a set of sampled points stored in memory can be controlled and the electronic circuitry generates the waveform passing repeatedly through the set. A block diagram of a simplified AWG is shown in Figure 1.1

![Block Diagram of AWG](image.png)

Figure 1.1: Block diagram of an arbitrary waveform generator.

Methods of interest for this research include not only lowering the stored samples of an entire sinusoidal sweep waveform, but also storing just one sampled cycle or sinusoid and repeatedly generating the waveform while increasing the time step between samples to create a sine sweep. This technique questions the necessary number of samples per cycle to produce a signal of high enough fidelity to accurately conduct modal analysis of the component under test.

There has been research conducted for direct digital synthesis waveform generation to increase bandwidth capabilities [25]. However, the architecture of the system was still constrained to a fixed clock rate and produced the signal based on the accumulator values and the sine function data stored in ROM. The output in that work was a constant frequency waveform.

The spectral components of waveforms generated from function generators and those created by digital signal processors (DSP) have also been compared for waveforms with low number of samples stored in memory. Based on the total harmonic distortion (THD) of generation signals, Sia concluded that eight (8) samples per cycle was adequate for accuracy. The THD of the DSP generated signal was comparable to commercially available signal generators with higher sampled waveforms [27]. However, these tests were also based on constant frequencies with a fixed clock rate, and only focused on the signal itself.
and not the corresponding response of a system.

Even though most current waveform generators are based on digital architecture and have variable clock rates, this doesn’t necessarily mean that the clock varies throughout the signal generation. Arbitrary waveform generators have variable clock rates, but this only allows the AWG to optimize the clock settings for the desired output frequency [10]. This allows for fewer points to be stored in memory and simply increases the update rate to increase the frequency of the signal. However, this modulation of clock speed does not occur during signal generation. Typically, AWG are used to repeat stored signals at a fixed clock rate, which is variable based on the frequency of the desired waveform.

The objective of this research is to develop a means of signal generation that is reliable and able to be produced using available hardware. The development and implementation of this algorithm is documented in the following sections.

1.3 Overview and Contributions

1.3.1 Overview

This thesis provides insight into vibration testing and how the response of a system can vary based on the fidelity of the excitation waveform. The understanding of these characteristics is very important in accurately determining the damping, mode shape, and fundamental frequencies of turbine engine components. Function generators used in vibration testing are based on a fixed horizontal resolution defined by the sampling frequency of the device.

1.3.2 Contributions

- This thesis establishes a new methodology for waveform generation using a variable update rate for frequency sweeps.
• An analytical procedure for obtaining reliable frequency and damping data for a single-degree-of-freedom and multiple-degree-of-freedom system models will be provided.

• Both analytical predictions and experimental results will demonstrate that it is possible to implement a variable update rate forcing function for accurate modal analysis.

• An adequate number of samples per cycle for accurate vibration testing will be determined and validated through experimental testing.
Background

2.1 Engine Order Excitation

Rotating components in a gas turbine engine, such as fans, compressors, and turbines, are subjected to various pressure disturbances caused by struts, vanes, stators, or other obstructions in the gas path. These disturbances create an engine order excitation, where the engine order refers to the number of equally spaced disturbances either upstream or downstream of the bladed disk [21]. The resulting blade vibration can lead to high cycle fatigue (HCF) failures as cycles quickly accumulate due to relatively high modal frequencies of turbine engine components.

The frequency of the excitation is related to the rotating speed of the engine and the number of struts or vanes, or the spatial content of the airflow distortion pattern. The fundamental harmonic of the excitation equals the product of the speed, $R$, in revolutions per minute, and the order of the excitation, $c$ (e.g., number of upstream vanes) [22]

$$f = \frac{cR}{60}$$

(2.1)

The method commonly used to analyze the vibration excitation of a turbine engine component is to create a Campbell diagram. The Campbell diagram is an overall view of the vibration excitation that can occur on an operating system [8]. An example Campbell
diagram plot is shown in Figure 2.2. Engine rotational speed is along the \( X \)-axis and the system frequency along the \( Y \)-axis. This form of design study is necessary when designing rotating components to determine what responses, for a given rotor assembly, are excited by a given frequency and its harmonics, or subharmonics [6].

As a simplified example, a compressor stage with a 400 Hz fundamental frequency and four downstream vanes, as shown in Figure 2.1, would experience a 4-engine-order driver (4EO), as well as possibly less intense drivers of 2EO and 8EO. The fundamental mode would be driven at 24,000 rpm for a 1 per revolution excitation, or 1EO.

\[
400 \text{ Hz} \times 60 = 24,000 \text{ rpm}
\]

It is apparent from the Campbell diagram that a forcing frequency of 6,000 rpm would also drive the 400 Hz natural frequency of the blade due to the 4 vanes behind the blade row causing a 4EO driver. This is calculated using Equation (2.1).

\[
400 \text{ Hz} \times 60 = (4EO \times 6,000 \text{ rpm}) = 24,000 \text{ rpm}
\]

Excitation would also be seen at the 2EO engine order at 12,000 rpm as a subharmonic of the fundamental frequency.

\[
400 \text{ Hz} \times 60 = (2EO \times 12,000 \text{ rpm}) = 24,000 \text{ rpm}
\]
Customarily, turbine engine components are designed to either avoid resonances in the operation speed range of the engine associated with known engine-order excitation sources or to place resonances in lower-speed regions and away from mission points in the cycle where sustained operation is expected, such as ground idle, flight idle, and max speeds.

The Campbell diagram is used to display the excitation frequencies due to known sources and compare those frequencies with the variation in vibratory frequencies of the blades with engine speed. Blade vibratory frequencies can change with engine speed due to temperature changes and centrifugal stiffening of the blade, as seen in the 8,000-12,000 rpm range in Figure 2.2. Centrifugal stiffening will cause an increase in frequency, while increased temperatures will decrease the frequencies of the blades. The example given has a dominating temperature effect as the modal frequencies decrease with higher rotor speeds. The intersection of engine-order excitation lines and blade-frequency lines indicates resonance at the crossing engine speed [22]. If, for the previous example, idle was at 6,000 rpm, and max speed was near 12,000 rpm, the blades would need to be redesigned to avoid these engine-order excitation regions.

Figure 2.2: Example Campbell diagram.
Once the design meets the design criteria for the vibration due to operating ranges based on the Campbell diagram, blades and rotors are manufactured and bench tests are conducted to check natural frequency bands. Strain gaged blades are also tested in rigs to determine the actual response and strain levels in the components.

Resonances in blades occur due to the upstream or downstream disturbances. The blade is excited as it rotates past the stationary vanes if the apparent frequency of pressure fields coincides with one of the natural frequencies of the airfoil. Components in the flow path of a turbine engine are exposed to unsteady aerodynamic forces, leading to forced vibration and potentially high cycle fatigue (HCF) that results from the temporally and/or spatially nonuniform flows [19]. These forces can cause excessive vibration stress that can lead to high cycle fatigue (HCF) failures. The ability to identify and validate system modes is crucial to turbine engine safety.

2.2 Traveling Wave Excitation (TWE)

The traveling wave system was designed to simulate engine order excitation in a stationary bladed disk to identify the forced response localization and amplification due to mistuning. It can excite the component according to the operating range and verify the predicted responses and critical engine order crossings in the Campbell diagram. The system can accommodate bladed disks and integrally bladed rotors of varying sizes and number of blades using either acoustic or magnetic excitation [15]. The traveling wave methodology and limitations of the current system are addressed in the following sections.

2.2.1 Methodology

Engine order excitation is simulated in a stationary bladed disk by applying harmonic excitation to all blades simultaneously. The forcing function is modeled as a sine function.
This simulates the driving excitation seen as the blades passes each of the disturbances. As an example, Figure 2.3(a) depicts a $12EO$ excitation as a sine wave wrapped around the rotor. It is evident there would be 12 drivers as the blades rotated through the disturbances.

The traveling wave system uses a fixed rotor, and mathematically rotates the excitation by changing the phase of each blade’s source by the inter-blade phase angle which is calculated by Equation (2.2).

$$\theta = 2\pi \frac{c}{n} \quad (2.2)$$

Again, $c$ is the desired engine order, and $n$ is the number of blades in the rotor. As an example, the excitation signals of a 5 bladed rotor to a $12EO$ driver are shown in Figure 2.3(b).

![Traveling wave around rotor](image1)

![Excitation signals for each blade](image2)

Figure 2.3: Traveling wave and signals for $12EO$ with 5 blades.

The current traveling wave system implements a phase shifting box to generate phase-shifted signals from sine and cosine inputs from a single function generator using the methodology explained in Appendix A. The flow of the current Traveling Wave Excitation (TWE) system is depicted in Figure 2.4. The function generator is used to produce the sine and cosine waveforms that are sent to the phase shifting box. This phase shifting box is
calibrated before the test and each channel is assigned a phase and voltage setting based on the desired engine order. These signals are all sent separately to individual amplifiers and continued through transformers to the excitation source. Typically, the excitation hardware used is non-contacting electromagnets [16]. These are used mainly for the frequency range and flexibility of usage.

The device used to measure the response of the system is a laser vibrometer. The laser vibrometer uses the Doppler effect to measure the velocity of the test specimen and returns this response to the laser controller. The laser controller uses the zero-degree phase sine waveform from the function generator as a reference signal, and performs a scan of all the specified points. The Frequency Response Function is then calculated by transforming the input-output (force-response) time data into the frequency domain using a Fast Fourier Transform (FFT) [4]. The FRF for each data point can then be exported to perform post test data reduction to determine system properties such as damping, mode frequencies, and mode shapes for the component under test.

The results of this post test processing can create an as-tested Campbell diagram to determine the validity of the design analysis and verify the frequencies of the actual component as compared to the design intent model that was used to create the ideal Campbell diagram.
This is a powerful, affordable method used by most turbine engine manufacturers as a step in the design validation process or to help determine root cause for failure in fielded systems. One downfall of the Traveling Wave system is that the frequency and damping effects of the internal operating temperatures, airflow, and centrifugal stiffening of the blade due to rotation is not captured in bench tests. But this is a very affordable method to validate the analytical models and ensure safety before spin rig tests or operational engine tests.

### 2.2.2 Future Design

The TWE system needs to be able to generate the excitation drivers seen in current and future gas turbine engines. This includes the ability to generate multiple groups of differing engine order signals. The current system must be duplicated to create a second engine order excitation group, and then again for each additional engine order desired. Duplicate systems would be an additional expense and occupy more laboratory space as more variables are desired and the system continues to expand. The next generation TWE system will address these requirements in a cost-effective system. By using LabVIEW to build and store, or stream, the phase shifted waveforms for each channel, the phase shifting box can be eliminated and replaced by a National Instruments card (PCI-6713/6731) already available and capable of analog output of waveform signals generated in LabVIEW. The function generator can also be eliminated since the laser controller outputs a TTL reference signal for each test sequence. This system is setup as shown in Figure 2.5.

The goal of the future system is to develop a method of waveform generation that maximizes output channels to support testing of high blade count rotors. The available hardware is state of the art analog output devices, but has limitations with regards to resolution, memory, and update rate. The problem then becomes determining the method of developing and generating the waveforms of necessary fidelity.
Figure 2.5: Multiple Engine Order TWE System diagram.

2.3 Waveform Fidelity

The complete measurement solution must be evaluated for both the acquisition system and the signal generator to have an effective design. Given the acquisition system in place, this research will concentrate on the design of the signal generator to complement the acquisition system.

There are certain aspects of waveform generation methodology that are important to understand when developing a signal generator. The first being to use analog or digital technology. Most signal generators today are based on digital technology [11]. This research investigates the limits to the fidelity of the signal and the corresponding response of the test article.

A mixed signal generator (arbitrary waveform generator or arbitrary/function generator) is needed to create the required sine sweep or periodic chirp waveform desired for this testing. The architecture of the analog signal generator will not be altered, but the stored data will be altered in an attempt to produce the same desired waveform for a linear sine sweep. This will be accomplished by introducing an external timing signal to drive the update clock of the AWG.

To investigate the fidelity of the waveform, the amplitude and timing attributes of the
signal, as well as the sampling rate necessary to avoid aliasing will be discussed in the following sections.

2.3.1 Vertical (Amplitude) Resolution

The resolution of an AWG is expressed as the resolution of the digital-to-analog converter. This is fixed based on the specified hardware and is measured by the unit of bits. The vertical resolution determines how accurately you can design the waveform with respect to its amplitude. For example, an $n$-bit system distinguishes amplitude changes in voltage step sizes equal to

$$\Delta V = \frac{V_{\text{max}}}{2^n} \quad (2.3)$$

where $V_{\text{max}}$ is the maximum amplitude of the waveform. In typical generators, this varies from 8 to 16 bits [17]. For a $2 \, V_{\text{pp}}$ waveform, using a generator with 16 bit vertical resolution, the amplitude changes are determined by Equation (2.3) where

$$\Delta V = \frac{V_{\text{max}}}{2^n} = \frac{2V}{2^{16}} = 30 \mu V$$

This $30 \, \mu V$ amplitude change for a $2 \, V$ waveform is consistent with the equipment used in this research listed in Table 4.1.
2.3.2 Horizontal (Timing) Resolution

The timing resolution is determined by calculating the inverse of the sampling frequency, \( f_{\text{samp}} \).

\[
T = \frac{1}{f_{\text{samp}}} \tag{2.4}
\]

The timing resolution is typically fixed for a given signal since the sampling frequency is fixed. However, the waveform generated through Single Cycle algorithm, as developed for this research, has a variable update rate resulting in effect a variable timing resolution. The update rate increases linearly with the frequency and will be discussed in more detail in Section 2.5.1.

2.3.3 Nyquist Sampling Theorem

The distortion of a waveform, called aliasing, occurs due to improper sampling of the signal. The Nyquist sampling theorem defines the nominal sampling interval required to avoid aliasing stating the sampling frequency must be more than twice that of the highest spectral frequency component of the generated signal to avoid aliasing [28]. This ensures the sampled waveform will have enough points to accurately produce the details of the desired signal while avoiding aliasing. In practice, 2.5 times is used and is stated mathematically as

\[
f_s = 2.5f_{\text{max}} \tag{2.5}
\]

where \( f_s \) is the sampling frequency, and \( f_{\text{max}} \) is the highest frequency contained in the signal.

This criteria will become critical when selecting a sampling rate to generate the timing signal for the waveforms with variable update rate. These details are discussed in Section 3.7.
2.4 Modal Analysis

It is necessary, once bench testing is complete, to reduce the data taken and properly identify the system’s vibratory characteristics. By identifying the system’s transfer function, the system’s vibratory properties can then be determined by characterizing the response of a system to forced vibrations by relating the input and output of the system.

2.4.1 Transfer Functions

Because the nature of this research is focused around a single mode of a system, it will be modeled as a single degree of freedom (SDOF) system illustrated in Figure 2.6. This will allow for simplified models to be run and reduce the computational requirement drastically. The SDOF model will be defined differently for each mode of interest by modifying the system’s properties to match the mode frequencies identified through beam theory addressed later. For the single degree of freedom (SDOF) system, the transfer function ($H(s)$) can be determined from the system’s equation of motion defined by

\[ m\ddot{x} + c\dot{x} + kx = F(t) \] (2.6)

![Figure 2.6: Single degree of freedom system.](image)
Transfer functions represent a method of solving linear ordinary differential equations for linear systems. The initial displacement and velocity of the system were set to zero. The equation of motion was transformed into the Laplace domain by the following relations

\[
x(t) = x_0 e^{st}
\] (2.7)

\[
\dot{x}(t) = sx_0 e^{st} = sx
\] (2.8)

\[
\ddot{x}(t) = s^2x_0 e^{st} = s^2x
\] (2.9)

Substituting these back into Equation (2.6) results in

\[
ms^2x + csx + kx = F
\] (2.10)

In order to describe the effect of the system on the input, the transfer function \(H(s)\) can be defined as the ratio of the output \(x\) to the input \(F\):

\[
H(s) = \frac{x}{F} = \frac{1}{ms^2 + cs + k}
\] (2.11)

This will be the transfer function used in Chapter 3 to create a single degree of freedom model for modal analysis simulations. From the response of the system, the Frequency Response Function (FRF) will be used to estimate system modal parameters.

### 2.4.2 Modal Assurance Criteria (MAC)

The Modal Assurance Criteria (MAC) was used to calculate the orthogonality of the identified mode shapes to the ideal mode shapes from the analytical estimations. The MAC was originally created because of the need to gage quality assurance of experimental mode
shapes [2]. To validate an experimental model, the experimental mode shapes, derived from the frequency response functions, are used in a weighted orthogonality check. This is completed by using an analytical mass or stiffness matrix as the weighting matrix. In this research, the experimental modal shapes and an analytically determined mass matrix are used as follows:

$$MAC = \frac{(\psi_l^T [M] \psi_m)^2}{(\psi_l^T [M] \psi_l) (\psi_m^T [M] \psi_m)}$$  \hfill (2.12)

This allows the orthogonality of the mode shapes to be compared between experimental and analytical results. For the analytical simulations in this research, the resulting mode shapes from both the ideal chirp and the degraded waveforms are compared.

### 2.5 Waveform Generation Theory

Typical vibration analysis employs a waveform generator as a forcing function for the system under testing. Function generators are based upon a constant update rate hard coded into the algorithm within the system. The function generator previously used had a maximum clock, or sample, rate of 40 mega-samples per second (MS/s). Due to some time domain inconsistencies of signals generated by this function generator, the internal function generator of the Polytec laser vibrometry system was used, which is limited to 1MS/s.

Function generators typically calculate and store the entire linear chirp waveform in a memory table. However, they are limited by a minimum and maximum sweep time.

The objective of this research is to develop a means of signal generation that is reliable and able to be produced using available hardware. The development and implementation of this code is shown in the following sections.
2.5.1 Linear Chirp Waveform

A linear chirp waveform is comprised of several inputs to build the waveform between two specified frequencies over a specified length of time. This waveform was chosen for this research because of its distribution of energy across the frequency range of the sweep as illustrated by the FFT of the signal. This is discussed later and is illustrated in Figure 3.17. These parameters include the beginning frequency, \( f_1 \), ending frequency, \( f_2 \), samples per cycle, \( N \), and the total sweep time, \( T_s \), or period of the sweep.

An example of a linear chirp waveform is shown in Figure 2.7. The signal shown is an actual capture of a Wavetek 195 function generator with a sampling frequency of 41 kHz. This is the equipment currently used for the waveform generation of the TWE system.

![Figure 2.7: First 3 cycles of a linear chirp waveform capture.](image)

Typical Waveform Algorithm

Typical methods used to create waveforms are based on an equal time interval between samples, or constant update rate. The entire waveform must be built and stored in the waveform generator’s memory in order to generate the entire sweep. This is memory intensive and
may become difficult to store and transmit for high fidelity waveforms of multiple engine orders; therefore, the total number of samples stored must be reduced.

To reduce the total number of samples stored, and maintain the integrity of the waveform, the minimum number of samples in a given cycle, \( N \), is chosen. This in turn sets a constant time interval, \( \Delta t \), between samples as shown in Equation (2.13). This constant \( \Delta t \) is required by the hardware’s internal clock during configuration as a fixed update rate for analog output. Because there is a constant \( \Delta t \) between all samples, there are more samples per cycle for the lower frequencies in a signal. This is evident in Figure 2.8, where the red lines visually illustrate the constant \( \Delta t \), and the effective number of samples in any given cycle decrease as the frequency increases over time. The waveform shown is a linear chirp from 1-11 Hz. These frequencies were chosen to illustrate the waveform in an observable manner, and is used throughout this and following sections as an example.

![Linear chirp waveform example with a constant update rate.](image)

Figure 2.8: Linear chirp waveform example with a constant update rate.

Several components must be calculated in order to build the waveform. Given the beginning and ending frequencies (\( f_1 \) and \( f_2 \) respectively), the update rate or time interval, \( \Delta t \), between samples is defined as the ratio of the period of the ending frequency, \( T_e \), by
the desired number of samples per cycle, \( N \).

\[
\Delta t = \frac{T_e}{N} \quad (2.13)
\]

The period of the ending frequency of the signal, \( T_e \), is the time, in seconds, it takes to complete one full cycle of the ending frequency, or its period, which is the inverse of the ending frequency.

\[
T_e = \frac{1}{f_2} \quad (2.14)
\]

Substituting Equation (2.14) into Equation (2.13) results in the time interval between samples being determined by the inverse of the product of the ending frequency and the minimum desired number of samples per cycle.

\[
\Delta t = \frac{1}{f_2 N} \quad (2.15)
\]

A sinusoidal wave is calculated by [9]

\[
Y(t) = \sin(2\pi [ f(t)] t + \phi) \quad (2.16)
\]

where \( \phi \) is the phase and \( f(t) \) is the frequency as a function of time. In order to define the waveform by the beginning and ending frequencies, the spectral frequency function in Equation (2.17) is first substituted back into Equation (2.16).

\[
f_{spectral}(t) = \left( \frac{f_2 - f_1}{t_2 - t_1} \right) t + f_1 \quad (2.17)
\]

Integrating this over the time of the sweep (\( t_1 \) to \( t_2 \)) results in Equation (2.18) as Irvine
proved using the accumulate cycle function [14]. The frequency, \( f(t) \), then becomes

\[
f(t) = \left[ \frac{1}{2} \left\{ \frac{f_2 - f_1}{T_s} \right\} t + f_1 \right]
\]  
(2.18)

where the total duration of the signal is equal to the ending time

\[
T_s = t_2
\]  
(2.19)

because \( t_1 = 0 \) for all waveforms.

Substituting (2.18) back into the Equation (2.16), the final amplitude of the waveform is defined in Equation (2.20).

\[
Y(t) = \sin \left( 2\pi \left[ \frac{1}{2} \left\{ \frac{f_2 - f_1}{T_s} \right\} t + f_1 \right] t + \phi \right)
\]  
(2.20)

The total number of samples, \( n_T \), in the waveform is determined by

\[
n_T = \frac{T_s}{\Delta t} + 1
\]  
(2.21)

where \( \Delta t \) refers to the amount of time between the output samples of the function generator. Figure 2.8 illustrates the waveform generated using this process, and the parameters corresponding to this waveform are listed in Table 2.1.

<table>
<thead>
<tr>
<th>Begin Freq</th>
<th>End Freq</th>
<th>Sweep Time</th>
<th>Samples/Cycle</th>
<th>Time Interval</th>
<th>Samples Stored</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f_1 ) (Hz)</td>
<td>( f_2 ) (Hz)</td>
<td>( T_s ) (s)</td>
<td>( N )</td>
<td>( \Delta t ) (s)</td>
<td>( n_T )</td>
</tr>
<tr>
<td>1</td>
<td>11</td>
<td>0.5</td>
<td>10</td>
<td>0.0090909</td>
<td>56</td>
</tr>
</tbody>
</table>
**Single Cycle Algorithm**

The Single Cycle Algorithm was developed for this research as a means to reduce the total number of samples stored. Reducing the number of samples stored allows for more channels to be generated based on memory limitations of hardware. This method offers two key advantages over the typical equal interval algorithm. First, only one cycle of data is required to be stored into memory, and second, the numbers of samples per cycle, \( N \), are consistent throughout the waveform, so the fidelity actually increases across the frequency range.

The first step to the algorithm is computing the stored amplitudes of each sample. This is achieved by dividing one complete sinusoidal cycle into \( N \) number of segments, shown as the green dashed line, which results in \( N + 1 \) samples as illustrated in Figure 2.9. The first and last amplitudes of the sinusoid are the same; therefore, only the first \( N \) samples, shown in blue in Figure 2.9, are necessary to be stored in the memory. Because this cycle is continually repeated to generate the waveform, storing the last amplitude of the sinusoid, shown in red, would cause leakage in the signal, which would drastically affect the results in the frequency domain and cause varying excitation to the component under test [7].

The system is then set to output these stored samples according to an external timing reference signal from another device. This allows for a variable update rate which will

![Figure 2.9: One cycle illustrating 10 samples/cycle.](image-url)
create the desired frequency sweep. As the update rate increases, the stored samples are output at the increased rate which increases the frequency of the signal.

A standard TTL signal may be used as a timing source, determined by the multiple of the desired sine sweep based on the desired samples per cycle, $N$. The beginning ($f_1$) and ending ($f_2$) frequencies are multiplied by the desired samples per cycle, $N$, to calculate the beginning ($f_{1\text{multi}}$) and ending ($f_{2\text{multi}}$) frequencies of the new signal.

$$f_{1\text{multi}} = f_1N \quad (2.22a)$$
$$f_{2\text{multi}} = f_2N \quad (2.22b)$$

For the 1-11Hz example, implementing Equations (2.22a) and (2.22b) results in the beginning and ending frequencies of

$$f_{1\text{multi}} = 1 \text{ Hz} \times 10 \text{ spc} = 10 \text{ Hz}$$
$$f_{2\text{multi}} = 11 \text{ Hz} \times 10 \text{ spc} = 110 \text{ Hz}$$

Substituting these into Equation (2.20) results in the multiple waveform depicted in Figure 2.10.

From this waveform, the resulting TTL waveform is then used as the timing reference signal. The TTL signal is either high or low depending upon whether the signal is positive or negative. Figure 2.11 shows how the TTL is high for positive voltages of the waveform, and low, or zero, for negative voltages of the waveform.

This waveform will have $N$ sinusoidal cycles for every cycle of the parent waveform which can be seen in the comparison of the TTL signal and the typical sine sweep waveform in Figure 2.12.

For each rising edge of the TTL signal, the system will update the output waveform voltage from the single cycle amplitudes stored in memory. The voltage remains the same
until the system receives another rising edge of the TTL signal. These waveforms are therefore stepped as shown for the 1-11 Hz example in Figure 2.13. The resulting waveform using this methodology is termed the SPC waveform since it is based on a specified number of samples/cycle (spc). The example SPC waveform is compared in Figure 2.14 to the ideal chirp signal with constant horizontal (timing) resolution as generated by a function generator.

The function generator is then set up with the desired beginning and ending frequen-
cies, sweep time ($T_s$) and required voltage settings. If the function generator cannot output a TTL reference signal, then the output can be changed to a square wave and the voltage set to 4V peak-to-peak with a 2V offset for the TTL logic to work properly. These parameters can be substituted into Equation (2.23) resulting in the multiplied frequency waveform.
which is used as the timing signal for the sample per cycle (SPC) waveform.

\[
Y_{multi}(t) = \sin \left( 2\pi \left[ \frac{1}{2} \left\{ \frac{f_{2multi} - f_{1multi}}{T_s} \right\} t + f_{1multi} \right] + \phi \right)
\]  

(2.23)

Each rising edge of the TTL signal prompts the analog output device to change the voltage to the value defined in the next sample stored in memory. Even though there are only \( N \) samples in memory, the sequence of amplitudes is repeated as long as the reference
clock gets a signal or the program is stopped. An example of the resulting waveform for excitation is shown in Figure 2.15 as the red stair-stepped signal.

The voltage changes are stair-stepped because the logic in the analog output device holds the current voltage until the next clock update references the new voltage. The parameters used for this waveform are listed in Table 2.2.

| Table 2.2: Parameters for the waveform of Figure 2.15 |
|-----------------|-----------------|-----------------|-----------------|-----------------|
| Begin Freq  \( f_1 \) (Hz) | End Freq  \( f_2 \) (Hz) | Samples/Cycle \( N \) | Time Interval \( \Delta t \) (s) | Samples Stored \( n_T \) |
| 1 | 11 | 10 | varies | 10 |

From Tables 2.1 and 2.2, it can be compared for a 1-11 Hz sweep of 10 samples/cycle, that using the Single Cycle Algorithm creates a 85% reduction in samples stored.

Additional waveform examples are shown in Appendix B for 3-10, 20, 50, 100, and 500 samples/cycle.
Analytical Research

3.1 Single Degree of Freedom Model

Single degree of freedom (SDOF) systems were created to predict the system’s response. The first system was created to model the first bending mode of the cantilevered beam, and the second to model the third bending mode of the cantilevered beam. Modal analysis was conducted on the simulated responses to understand the effect of using the SPC waveform. The SDOF system is shown in Fig 2.6.

The material and geometrical properties of the SDOF systems are listed in Table 3.1 and were based upon the cantilevered beam used in this research. The beam material is Hastelloy® X, from which the mass was determined by volume and published typical physical properties. The mass could not be determined by weight due to the thick root used for clamping as illustrated later in Figure 4.3. The damping, $c$, was determined by an actual sine dwell bench test set around the modal frequency of the third bending mode. The stiffness was then set to match the modal frequency for each system. Setting each SDOF model to match the first and third bending modes will allow correlation to later experimental research.
Table 3.1: SDOF properties to match beam modal frequencies

<table>
<thead>
<tr>
<th>Mode</th>
<th>Mass ( m ) (lbm)</th>
<th>Stiffness ( k ) (lbf/in)</th>
<th>Damping ( c ) (lbf s/in)</th>
<th>Frequency (Hz)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.000419</td>
<td>33.39</td>
<td>52.9e-6</td>
<td>44.9</td>
</tr>
<tr>
<td>3</td>
<td>0.000419</td>
<td>10322</td>
<td>52.9e-6</td>
<td>789.4</td>
</tr>
</tbody>
</table>

3.2 Multiple Degree of Freedom Model

This research investigated the application of the Single Cycle algorithm to traveling waveform excitation. The TEFF lab currently does not have the capability to test high numbers of channels using SPC waveforms due to current hardware availability. Therefore, this research includes a multiple degree of freedom model to simulate a bladed disk. Following a successful conclusion from the model, a new system of more channels can be pursued to apply this methodology to future testing. The bladed disk model was created to study the accuracy of identifying natural frequencies, damping ratios, and mode shapes when using SPC waveforms for excitation.

The simulated bladed disk was designed to demonstrate the blade-dominated and disk-dominated modes. The interaction of the two can be excited by applying the traveling wave excitation for specific engine orders.

A beam model shown in Figure 3.1(a) was created in ANSYS to the bladed disk model. The dimensions of the beam and blade both match the dimensions and properties of the cantilevered beam used for experimental research so the results could be correlated.

3.3 Beam Mesh Convergence

In order to validate the model, a mesh convergence study through the length and width of the beam was conducted. This assessment was conducted similar to the method defined by Runyon in 2004 [26]. Beginning with eight elements through the length, one across the width, and one element through the thickness, the number of length and width divi-
Although only modes 1 and 3 are of interest for the beam, the first 10 modes were investigated for trends. The resulting modal frequencies are plotted in Figure 3.3(a). It is evident that the more dense the mesh, the more converged the frequencies become. The percent difference was calculated by subtracting the smaller mesh from the coarse mesh, and dividing by the more coarse mesh. This results in negative percent differences as the modal frequencies decrease. This is as expected due to lower stiffness in the model as the number of elements, and therefore degrees of freedom, increase. The percent difference of each mode approached zero as the element density increased. From 16x2 to 32x4, mode 2 frequency difference was still -1.18%. The 32x4 mesh density was chosen because for Modes 1-4, the frequency only changes less than -0.64%. Although the more densely

Figure 3.1: ANSYS models for multiple degree of freedom simulations.

Figure 3.2: ANSYS meshes for beam frequency convergence study.
meshed models converged better for higher modes, the 32x4 mesh was chosen for its accuracy for the modes of interest and to save computational time in the models. An even number of elements across the width was also chosen because the nodes along the neutral axis of the beam are chosen for later analysis. Having an even number allows for only those nodes to be selected. This will aid in eliminating the torsional modes in the analysis.

The number of element divisions through the thickness was also studied from one to three elements through the thickness for the 32x4 mesh for the beam. As expected, the number of elements through the thickness only slightly affected the frequency results. As the number of elements through the thickness increases, so did the frequencies. This may be attributed to the excessive element aspect ratios with the thinner elements as Runyon concluded [26].

### 3.4 Beam Model Reduction

The System Equivalent Reduction Expansion Process (SEREP) was performed to reduce the size of the larger analytical model to develop a more efficient model for further analytical studies. SEREP was chosen because, as identified by Avitabile [3], a reduced system’s
Figure 3.4: Convergence study of elements through thickness.

eigensolution remains the same as the full system. This reduction is performed by identifying the active and inactive degrees of freedom to create the transformation matrix. The degrees of freedom chosen were the nodes in the middle of the top of the beam. Of the 33 nodes available, every third node was kept from the root to the tip of the beam. From these eleven nodes, only the degrees of freedom in the bending direction were kept since the first and third bending modes are the only modes of interest for this research. Because a total of eleven degrees of freedom are desired to be kept as active degrees of freedom in the reduced model, the first eleven bending modes of the full finite element model were kept. These modes were chosen so that the reduced mass and stiffness matrices would have full rank. Rank deficient matrices result in incorrect eigensolution.

The eigensolution of the full finite element model is used to map between the full set of degrees of freedom and the reduced set of degrees of freedom. The eigensolution of the original model yields a full set of eigenvectors, $U$, which is reduced to a smaller matrix of just the 11 kept bending modes, $U_n$. That is then partitioned into two separate matrices of the eleven active degrees of freedom, $U_a$, and the inactive degrees of freedom, $U_d$, eigenvectors of the eleven bending modes.

The relationship between the active and deleted degrees of freedom are written in the
The modal transformation can be rewritten in terms of the active and deleted sets of degrees of freedom as

\[
\{x_n\} = \begin{bmatrix} x_a \\ x_d \end{bmatrix} = [T]\{x_a\} = \begin{bmatrix} U_a \\ U_d \end{bmatrix}\{p\}
\] (3.2)

where the relationship for the active set of degrees of freedom is written as

\[
\{x_a\} = [U_a]\{p\}
\] (3.3)

Solving for \(\{x_a\}\) involves a generalized inverse since the number of unknowns is not equal to the number of equations needed to achieve the solution. The least squares solution for the generalized inverse is used since the number of master degrees of freedom is far greater than the number of kept modes in the reduced system, so the generalized inverse is used to determine the modal displacement and is defined as

\[
\{p\} = ([U_a][U_a]^T)^{-1}[U_a]^T\{x_a\} = [U_a]^\dagger\{x_a\}
\] (3.4)

where the \(^\dagger\) denotes the generalized inverse. Now, the equation for modal displacement can be substituted into the modal transformation equation, Equation (3.2), resulting in

\[
\{x_n\} = [U_n][U_a]^\dagger\{x_n\} = [T_u]\{x_a\}
\] (3.5)

where the SEREP transformation matrix is determined to be

\[
[T_u] = [U_n][U_a]^\dagger
\] (3.6)
Using the SEREP transformation matrix, the reduced mass and stiffness matrices can be written as

\[
[M_a] = [T_u]^T [M_n] [T_u] \tag{3.7}
\]

\[
[K_a] = [T_u]^T [K_n] [T_u] \tag{3.8}
\]

For the MDOF model of the beam, these matrices are now 11x11 matrices since 11 degrees of freedom were kept, and are of full rank due to keeping the first 11 eigenvectors correlating to the 11 bending modes of the original system.

To validate the reduced model of the beam, the compliance frequency response function was compared to the previous single degree of freedom model. The input and output used to compute the frequency response function was the points at the tip of the beam. The resulting compliance frequency response functions are shown in Figure 3.5 illustrating the similarity between the reduced model of the beam for Modes 1 and 3 to the single degree of freedom model of both modes.

![Compliance Transfer Function](a) Mode 1  
![Compliance Transfer Function](b) Mode 3

Figure 3.5: Comparison of compliance FRF between SDOF and MDOF model.

The resulting mode shapes of the beam reduced model were also compared to the original full model. Figure 3.6(a) shows the same degrees of freedom from the original model plotted above the resulting mode shape of the reduced beam model. The mode
shapes are exact as the calculated MAC was 1 for all retained modes as shown in Figure 3.6(b).

Comparing the modal frequency results of the full and reduced model for the beam in Table 3.2 shows the exact eigensolution for the reduced model. This validates the SEREP reduction for the beam, as the eigenvectors and eigenvalues are identical to those of the full model.

![Figure 3.6: Comparison of normalized mode shapes for full and reduced beam model.](image)

<table>
<thead>
<tr>
<th>Mode</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Full Beam Frequency (Hz)</td>
<td>44.6</td>
<td>280.2</td>
<td>788.1</td>
</tr>
<tr>
<td>Reduced Beam Frequency (Hz)</td>
<td>44.6</td>
<td>280.2</td>
<td>788.1</td>
</tr>
</tbody>
</table>

To validate the simulated bladed disk model, the entire disk and all but one blade were constrained, leaving just one blade acting as a cantilevered beam. The blades were meshed in the same fashion as the beam using 32 elements through the length, four elements across the width, and one element through the thickness since the beam and blades are the same dimensions. The root of cantilevered blade is slightly curved due to the curvature of the disk as compared to the cantilevered beam with a flat boundary at the fixed end. The resulting modal frequencies of the cantilevered blade were just slightly lower due to this difference.
at the clamped end. Table 3.3 lists the first three modal frequencies, but the same held true for all eleven of the kept bending modes. Figure 3.7 illustrates the mass normalized mode shapes and resulting MAC calculation for kept modes.

Table 3.3: Modal frequencies of cantilevered beam and single blade

<table>
<thead>
<tr>
<th>Mode</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Beam Frequency (Hz)</td>
<td>44.6</td>
<td>280.2</td>
<td>788.1</td>
</tr>
<tr>
<td>Blade Frequency (Hz)</td>
<td>44.6</td>
<td>279.9</td>
<td>787.2</td>
</tr>
</tbody>
</table>

![Mode shapes](image1)

![Mass normalized MAC](image2)

(a) Mode shapes (b) Mass normalized MAC

Figure 3.7: Comparison of mode shapes for beam and single blade models.

### 3.5 Disk Mesh Convergence

The ANSYS model of the simulated bladed disk was generated from one disk sector and one blade as shown in Figure 3.8. Once the volumes were created and meshed, the volumes and meshes were repeated in sectors around the origin and the conjoining nodes were coupled. This was chosen over using symmetry so that the node numbers and positions could be exported for use in MATLAB.

Similar to that of the beam, the mesh was studied to converge on the frequency solutions for the number of element divisions across the arc of the disk, and radially through
the disk. To match the mesh in the blade, the middle of the disk sector was set to the same number of element divisions, four, as the blade. The other two arcs of the sector were set to 1, 2, 3, 4 and 5 element divisions. The number of element divisions radially through the disk were incrementally increased from two to six with the arc increments respectively as seen in Figure 3.8. Only one sector of the disk is shown, but the convergence study included the entire disk.

![Figure 3.8: ANSYS meshes for beam convergence study.](image)

The bladed disk was constrained for all degrees of freedom at the inner diameter and the resulting modal frequencies were compared for the first 75 modes. The resulting frequencies are shown in Figure 3.9(a) and percent difference in Figure 3.9(b). The ANSYS model was solved for the first 75 modes containing flexural, torsional, and rigid body modes. Of these, the 30 flexural modes were retained to reduce the 30 degree of freedom model. The percent difference between estimated modal frequencies from three element divisions across the arc and four elements radially across the disk (3x4) to 4x5 was less than 0.7%. Therefore, to save computational time and space, the 3x4 element divisions were used for the full model of the bladed disk.

### 3.6 Disk Model Reduction

The unconstrained ANSYS model of the bladed disk was solved using modal analysis. The resulting mass and stiffness matrices were then extracted from ANSYS and MATLAB was
used to apply the constraints by removing the rows and columns of the matrices pertaining to the constrained degrees of freedom. Using the SEREP transformation matrix in Equation (3.6), and the methodology discussed in Section 3.4, the matrices were then parsed by retained and removed nodes and degrees of freedom.

The model was first reduced to 30 degrees of freedom, keeping 3 nodes per blade and only the z-degree of freedom pertaining to a degree of freedom in which the blade flexes during bending modes. This was analyzed for the first 30 modes to ensure the eigensolution matched the original model and for visual confirmation of the modes. The resulting MAC was 1 for every kept mode as expected. The multiple nodes per blade were initially kept and then compared to the 10 degree of freedom resulting mode shapes. The green nodes of Figure 3.10(b) are the retained nodes and the resulting mode shapes are plotted in blue in Figure 3.12. The 30 degree of freedom model was too large to compute high fidelity simulations, so a smaller 10 degree of freedom model was created.

Traveling wave experiments are typically performed using one point of excitation near the tip of each blade. To mimic this and further save computational space, a reduced model of ten degrees of freedom was created. The tip nodes and the z-degrees of freedom were retained resulting in a total of 10 active degrees of freedom. Also, the first 10 flexural modes
Figure 3.10: Nodes for reduced model validation for 30 kept degrees of freedom.

were retained from the full model. This was the reduced model used for the traveling wave excitation simulations. Keeping 10 degrees of freedom allows for 10 unique inputs while keeping the exact eigensolution. The full model was fixed at the inner diameter of the disk as shown in Figure 3.11(a). SEREP was then used to reduce the full model shown in Figure 3.11(d) to just the retained nodes in Figure 3.11(b).

Four of the resulting mode shapes are plotted in Figure 3.12. The blue lines represent the 30 degree of freedom reduced model, and the green represents the 10 degree of freedom reduced model. The blue line connecting the inner nodes is actually the outer diameter of the disk, as well as the green dashed line connecting the tip nodes is merely added for
The first and second bending modes for the engine order excitations of 0EO and 2EO were picked to plot, however, the MAC for all the modes was 1 and is shown in Figure 3.13.

This 10 degree of freedom model will be used in the following analytical research to simulate the response of a simulated bladed disk to traveling wave excitation, using the single cycle algorithm, to investigate whether there is an adverse effect on the system response due to lower fidelity waveforms.

Figure 3.11: Nodes for reduced model validation for 10 kept degrees of freedom.
Figure 3.12: Modeshape comparison between 30 and 10 active degrees of freedom.

Figure 3.13: MAC between 30 and 10 active degrees of freedom reduced models.
3.7 Waveform Generation

To simulate the typical excitation signal of the function generator to be used in bench testing, the forcing functions are initially defined by the sine sweep represented by Equation (2.20). This will be the baseline for comparison for each SPC waveform.

The signal parameters were determined using the same methodology as the Polytec Laser Vibrometry system. The forcing functions of the analytical simulations are identical to the experimental excitations, thus, the resulting responses were directly comparable. The chirp signal is defined by the beginning and ending frequencies \( f_1 \) and \( f_2 \) respectively, the number of FFT lines \( N_{\text{fft}} \), and the bandwidth \( b \). The remaining parameters are then calculated based on these inputs. The time of the sweep \( T_s \) is the ratio of the number of FFT lines to the bandwidth as follows:

\[
T_s = \frac{N_{\text{fft}}}{b}
\]  

(3.9)

To avoid aliasing as identified in Section 2.3.3, the sampling rate is set based on the TTL signal that will be used for the external trigger. In the analytical model, the TTL signal must be sampled at an adequate rate such that aliasing does not occur at higher frequencies for SPC waveforms with high samples/cycle. For these cases the time between updates is at its minimum. When creating the TTL signal, under-sampling results in generating the incorrect TTL signal and thus an erroneous or even phase-shifted waveform. An example of the resulting SPC waveform for well sampled timing signal is shown in Figure 3.14(a) and a poorly sampled timing signal in Figure 3.14(c) for otherwise identical settings. It is clear that the signal is incorrect because the frequency shift seems to decrease near the end of the sweep. This fact is evident in the resulting FFT of the waveform. Figures 3.14(b) and 3.14(d) show the good and poor distribution over the intended frequency range for the sampled signals. The FFT validates that the frequency distribution over the intended range was not met. This will be avoided in this research by verifying, for all waveforms, that the
Nyquist sampling theorem is met.

The sampling rate was originally set to match the maximum 1MS/s output of the function generator. However, the computed waveforms were too large to store and perform calculations on. To save storage space and calculation time, the minimum required sampling rates were used to reduce computer storage and memory requirements and avoid aliasing. The minimum sufficient sampling rate was determined using the Nyquist sampling theorem as addressed in Section 2.3.3 and is based on the minimum time between samples. This occurs at highest frequency of the chirp. The minimum sampling rate is calculated by Equation (3.10a) for Normal FFT tests. The minimum sampling rate for the TTL signal was
calculated by Equation (3.10b) for Zoom FFT tests centered about the modal frequency.

\[
f_s = \frac{5N_{spc} f_{\text{max}}}{2} \quad (3.10a)
\]

\[
f_z = \frac{5N_{spc} (f_n + \frac{b}{2})}{2} \quad (3.10b)
\]

The multiplier satisfies the Nyquist criteria, \( f_s \) and \( f_z \) are the sampling rates, \( f_n \) is the modal frequency, \( f_{\text{max}} \) is the largest frequency in the sweep, and \( N_{spc} \) is the number of samples/cycle being used. Based on the analytical test matrix, the minimum sampling frequency for Mode 1 is calculated using Normal FFT test at 100Hz bandwidth around the modal frequency at 100 samples/cycle. The parameters used for Mode 3 were a Zoom FFT test with 32Hz bandwidth around the modal frequency at 100 samples/cycle. The minimum sampling frequencies calculated are shown in Table 3.4

<table>
<thead>
<tr>
<th>Mode Number</th>
<th>Highest Frequency (Hz)</th>
<th>Highest Sampling Rate, ( f_s ) (samples/cycle)</th>
<th>Sampling Rate, ( f_z ) (KS/s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>100</td>
<td>100</td>
<td>25</td>
</tr>
<tr>
<td>3</td>
<td>804</td>
<td>100</td>
<td>201</td>
</tr>
</tbody>
</table>

Using the methodology described in Section 2.5.1 for the typical waveform generation, and Section 2.5.1 for the Single Cycle waveform generation, the SDOF system response was determined as a function of displacement. An example of the waveforms created, for 1600 FFT lines at 50Hz bandwidth, are shown in Figure 3.15.

After computing the SPC-multiple waveform, the crossings of this signal are determined and a TTL signal is simulated. From the TTL signal, the algorithm iteratively steps through every other crossing, or rising edge of the TTL signal, and creates the next segment of the SPC waveform with the corresponding amplitude. The amplitude is held constant until the next crossing.

The sampling frequency must be high in order to accurately capture the crossings,
otherwise aliasing would occur and the resulting waveform would be inaccurate. This became computationally demanding to simulate for high resolution waveforms; however, experimentally this results in saving memory. When performing experiments, the TTL signal is created by the function generator, and the sampling rate of the acquisition system can be set based on the frequency of the excitation signal, not the timing signal as performed here analytically.

3.8 Linear Time-Invariant Simulation

3.8.1 Single Degree of Freedom Simulation

The transfer function, as described in Equation (2.11), is then used as an input into a Linear Time-Invariant (LTI) simulation to predict the response of the system. The simulation was conducted using the LSIM MATLAB function in the Control System Toolbox. Using the LSIM function enables simulation of the time response of the discrete linear system to any arbitrary forcing function specified.

Using the mass, damping, and stiffness variables defined from Table 3.1, and the SPC
waveform and time vector, the inputs into the algorithm for the single degree of freedom simulation are inputs to the LSIM function to simulate the response of the system.

A first-order hold was used in the simulation to mathematically model the reconstruction of sampled signals conventionally executed through the digital-to-analog converter (DAC) of the analog output device. Applying this to a linear, time-invariant system results in the correct piecewise linear function in the output [20]. The output of the LSIM function is the resulting response of the single degree of freedom to the given input to the model as shown in Figure 3.16.

![Figure 3.16: Time data for forcing function and SDOF system response.](image)

### 3.8.2 Multiple Degree of Freedom Simulation

To study the effect of the Single Cycle waveform on the traveling wave application, the 10 degree of freedom model of the bladed disk was used. This bladed disk was modeled using state space representation as a linear time-invariant (LTI) system. The state space approach was used because its time-domain formulation is necessary for simulation of responses in the time-domain. This method also has significant advantages computationally and is well-suited for modeling multiple-input, multiple-output (MIMO) systems [29].

This representation of the physical system is mathematically defined as a set of inputs,
outputs, and state variables related by the state equations. The general form of a linear time-invariant system is a first-order differential equation which Williams described as [29]:

\[ \dot{x}(t) = Ax(t) + Bu(t) \]  \hspace{1cm} (3.11)

\[ y(t) = Cx(t) + Du(t) \] \hspace{1cm} (3.12)

in which \( x(t) \) is the \( n \)-dimensional state vector

\[ x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} \]

whose \( n \) scalar components are the state variables. Similarly, the \( m \)-dimensional input vector, \( u(t) \), and the \( p \)-dimensional output vector, \( y(t) \), are written as

\[ u(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_n(t) \end{bmatrix} \]

\[ y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_p(t) \end{bmatrix} \]

For damped mechanical systems, the state space coefficient matrices \( A, B, C, \) and \( D \) are
defined as follows

\[
A = \begin{bmatrix}
0 & I \\
-M_a^{-1}K & -M_a^{-1}C_d
\end{bmatrix}
\]  \hspace{1cm} (3.13)

\[
B = \begin{bmatrix}
0 \\
M_a^{-1}
\end{bmatrix}
\]  \hspace{1cm} (3.14)

\[
C = [I] \hspace{1cm} (3.15)
\]

\[
D = [0] \hspace{1cm} (3.16)
\]

where the viscous damping matrix, \(C_d\), as defined by Adhikari [1] using generalized proportional damping, is written as

\[
C_d = [U_n^T]^{-1} \begin{bmatrix}
\cdots \\
2\zeta_i\omega_i \\
\cdots
\end{bmatrix} [U_n]^{-1}
\]  \hspace{1cm} (3.17)

The eigenvectors, \(U_n\), and modal frequencies, \(\omega_i\), are those of the reduced system previously determined. The damping ratios, \(\zeta_i\), were set to match that of the beam so the results would be comparable.

The simulation was run identically to that of the beam, using a first order hold, but the forcing function included a phased excitation signal for each degree of freedom based on the inter-blade phase angle depending on the engine order excitation desired. Examples of these signals are shown in Appendix C for the 10 bladed disk at 4EO excitation.
3.9 Modal Analysis

The FRF was used to identify the vibratory characteristics of the system. The first step in computing the FRF is calculating the FFT of the input \( f(t) \) and response \( x(t) \) if first calculated as follows:

\[
X_f(f) = \mathcal{F}(f(t)) \tag{3.18}
\]

\[
X_x(f) = \mathcal{F}(x(t)) \tag{3.19}
\]

For the input and response time data in Figure 3.16 are shown in Figure 3.17. The \( H_2 \)

![Figure 3.17: FFT of forcing function and SDOF system response.](image)

Transfer Function of the system was used for its ability to best represent the system with noise in the response, and underestimates for both an input and output with noise [18]. It is determined using the following definitions:

\[
H_2(f) = \frac{S_{xx}}{S_{xf}} \tag{3.20}
\]

where \( S_{xx} \) is the auto spectrum density, and \( S_{xf} \) is the cross spectrum density. The auto spectrum density describes how the power of the signal is distributed with frequency and is
defined as
\[ S_{xx} = \frac{X_x X_x^*}{\eta^2} \quad (3.21) \]

where \( X_x^* \) is the complex conjugate of \( X_x \). The cross spectrum density relates, in the frequency domain, the time signal of the input and the time signal of the response and is defined as
\[ S_{xf} = \frac{X_f X_x^*}{\eta^2} \quad (3.22) \]

The auto spectrum density can also be defined as the Fourier transform of the autocorrelation function, or the cross spectrum density of the signal and itself. An example of the resulting transfer function is shown in Figure 3.18.

![Figure 3.18: Resulting Frequency Response Function of SDOF system.](image)

The damping and modal frequency were then calculated from the \( H_2 \) transfer function using the single degree and multiple degree of freedom curve fit functions of the MATLAB Vibration Toolbox. These functions determine the damping coefficient, \( \zeta \), and the modal frequency, \( \omega \), by calculating the poles and residues of the FRF.

The damping parameter typically used in the TEFF to measure the damping of a system is the Q factor, or \( Q \). This will be the parameter used to compare system properties throughout this research. The Q factor is related to the damping ratio, \( \zeta \), through the fol-
following relationship

\[ Q = \frac{1}{2\zeta} \]  

(3.23)

The Q factor is dimensionless because the damping ratio is also dimensionless.

These procedures were iterated through the full range of bandwidth and resolution settings for the laser vibrometer system. The full analytical test matrix is discussed in the following section, and the results are shown in Section 5.1.1 for the ideal chirp waveforms, and in Section 5.1.2 using the SPC waveforms.

### 3.10 Analytical Test Matrix

Tables 3.5 - 3.6 list the Mode 1 test matrix, and Table 3.7 lists the Mode 3 test matrix completed for the analytical research of the single degree of freedom model. These tables are grouped by the bandwidth and number of FFT lines used for each simulation. For each setting group, the frequency resolution, sweep rate, and sweep time are listed. The Mode 3 test matrix was reduced using lessons learned from the Mode 1 results to eliminate unnecessary simulations.

The simulation parameters are picked to match the Polytec vibrometry system. Although some of the simulations were beyond the capability of the Polytec system used, another newer Polytec vibrometry system, which was not available for use in this research, does have the capability to reach these increased resolutions. These tests settings were chosen to show convergence of the data as the number of FFT lines was increased as well as investigate the capability of the new vibrometry system. Some of the tests frequency resolutions are below what is typically used in practice for modal analysis, but were chosen to study if the SPC waveforms perform any different at these settings than the ideal chirp signals.

Table 3.8 lists the complete test matrix for the simulated bladed disk multiple degree
of freedom model. Each of these test settings were simulated in MATLAB for the ideal chirp waveform and the SPC waveforms for 3-100 samples/cycle. Only the performance of the current vibrometry system was simulated due to the convergence of the results within the frequency resolutions capability.

There were 5,940 tests for Mode 1, and 1,782 for Mode 3, and 5,940 for the simulated bladed disk for a total of 13,662 simulations completed for this analysis. The results of these simulations are in Section 5.1 for the SDOF, and Section 5.3 for the simulated bladed disk.

<table>
<thead>
<tr>
<th>Bandwidth (Hz)</th>
<th>FFT lines</th>
<th>Frequency Resolution (mHz)</th>
<th>Sweep Rate (mHz/sec)</th>
<th>Sweep Time (sec)</th>
</tr>
</thead>
<tbody>
<tr>
<td>50</td>
<td>2000</td>
<td>200</td>
<td>0.5</td>
<td></td>
</tr>
<tr>
<td>100</td>
<td>1000</td>
<td>100</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>200</td>
<td>500</td>
<td>50</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>400</td>
<td>250</td>
<td>25</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td>800</td>
<td>125</td>
<td>12.5</td>
<td>8</td>
<td></td>
</tr>
<tr>
<td>1600</td>
<td>63</td>
<td>6.3</td>
<td>16</td>
<td></td>
</tr>
<tr>
<td>3200</td>
<td>31</td>
<td>3.1</td>
<td>32</td>
<td></td>
</tr>
<tr>
<td>6400</td>
<td>16</td>
<td>1.6</td>
<td>64</td>
<td></td>
</tr>
<tr>
<td>12800</td>
<td>7.8</td>
<td>0.8</td>
<td>128</td>
<td></td>
</tr>
<tr>
<td>25600</td>
<td>3.9</td>
<td>0.39</td>
<td>256</td>
<td></td>
</tr>
<tr>
<td>51200</td>
<td>2.0</td>
<td>0.20</td>
<td>512</td>
<td></td>
</tr>
<tr>
<td>102400</td>
<td>1.0</td>
<td>0.10</td>
<td>1024</td>
<td></td>
</tr>
<tr>
<td>250</td>
<td>5000</td>
<td>1250</td>
<td>0.20</td>
<td></td>
</tr>
<tr>
<td>100</td>
<td>2500</td>
<td>625</td>
<td>0.40</td>
<td></td>
</tr>
<tr>
<td>200</td>
<td>1250</td>
<td>313</td>
<td>0.80</td>
<td></td>
</tr>
<tr>
<td>400</td>
<td>625</td>
<td>156</td>
<td>1.6</td>
<td></td>
</tr>
<tr>
<td>800</td>
<td>313</td>
<td>78</td>
<td>3.2</td>
<td></td>
</tr>
<tr>
<td>1600</td>
<td>156</td>
<td>39</td>
<td>6.4</td>
<td></td>
</tr>
<tr>
<td>3200</td>
<td>78</td>
<td>20</td>
<td>12.8</td>
<td></td>
</tr>
<tr>
<td>6400</td>
<td>39</td>
<td>10</td>
<td>25.6</td>
<td></td>
</tr>
<tr>
<td>12800</td>
<td>20</td>
<td>4.9</td>
<td>51.2</td>
<td></td>
</tr>
<tr>
<td>25600</td>
<td>10</td>
<td>2.4</td>
<td>102</td>
<td></td>
</tr>
<tr>
<td>51200</td>
<td>4.9</td>
<td>1.2</td>
<td>205</td>
<td></td>
</tr>
<tr>
<td>102400</td>
<td>2.4</td>
<td>0.6</td>
<td>410</td>
<td></td>
</tr>
<tr>
<td>Bandwidth (Hz)</td>
<td>FFT lines</td>
<td>Frequency Resolution (mHz)</td>
<td>Sweep Rate (mHz/sec)</td>
<td>Sweep Time (sec)</td>
</tr>
<tr>
<td>---------------</td>
<td>-----------</td>
<td>---------------------------</td>
<td>---------------------</td>
<td>-----------------</td>
</tr>
<tr>
<td>500</td>
<td>50</td>
<td>10000</td>
<td>5000</td>
<td>0.10</td>
</tr>
<tr>
<td></td>
<td>100</td>
<td>5000</td>
<td>2500</td>
<td>0.20</td>
</tr>
<tr>
<td></td>
<td>200</td>
<td>2500</td>
<td>1250</td>
<td>0.40</td>
</tr>
<tr>
<td></td>
<td>400</td>
<td>1250</td>
<td>625</td>
<td>0.8</td>
</tr>
<tr>
<td></td>
<td>800</td>
<td>625</td>
<td>313</td>
<td>1.6</td>
</tr>
<tr>
<td></td>
<td>1600</td>
<td>313</td>
<td>156</td>
<td>3.2</td>
</tr>
<tr>
<td></td>
<td>3200</td>
<td>156</td>
<td>78</td>
<td>6.4</td>
</tr>
<tr>
<td></td>
<td>6400</td>
<td>78</td>
<td>39</td>
<td>13</td>
</tr>
<tr>
<td></td>
<td>12800</td>
<td>39</td>
<td>19.5</td>
<td>26</td>
</tr>
<tr>
<td></td>
<td>25600</td>
<td>20</td>
<td>9.8</td>
<td>51</td>
</tr>
<tr>
<td></td>
<td>51200</td>
<td>9.8</td>
<td>4.9</td>
<td>102</td>
</tr>
<tr>
<td></td>
<td>102400</td>
<td>4.9</td>
<td>2.4</td>
<td>205</td>
</tr>
<tr>
<td>1000</td>
<td>50</td>
<td>20000</td>
<td>20000</td>
<td>0.05</td>
</tr>
<tr>
<td></td>
<td>100</td>
<td>10000</td>
<td>10000</td>
<td>0.10</td>
</tr>
<tr>
<td></td>
<td>200</td>
<td>5000</td>
<td>5000</td>
<td>0.20</td>
</tr>
<tr>
<td></td>
<td>400</td>
<td>2500</td>
<td>2500</td>
<td>0.4</td>
</tr>
<tr>
<td></td>
<td>800</td>
<td>1250</td>
<td>1250</td>
<td>0.8</td>
</tr>
<tr>
<td></td>
<td>1600</td>
<td>625</td>
<td>625</td>
<td>1.6</td>
</tr>
<tr>
<td></td>
<td>3200</td>
<td>313</td>
<td>313</td>
<td>3.2</td>
</tr>
<tr>
<td></td>
<td>6400</td>
<td>156</td>
<td>156</td>
<td>6</td>
</tr>
<tr>
<td></td>
<td>12800</td>
<td>78</td>
<td>78</td>
<td>13</td>
</tr>
<tr>
<td></td>
<td>25600</td>
<td>39</td>
<td>39</td>
<td>26</td>
</tr>
<tr>
<td></td>
<td>51200</td>
<td>19.5</td>
<td>20</td>
<td>51</td>
</tr>
<tr>
<td></td>
<td>102400</td>
<td>9.8</td>
<td>9.8</td>
<td>102</td>
</tr>
<tr>
<td>2000</td>
<td>50</td>
<td>40000</td>
<td>80000</td>
<td>0.03</td>
</tr>
<tr>
<td></td>
<td>100</td>
<td>20000</td>
<td>40000</td>
<td>0.05</td>
</tr>
<tr>
<td></td>
<td>200</td>
<td>10000</td>
<td>20000</td>
<td>0.10</td>
</tr>
<tr>
<td></td>
<td>400</td>
<td>5000</td>
<td>10000</td>
<td>0.20</td>
</tr>
<tr>
<td></td>
<td>800</td>
<td>2500</td>
<td>5000</td>
<td>0.40</td>
</tr>
<tr>
<td></td>
<td>1600</td>
<td>1250</td>
<td>2500</td>
<td>0.80</td>
</tr>
<tr>
<td></td>
<td>3200</td>
<td>625</td>
<td>1250</td>
<td>1.60</td>
</tr>
<tr>
<td></td>
<td>6400</td>
<td>312.5</td>
<td>625</td>
<td>3.20</td>
</tr>
<tr>
<td></td>
<td>12800</td>
<td>156.25</td>
<td>312.5</td>
<td>6.40</td>
</tr>
<tr>
<td></td>
<td>25600</td>
<td>78.125</td>
<td>156.25</td>
<td>12.80</td>
</tr>
<tr>
<td></td>
<td>51200</td>
<td>39.0625</td>
<td>78.125</td>
<td>25.60</td>
</tr>
<tr>
<td></td>
<td>102400</td>
<td>19.5313</td>
<td>39.0625</td>
<td>51.2</td>
</tr>
</tbody>
</table>
### Table 3.7: Complete test matrix of waveforms for analytical SDOF model (Mode 3)

<table>
<thead>
<tr>
<th>Bandwidth (Hz)</th>
<th>FFT lines</th>
<th>Frequency Resolution (mHz)</th>
<th>Sweep Rate (mHz/sec)</th>
<th>Sweep Time (sec)</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>50</td>
<td>160</td>
<td>1.3</td>
<td>6.25</td>
</tr>
<tr>
<td></td>
<td>100</td>
<td>80</td>
<td>0.64</td>
<td>13</td>
</tr>
<tr>
<td></td>
<td>200</td>
<td>40</td>
<td>0.32</td>
<td>25</td>
</tr>
<tr>
<td></td>
<td>400</td>
<td>20</td>
<td>0.16</td>
<td>50</td>
</tr>
<tr>
<td></td>
<td>800</td>
<td>10</td>
<td>0.08</td>
<td>100</td>
</tr>
<tr>
<td></td>
<td>1600</td>
<td>5</td>
<td>0.04</td>
<td>200</td>
</tr>
</tbody>
</table>

### Table 3.8: Complete test matrix of TWE waveforms for analytical MDOF model

<table>
<thead>
<tr>
<th>Bandwidth (Hz)</th>
<th>FFT lines</th>
<th>Frequency Resolution (mHz)</th>
<th>Sweep Rate (mHz/sec)</th>
<th>Sweep Time (sec)</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>50</td>
<td>160</td>
<td>1.3</td>
<td>6.25</td>
</tr>
<tr>
<td></td>
<td>100</td>
<td>80</td>
<td>0.64</td>
<td>13</td>
</tr>
<tr>
<td></td>
<td>200</td>
<td>40</td>
<td>0.32</td>
<td>25</td>
</tr>
<tr>
<td></td>
<td>400</td>
<td>20</td>
<td>0.16</td>
<td>50</td>
</tr>
<tr>
<td></td>
<td>800</td>
<td>10</td>
<td>0.08</td>
<td>100</td>
</tr>
<tr>
<td></td>
<td>1600</td>
<td>5</td>
<td>0.04</td>
<td>200</td>
</tr>
</tbody>
</table>
Experimental Research

4.1 Testing Equipment

This research was conducted with the use of the equipment in the Turbine Engine Fatigue Facility (TEFF) at Wright Patterson Air Force Base in Dayton, Ohio. The equipment was setup as depicted in Figure 4.1. The details of each device, or specimen, will be discussed in more detail in the following sections.

![Test Setup](image1.png)  ![Beam Excitation](image2.png)

Figure 4.1: Test equipment setup in TEFF laser vibrometry laboratory.

4.1.1 Scanning Laser Vibrometer

To capture the response of the beam, each test was conducted using the Polytec scanning laser vibrometer system (PSV 300) consisting of an OFV-056 vibrometer scanning head, an
OFV-3001S laser controller, a junction box and workstation. The scanning laser vibrometer allowed an array of points along the beam to be sampled throughout each test. The vibrometer captured the velocity of each data point and through the Polytec software, was able to build the recorded mode shape based on the Frequency Response Function of each data point.

![Diagram of Polytec scanning laser vibrometer](image)

Figure 4.2: Polytec scanning laser vibrometer.

The limits of the Polytec acquisition system depend on the test. For both FFT and Zoom FFT tests using linear chirp signals, the number of available FFT lines depends on the bandwidth chosen. Because the laser is limited to 20 mHz frequency resolution \(f_{\text{res}}\), depending on the bandwidth \(b\) selected, the available FFT lines \(N_{\text{fft}}\) are determined by

\[
N_{\text{fft}} = \frac{b}{f_{\text{res}}}
\]  

Because the light source of the PSV is a multi-mode helium neon laser, two modes can exist. The interference of the two modes leads to the intensity of the resulting optical signal varying periodically and can cause erroneous data acquisition [23]. To prevent this interference, the scanning head was positioned using the published optimal stand-off distance, \(d\), defined by

\[
d = 14 \text{ mm} + n \times 203 \text{ mm}, \quad n = 0; 1; 2; \ldots
\]
Setting the value of \( n \) to four, the resulting stand-off distance is 623mm, or 24.5in. This distance kept the entire beam within the range of the laser and camera and was set for each test.

The Polytec workstation has a National Instruments PCI-4452 data acquisition board and PCI-6711 analog output card acting as the internal function generator integrated into the computer. The PCI-6711 is very similar to the PCI-6713/31 being tested in this research. The specifications and comparison are shown below in Table 4.1. The PCI-6711 is used for all of the baseline tests as the integrated signal generator, the PCI-6731 is used for all of the SPC waveform signals controlled by a separate computer, and the PCI-6713 is the device that will potentially be used in the new Traveling Wave Excitation system due to the eight analog output (AO) channels.

<table>
<thead>
<tr>
<th>AO Channels</th>
<th>PCI-6711</th>
<th>PCI-6713</th>
<th>PCI-6731</th>
</tr>
</thead>
<tbody>
<tr>
<td>FIFO buffer</td>
<td>8,192</td>
<td>16,384</td>
<td>16,384</td>
</tr>
<tr>
<td>Resolution</td>
<td>12 bits</td>
<td>12 bits</td>
<td>16 bits</td>
</tr>
<tr>
<td>Max Update Rate</td>
<td>1MS/s</td>
<td>740KS/s - 1MS/s</td>
<td>1MS/s</td>
</tr>
</tbody>
</table>

The Polytec software also exhibits the capability to export data files in the universal file format (UFF). This capability allows the data to be imported into MATLAB for further data reduction.

### 4.1.2 Clamp Fixture Tower

The beam was clamped and cantilevered from the fixture using a previously existing tower mounted on an optical table for stability. The use of a 1.5 inch block fixture atop the Inconel tower clamped the specimen using two 3/4 inch bolts on either side to evenly distribute the load onto the root of the beam. A 1/2 inch bolt was placed through the block and beam
to center the beam in the fixture prior to torquing the fixture. The torque on the 3/4 inch bolts was set to 100 ft-lbs for each test based on previously completed repeatability tests for torques greater than 80 ft-lbs using the same clamp and beam as was used in this research.

4.2 Beam Specimen

The beam used in this research is a rectangular beam made of a nickel-based alloy, Hastelloy® X, which has various turbine engine applications. The beam has a thick root which is a design used by TEFF lab for its repeatability alleviating the difficulties with mounting the beam squarely in fixture each time. It also helps produce repeatable damping values for lightly damped systems. Also, a 3-dimensional rendering of the beam is shown in Figure 4.3.

![Figure 4.3: Rendering of thick root beam used in testing.](image)

4.2.1 Specifications and Properties

The measured dimensions of the cantilevered beam are listed in Table 4.2, and the material properties of beam are listed in Table 4.3.

<table>
<thead>
<tr>
<th>Table 4.2: Beam geometry</th>
</tr>
</thead>
<tbody>
<tr>
<td>Length (in)</td>
</tr>
<tr>
<td>8.0</td>
</tr>
</tbody>
</table>
Table 4.3: Beam material properties

<table>
<thead>
<tr>
<th>Modulus ($10^6$ psi)</th>
<th>Density ($lb/in^3$)</th>
<th>Poisson’s Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>29.8</td>
<td>.297</td>
<td>.320</td>
</tr>
</tbody>
</table>

4.2.2 Modal Predictions

The length to width ratio of the cantilevered beam is greater than 8:1, so beam theory applies well for the lower order modes. The theoretical mode shapes were calculated using beam theory, as defined by Inman [13]. The first and third bending modes are shown in Figure 4.4 since these are the modes of interest for this research. The associated modal frequencies are listed in Table 4.4. These will be used to compare the analytical and experimental results.

![Figure 4.4: First and third bending mode shapes using beam theory.](image)

Table 4.4: Beam Theory Modal Frequencies

<table>
<thead>
<tr>
<th>Mode</th>
<th>Frequency (Hz)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>44.9</td>
</tr>
<tr>
<td>3</td>
<td>787.9</td>
</tr>
</tbody>
</table>
4.3 Experimental Procedure

The testing procedure is very similar to that of the analytical research, but since the Polytec system has limitation based on hardware capabilities, and based on the results from the analytical model, the testing matrix for the beam is greatly reduced, only testing specifications that are capable and relevant. The full test matrix is listed in Table 4.5.

4.3.1 General Setup

The test setup was all on one optical table comprised of the Polytec PSV-300 system, a single channel amplifier and transformer, a stand-alone computer running LabVIEW, the fixture, magnetic exciter, and the beam itself.

The baseline tests used the Polytec internal signal generator to create and output the linear chirp waveform signals for excitation. To create the SPC waveforms, the internal signal generator was turned off through the Polytec software. The PCI-6711 device was then separately controlled through LabVIEW to create the SPC-multiple waveform and output the corresponding TTL signal.

The TTL signal from the Polytec PCI-6711 signal generator was connected to the PCI-6731 card on the stand-alone computer to serve as the external clock. The associated trigger from the Polytec PCI-6711 was connected to the PCI-6711 to initiate signal generation and synchronize the output with the Polytec acquisition system.

The stand-alone computer ran a LabVIEW algorithm that calculated the Single Cycle amplitudes and stored it on-board the PCI-6731 card’s local FIFO (First In First Out) buffer. Once initiated by the trigger, the resulting waveform is output to the input of the Polytec Junction Box.

Both the baseline linear chirp and SPC waveforms were routed through an amplifier and transformer before being sent to the magnetic exciter.
4.3.2 Testing Procedure

Prior to beginning a test, the Polytec system is initialized by calibrating the laser, and setting up the test parameters for frequency, bandwidth, and resolution in the Polytec software.

Because the bending modes were of interest for this research, the scan points were placed along the center axis of the beam to reduce the measurement of any torsional responses. The 25 scan points were equally spaced beginning 0.1 inches from the root and ending 0.1 inches from the tip.

A small, circular, cobalt disk was placed on the underneath side of the beam to transfer the excitation from the magnetic exciter because Hastelloy® X is non-magnetic. The gap between the disk and the exciter was set to 0.035in for all tests conducted.

An initial check of the test setup was then conducted by running a continuous scan on a single data point to ensure all of the parameters were set correctly. Once complete, a scan through each point down the beam is made manually to ensure there aren’t any overage signals or errors, and a final check to make sure the laser hadn’t drifted and is still centered and the end points are at the correct locations is made. Then the test is initiated using the baseline Polytec internal function generator.

Then according to the test matrix, Table 4.5, the associated SPC waveform tests are conducted. This was then repeated through each bandwidth and resolution setting for the entire test matrix.

4.3.3 Modal Analysis

Once the test was complete, a UFF file was exported with the resulting FRF and mode shape data from the scan. This data was then reduced in MATLAB to determine the damping, frequency, and MAC of the resulting mode shape of the beam. This was accomplished using the same methodology as in the Analytical Research with the addition of calculating the MAC.
As discussed in Section 2.4.2, the mass normalized MAC is calculated by Equation (2.12). The inputs to the equation were $\psi_l$ (the mode shape determined by beam theory) and $\psi_m$ (the experimentally determined mode shape from the beam). The mass matrix is determined from a reduced finite element model for the 25 nodes.

### 4.4 Experimental Test Matrix

Table 4.5 lists the entire test matrix completed for the experimental testing using the cantilevered beam, the Polytec system, and the SPC waveforms generated in LabVIEW. A total of 1,173 bench tests were completed for this research to validate the effectiveness of the SPC waveforms.

The test matrix is setup identically to the analytical test matrices, identifying the samples/cycle, the frequency resolution, sweep rate, and sweep time used for each setting. The samples/cycle vary from test to test due to the physical memory limitations of the computer used. Because the Polytec internal signal generator was used to generate the TTL timing signal as well as used to run the Polytec acquisition software, the physical memory was often reached. This created the inability to reach 20 samples/cycle or more in some test cases. The Polytec internal generator was used as the baseline and comparison for each of the test settings.

The results of these tests are shown in Section 5.2.
<table>
<thead>
<tr>
<th>Bandwidth (Hz)</th>
<th>FFT lines</th>
<th>Samples/Cycle</th>
<th>Chirp</th>
<th>Frequency Resolution (mHz)</th>
<th>Sweep Rate (mHz/sec)</th>
<th>Sweep Time (sec)</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>100</td>
<td>3-10, 20</td>
<td></td>
<td>80</td>
<td>0.64</td>
<td>12.5</td>
</tr>
<tr>
<td></td>
<td>200</td>
<td>3-10</td>
<td></td>
<td>40</td>
<td>0.32</td>
<td>25</td>
</tr>
<tr>
<td></td>
<td>400</td>
<td>3-10</td>
<td></td>
<td>20</td>
<td>0.16</td>
<td>50</td>
</tr>
<tr>
<td>16</td>
<td>100</td>
<td>3-10, 20, 50</td>
<td></td>
<td>160</td>
<td>2.56</td>
<td>6.25</td>
</tr>
<tr>
<td></td>
<td>200</td>
<td>3-10, 20</td>
<td></td>
<td>80</td>
<td>1.28</td>
<td>12.5</td>
</tr>
<tr>
<td></td>
<td>400</td>
<td>3-10, 20</td>
<td></td>
<td>40</td>
<td>0.64</td>
<td>25</td>
</tr>
<tr>
<td></td>
<td>800</td>
<td>3-10</td>
<td></td>
<td>20</td>
<td>0.32</td>
<td>50</td>
</tr>
<tr>
<td>32</td>
<td>100</td>
<td>3-10, 20</td>
<td></td>
<td>320</td>
<td>10.24</td>
<td>3.125</td>
</tr>
<tr>
<td></td>
<td>200</td>
<td>3-10, 20</td>
<td></td>
<td>160</td>
<td>5.12</td>
<td>6.25</td>
</tr>
<tr>
<td></td>
<td>400</td>
<td>3-10, 20</td>
<td></td>
<td>80</td>
<td>2.56</td>
<td>12.5</td>
</tr>
<tr>
<td></td>
<td>800</td>
<td>3-10, 20</td>
<td></td>
<td>40</td>
<td>1.28</td>
<td>25</td>
</tr>
<tr>
<td></td>
<td>1600</td>
<td>3-10</td>
<td></td>
<td>20</td>
<td>0.64</td>
<td>50</td>
</tr>
</tbody>
</table>
Results and Discussion

5.1 Analytical Single Degree of Freedom Results

5.1.1 Ideal Chirp Waveform Results

SDOF Mode 1 Results

The SDOF model for Mode 1 simulated the response of the system to a given input as discussed in Chapter 3. To baseline the test, the model was initially run using an ideal chirp waveform from 1-100 Hz to simulate the signal of a function generator. The results of the identified damping are shown in Figure 5.1. To clearly understand the effect frequency resolution and bandwidth have on estimating the damping of a system, the results are plotted in two different manners, first with respect to bandwidth, Figure 5.1(a), and secondly with respect to the number of FFT lines in Figure 5.1(b).

It can be seen from Figure 5.1(a) that very high resolution is necessary to accurately identify the damping of the system for Mode 1. According to the Mode 1 simulation results, this system requires a minimum number of 12800 FFT lines for a 100 Hz bandwidth. The current Polytec system is limited to 6400 FFT lines; therefore, the ability to accurately determine the damping of the system is impossible for this low order mode.

However, the damping, $Q$, did converge to 2238 which was the actual damping of the system. All of the bandwidths simulated converged to the actual value as seen in Figure
By these results, it can be stated the damping converges more quickly the lower the bandwidth. The modal frequency was also determined from the SDOF model for Mode 1. Unlike the damping estimation, the modal frequency of the mode was less dependent on bandwidth and number of FFT lines. The frequency was correctly identified within 0.1% of the actual value of 44.9 Hz for all by one of the simulations.
This result was due to the low number FFT lines and high bandwidth used resulting in a total sweep time of 0.003 seconds as seen in the test matrix in Table 3.6. This was not nearly enough time to adequately excite the system and measure the simulated response.

Because the frequency resolution is related to both the bandwidth and number of FFT lines through Equation (4.1), the results will be plotted against frequency resolution for the remaining ideal chirp waveform results. This allows for the effect of both to be captured, and the damping and frequency results to be plotted concisely. The results for Mode 1 are shown in this manner in Figure 5.3.

![Figure 5.3: SDOF Mode 1 results using ideal chirp waveforms.](image)

As expected, tests converged to the actual system properties as the resolution increased, and the smaller bandwidth tests converged the quickest. The effect of frequency resolution did not play as crucial a factor in estimating the frequency of the system. All of the simulations converged within 0.1% of the answer (44.9 Hz) by 10 Hz frequency resolution. Identifying the modal frequency for Mode 1 is well within the capability of the laser system used in this research. However, because the system was not capable of accurately estimating damping, Mode 3 was also simulated.

The damping and modal frequency results contained in this section are used as the baseline for comparison to the Mode 1 results using the SPC waveforms. These results are
found in Section 5.1.2.

**SDOF Mode 3 Results**

Because damping could not be accurately identified within the frequency resolution capability of the laser, another single-degree-of-freedom model was created to simulate the third bending mode of the beam used in the experimental research. The simulation was that of a Zoom FFT around the modal frequency of interest. Based on the beam theory modal frequencies identified in Table 4.4, the simulations were conducted to capture the accuracy of the excitation signals for the entire test matrix for Mode 3. Identical to the Mode 1 research, the simulations were first completed for the ideal chirp waveform to simulate the function generator signal currently used in testing.

![Graphs](image_url)

**Figure 5.4: SDOF Mode 3 results using ideal chirp waveforms.**

First, the accuracy of damping identification was assessed. Figure 5.4(a) illustrates the ability to accurately identify damping only occurs at frequency resolutions of 160 mHz or less for the three bandwidths of interest: 8, 16, and 32 Hz. For instance, for 50 FFT lines, the only accurate setting was 8 Hz bandwidth. 100 FFT lines were necessary for the 16 Hz bandwidth simulation, and a minimum of 200 FFT lines were required for the 32 Hz bandwidth simulation to accurately identify damping for Mode 3. At the worst setting (32
Hz bandwidth and 50 FFT lines) the percent error was only 6% and quickly came within 1% error by the 100 FFT line simulation.

As predicted, the accuracy of identifying the damping decreases the shorter the sweep time. When the test is performed at sweep rates greater than 5 Hz/s, the system does not get enough excitation at the modal frequency to excite the response. With low responses, the identification of damping cannot be accurately assessed.

The ability to accurately identify the modal frequency was then assessed. Figure 5.4(b) depicts the dependency of modal frequency identification to frequency resolution. Although the results convergence to the correct modal frequency for lower bandwidth and higher FFT line settings, even the worst tests are within 0.001% error. At frequency resolutions greater than 160 mHz, the frequency identification was within 0.001% error where the damping estimation was less accurate at 6% error for the same simulation. The modal frequency can be estimated with much greater accuracy for the same test settings than damping can be estimated. Damping requires more fidelity in the peak of the Frequency Response Function, where the modal frequency only needs to estimate the location of the peak.

5.1.2 Single Cycle Waveform Results

SDOF Mode 1 Results

Based on the results using the ideal linear chirp waveforms, the SPC waveforms are compared to these results for each respective test setting, and not to the actual SDOF parameters themselves. As the number of samples/cycle increases, the shape of the waveform approaches the shape of the ideal chirp signal. Therefore, it was hypothesized that the results will converge to that of the ideal chirp waveform as the samples/cycle increases and not necessarily to the actual properties of the system.

Figure 5.5(a) depicts the results of the SDOF Mode 1 system using the SPC waveforms
at 100 Hz bandwidth. The dashed line is the result from the ideal chirp waveform at the 100 Hz bandwidth setting using 25600 FFT lines as depicted previously in Figure 5.1(a) and is shown to illustrate the convergence. The SPC waveforms converge quickly to the level of accuracy of the ideal chirp waveform of the same resolution for all tests as shown in Figure 5.5(b). The worst results (3 samples/cycle) were within 10% error of the ideal chirp waveform for the 100 Hz bandwidth simulations. By 30 samples/cycle, all the results were consistent with the ideal chirp identified values, but by 10 samples/cycle the error was acceptable.

![Graphs](image)

(a) Damping vs Samples/Cycle  
(b) Damping Error vs Samples/Cycle

Figure 5.5: Damping results for SDOF Mode 1, 100Hz bandwidth.

Figure D.3 presents the damping results for the remaining 250-2000 Hz tests. For all the simulations, the identified damping quickly converged to the values identified using the ideal chirp waveforms. Unlike the solutions for damping, the frequency results from the single degree of freedom model had very little variation. The modal frequency results for the 100 Hz bandwidth simulation using 12800 FFT lines is shown in Figure 5.6(a). The highest error occurred at the lowest number of samples/cycle, but converged quickly for all lines of resolution as shown by the percent error in Figure 5.6(b). Regardless of the samples/cycle, the results were within 0.2% error of the estimations for the modal frequency using the respective ideal chirp waveforms.
The remaining modal frequency results are shown in Figure D.4 for the simulations of 250, 500, 1000, and 2000 Hz bandwidth. As expected, the estimate of modal frequency for larger bandwidth simulations are less accurate, yet they still converged to the identified modal frequency of each respective ideal chirp waveform. The lowest resolution simulation resulted in the least accurate result in each test case. For the largest bandwidth settings, the SPC waveform converged only at the highest samples/cycle.

**SDOF Mode 3 Results**

Due to the frequency resolution limitations of the Polytec system, Mode 1 could not be accurately assessed. Mode 3 was chosen for modal analysis, and its results are discussed in this section.

Based on the setup of the Polytec vibrometry system, the SDOF simulation was revised to perform a Zoom FFT around the third bending mode. This was performed in 8, 16, and 32 Hz bandwidths around the actual modal frequency of the system. The results were as predicted based on the previous results. The simulations with the smaller bandwidth and larger number of FFT lines converged more quickly to the result of the ideal chirp waveform. The damping results for the 8 Hz, 1600 FFT line simulation are plotted in
Figure 5.7(a). It is evident that the damping estimation converges to the result determined using the ideal linear chirp waveform. Figure 5.7(b) shows the percent error of every 8 Hz bandwidth test. All of these simulations, regardless of samples/cycle, were less than 0.003% error.

The remaining damping results for 16 and 32 Hz bandwidth using the SPC waveforms in shown in Figures D.6(a) and D.6(b), respectively.

![Figure 5.7(a): Damping vs Samples/Cycle](image1)

![Figure 5.7(b): 8 Hz bandwidth](image2)

Figure 5.7: Damping results for SDOF Mode 3 results using SPC waveforms

Unlike the results for Mode 1, the simulations using the SDOF model for Mode 3 had very little variation in the modal analysis using the SPC waveforms as the source of excitation. As seen for the 8 Hz bandwidth 1600 FFT line simulation in Figure 5.8(a), the SPC waveforms, even for three samples/cycle, estimated the modal frequency within 0.11% difference of the respective ideal chirp waveform solution.

The remaining modal frequency results for 16 and 32 Hz bandwidth using the SPC waveforms in shown in Figures D.6(a) and D.6(b), respectively. The 50 FFT line solution had the largest error in each case and took the most number of samples/cycle to converge, yet converged within 1% error at seven samples/cycle.

Using the Zoom FFT approach around higher modes allowed for better frequency resolution which resulted in more accurate analysis. This approach will be used in the
Figure 5.8: Modal frequency results for SDOF Mode 3 results using SPC waveforms

experimental research of the beam to which this single degree of freedom model was built to represent. The experimental results are discussed in the following section.

5.2 Experimental Results

The following sections present the results from the bench testing of the cantilevered beam identified in Chapter 4. These tests were centered on the third bending mode of the cantilevered beam to maximize the capability of the acquisition system. As in the analytical research, the experimental research was performed in 8, 16, and 32 Hz bandwidths for a multitude of number of FFT lines as listed in Table 4.5. A Zoom FFT test was performed for all of these tests on the Polytec laser vibrometry system and the results are discussed in this section.

The damping results for all the bandwidth settings are plotted in Figure 5.9(a). The data is also broken down by bandwidth and FFT settings in Figures D.7-D.10. The majority of the data is accurate with exception of the lowest 2 FFT line settings for the 32 Hz bandwidth test. These two tests were at 160 mHz or greater frequency resolution, so the larger error in damping estimation is as predicted from the analytical simulations.
The percent error shown in Figure 5.9(b) was calculated, for each test setting, against the result of the respective chirp waveform produced by the Polytec internal function generator. With exception of the two outlying tests, the error was less than 3% for any of the SPC waveforms. The results converged to that of the experimental chirp waveform. The 32 Hz bandwidth setting shows this convergence the best as the 50 samples/cycle waveform was able to be tested.

Figure 5.9: Experimental damping results for Mode 3 using SPC waveforms.

The 8 Hz bandwidth tests resulted in the lowest percent error from the actual solution and the corresponding Polytec waveform solution for both damping and frequency. From 5-10 samples/cycle, the damping converged within 2% of the experimental chirp result. The damping prediction of the beam excited by the ideal chirp waveform was less than 0.09% different than the actual measured response using the Polytec signal.

The results for the modal frequency estimations for Mode 3 are shown in Figure 5.10(a). These results were much closer the to both the predicted frequency, and the result of the experimental chirp signals. The results are all very precise as shown by the percent error in Figure 5.10(b). All of the SPC waveforms were within 0.03% error of the respective experimental chirp waveform, but most were within 0.01% of the baseline value.

Both the damping and frequency results were more accurate than the analytically pre-
dicted accuracy for the respective test parameters. This is most likely due to the noise induced in the signal as it passes through the cables, amplifier, and transformer. The hard edges of the signal are softened and it more quickly begins to take the form of the experimental chirp signal. Especially at higher frequencies, the horizontal resolution is less effected because the change in the amplitude of the signal is significantly more.

Due to the successful generation and accuracy of the SPC waveforms on a cantilevered beam, further analytical analysis was conducted and is discussed in the following section.

5.3 Analytical Simulated Bladed Disk Results

To determine the possibility of implementing SPC waveform generation for traveling wave excitation, a multiple degree of freedom model was created. This model simulated a 10 finger disk and was excited using the traveling wave simulated excitation. A signal was applied to the tip of each blade and was offset from the previous by the inter-blade phase angle determined by Equation (2.2). This replicated engine order excitation to excite specific nodal diameter modes of the system. Based on the previous results, all the simulations for the finger disk were set at a bandwidth of 8 Hz for maximum accuracy, and to match
potential future tests. It gave the most accurate results for the other systems, so it was chosen for this set of tests.

5.3.1 Traveling Wave Excitation Using Theoretical Waveforms

The first simulation using the traveling wave was performed using a linear ideal chirp waveform as before. This set the baseline for the system’s response to which the SPC waveforms were compared. Similar to the results from the single and multiple degree of freedom models of the beam, Figure 5.13 shows the necessity for high resolution to accurately capture the first bending modes. This was a similar effect for each of the nodal diameter responses for first bending. The first and second nodal diameter responses estimated higher damping, while the other nodal diameters slightly underestimated the system’s damping. Also, only the simulations of 10 and 5 mHz frequency resolution resulted in accurate estimations for damping. These were produced by the 800 and 1600 FFT line simulations. However, the next higher mode was accurate for even 80 mHz frequency resolution as shown in Figure 5.11(b).

![Figure 5.11: Damping results for finger disk using ideal chirp waveforms.](image)

(a) Damping for 1B modes  (b) Damping for 2B modes

The second bending mode converged quickly. Even for the lowest resolution tests the estimated damping value was within 20%. Tests of these lower resolutions are typically performed initially to confirm the setup of the system and calculate some initial values,
but then higher resolution tests are conducted to get more accurate results. The higher resolution tests come at a price of sweep time, so simulations such as this give insight as to the necessary settings of the acquisition system.

![Diagram of Modal Frequencies for 1B modes and 2B modes](image)

**(a) Modal Frequencies for 1B modes**

**(b) Modal Frequencies for 2B modes**

Figure 5.12: Modal frequency results for finger disk using ideal chirp waveforms.

The estimations of the modal frequency for each of the nodal diameters was very accurate as shown in Figure 5.14. Every simulation setting resulted in accurate estimation. These along with the damping estimation will be used as the baseline for comparison to the SPC waveforms in the following section.

### 5.3.2 Traveling Wave Excitation Using Single Cycle Waveforms

The 10 finger disk was analyzed for response characteristics when the phased excitation was generated using the Single Cycle waveform. Figure 5.13 captures the effect of the Single Cycle waveform for the 800 FFT line simulation on damping. Both the damping and frequency results of the lower resolution simulations are in Appendix D in Section D.3.

The results show there is a negligible effect of the SPC waveforms no matter how few or how many samples/cycle are chosen. All of the damping estimations converged as shown in Figure 5.13(a) for Mode 1 nodal diameters, and in Figure 5.13(b) for Mode 2 nodal diameters. For the second bending modes, the first and third nodal diameters were
slightly over-estimated at first, but again, all converged quickly.

The percent error of the damping results for both the first and second bending modes are shown in Figures 5.13(b) and 5.13(d), respectively. For the first bending modes, the largest error was 2.8% difference from the ideal chirp waveform result, but all results converged to less than 1% error by 8 samples/cycle. For the second bending modes, the error was significantly less with a maximum of 1.1% error, and converged within -0.4% by 10 samples/cycle.

Figure 5.13: Damping results for finger disk using SPC waveforms.

The frequency results for both the first and second bending modes are shown in Figure 5.14(a) and 5.14(c), respectively. The largest error was less than 0.001% difference from the ideal chirp waveform result. All of the frequency results converged immediately for
both modes for all of the nodal diameters. This is similar to the previous modal frequency results, that it is less affected by the test parameters.

Figure 5.14: Modal frequency results for finger disk using SPC waveforms.

The MAC results for both the first and second bending modes are shown in Figure 5.15(a) and 5.15(c), respectively. The largest error was less than 0.09% difference from the ideal waveform results shown in Figures 5.15(b) and 5.15(d). All of the MAC results converged for both modes for all of the nodal diameters.
Figure 5.15: MAC results for finger disk using SPC waveforms.
Conclusion

6.1 Summary

A methodology for creating variable update rate waveforms, and a study of the effects of such waveforms, for both single channel modal analysis and traveling wave excitation, has been provided. Both single and multiple degree of freedom models were investigated to predict the effects of linearly variable update rate waveforms. Additionally, the influence of traveling wave excitation using phased Single Cycle waveforms was studied. Reduced models using SEREP were constructed to analyze the modal response of the multiple degree of freedom models. These predictions were validated experimentally for the cantilevered beam.

The analytical predictions using both single and multiple degree of freedom models for the cantilevered beam proved accurate. The experimental results were within 1% of the predicted values for the third bending mode of the beam. The investigation of lower resolution tests resulted in inaccurate results as expected. The SPC waveform performed only as well as the ideal chirp waveform, and there was no benefit in lower resolution tests using these waveforms. As hypothesized, the results using the SPC waveforms converged quickly to the result of the ideal chirp waveform. The higher resolution tests converged more quickly thus requiring fewer samples/cycle to create the SPC waveform. The lower resolution tests did not adequately excite the system and therefore the modal analysis results were inaccurate for both the ideal chirp and SPC waveforms.
The methodology of storing just one sinusoidal cycle, and using a linearly varying timing signal to create a full excitation signal for modal analysis proved feasible and accurate. Further, the lower fidelity signal did not affect the phased characteristics of traveling wave excitation simulations. The damping results when using SPC waveforms of 10 samples/cycle were accurate for all of the systems analyzed. Also, identifying the modal frequency was accurate for even 3 samples/cycle for all of the systems analyzed. However, if physical memory allows, more samples/cycle should be used as the results converge more to the ideal chirp waveform with more samples/cycle.

6.2 Recommendations

The Single Cycle algorithm provides potential improvement to the traveling wave system. At this point, this research is applicable to the limited set of conditions studied. The only experimental validation conducted was for bending modes of a cantilevered beam. The results are promising, but caution should be taken when applying these methods beyond the demonstrated configurations.

The experimental testing was limited by the frequency resolution of the acquisition system. A higher resolution acquisition system may provide better validation capabilities for lower frequency modes and also provide the same accuracy for shorter sweep times.

Due to hardware limitations of available function generators, not all of the tests would have been able to be performed due to the required length of the signal. By turning off the internal signal generator of the Polytec system, the necessary TTL timing signal could be created and delivered to the other analog output card to create the SPC waveform. This however, became cumbersome at times due to routing multiple signals in and out of the same computer. It is recommended that a separate signal generator card is used to create the required TTL timing signal to eliminate additional computational requirements of the acquisition system.
6.3 Future Work

The application of the Single Cycle waveform to traveling wave excitation would be very significant. Building a new traveling wave system with multiple analog output cards and synchronized clocks would be able to produce multiple engine orders simultaneously without demanding large amounts of memory for storing waveforms.

In order to more fully evaluate the effect of Single Cycle waveforms, a thorough investigation of the synchronization requirements and response of a bladed disk system should be considered. While this research focused on bending modes, it would be of interest to evaluate the influence on torsional and stiff-wise modes as well.


Appendix A

Traveling Wave Signal Generation

This appendix is a verbatim extraction (except for renumbering of equations and references) from previously published results [15].

A.1 Methodology

Engine order excitation can be simulated in a stationary bladed disk by applying harmonic excitation to all blades where the excitation differs from blade to blade by a constant inter-blade phase angle $\theta$:

$$F_i = A \sin (\omega t + i\theta)$$  \hspace{1cm} (A.1)

$$i = 1 \ldots n - 1$$  \hspace{1cm} (A.2)

$$\theta = \frac{2\pi c}{n}$$  \hspace{1cm} (A.3)

where $F_i$ is the forcing function on each blade, $A$ is the force amplitude, $\omega$ is frequency, $t$ is time, $i$ is the blade number, $c$ is the engine order excitation, and $n$ is the number of blades. This type of excitation in a stationary bladed disk is referred to as traveling wave excitation. The TEFF traveling wave system produces traveling wave excitation by mixing a sine and a cosine wave. This approach reduces the cost over purchasing separate phase locking signal sources by at least an order of magnitude. A two-channel function generator outputs a sine and cosine wave in either constant tone or sweep mode. The triggered
sweep mode significantly speeds data acquisition over a stepped sine test technique. A phase shifter was built to produce 24 phase-shifted signals from the sine and cosine inputs. The phase-shifted sine waves required for traveling wave excitation in equation (A.1) are created using the following trigonometry identity:

\[
A \sin(\omega t + i\theta) = B \cos(\omega t) + C \sin(\omega t) \quad (A.4)
\]

\[
B = A \sin(i\theta) \quad (A.5)
\]

\[
C = A \cos(i\theta) \quad (A.6)
\]
Appendix B

Single Cycle Waveform Examples

Figure B.1: Example waveform for 1-11 Hz sweep using 3 samples/cycle.

Figure B.2: Example waveform for 1-11 Hz sweep using 4 samples/cycle.
Figure B.3: Example waveform for 1-11 Hz sweep using 5 samples/cycle.

Figure B.4: Example waveform for 1-11 Hz sweep using 6 samples/cycle.

Figure B.5: Example waveform for 1-11 Hz sweep using 7 samples/cycle.
Figure B.6: Example waveform for 1-11 Hz sweep using 8 samples/cycle.

Figure B.7: Example waveform for 1-11 Hz sweep using 9 samples/cycle.

Figure B.8: Example waveform for 1-11 Hz sweep using 10 samples/cycle.
Figure B.9: Example waveform for 1-11 Hz sweep using 20 samples/cycle.

Figure B.10: Example waveform for 1-11 Hz sweep using 50 samples/cycle.

Figure B.11: Example waveform for 1-11 Hz sweep using 100 samples/cycle.
Figure B.12: Example waveform for 1-11 Hz sweep using 500 samples/cycle.
Appendix C

Single Cycle Traveling Wave Examples

All example waveforms plotted in this appendix are for a system with 10 blades and excitation of 4EO. Only the first 5 signals are plotted as signals 6-10 would be identical to signals 1-5.

(a) Single Cycles

(b) Generated Waveforms

Figure C.1: Example TWE waveforms for 1-11 Hz sweep using 3 samples/cycle.
Figure C.2: Example TWE waveforms for 1-11 Hz sweep using 4 samples/cycle.

Figure C.3: Example TWE waveforms for 1-11 Hz sweep using 5 samples/cycle.
Figure C.4: Example TWE waveforms for 1-11 Hz sweep using 6 samples/cycle.

Figure C.5: Example TWE waveforms for 1-11 Hz sweep using 7 samples/cycle.
Figure C.6: Example TWE waveforms for 1-11 Hz sweep using 8 samples/cycle.

Figure C.7: Example TWE waveforms for 1-11 Hz sweep using 9 samples/cycle.
Figure C.8: Example TWE waveforms for 1-11 Hz sweep using 10 samples/cycle.

Figure C.9: Example TWE waveforms for 1-11 Hz sweep using 20 samples/cycle.
(a) Single Cycles

(b) Generated Waveforms

Figure C.10: Example TWE waveforms for 1-11 Hz sweep using 50 samples/cycle.

(a) Single Cycles

(b) Generated Waveforms

Figure C.11: Example TWE waveforms for 1-11 Hz sweep using 100 samples/cycle.
Figure C.12: Example TWE waveforms for 1-11 Hz sweep using 500 samples/cycle.
Appendix D

Additional Results

This Appendix contains the additional results referenced from the main document in Chapter 5. These results are additional test settings that were studied to completely understand the effect bandwidth and number of FFT lines had on the accuracy of the degraded waveforms. Similar to the results in Chapter 5, the SPC waveforms in these additional tests converged to the identified modal parameters using the ideal chirp signal.
D.1 Analytical Single Degree of Freedom Results

D.1.1 Ideal Chirp Waveform Results

SDOF Mode 3 Results

(a) Bandwidth vs Identified Damping
(b) FFT lines vs Identified Damping

Figure D.1: SDOF Mode 3 damping results using ideal chirp waveforms.

(a) Bandwidth vs Identified Frequency
(b) FFT lines vs Identified Frequency

Figure D.2: SDOF Mode 3 frequency results using ideal chirp waveforms.
D.1.2 Single Cycle Waveform Results

SDOF Mode 1 Results

Figure D.3: Damping results for SDOF Mode 1 using SPC waveforms.
Figure D.4: Modal frequency results for SDOF Mode 1 using SPC waveforms.
SDOF Mode 3 Results

Figure D.5: Damping results for SDOF Mode 3 results using SPC waveforms

Figure D.6: Modal frequency results for SDOF Mode 3 results using SPC waveforms

D.2 Experimental Results
Figure D.7: Experimental results for Mode 3, 8 Hz bandwidth.
(a) Damping results 100 FFT lines  
(b) Modal frequency results 100 FFT lines  
(c) Damping results 200 FFT lines  
(d) Modal frequency results 200 FFT lines  
(e) Damping results 400 FFT lines  
(f) Modal frequency results 400 FFT lines  

Figure D.8: Experimental results for Mode 3, 16 Hz bandwidth.
Figure D.9: Experimental results for Mode 3, 32 Hz bandwidth, 100-400 FFT lines.
Figure D.10: Experimental results for Mode 3, 32 Hz bandwidth, 800 FFT lines.
D.3 Traveling Wave Excitation Using Single Cycle Waveforms

Figure D.11: Damping results for finger disk using SPC waveforms, 50 FFT lines.
Figure D.12: Damping results for finger disk using SPC waveforms, 100 FFT lines.
Figure D.13: Damping results for finger disk using SPC waveforms, 200 FFT lines.
Figure D.14: Damping results for finger disk using SPC waveforms, 400 FFT lines.
Figure D.15: Modal frequency results for finger disk using SPC waveforms, 50 FFT lines.
Figure D.16: Modal frequency results for finger disk using SPC waveforms, 100 FFT lines.
Figure D.17: Modal frequency results for finger disk using SPC waveforms, 200 FFT lines.
Figure D.18: Modal frequency results for finger disk using SPC waveforms, 400 FFT lines.
(a) MAC for 1B modes

(b) Error for 1B modes

(c) MAC for 2B modes

(d) Error for 2B modes

Figure D.19: MAC results for finger disk using SPC waveforms, 50 FFT lines.
Figure D.20: MAC results for finger disk using SPC waveforms, 100 FFT lines.
Figure D.21: MAC results for finger disk using SPC waveforms, 200 FFT lines.
Figure D.22: MAC results for finger disk using SPC waveforms, 400 FFT lines.