Fundamental Understanding of Blisk Analytical Response

Joseph A. Beck
Wright State University

Follow this and additional works at: https://corescholar.libraries.wright.edu/etd_all
Part of the Engineering Commons

Repository Citation
https://corescholar.libraries.wright.edu/etd_all/709

This Dissertation is brought to you for free and open access by the Theses and Dissertations at CORE Scholar. It has been accepted for inclusion in Browse all Theses and Dissertations by an authorized administrator of CORE Scholar. For more information, please contact library-corescholar@wright.edu.
Fundamental Understanding of Blisk Analytical Response

A dissertation submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy

By

Joseph A. Beck
B.S.M.E, United States Air Force Academy, 2006
M.S.E, Wright State University, 2010

2013
Wright State University
WRIGHT STATE UNIVERSITY
GRADUATE SCHOOL

April 15, 2013


________________________________________
Joseph C. Slater, Ph.D.
Dissertation Director

________________________________________
Ramana V. Grandhi, Ph.D.
Director, Ph.D. in Engineering Program

________________________________________
R. William Ayres, Ph.D.
Interim Dean, Graduate School

Committee on Final Examination

________________________________________
Joseph C. Slater, Ph.D.

________________________________________
Ramana V. Grandhi, Ph.D.

________________________________________
Jeffrey M. Brown, Ph.D.

________________________________________
Charles J. Cross, Ph.D.

________________________________________
Ha-Rok Bae, Ph.D.

________________________________________
Richard G. Cobb, Ph.D.
ABSTRACT


This effort seeks to increase the reduced-order model fidelity for mistuned Integrally Bladed Rotor (IBR) and Dual Flow-path Integrally Bladed Rotor (DFIBR) response prediction by explicitly accounting for blade geometric and material property deviations. These methods are formulated in a component mode synthesis (CMS) framework utilizing secondary modal reductions in a cyclic symmetry format. The resulting reduced-order models (ROMs) capture perturbations to both blade natural frequencies and mode shapes resulting from geometric deviations. Furthermore, the secondary modal reductions and cyclic symmetry format offer significant computational savings over traditional component mode synthesis methods that give a further reduction in model size. The first formulation for IBRs assumes a tuned disk-blade connection and presents two methods that explicitly model blade geometry surface deviations by performing a modal analysis on different degrees of freedom of a parent reduced-order model. The parent ROM is formulated with Craig-Bampton component mode synthesis (CB-CMS) in cyclic symmetry coordinates for an IBR with a tuned disk and blade geometric deviations. The first method performs an eigen-analysis on the constraint-mode degrees of freedom (DOFs) that provides a truncated set of Interface modes while the second method includes the disk fixed-interface normal modes in the eigen-analysis to yield a truncated set of Ancillary modes. Both methods can utilize tuned or mistuned modes, where the tuned modes have the computational benefit of being computed in cyclic symmetry coordinates. Furthermore, the tuned modes only need to be calculated once, which offers significant computational savings for subsequent mistuning studies. Each geometric mistuning method relies upon the use of geometrically mistuned blade modes in the component mode framework to provide a very accurate ROM. Free and forced response results are compared to both the full finite element model (FEM) so-
lutions and a traditional frequency-based approach used widely in academia and the gas turbine industry. It is shown that the developed methods provide highly accurate results with a significant reduction in solution time compared to the full FEM and parent ROM. An investigation into the assumed tuned disk-blade connection is then performed. Two types of disk-blade connection mistuning are investigated: as-measured principal component deviations and random perturbations to the inter-blade spacing. Finally, these methods are extended to ROM methodologies for DFIBRs to assess the susceptibility of these new designs to mistuning and to be able to efficiently and effectively predict response amplification. Two main approaches are presented: first, a frequency-based method that is analogous to traditional mistuning approaches for IBRs, and second, geometric approaches that explicitly model blade geometry surface deviations. These methods help characterize DFIBR dynamic response and investigate the unique aspects that differentiate these advanced components from IBRs. In all methods, free and forced response results are compared to both the full FEM solutions and the traditional frequency-based approaches. It is shown that the developed methods provide highly accurate results with a significant reduction in solution time compared to the full FEM and parent ROM.
List of Symbols

Main Variables

\( C \) = Engine order excitation
\( E \) = Real-valued Fourier matrix
\( f \) = Physical space force vector
\( \mathcal{F} \) = Force vector in CB-CMS modal space
\( h \) = Harmonic index
\( \lambda, \Lambda \) = CB-CMS fixed-interface normal eigenvalues and spectral matrix
\( M, K \) = Mass and stiffness matrix is physical space
\( \mathcal{M}, \mathcal{C}, \mathcal{K} \) = Mass, damping, and stiffness matrix in CB-CMS modal coordinates
\( N \) = Number of cyclic sectors and blades
\( N_{<>} \) = Number or size, where \( \cdot \) is a placeholder for any variable
\( P \) = Forcing phase vector
\( p \) = Generalized CB-CMS modal coordinate vector
\( \Phi, \phi \) = CB-CMS fixed-interface normal modes
\( \psi \) = Inter-blade phase angle
\( \Psi \) = CB-CMS constraint modes
\( q \) = Secondary modal coordinate vector
\( U \) = CB-CMS coordinate transformation matrix
\( x \) = Physical DOF vector
\( \zeta \) = Modal damping coefficient
\( \Delta X \) = Matrix of measured coordinate deviations
\( \bar{\pi} \) = Variable mean
\( \Sigma \) = First order covariance matrix of \( \Delta X \)
\( \Psi \) = Matrix of eigenvectors
\( Z \) = Principal component score matrix
\( \xi \) = Fraction of critical damping

Hats

\( \sim \) = Cyclic coordinates

Left Superscripts

\( A \) = Blade components
\( CB \) = CB-CMS matrices
\( D \) = Disk component
\( R \) = Ring component

Right Superscripts

\( (< , >) \) = Harmonic or component index place holder
\( a \) = Blade index
\( \tau \) = Matrix transpose

Right Subscripts

\( c \) = CB-CMS constraint mode DOF
\( n \) = CB-CMS fixed-interface normal mode DOF
\( \beta, \alpha, \tau, \sigma, \Gamma \) = DOF index notation
<table>
<thead>
<tr>
<th>Acronym</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>ADLARF</td>
<td>Augmented Damping Low Aspect Ratio Fan</td>
</tr>
<tr>
<td>Blisks</td>
<td>Bladed Disks</td>
</tr>
<tr>
<td>CB-CMS</td>
<td>Craig-Bampton Component Mode Synthesis</td>
</tr>
<tr>
<td>CMM</td>
<td>Coordinate Measurement Machine</td>
</tr>
<tr>
<td>CMS</td>
<td>Component Mode Synthesis</td>
</tr>
<tr>
<td>DFIBR</td>
<td>Dual Flow-path Integrally Bladed Rotor</td>
</tr>
<tr>
<td>DOF</td>
<td>Degree of Freedom</td>
</tr>
<tr>
<td>EO</td>
<td>Engine Order</td>
</tr>
<tr>
<td>EOM</td>
<td>Equation of Motion</td>
</tr>
<tr>
<td>EVP</td>
<td>Eigenvalue Problem</td>
</tr>
<tr>
<td>FEA</td>
<td>Finite Element Analysis</td>
</tr>
<tr>
<td>FEM</td>
<td>Finite Element Model</td>
</tr>
<tr>
<td>HCF</td>
<td>High Cycle Fatigue</td>
</tr>
<tr>
<td>HOM</td>
<td>Higher Order Mode</td>
</tr>
<tr>
<td>IBR</td>
<td>Integrally Bladed Rotor</td>
</tr>
<tr>
<td>LEO</td>
<td>Low Engine Order</td>
</tr>
<tr>
<td>MAC</td>
<td>Modal Assurance Criterion</td>
</tr>
<tr>
<td>ND</td>
<td>Nodal Diameter</td>
</tr>
<tr>
<td>OEM</td>
<td>Original Engine Manufacturer</td>
</tr>
<tr>
<td>PCA</td>
<td>Principal Component Analysis</td>
</tr>
<tr>
<td>PC</td>
<td>Principal Component</td>
</tr>
<tr>
<td>ROM</td>
<td>Reduced-order Model</td>
</tr>
</tbody>
</table>
## Contents

1 Introduction ................................. 1
   1.1 Turbine Engines ........................... 1
   1.2 High Cycle Fatigue ......................... 2
   1.3 IBRs and DFIBRs .......................... 7
   1.4 Mistuning and Mode Localization .......... 8
   1.5 Overview of the Dissertation ............. 9

2 Literature Review ........................... 12
   2.1 Lumped Parameter Models ................ 12
   2.2 Nominal Mode Mistuning Reduced-Order Models .... 13
   2.3 Geometric Mistuning Reduced-Order Models .... 20
   2.4 Crack-Induced Mistuning .................. 23
   2.5 Probabilistic Mistuning Methods .......... 25
   2.6 Mistuning Identification .................. 27
   2.7 Management of Mistuning for Turbine Engines ...... 28
   2.8 Aerodynamic Effects ...................... 29
   2.9 Summary .................................. 30

3 Research Overview ........................... 31
   3.1 Research Need ............................. 31
   3.2 Research Contribution .................... 33
   3.3 Summary .................................. 35

4 Dynamics of Rotationally Periodic IBRs and DFIBRs ....... 37
   4.1 Mode Shapes .............................. 38
   4.2 Cyclic Constraints ......................... 40
6.10.1 CC and CA Tuned and Mistuned Mode Comparison ............... 98
6.10.2 Mistuned Free Response Results .................................. 101
6.10.3 Mistuned Forced Response Results ............................... 104
6.10.4 Model Size and Solution Times .................................... 108
6.11 Conclusions .............................................................. 113

7 Geometric Mistuning ROM for IBRs with Mistuned Disk-Blade Connections 115
7.1 Introduction ............................................................... 116
7.2 Reduced-Order Model Formulation ................................. 118
  7.2.1 Excitation Force .................................................... 122
  7.2.2 Tuned Constraint Mode Approximation .......................... 123
  7.2.3 Interface Mode Reduction ...................................... 124
    7.2.3.1 Tuned Interface Modes ................................... 125
    7.2.3.2 Mistuned Interface Modes ................................ 126
7.3 Mistuning Models ...................................................... 126
7.4 Disk-Blade Connection Modeling ................................... 128
  7.4.1 Mistuned Principal Component Configuration ...................... 128
  7.4.2 Mistuned Inter-blade Spacing .................................. 129
7.5 Results ................................................................. 131
  7.5.1 Constraint Mode Comparison ................................... 133
  7.5.2 Interface Mode Comparison .................................... 134
  7.5.3 Natural Frequency Comparison .................................. 137
  7.5.4 Forced Response Comparison ................................... 137
  7.5.5 Mistuned and Tuned Disk-Blade Connection Comparison .......... 140
  7.5.6 Model Size and Solution Times ................................ 142
7.6 Conclusions .............................................................. 144

8 Mistuning ROMs for Dual Flow-path Integrally Blade Rotors 145
8.1 Introduction .............................................................. 146
8.2 Cyclic Component Matrices .......................................... 150
8.3 Blade Component Matrices ........................................... 152
8.4 Nominal Mode Approaches ........................................... 153
  8.4.1 Formulation for all Blades .................................... 153
  8.4.2 Rearrange Blade Matrices ...................................... 154
  8.4.3 Component Coupling ............................................ 158
  8.4.4 Nominal Method CB-CMS ROM ................................ 162
8.5 Secondary Modal Reduction ROMs .................................. 163
Bibliography 214

Appendices

A  Cyclic Constraint Formulation 229
B  Circulant Matrices 236
C  Kronecker Product 238
## List of Figures

1.1 The fundamental turbine engine process where air enters at the left through the intake and undergoes compression, fuel is added in the combustor, and then expands through the turbine before exiting on the right [1].

1.2 Typical IBR Campbell diagram showing selected mode crossings [6].

1.3 Schematic of a constant life diagram that illustrates safety factors and HCF margin [5].

1.4 Schematic showing HCF crack length as a function of fatigue life and how the majority of HCF life falls below inspection capabilities [5].

1.5 Illustration comparing a tuned and mistuned IBR natural mode of vibration.

2.1 Schematic of a lumped parameter model that illustrates the coarse representation of an IBR through various connections and lumped masses [26].

4.1 Partitioned IBR index notation used in the mathematical formulation.

4.2 Full ADLARF IBR test case.

4.3 Nodal diameter plot illustrating natural frequencies versus nodal diameters. The investigated frequency ranges at the specific EO excitations are circled.

4.4 Cantilevered ADLARF blade modes shape types describing the blade motion for each mode family of Fig. 4.3 (HOM = Higher-Order Mode).

4.5 Full DFIBR test case.

4.6 The nodal diameter plot of the DFIBR illustrating the tuned system natural frequencies versus the harmonic index.

4.7 Modal Stresses for Family 4 illustrating the transition from primarily outer-blade motion through a system mode to primarily inner-blade motion.

4.8 First system mode at Harmonic 0 illustrating the inner-blade cantilevered first bend mode under fixed-fixed blade root and tip conditions.
4.9 Illustration how different EO excitations between the inner- and outer-blades causes non-constant force magnitudes around the DFIBR and the disruption in the phase of Eq. 4.27 ........................................ 57
4.10 Individual and Total Variance Explained by Principal Components used for the ADLARF Rotor ................................................................. 60
4.11 Random ADLARF blade (pressure side) x-direction surface deviations ... 61

5.1 System substructuring process of an industrial turbine engine fan illustrating the 16 blade components and one disk component ....................... 64

6.1 Modal Assurance Criterion for tuned and mistuned Interface modes ....... 99
6.2 Modal Assurance Criterion for tuned and mistuned Ancillary modes ...... 100
6.3 IBR natural frequency error for each ROM compared against full FEM predictions ................................................................. 102
6.4 Modal participation factors for the EO excitations and forcing frequency range on interest from Fig. 4.3 on page 47. ................................. 103
6.5 Comparison of Rotor A IBR mistuned mode 121 against the full FEM ... 105
6.6 Comparison of Rotor B IBR mistuned mode 124 against the full FEM ... 106
6.7 IBR peak forced response levels over excitation frequency range .......... 107
6.8 Comparison of Rotor A peak blade-to-blade mistuned forced response levels against the full FEM predictions ........................................ 109
6.9 Comparison of Rotor B peak blade-to-blade mistuned forced response levels against the full FEM predictions ........................................ 110
6.10 Peak blade-to-blade mistuned forced response error .......................... 111

7.1 Index notation for the IBR partitionment of a single sector ................. 118
7.2 Histogram of the magnitudes of Euclidean distance geometric points between a tuned disk-blade connection and mistuned configuration A1 ... 129
7.3 Superposition of all 16 disk-blade connections for mistuned A3 configuration that illustrate upper and lower surface deviations ................. 130
7.4 Histogram of the magnitudes of Euclidean distance geometric points between a tuned disk-blade connection and mistuned configuration A4 ... 131
7.5 Superposition of all 16 disk-blade connections for configuration A4 that illustrate inter-blade spacing deviations ................................. 132
7.6 Worst-case magnitude of nodal displacement deviations between a tuned and mistuned constraint mode ................................................. 135
7.7 Modal Assurance Criterion for tuned and mistuned Interface modes ....... 136
7.8 IBR natural frequency error for each ROM compared against full FEM predictions for configurations A1, A3, and A4 .................................................. 138
7.9 Peak blade-to-blade response predictions for each ROM compared against full FEM predictions for configurations A1 and A4 .......................... 139
7.10 Peak blade-to-blade response error for each ROM compared against full FEM predictions for configurations A1, A3, and A4 .......................... 141
7.11 MIMCM peak blade-to-blade response error compared against those assuming a tuned disk-blade connection ........................................ 143

8.1 Dual Flow-Path Integrally Bladed Rotor consists of an inner IBR and an integral outer ring with a second set of blades ...................................... 147
8.2 Partitioned IBR index notation used in the mathematical formulation .................. 149
8.3 Illustration how different EO excitations between the inner- and outer-blades causes non-constant blade displacement magnitudes around the DFIBR and the disruption in the phase .............................................. 173
8.4 ROM natural frequency prediction error of the two mistuned DFIBR test cases when compared against full FEM solutions .............................. 175
8.5 Largest participating system mode plotted for the inner- and outer-blades for DFIBR A with the excitation conditions described in Table 8.2 ........ 176
8.6 Largest participating system mode plotted for the inner- and outer-blades for DFIBR B with the excitation conditions described in Table 8.2 ........ 177
8.7 Modal participation factors for DFIBRs A and B with the excitation conditions described in Table 8.2 ......................................................... 179
8.8 Peak DFIBR response for test case A with the excitation conditions described in Table 8.2 ................................................................. 181
8.9 Peak DFIBR response for test case B with the excitation conditions described in Table 8.2 ................................................................. 182
8.10 Peak blade response for test case A with the excitation conditions described in Table 8.2 ................................................................. 183
8.11 Peak blade response for test case B with the excitation conditions described in Table 8.2 ................................................................. 184
8.12 Tuned DFIBR system modes in the frequency range of interest at harmonic index \( h = 0 \) ................................................................. 200
8.13 Modal participation factors for the EO excitations \( I^C = O^C = 0 \) over an excitation frequency range of 1450 – 1750 Hz ......................... 202
8.14 Comparison of predicted mistuned mode 109 against the full FEM solution
8.15 CCN and CMS blade-to-blade forced response predictions for the mistuned DFIBR compared against full FEM solutions ................. 205
8.16 Interface (CC) and Ancillary (CA) reduced ROMs’ blade-to-blade forced response predictions for the mistuned DFIBR compared against full FEM solutions .................................................. 206
8.17 Interface (CC) and Ancillary (CA) reduced ROMs’ blade-to-blade forced response prediction percent error for the mistuned DFIBR compared against full FEM solutions .................................................. 207
8.18 ROM peak DFIBR forced response predictions of the inner-blades as compared to the full FEM solution ................................. 208
List of Tables

4.1 Basic FEM size data for the ADLARF rotor ........................................ 47
4.2 Basic FEM size data for the DFIBR partitioning defined in Fig. 8.2 ........ 51

6.1 ROM names and associated EOM ......................................................... 95
6.2 Mistuning ROM sizes ................................................................. 96
6.3 Secondary Modal Analysis Solution times ........................................... 112
6.4 Mistuning ROM sizes ................................................................. 113
6.5 ROM Eigen-Problem Solution times .................................................. 113

7.1 Inter-blade spacing deviations from the nominal placement ................. 133

8.1 Mistuning ROM sizes ................................................................. 170
8.2 Engine order excitation conditions for the two mistuned DFIBR test cases . 174
8.3 Developed ROM sizes ................................................................. 185
8.4 Developed ROM Size Comparisons .................................................. 186
8.5 Developed ROM sizes ................................................................. 209
8.6 Developed ROM Size Comparisons .................................................. 210
Acknowledgment

I would first like to thank Dr. Slater for his guidance and technical expertise throughout this research effort. We’ve had many fruitful discussions that were great learning opportunities of the research process. I’m very grateful for his contribution throughout my whole academic experience.

I would also like to acknowledge the Engine Integrity Branch, under the direction of Dr. Charles Cross, for their support and technical guidance as well. They have been instrumental in helping understand the research need within academia, government, and industry. This branch is truly one of the most instrumental in maintaining reliable engines since it is composed of technical experts with vast knowledge in fundamental, applied, and transitioning research.

I am also grateful for my Committee Members and the direction they have provided throughout this research effort. Their probing questions and insight helped guide and solidify the foundation of this work and I thankful for their time spent with me as I navigated the Ph.D. degree requirements.

Alysoun Taylor-Hall, the Ph.D. in Engineering Program Coordinator, was also instrumental in completing this document and the Ph.D. program. I would like to thank her for the countless times she helped me coordinate the degree requirements. I would have been lost without her!

I would also like to thank the Dayton Area Graduate Studies Institute for their financial support. Without it, I never would have completed the degree. This program has been instrumental in growing and developing engineering workforce talent for the area and I hope it continues long into the future.

Dr. Jeff Brown was also fundamental in completing this work. He has been a great source of technical expertise as well as an ear to bend during the less-than-fun times of the degree. Since my arrival in his research group, he has been instrumental in my growth as a researcher.
I would like to thank my wife, Melissa, for the countless sacrifices she has made to make this degree a reality. Throughout the whole degree she has been working full-time, raising two beautiful boys, and maintaining the household. I’m indebted to her for the amount of work she put in behind the scenes so I could study, read, and write. The degree means so much more to me because she is able to share in its completion.

Lastly, I would like to thank my two boys, Sam and Matt, who were a constant reminder during the stressful times of what is really important in life. I look forward to spending much more time with you both and watching you grow and learn. I love you both.
Dedicated to

My Boys:

Samuel, 4
Mathew, 2
Introduction

1.1 Turbine Engines

The capabilities of the turbine engine since its inception in the 1940’s rendered it one of the most well-recognized and versatile power systems. From generating thrust for commercial and military aerospace platforms to producing torque for electrical generators, the turbine engine represents a great achievement over a multitude of complex engineering problems requiring complex solutions. These problems encompass many different technical disciplines, each engulfing a vast amount of research. This work will use these past research efforts to model and better understand physical phenomena that will aid and improve the design of turbine engine components from a structural dynamics standpoint.

The turbine engine is composed primarily of three constituents: a compressor, combustor, and turbine. Figure 1.1 illustrates these primary modules with a cross-sectional view of a turbine engine. The process first begins with the compressor where rotating components exchange mechanical energy to an air stream. The airflow enters the engine and passes through a decreasing annulus where the flow is compressed by successive stages of stationary and rotating components. The stationary components are referred to as stators and diffuse the airflow. The rotating components are referred to as rotors, where each is comprised of blades attached to a disk. The blades are either inserted or the disk/blade structure is machined from a single metal forging or manufactured through welding blades.
Figure 1.1: The fundamental turbine engine process where air enters at the left through the intake and undergoes compression, fuel is added in the combustor, and then expands through the turbine before exiting on the right [1].

to the outer diameter of the disk. The latter type is referred to as Integrally Bladed Rotors (or Bladed Disks (Blisks), or generically rotors) and Dual Flow-path Integrally Bladed Rotors (DFIBRs). The research conducted herein further focuses on IBR and DFIBR components.

After the air is compressed, fuel is added and the mixture is burned in the combustor. In this stage of the propulsion system, the thermal energy of the flow stream is increased by the exothermic chemical reaction between the fuel and oxygen in the airflow. The subsequent increase in energy in the flow is extracted via the turbine section. The turbine is fundamentally the reverse process of the compressor, where the garnered energy powers the compressor. The remaining amount of flow energy is either left to produce thrust for aircraft applications or torque through further turbine expansion.

1.2 High Cycle Fatigue

High Cycle Fatigue (HCF) failure of turbomachinery components, particularly flow-path components, manifests itself through increased engine development time and costs for original engine manufacturers (OEMs) and increased maintenance costs and reduced engine
readiness for the U.S. government. Between 1982 and 1996, HCF accounted for 56% of Class A engine related failures. In 1994, the required maintenance man-hour expenditures for risk management inspections surpassed 850K hours- carrying a financial burden of $400M per year [2]. Furthermore, even though 90% of potential HCF issues are identified during engine development testing, the remaining 10% accounts for almost 30% of the total development cost [3]. While a significant investment was made by OEMs and the U.S. government since these numbers were published to re-evaluate design and life management procedures for HCF, this failure mechanism continues to be problematic even for new engine programs [4]. Not surprisingly, these new engine designs have an increased demand for higher thrust-to-weight ratios and reduced specific fuel consumption that require higher pressure ratio fans, compressors, and turbine stages, shroudless low aspect ratio blades, and integrally bladed disks. These requirements have resulted in highly stressed blades with high modal densities and lower damping operating in unavoidable and precarious operating ranges with increased aerodynamic excitation with synchronous and asynchronous components, and mistuning - all precursory indicators of potential HCF issues.

HCF materializes from a lack of detailed understanding of vibratory loading and component dynamic response as well material capabilities. To further compound the problem, Fig. 1.4 illustrates a major difficulty in identifying and detecting HCF damage at the onset of fatigue [5]. While the Low Cycle Fatigue failure mode has a relatively long propagation-to-failure life after crack detection, HCF has a relatively larger crack initiation-to-propagation life before the crack length reaches a detectable size, with a small fraction of propagation-to-failure life. When such a small propagation-to-failure life is coupled with resonant driving frequencies ranging into the KHz regime, failure can occur in minutes. Blade designers seek to avoid these resonant frequencies by first identifying typical strong excitation sources driven by upstream vanes, downstream struts, and lower orders of engine revolutions. Component resonances are then compared to these excitation drivers on a

\[^1\]>$2M in damage or loss of life
Campbell diagram (illustrated in Fig. 1.2), where low-order crossings are “designed out” of key operating ranges. However, low aspect ratio blades can have a high modal density that result in unavoidable crossings of resonant vibration modes and known excitation drivers. These conditions are usually higher-order modes with presumed weaker drivers that can be further analyzed to determine the alternating and steady stresses [6]. Plotting the vibratory stress as a function of mean stress provides a constant life diagram, or sometimes mistakenly referred to as a Goodman or modified Goodman diagram, (illustrated in Fig. 1.3) that renders the component’s HCF margin.

The constant life diagram is constructed by plotting available material alternating, or vibratory, stress versus mean stress for a constant design life of $10^7$ cycles or more. In the absence of test data, the diagram is more easily constructed by drawing a straight line connecting the fully reversed loading condition on the ordinate ($R = -1$) to the ultimate tensile strength (UTS), or sometimes yield strength, of the material on the abscissa. This line is usually referred to as the Goodman limit. Variability in material data that populates this diagram is handled through statistical analysis that establishes a lower bound on this limit. Furthermore, a safety factor is usually imposed for turbine engines that limits the allowable alternating stresses to 60% of the Goodman limit. In addition, design practices can place a maximum allowable vibratory stress that is completely independent of the mean stress. This provides a bounded operating area, where any operating conditions that fall within the shaded region in Fig. 1.3 are considered safe from HCF as long as the maximum number of vibratory cycles does not exceed the constant design life for which the diagram was constructed.

A core requirement of determining a component’s HCF margin is the determination of the vibratory stress. While known drivers can be analyzed, existence of vibratory stresses from unknown drivers and unknown component dynamic response still pose a challenge for HCF prediction. Blade designers accommodate this issue through conservative design
Figure 1.2: Typical IBR Campbell diagram showing selected mode crossings [6].

Figure 1.3: Schematic of a constant life diagram that illustrates safety factors and HCF margin [5].
practices that utilize minimum material properties and/or the introduction of conservative allowable vibratory stresses, as previously discussed. Such conservatism, while producing safe designs, will negatively impact weight, performance, and cost - a detriment that conflicts with the increased demand for higher thrust-to-weight ratios and reduced specific fuel consumption in new engines. In addition to design conservatism, subsequent engine tests attempt to “flush out” the impact of any unknowns with durability testing protocols that will demonstrate HCF tolerance. Unfortunately, if problems are found late in the design process, design changes can be time consuming and costly. This has driven the need for a better understanding of vibratory loading and component dynamic response in early design stages. This work specifically addresses the lack of detailed understanding in component dynamic response, particularly IBRs. These components are discussed further in the following section.
1.3 IBRs and DFIBRs

Integrally Bladed Rotors and DFIBRs are rotating engine components that are either manufactured from a single monolithic alloy forging or by welding blades to the outer diameter of a disk. The fundamental tenant is that the blades and disk comprise an integral component; this is in contrast to traditional inserted blade designs where a blade slides into a machined dovetail slot on the disk outer-diameter. The dynamic response of IBRs and DFIBRs is driven by their interaction with the surrounding flow field. As the airflow passes through the stationary stators, the flow field is distorted with regions of high and low pressure. Additionally, since the stators are spaced symmetrically around the circumference of the annulus, these high-low pressure regions are also symmetrically spaced that impart a harmonic forcing function that is a function of the rotational speed. There is also a characteristic shape to this steady-state forcing function since all blades are experiencing a constant, out-of-phase force amplitude. Other flow-induced excitation sources arise from Low-Engine-Order (LEO) excitation mechanisms that occur at lower frequencies in the operating range due to stator throat width variations, flow exit angle excitations, passage cooling flow perturbations, or temperature distortions [7]. As the symmetric high-low pressure distortions and LEO resonant conditions are approached in the operating range, forced response amplitude dramatically increases. The excited modes are often characterized by the shape of the excited cantilevered blade mode (e.g. flexural, torsion, stripe or chord-wise bending, and edge-wise bending), but the dynamic characteristics of an IBR differ from an individual blade due to the coupling of blades through the disk and/or shrouds [8]. Hence, it is critical to accurately model the disk-blade connection and cantilevered blade modes. This is a crux of IBR and DFIBR dynamic response prediction, particularly mistuned response, and will be expanded upon in later sections.

Modeling IBR and DFIBR dynamic response with nominal design parameters yields a rotationally periodic structure that mandates each disk-blade sector be an exact replica of its neighboring sectors. Consequently, the entire IBR or DFIBR can be modeled by a single
fundamental sector. The significance of this cyclic symmetry assumption is utilized particularly in Finite Element Analysis (FEA) because the size of the model and computational requirements will be significantly reduced [9,10]. As a result, solution time is dramatically decreased and the design space can be explored to a much greater extent.

Unfortunately, there are small irregularities in the geometric and material characteristics of individual blades, referred to as mistuning, causing the rotational periodicity to break down. The ramifications of mistuning are two-pronged. First, individual blades can experience a localization of vibration energy causing forced response levels greater than predicted in a tuned, cyclic symmetry analysis. The localization phenomenon has been studied extensively and is reviewed in the following section. The second ramification is an increase in model size and computational burden because the entire component must now be solved in an FEA application, as opposed to a fundamental sector. This fact has driven the need for physics-based ROMs to effectively and efficiently predict mistuned response, particularly for Monte Carlo simulations seeking to characterize the full mistuned response distribution. This work revolves around this need and will provide an advantageous alternative to traditional mistuned response prediction methods in coming chapters.

1.4 Mistuning and Mode Localization

As introduced in the previous section, mistuning refers to the perturbations that disturb the rotational periodicity of an IBR. These perturbations arise from geometric deviations during manufacturing (e.g. resulting from tool wear), material property disparity (e.g. property variations through a forging), non-uniform component wear (e.g. foreign object damage), and fatigue (e.g. existence of cracks [11, 12]); perturbations are therefore random and unavoidable. Furthermore, even small mistuning that falls within strict manufacturing or return-to-service tolerances can have a dramatic effect on the IBR dynamic response through realizations of mode localization and eigenvalue loci veering [13]. This localiza-
tion of vibration energy has been studied quite extensively, first occurring in the study of solid state physics [14] and later in structural dynamics [15]. A thorough review of the topic for structural dynamics and other applications can be found in [16] and the references contained therein.

Mode localization is manifested as stress amplification that can be significantly greater than predicted in a tuned, cyclic symmetry analysis; the maximum theoretical limit of response amplification determined by Whitehead is calculated as $\frac{1}{2} \left(1 + \sqrt{N_A}\right)$, where $N_A$ is the number of blades [17, 18]. IBRs are composed of coupled sectors that have closely spaced eigenvalues that make them highly sensitive to small irregularities (mistuning). This is the genesis of mode localization since mistuning splits double system modes; an excitation frequency near the tuned natural frequency will excite multiple modes in the mistuned system. This phenomenon occurs because modes dispersing from the energy source are reflected at boundaries of nearly similar disk-blade sectors making up the nearly cyclically symmetric IBR. The resulting confinement in energy causes amplitudes higher than predicted by an analysis with assumed periodicity, with possible catastrophic consequences in HCF. The mode localization phenomenon is illustrated and compared to the tuned response in Fig. 1.5, where red indicates large displacement and blue is minimal displacement. Note that blade mode shapes are symmetric around the rotor in the tuned response in Fig. 1.5a, while the energy is localized to a single blade in the mistuned response in Fig. 1.5b.

1.5 Overview of the Dissertation

This document is organized in the following manner: first, a current literature review is given in Chapter 2 that investigates previous research efforts in the area of IBR and DFIBR dynamic response, particularly mistuned response prediction. This search begins with early, fundamental research but focuses more on robust prediction methods that came with the advent of finite element modeling. The purpose of this search is to provide a
Figure 1.5: Illustration comparing a tuned and mistuned IBR natural mode of vibration.
solid foundation for establishing a research need and approaches for incrementing the existing body of research. These needs are identified and elaborated in Chapter 3, which focuses the remaining body of this document. Chapter 4 provides a mathematical description of cyclic symmetry modeling and highlights the important dynamic characteristics of rotationally periodic IBRs and DFIBRs. Particularly, the IBR and DFIBR models are described in detail and a description of their inherent dynamic properties is given that help tailor the modeling approaches developed in the subsequent chapters. An overview of the parent Craig-Bampton CMS reduced-order modeling approach is then given in Chapter 5 that provides the foundation for the proposed modeling approaches. Chapter 6 then describes the accurate, highly reduced-order models for mistuned response prediction. These models assume the disk-blade connection is tuned. Chapter 7 investigates this assumption and provides a highly reduced-order modeling capability to incorporate geometric deviations at this connection. Chapter 8 extends the developed modeling methods of IBRs to DFIBRs. These models help characterize free and forced dynamic response characteristics of both tuned and mistuned components. Lastly, a brief conclusion of the document and areas of future work are discussed in Chapter 9.
Literature Review

The mistuning phenomenon is a heavily researched topic. Excellent reviews of the literature base can be found in the works by Slater, et al. [19] and Castanier and Pierre [20]. This chapter adds to these reviews and provides an up-to-date survey of the existing literature. Early mistuning research is briefly discussed in Section 2.1 just to highlight some of the foundational work. A thorough review of nominal mode mistuning methods and geometric mistuning methods is given in Sections 2.2 and 2.3, respectively. These sections are perhaps the most important to this current research effort since they identify previous attempts at developing mistuning ROMs. Section 2.4 then identifies some recent efforts to develop models capable of predicting forced response levels with the presence of a crack in the blade. Another important area of mistuning is reviewed in Section 2.5 that discusses probabilistic mistuning efforts that sought distributions of forced response levels. The last three sections, 2.6-2.8, briefly discuss other mistuning research areas.

2.1 Lumped Parameter Models

Fundamental mistuning research began with lumped parameter models, such as that in Fig. 2.1 that demonstrated blade response amplification due to minor dimensional variations [21–25]. These models varied from author to author, but common to each is the coarse representation of disk and blade dynamic interaction through the use of lumped parameters. While these papers effectively identified that mistuning could be the culprit of
Figure 2.1: Schematic of a lumped parameter model that illustrates the coarse representation of an IBR through various connections and lumped masses [26].

a few “rogue” blade failures, the magnitude of blade response amplification predicted by each author varied greatly. Furthermore, they required the arduous task of parameter identification for the lumped components. Better models were needed for accurate mistuned response prediction.

2.2 Nominal Mode Mistuning Reduced-Order Models

Lumped parameter model inaccuracies were addressed with the advent of FEA since ROMs could be systematically deduced from a parent finite element model. However, these new ROMs sought to maintain the computational efficiencies of their lumped model predecessors through approximations. The most prevalent approximation is that mistuned response can be represented as a linear sum of tuned modes. In this assumption, mistuning is presumed to be manifested as perturbations to blade natural frequencies while the mode shapes remain unaffected. This section highlights previous mistuned response modeling efforts
that either directly use this assumption or are loosely based on this assumption. Main highlights and shortfalls of the efforts are illustrated that will later be summarized into a research need in Section 3.1.

A ROM technique was introduced by Castanier, et al. [27] that utilized a modified component mode synthesis technique with the constraint modes removed further reduce the size of the model. This approach became known as REDUCE. The model described the disk motion by mode shapes of a disk with a massless blade attached. The blade motion was described by a summation of the deflections caused by the disk modes and those of a cantilevered blade fixed at the disk-blade connection. As a result, the motion between the disk and blade is described without the use of constraint modes. The approach provides access to the blade modal stiffnesses, which are then perturbed to simulate mistuning. Results indicated that the ROM was overly stiff. The authors circumvented this by artificially modifying the blade modal stiffness to match the parent FEM. This method was latter modified by Bladh et al. to include an IBR with shrouds [28]. Furthermore, the technique was used by Kruse and Pierre [29] to investigate the localization phenomena and the inter-blade structural coupling for an academic rotor. The same work also used the REDUCE method in a Monte Carlo simulation of forced response levels to show the distribution in mistuned response. The same authors used the approach again in a study with a real turbomachinery rotor and investigated aerodynamic coupling of vibration energy in addition to the usual structural coupling through the disk [30].

A ROM using a subset of nominal modes was investigated by Yang and Griffin [31] that became known as SNM or the Modal Domain Approach (MDA). The term “subset of nominal modes” refers to predicting the mistuned modes as a limited sum, or subset, of “nominal” IBR modes. This method differs from earlier reduced-order modeling approaches in that there is no partitioning of the IBR into components as done in component mode techniques. The mathematical form of SNM was written in state-space form for the inclusion of aerodynamic and gyroscopic forces, however, the effort didn’t include these ef-
fects in the results. The DOFs were then reduced by limiting the number of nominal modes in the modal representation of the system. Mistuning was implemented by perturbing the nominal stiffness matrix by a constant (e.g. Elastic modulus perturbations), which provides a proportional mistuned stiffness matrix. If mistuning was not implemented, the ROM could reproduce the exact results of the tuned parent FEM. However, since the method relies on using a tuned modes in the ROM, there is a degree of approximation. The authors relied upon their work in [32] to argue that the tuned and mistuned modes span the same space and the approximation is accurate. SNM produced accurate mistuned system natural frequencies and mode shapes, as well as accurate forced response predictions for an academic rotor model. Furthermore, the results were much more accurate than the authors’ previous effort, Linear Mistuning Computer Code (LMCC), that was formulated from a receptance technique [33].

The SNM code was further simplified by Feiner and Griffin [34] to produce the Fundamental Mistuning Model (FMM). In FMM, the subset of nominal modes retained is limited to a single family of modes. Furthermore, this single family of modes must be an isolated set of modes in which the strain energy is primarily in the blades, i.e. there must be minimal disk participation in the mode shapes. This provides the opportunity to approximate a disk-blade sector modes with the corresponding cantilevered blade modes. This simplification creates a ROM that only requires the nominal system modes and the blade frequency deviations as inputs. Excellent results were reported for a simple academic model and a realistic turbomachinery rotor for the isolated, first-bending blade mode family. The method also showed promise for modeling higher frequency ranges where the blade motions are similar at higher nodal diameter patterns. The FMM approach was also used for mistuning identification [35, 36].

A comparison between two reductions techniques was carried out by Moyroud, et al. [37], where the first approach employed tuned system modes from a classical modal analysis that is similar to SNM [31] and the second utilized tuned Craig-Bampton compo-
component mode synthesis modes. Elastic modulus mistuning was implemented in cyclic coordinates and free and forced response comparisons were made for three models with increasing complexity. It was found that both reduction methods were computationally tractable and both predicted the mode localization equally well. In each approach, mistuning is projected onto tuned modes, where it is assumed the mistuned response can be approximated as a linear sum of tuned modes.

Lim, et al. [38], investigated two techniques for generating ROMs from parent FEMs using a component-based modeling technique. The effort partitioned a mistuned IBR into a tuned system and a set of virtual blade mistuning components that are then re-assembled using a basis of tuned-system normal modes and attachment modes in a component mode synthesis approach. The first technique is applicable to blisks with geometric mistuning, i.e. large perturbations in mass and stiffness matrices that result in large differences between tuned and mistuned system mode shapes. For the first technique, all attachment modes need to be retained, which are as numerous as the number of DOF per blade times the number of blades with geometric deviations. This study considered an IBR with large geometric deviations to a single blade and results were excellent. However, as the number of blades with geometric mistuning increase, the size of the ROM becomes large. The second technique made simplifying assumptions to the first technique by using the findings from [32] to neglect the large number of attachment modes. Furthermore, it is assumed that the mistuned modes can be approximated by a subset of tuned modes. Stiffness mistuning was implemented in physical coordinates was projected onto the normal modes of a cantilevered blade, as done in [28]. This approach was shown to work well when disregarding off-diagonal terms in the mistuning projection matrix, but the motion of a blade in the mistuned system should dominated by one mode of the tuned, cantilevered blade frequency. When multiple, dominant cantilevered blade modes are present, the corresponding mistuning values are needed. This final approach was termed the Component Mode Mistuning (CMM) model was shown to be very compact, and capable of capturing mistuning effects.
of a parent FEM with proportional and non-proportional mistuning.

Bladh, et al. [39], formulated two ROMs for mistuned response prediction. The first was formulated from a component mode synthesis method with a cyclic symmetry description of the tuned disk. The IBR was substructured into components consisting of a disk and individual blades. Since mistuning was only considered in the blades, this substructuring approach alleviates the need to perform any computations pertaining to the disk, outside of the initial calculations of the disk component modes. Retaining only a subset of the component fixed-interface normal modes for each blade and the disk results in ROM developed from a parent FEM. The accuracy of the ROM approaches that of the parent FEM as the number of modes retained approaches the complete set for each component. The second ROM introduced is a straightforward, non-component mode synthesis technique that uses a generalized version of the mistuning projection method developed in [28]. This approach is analogous to the SNM method [31]. Lastly a secondary modal analysis reduction technique (SMART) was introduced that performs a classic modal analysis on a tuned, intermediate model formulated from the component mode synthesis approach outlined earlier in the same work. Tuned modes of interest are then retained for building the ROM that is mistuned using the mistuning projection technique. Again, the method is analogous to SNM, with the exception that the tuned modes are found from an intermediate ROM. This creates a ROM of an absolute minimum size. Accurate results were shown in [40].

An asymptotic expansion method, referred to as the Asymptotic Mistuning Model (AMM), was developed by Martel and Corral [41] to investigate the maximum mistuning amplification that can be produced by adding small mistuning to an IBR. AMM presents a further reduction to FMM [34] for when all the modes of a family do not share the same approximate frequency. An asymptotic perturbation technique is then used to evaluate the first-order effects of small mistuning in forced response amplification. AMM found for isolated modes that an amplification factor of approximately 20% can be realized, as shown in earlier work by Macbain and Whaley [42] and Kenyon and Griffin [43]. When clustered
modes are investigated, a closed form expression for the maximum mistuning amplification is found that is a function of the number of active modes. This expression provided closer results to those found by Petrov and Ewins [44], over the traditional Whitehead closed-form expression [17, 18].

Vargiu, et al. [45], developed a ROM, referred to as the Integral Mode Mistuning (IMM) model that is an extension of the Component Mode Mistuning (CMM) model created by Lim, et al. [38]. Where the CMM model considered only blade frequency mistuning, the IMM model was expanded to include a disk-blade sector mistuning that incorporated i) blade frequency mistuning applied through elastic modulus perturbations and ii) disk-blade connection mistuning through adjustments to the nodal coupling between the blade and the disk. This latter form of mistuning is particularly useful for inserted blade-type, or non-integrally bladed rotors, where the blades are inserted into the disk through dove-tail type slots. The amount of disk-blade coupling at the interface is then a function of the centrifugal load on the blade. The test case used for the IMM model provided a ROM that is only a fraction of a percent of the full FEM size. The authors investigated frequency error, frequency splitting error, forced response error, and a modal assurance criterion (MAC) error of the IMM approach compared against a full FEM mistuned with the same methods. Results looked promising for the investigated first family of modes. However, inaccuracies arose at lower frequencies due to the approximation in sector mistuning frequencies. Furthermore, the approach is restricted to investigating a modal family of interest that is sufficiently isolated. In addition, the method assumes that blade mistuning only results in blade frequency perturbations and neglects impacts to the mass matrix and mode shapes. However, the approach did confirm the findings in [46] that disk-blade interface coupling is important in determining the total mistuning amplification factor for inserted blade-type rotors.

Tran [47] developed a modified component mode synthesis method with partial Interface modes. This generalization of the classical component mode synthesis methods and
those using Interface modes where certain DOF at the interface are retained as physical
DOF instead of using a complete modal basis to represent the interface displacements. The
approach was demonstrated on a simple academic model with simple geometry and rather
large proportional mistuning. Accurate results were obtained for this simple test case for
the many different proposed methods.

An Artificial Neural Network (ANN) and Genetic Algorithm (GA) were used by
Scarelli and Lecce [48] to predict a blade tip displacement for a mistuned rotor given a
mistuned rotor configuration. While not a nominal mode method in a dynamic sense, the
mistuning implementation relied on blade elastic modulus perturbations and did not explic-

ty account for geometric mistuning effects. The ANN was trained with a small population
of mistuned rotors for different configurations of ANNs. Results proved to be rather inac-
curate. The GA was used to predict the worst mistuning pattern in the rotor. This analysis
showed that the GA is capable of performing such searches.

Baik, et al. [49] developed an early mistuning assessment design tool using a power
flow analysis to characterize tuned disk-blade dynamic interaction that is usually indicative
of high mistuning. By calculating the energy input into the blades, energy dissipated in
the blades, and energy coupled to the disk the authors showed that power flow analysis
is capable of identifying mode localization in a mistuned bladed disk. This metric was
termed the Tuned Coupling Power Indicator was developed that described the systematic
estimation of disk-blade dynamic interaction. Furthermore, a Stress Amplification Factor
was developed that provided an approximate index for estimating blade stress amplification
directly from displacements without have to perform a finite element stress analysis. Ulti-
mately, the effort showed that a Tuned Coupling Power Indicator is capable of predicting
the blade stress amplification calculated from the Stress Amplification Factor on different
bladed disks. In each, the test cases were mistuned via elastic modulus and mass density
perturbations. As in the work by Scarelli and Lecce [48], the power flow analysis isn’t a
nominal mode approximation, but it is derived from a nominal model.
2.3 Geometric Mistuning Reduced-Order Models

Modeling methods outlined in the previous section have provided computationally efficient approaches for predicting mistuned response. However, the tuned mode approximation discussed at the beginning of Section 2.2 can lead to significant errors [50, 51]. To address these inaccuracies, new methods were developed to account for geometric and material property perturbations beyond the traditional approaches of Section 2.2. This section highlights these works and briefly discusses the approaches used and significant findings. Identified shortfalls are used to formulate a research need discussed in Section 3.1 on page 31.

A non-component mode synthesis was formulated by Lim, et al. [52] for large geometric mistuning that utilized mode-acceleration methods with static mode compensation. Here, the static modes account for mistuning as if they were produced by external forces and are used to compensate the the tuned-system normal modes. This new set of basis vectors span an approximate space as those of the mistuned system. Calculation of these modes were found to be less intensive than a full modal analysis of the mistuned system and showed better convergence than using only tuned-system modes. Furthermore, quasi static modes that account for inertia effects are included in higher frequency ranges. Results compared well to a full FEM solution, even as the mode shape changes due to mistuning became large.

An exact method to calculate the mistuned forced response of an IBR was formulated by Petrov, et al. [53]. This approach is formulated from the exact relationship between the tuned and mistuned forced response levels through use of the Woodbury-Sherman-Morrison inverse of a perturbed matrix formulation [54, 55]. A reduction of the system model to a manageable size without any approximations or loss of accuracy is capable with this approach. This is done by selecting only a subset of nodes of interest from the parent FEM. These DOFs of interest, or active DOFs, include those perturbed from the nominal IBR. The Woodbury-Sherman-Morrison formulation still requires the inversion of a matrix of size equal to the number of active DOF and hence, the authors formulated a recurrence...
scheme for updating the forced response levels that avoids doing this matrix inversion. The only approximations of the approach is the number of modes retained in calculating the FRFs of the tuned system. However, as the number of active DOF increase in size, the more computationally intensive the approach becomes and the reduced size of the model becomes ambiguous. Note that this method does not require a modal expansion, but is included in this section since the solution approaches that of a full FEM solution as the number of tuned modes used in the calculation of the FRFs increases.

A modification to the Modal Domain Approach [31] was made by Sinha to account for geometric mistuning [56]. In this effort, the author sought to account for both perturbations to the blade’s mass and stiffness matrices, without assuming the mistuned forced response is a linear combination of tuned system modes. To do this, the author incorporated MDA to obtain a set of tuned system modes and tuned modes of the system with all blades being perturbed by a single geometry component. These components were proper orthogonal decomposition (POD) features of coordinate measuring machine data of the actual geometry [57]. The resulting method was termed the Modified Modal Domain Approach (MMDA) and was shown to provide accurate results on an academic model. A drawback of the method occurs when a large number POD features are needed to describe the blade geometry. When this occurs, a large number of modes need to be retained, since the tuned system modes are augmented with the same number of modes for each POD feature. As a result, the model size can become large. MMDA was further refined to incorporate approximate deviations in mass and stiffness matrices computed by Taylor series expansions with respect to the POD features [58]. It was shown that first-order approximations were inaccurate, but significant improvements were made with the second-order approximations. The method was again refined for a multistage academic model and showed promising results, but is still limited by the number of POD features that represent the geometric mistuning [59].

Ganine, et al. [60, 61], reviewed the SMC method proposed by [52] to investigate its
limitations and extend the method for geometric mistuning problems. This was done by using inexact solutions of the linear Jacobi-Davidson correction equations in place of the quasi-static set of modes used in SMC. The authors presented an adaptation of Jacobi-Davidson iterative solver by exploiting the matrix structure of block circulant matrices of cyclic structures undergoing a structured perturbation. Particularly, these perturbations were considered to be due to a large geometric deviations and of small rank, i.e. the perturbations were confined to a small location of the system matrices. To meet these assumptions, a 29-blade IBR test case was used with large deformations to a single blade. The proposed algorithm was applied to a frequency range of interest where the SMC approach lacked accuracy and was shown to produce favorable results for system frequencies and mode shapes. Computational times were not compared between the new approach and SMC since the methods were written in different code languages, however, the authors made some conclusive finds. SMC is a good choice for moderate order FEMs with a localized, low rank perturbation for narrow areas of the spectrum. However, for very large models with modal interaction, the new approach is favorable. The approach requires further investigation to determine the effectiveness for large rank, small mistuning that occurs for small geometric deviations for a large set of blades.

A new reduction method for geometric mistuning based on cyclic modes of different sectors was proposed by Mbaye, et al. [62], for intentional mistuning of blade shapes. For each blade shape a tuned IBR is built with cyclic symmetry conditions and the sector cyclic modes are computed. A matrix of cyclic mode shapes corresponding to each blade type is then used as a projection basis. Displacement continuity between sectors with different blade types by constraining the redundant interface DOF of each sector. A phase correction is also performed to ensure a proper alignment of nodal diameters for independent calculations of cyclic modes. The approach was applied to a 23-blade industrial IBR with geometric modifications to two blades. A low order ROM is obtained and showed accurate frequency predictions and forced response predictions over a given frequency spectrum.
Further investigation is required to determine how reduced the final model is for a large set of mistuned blades.

A similar method that that of [62] was performed by Madden, et al. [63] that makes use of a pristine-rouge-interface modal expansion (PRIME) to model both large and small mistuning. For this approach, the finite element model is partitioned into disk-blade sectors where those sectors containing the large geometric mistuning, e.g. partial blade loss, are called rogue sectors and the remaining sectors are pristine. The PRIME reduction basis is calculated only from sector-level models and hence reduce computational requirements and results in three partitions of modal DOF: the pristine portion that is a nominal system; the rogue portion; and the interface portion that couples the pristine and rouge DOF of the system. Model reduction then occurs by projecting the system matrices onto a truncated set of the new modal basis. The authors noted that the approach is susceptible to being over-constrained as well as suffering from rank deficiencies. This problem was alleviated by performing PRIME conditioning that consisted of additional eigen-problems to approximate the null space. It is important to note that while the rogue DOF use the geometrically perturbed modes, any small mistuning in the pristine DOF is projected onto a set of tuned modes as is done in the traditional nominal mode approximation methodologies.

2.4 Crack-Induced Mistuning

Hou [64] conducted a study on cracking induced mistuning and resulting mode localization for different coupling ratios between simple lumped parameter blades. This investigation sought to identify possible correlations between dynamic characteristics of the model and crack parameters, and any response amplification of other pristine blades due to having a cracked blade. The model used in the analysis was a simple lumped parameter model composed of blades with a lumped mass supported by a massless beam. Cracking was simulated by a through-crack in the beam using a flexibility matrix method. Results indicate
that a presence of a crack can result in response amplification of pristine blades and is not localized to the blade with damage. Furthermore, the effort identified a connection between the cracked blade and its vibration response that the authors suggested could be used for crack detection procedures.

A nonlinear ROM was developed by Saito, et al. [65], to predict the forced response of an IBR with the presence of a cracked blade. This approach utilized a hybrid-interface component mode synthesis method that generates a reduced model with crack interface DOF retained as physical DOF. This allowed direct calculation of the crack interface contact forces. The developed ROM was integrated into a nonlinear forced response analysis that calculated the steady-state solution using a hybrid time/frequency domain method. The method was applied to an IBR FEM and illustrated that cracks can result in mode localization and response amplification. Furthermore, the authors presented results from a linear forced response analysis and illustrated that this approach leads to inaccurate predictions with the presence of a crack.

An linear ROM approach to include cracks in blades was conducted by Marinescu, et al. in [66]. The proposed approach model crack interface DOF in a relative coordinate system obtained from Craig-Bampton Component Mode Synthesis [67]. This allowed the use of representing the components in cyclic coordinates to reduce computational expenses. In addition to the crack, proportional mistuning is implemented through perturbations to blade elastic moduli. The dynamic response of this mistuning is then modeled by the CMM ROM developed by [38]. The reduction procedure adapted in the same work to incorporate large cracks or multiple cracks per sector by ensuring the transformation matrix from physical to reduced-order coordinates includes the subspace spanned by the cantilevered cracked blade mode shapes. Two test case models were used for validation: one with a small crack and the other with multiple large cracks. Accurate results were obtained and illustrated that the existence of a crack impacts the dynamic response of the IBR.

Mistuning induced by cracking in a multi-stage bladed-disk was investigated by D’Souza,
et al. [68]. The effort consisted of doing a stage-wise cyclic symmetric symmetry component mode synthesis with inter-stage coupling through the use of harmonic shape functions. This framework provides the possibility to include both nonlinearities due to crack opening and closing as well as small mistuning. Furthermore, the harmonic shape function coupling of the stage interfaces is able to incorporate stages that are composed of differing number of sectors and also when the stage ROMs are formulated using different methodologies. The approach was demonstrated on a two-stage model that is representative of industrial designs. Results demonstrated the interaction between the two-stages as well as the importance for conducting a nonlinear analysis in crack modeling.

2.5 Probabilistic Mistuning Methods

An early effort to investigate the statistical implications of mistuning was carried out by Griffin and Hoosac [26]. In their work, a simple lumped parameter model was used to identify mistuning dependencies and methods to reduce mistuned response amplification. Perhaps one of the unique aspects of their effort is the inclusion of engine test data for model comparison. The lumped parameters of the model were estimated from test data in an effort to replicate the engine response. Findings reiterated that the highest responding blade isn’t necessarily the worst mistuned blade and that maximum responses usually occurred near the tuned system frequency. The last point is significant since it provides justification for simple resonance avoidance for mistuned bladed disks in engine tests.

Sinha [69] developed an analytical technique for calculating the probability density functions of the mistuned response of bladed disks. Using this technique, the author was able to determine the probability that a blade’s response is below some predetermined value for a system with any number of blades, as long as the modal parameters of the model were assumed to be Gaussian. A lumped parameter model with a single DOF per blade was used to show the accuracy of the approach. Results were very good for systems with high damp-
ing, however, as the damping decreased the accuracy was diminished. The method was later modified with a higher-order technique [70] and then modified again to incorporate non-Gaussian modal parameter distributions [71]. The latter found that the type of modal parameter distribution had little effect on the mistuned response distribution. These efforts all assumed sinusoidal excitation, so Cha and Sinha [72] examined the impact of mistuning when the excitation is either white noise or narrowband random excitation. The mean and variance of the mistuned response were calculated from a state-space approach for a simple lumped parameter model. Results indicate that the method is fairly accurate as compared to numerical simulations. Sinha continued his work in determining the distribution of mistuned response by examining the statistical distribution of the mistuned peak maximum amplitude [73]. Here the author sought to characterize the type of this distribution as well as parameterize the distribution so a functional relationship between parameters to quickly evaluate the impact of mistuning on a design. Results from a Monte Carlo simulation for a lumped parameter model showed that the distributions are not Weibull and that a multi-layer neural network is incapable of mapping input parameters to the distribution. Sinha later used polynomial chaos theory to compute the statistics of the response distribution [74].

Mignolet and Lin [75] developed a closed-form perturbation approach for approximating the probability density function of the forced response for a randomly mistuned bladed disk subjected to harmonic excitation. This closed-form function is then approximated by deterministic perturbation methods that is then validated on a simple lumped parameter model, with a single degree of freedom for each blade. Results were shown to be more accurate than that of Sinha and Chen [70, 71] when compared to full Monte Carlo results, but the accuracy of the approach had a dependency on the engine order (EO) excitation chosen. An adaptive perturbation approach was later developed by the same authors that analyzed the mistuned response using a modal transformation matrix of a tuned system or a decoupled system, depending on the level of blade-to-blade coupling [76]. The method
was again used for a simple lumped parameter model and showed that accuracy increased as more natural frequencies close to the excitation frequency were retained. Mignolet, et al. [77] would later use the adaptive perturbation approach to derive a closed-form mistuned response model for the probability density function.

Bah, et al. [78] developed an approach to determine mistuned forced response statistics based on stochastic reduced basis methods (SRBMs) that approximate the random solution process using terms of a preconditioned stochastic Krylov subspace for basis vectors. This approach enables explicit expressions for the mistuned system response as a function of the random mistuning parameters. Results for a simple, 10-blade lumped parameter model show that the SRBMs have significant accuracy over perturbation techniques previously developed in other works near resonance conditions. However, no characterization of the response probability distribution function were given.

Lee, et al. [79] investigated the accuracy and efficiency of using multiple probabilistic methods, including most probable point (MPP) approaches, response surface methods with a moving least squares approach, a radius-based importance sampling method, and accelerated Monte Carlo simulation. A 29-blade lumped parameter model with two DOF per blade was used for each approach. The MPP methods and response surface models failed to capture the statistics of the maximum resonant response of the mistuned rotor. The accelerated Monte Carlo method was based on the property that the maximum rotor response distribution asymptotically approaches a Weibull distribution and was shown to be more accurate and efficient compared to the other approaches discussed in the paper.

2.6 Mistuning Identification

A mistuning identification procedure was developed by Judge, et al. [80] that is based on the mistuning ROM developed in [39] that was adapted to both free and steady-state forced response measurements. Free response measurements were shown to work well with low
damping and a low modal density. Forced response measurements were found to be needed for specimens with high modal densities or high damping because resonance peaks would merge and overlap. Experimental validation provided consistent results with numerical simulation and showed the prosed methods were capable of predicting known mistuning patterns.

Madden, et al. [81, 82] considers the Component Mode Mistuning approach of [38] and mistuning identification methods to investigate enhancements to the model and experimental approach. A Selection Ratio parameter that is based on cantilevered blade participation factors and disk-blade interface motion is developed to select tuned system normal modes for inclusion in the identification process. Furthermore, a procedure was developed that builds an inverse reduced-order model used for the inverse identification problem and is incorporated with a representation of mode shapes measured with a limited measurement DOF. The method was validated for mistuning identification at relatively low excitation frequencies. The approach was extended by the same authors in [83] to mistuning identification in higher frequency regions. Another related effort by Holland, et al. [84] used the CMM approach and focused on the testing and calibration procedures for mistuning identification using a traveling wave excitations system. The effort focused on three objectives: selection of modes of a tuned model for measurement DOF selection; identification of testing procedure; and an iterative blade force and phase calibration procedure.

2.7 Management of Mistuning for Turbine Engines

A two-part paper by Chan and Ewins [85, 86] revisited mistuning physics in a robust design concept that involved parameter design and tolerance design. This design concept focused on managing the blade mistuning problem instead of solving it. The authors used parameter design to determine settings of physical parameters that limited that maximum amplification factors. However, results showed that the robustness of some designs is not
improved after small design changes and tolerance design concepts that control the mistuning parameters of individual IBRs needs to be incorporated. It was determined that limiting the amount of small mistuning of each blade, or intentionally mistuning a select few blades, have reduced the maximum mistuning amplification factors.

2.8 Aerodynamic Effects

The CMM model developed by [38] was used by He, et al. [87] to investigate using tuned system modes to calculate unsteady aerodynamic forces instead of the traditional approach of using cantilevered blade normal modes. The case study illustrated strong aerodynamic coupling between the bladed disk and the aeroelastic configuration that changed the tuned and mistuned responses. Furthermore, mistuned mode shapes were found to be dependent upon this coupling. Differences in results between the system mode and cantilevered blade normal mode approaches were identified.

An investigation into the effect of aerodynamic loading on mistuned resonant response was carried out by Choi, Lawless, and Fleeter [88, 89]. Experimental measurements were taken at different aerodynamic loading conditions for a baseline bladed disk and three mistuned bladed disks. Mistuning was implemented through drilling radial holes into the blade tips as to not alter blade profiles. Blade frequency deviations were then determined by ping testing of each blade while mounted on the shaft. Results indicated intentional mistuning and steady-state aerodynamic loading have a significant effect on the maximum resonant responses. Furthermore, the work altered which blades were the high and low responders for each loading condition for the baseline and intentionally mistuned rotors.

Choi and Fleeter [90] and Choi, et al. [91] developed a mathematical model to investigate the effects of aerodynamic damping on the maximum amplification factor of mistuned bladed disks. The effort incorporated aerodynamic damping influence coefficients calculated from an inviscid linearized unsteady aerodynamic damping code in a lumped-
parameter, partial mistuning model. This partial model reduces the number of blades required on the row by assuming that mode localization restricts response amplification to a small subset of blades. The effort found that the size of remaining blades should be approximately half the total number of blades. Results indicate that aerodynamic damping has a large effect on mistuned response amplification when the amount of aerodynamic damping is non-negligible compared to the amount of structural damping. Furthermore, it was shown that an optimal mistuning pattern may not be optimal in the operating environment when unsteady aerodynamic effects are present.

2.9 Summary

This chapter reviewed previous mistuning work that exists within the literature. Of primary interest to this current effort are those modeling approaches that sought to accurately predict mistuned response with reduced-order modeling formulations. The next chapter will identify general shortfalls of these previous modeling efforts and propose a research need that is derived from these shortfalls. Identified research needs provide the justification for this current effort.
Research Overview

This chapter encapsulates the findings from the first two chapters and specifically scopes the problem that is researched. The objective of the effort is discussed and the proposed procedures and methods are briefly outlined. How this original research effort augments the main body of research is identified and major contributions are listed. These major contributions are defined as complete, stand-alone results that address research needs with unique solutions that have not been previously identified in the literature.

3.1 Research Need

High Cycle Fatigue continues to be problematic for IBRs because of inaccurate dynamic response predictions. Traditional design approaches that move high responding modes outside of critical operating ranges can introduce excessive design conservatism that negatively impacts performance. Furthermore, new blade designs have high modal densities with complex modal interactions that makes it extremely difficult to avoid some operating conditions. While these conditions are presumed to be weak drivers, mistuning can cause response amplification that can have catastrophic consequences for HCF. While full finite element models produce accurate predictions, these models can be prohibitively large and are greatly limited in producing a distribution of response. As a result, predicting mistuned response amplification with reduced-order models has become a significant body of research. Traditional approaches, and those used predominantly by industry, assume mis-
tuning manifests only in perturbations to blade natural frequencies and neglects alterations of blade mode shapes. These approaches are limited by assuming the mistuned response can be approximated as a linear sum of tuned modes. This approximation can result in significant errors, particularly for predicting a full mistuned response distribution and Type I\(^1\) and Type II\(^2\) statistical errors. Currently developed models that seek to account for geometric and material perturbations beyond the traditional tuned mode approach are also limited. Many of these approaches restrict geometric mistuning to only a small subset of blades, located on a small portion of the blade. While this is relevant for large mistuning (e.g. partial blade loss, foreign object damage, etc.), small mistuning on all blades resulted in rather large ROMs. Furthermore, some of the approaches still relied upon tuned blade modes for these small geometric deviations. Therefore, modeling approaches are needed that account for large and small mistuning on all blades and provide high fidelity approximations while still providing significant computational savings for IBRs and DFIBRs.

Furthermore, the push for advanced performance in turbine engines has resulted in DFIBR designs. There is a significant lack of research on the dynamic response of these new components. The closest related topic is IBRs with shrouds, however, DFIBRs are significantly different from shrouded rotors in many ways. First, shrouds are common to inserted blade designs while uncommon to IBRs. Second, shrouds lack an integral connection to each other while DFIBRs have an integral ring. Third, shroud locations can vary along the length of the blade while DFIBRs have the integral ring at the blade tips. Fourth, shrouded rotors always have the same number of blades beyond the shroud (really it is just the same blade) while DFIBRs can have completely different blades of varying amount after the ring. Lastly, the blades on shrouded rotors will see the same EO excitation while DFIBRs can have different EO excitations between the interior blades and the exterior blades. These new designs are susceptible to mistuning as well and require high fidelity ROMs.

\(^1\)Incorrect rejection of the true null hypothesis
\(^2\)Failure to reject a false null hypothesis
3.2 Research Contribution

This effort increased reduced-order model fidelity for mistuned response prediction by explicitly accounting for blade geometric and material property deviations. These methods were formulated in a component mode synthesis framework utilizing secondary modal reductions in a cyclic symmetry format. The resulting ROMs capture perturbations to both blade natural frequencies and mode shapes resulting from geometric deviations. Furthermore, the secondary modal reductions and cyclic symmetry format offered significant computational savings over traditional component mode synthesis methods that give an ultimate reduction in model size.

Chapter 6 describes a formulation that assumes a tuned disk-blade boundary and presents two methods that explicitly model blade geometry surface deviations for mistuning prediction in Integrally Bladed Rotors by performing a modal analysis on different degrees of freedom of a parent reduced-order model. The parent ROM is formulated with Craig-Bampton component mode synthesis in cyclic symmetry coordinates for an IBR with a tuned disk and small blade geometric deviations. The first method performs an eigenanalysis on the constraint-mode DOF that provides a truncated set of Interface modes while the second method includes the disk fixed-interface normal mode in the eigen-analysis to yield a truncated set of Ancillary modes. Both methods can utilize tuned or mistuned modes, where the tuned modes have the computational benefit of being computed in cyclic symmetry coordinates. Furthermore, the tuned modes only need to be calculated once which offers significant computational savings for subsequent mistuning studies. Each geometric mistuning method relies upon the use of geometrically mistuned blade modes in the component mode framework to provide a very accurate ROM. Free and forced response results are compared to both the full FEM solutions and a traditional frequency-based approach used widely in academia and the gas turbine industry. It is shown that the developed methods provide highly accurate results with a significant reduction in solution time compared to the full FEM and parent ROM.
Chapter 7 develops a formulation that captures a mistuned disk-blade boundary and assesses the accuracy of assuming this boundary is tuned (as done in Chapter 6). The new geometric mistuning approaches for IBRs are developed for incorporating geometric perturbations to a fundamental disk-blade sector, particularly the disk-blade boundary. The developed ROMs are formulated from a Craig-Bampton component mode synthesis framework that is further reduced by a truncated set of Interface modes that are obtained from an eigen-analysis of the CB-CMS constraint degrees of freedom. An investigation into using a set of tuned Interface modes and tuned Constraint modes for model reduction is then performed. A tuned mode approximation has the added benefit of being only calculated once, which offers significant computational savings for subsequent analyses. Two types of disk-blade connection mistuning are investigated: as-measured principal component deviations and random perturbations to the inter-blade spacing. Furthermore, the perturbation sizes are amplified to investigate the significance of incorporating mistuned disk-blade boundaries. Free and forced response results are obtained for each ROM and each disk-blade connection type and compared to full finite element model solutions. It is shown that the developed methods provide highly accurate results with a significant reduction in solution time compared to the full FEM. In addition, results indicate that the inclusion of a mistuned disk-blade connection becomes significant as the size of the geometric deviations become large.

Chapter 8 then develops ROM methodologies for DFIBRs to assess the susceptibility of these new designs to mistuning and to be able to efficiently and effectively predict response amplification. This formulation will utilize the findings of Chapters 6 and 7 in the reduced-order model formulation. Two main approaches are presented: first, a frequency-based method that is analogous to traditional mistuning approaches for IBRs, and second, geometric approaches that explicitly model blade geometry surface deviations. In both approaches, the model size is reduced by performing a modal analysis on different degrees of freedom of a parent reduced-order model that is an extension of the methods developed
in Chapter 6. Both Interface and Ancillary mode methods can utilize tuned or mistuned modes, where the tuned modes have the computational benefit of being computed in cyclic symmetry coordinates. These methods help characterize DFIBR dynamic response and investigate the unique aspects that differentiate these advanced components from IBRs. Free and forced response results are compared to both the full FEM solutions and the traditional frequency-based approach. It is shown that the developed methods provide highly accurate results with a significant reduction in solution time compared to the full FEM and parent ROM.

### 3.3 Summary

Specific contributions to the existing body of research can be summarized as:

**Major Contribution 1** A new geometric mistuning ROM for IBRs with tuned disk-blade boundaries (Chapter 6)

- **Minor Contribution 1.1** A model with *Interface Mode* reductions
- **Minor Contribution 1.2** A model with *Ancillary Mode* reductions
- **Minor Contribution 1.3** An investigation of error with a nominal mode approach

**Major Contribution 2** A new geometric mistuning ROM for IBRs with mistuned disk-blade boundaries (Chapter 7)

- **Minor Contribution 2.1** A model with *Interface Mode* reductions
- **Minor Contribution 2.2** A model with tuned or mistuned *Constraint Mode* approximations
- **Minor Contribution 2.3** An assessment on the importance of using a mistuned disk-blade boundary
**Major Contribution 3**  A new nominal mistuning ROM for DFIBRs and assessment of mistuned response amplification (Chapter 8)

**Minor Contribution 3.1**  A model with *Interface Mode* reductions

**Minor Contribution 3.2**  A model with *Ancillary Mode* reductions

**Minor Contribution 3.3**  A detailed investigation of tuned and mistuned DFIBR harmonic response subject to different excitation conditions

**Major Contribution 4**  A new geometric mistuning ROM for DFIBRs and assessment of mistuned response amplification (Chapter 8)

**Minor Contribution 4.1**  A model with tuned *Interface Mode* reductions

**Minor Contribution 4.2**  A model with tuned *Ancillary Mode* reductions

**Minor Contribution 4.3**  An investigation of error with a nominal mode approach
Dynamics of Rotationally Periodic IBRs and DFIBRs

This chapter introduces fundamental characteristics of tuned IBR and DFIBR dynamic response. These tuned structures are rotationally periodic structures, or cyclically symmetric structures, that exhibit particular free and forced response characteristics. Section 4.1 first introduces what rotational periodicity is and the types of modes that compose the free response. Section 4.2 then describes how to use the cyclic symmetry properties to reduce the fully cyclic structure model size to that of just a fundamental sector using cyclic constraints. Then, specific response characteristics of IBRs and DFIBRs are given in Sections 4.3 and 4.4, respectively, that are also used as test cases for the developed ROM capabilities. Finally, Section 4.5 introduces how each blade is geometrically mistuned from the design intent, or tuned, structure using as-manufactured measurements from a coordinate measurement machine.
4.1 Mode Shapes

The rotational periodicity of an IBR is defined when the geometry at an axial and radial location for an angle, $\theta$, is identical to the geometry at the remaining $(\theta + \psi)$ angles, where

$$\psi = \frac{2\pi n}{N}$$

(4.1)

where $N$ is the number of blades and $n$ is restricted to an integer $n = 0, 1, 2, \ldots, N - 1$. This angle, $\psi$, is commonly referred to as the inter-blade phase angle and will have $N$ independent values before the angle $\theta$ is repeated. This repetitive geometry is typically referred to as a disk-blade sector and is exploited particularly in finite element analysis where a series of analyses on one sector can yield a complete solution of the full 360 degree model with no more approximation than what is already inherent in finite element theory.

A description of eigenvectors for an IBR begins with partitioning the degrees-of-freedom (DOF) for the entire IBR according to

$$\mathbf{x} = \{ x^{(1)} x^{(2)} \ldots x^{(N)} \}^T$$

(4.2)

where $x^{(i)}$ is a vector of length $M$ that contains the real displacements DOF for the $n$th disk-blade sector. It then follows that there are a total of $NM$ DOF in the entire IBR.

The eigenvectors of the full IBR will fall into one of three categories, noting that $\mathbf{x}^T \mathbf{x} = 1$:

1. Each disk-blade sector will have displacements in-phase with their counterparts at all angles $(\theta + \frac{2\pi n}{N})$, i.e. $x^{(i)} = x^{(i+1)}$ for all $i = 1, 2, \ldots, N$, and Eq. 4.2 can be written as $\mathbf{x} = \{ x^{(1)} x^{(1)} \ldots x^{(1)} \}^T$.

2. Each disk-blade sector will have displacements in anti-phase with their counterparts at all angles $(\theta + \frac{2\pi n}{N})$, i.e. $x^{(i)} = -x^{(i+1)}$ for all $i = 1, 2, \ldots, N$. This class of
eigenvectors will only occur with even $N$.

3. Remaining eigenvectors that occur in degenerate pairs. In particular: $x^{(i)} \neq x^{(i+1)}$ and $x^{(i)} \neq -x^{(i+1)}$.

Further description of case (3) is now given. Consider the deflections described by $x$ after it has been rotated through an angle $(\theta + \frac{2\pi}{N})$ to the next disk-blade sector where the entire eigenvector displacements are now ordered by

$$x' = \{x^{(N)} \, x^{(1)} \ldots \, x^{(N-1)}\}^T$$

(4.3)

This rotated vector is also eigenvector that is distinguishable from $x$ and shares the same eigenvalue, $\lambda$, but the two vectors are not orthogonal to each other. For this to occur, there exists another eigenvector $\bar{x}$ that is orthogonal to $x$ at the same eigenvalue, $\lambda$, to enable $x'$ to exist at $\lambda$. Subsequently, $x'$ must be a linear combination of the two orthogonal vectors,

$$x' = cx + s\bar{x}$$

(4.4)

where $c$ and $s$ are constants. Furthermore, consider another eigenvector $\bar{x}'$ that is orthogonal to $x'$ defined by

$$\bar{x}' = -sx + cx$$

(4.5)

where $x'^T\bar{x}' = 0$ since $x^T\bar{x} = 0$. Likewise, $x'^T x' = 1$ and utilizing Eq. 4.4, $c^2 + s^2 = 1$ where $c = \cos \psi$ and $s = \sin \psi$ with $\psi$ being the inter-blade phase angle given in Eq. 4.1. The rotated eigenvectors, $x'$ and $\bar{x}'$ of Equations 4.4 and 4.5 can now be expressed by the transformation
\[
\begin{bmatrix}
    \mathbf{x}' \\
    \mathbf{\bar{x}}'
\end{bmatrix}
= 
\begin{bmatrix}
    c_{NM} & s_{NM} \\
    -s_{NM} & c_{NM}
\end{bmatrix}
\begin{bmatrix}
    \mathbf{x} \\
    \mathbf{\bar{x}}
\end{bmatrix}
\]

(4.6)

where \( I_{NM} \) is an identity matrix of size \( NM \). This transformation matrix simply rotates \( \mathbf{x} \) and \( \mathbf{\bar{x}} \) around \( n \) disk-blade sectors.

Much like \( \mathbf{x}' \) of Eq. 4.4 is a linear combination of the two orthogonal eigenvectors \( \mathbf{x} \) and \( \mathbf{\bar{x}} \) at the same eigenvalue, any other linear combination is also valid. Consider the complex vector

\[
\mathbf{z} = \mathbf{x} + i\mathbf{\bar{x}}
\]

(4.7)

that is also a valid eigenvector that allows Eq. 4.6 to be expressed as

\[
\mathbf{z} = e^{i\psi}
\]

(4.8)

that will be used as a cyclic constraint outlined in the following section.

### 4.2 Cyclic Constraints

The single sector formulation begins by partitioning a cyclic sector with a CB-CMS partitionment according to Fig. 4.1, where \( \Gamma \) are interface DOF, \( \alpha \) are independent interface DOF, \( \beta \) are the dependent interface DOF, and \( \sigma \) are non-interface DOF. A DOF vector for the cyclically symmetric disk or ring component, \( \mathbf{x} \), corresponding stiffness matrix, \( K \), and
force vector, \( f \), are given by

\[
\begin{align*}
\mathbf{x} &= \begin{pmatrix}
x_\sigma \\
x_\Gamma \\
x_\alpha \\
x_\beta 
\end{pmatrix}, \\
K &= \begin{bmatrix}
K_{\sigma\sigma} & K_{\sigma\Gamma} & K_{\alpha\sigma} & K_{\beta\sigma} \\
K_{\sigma\Gamma}^T & K_{\Gamma\Gamma} & K_{\alpha\Gamma} & K_{\beta\Gamma} \\
K_{\alpha\sigma}^T & K_{\alpha\Gamma}^T & K_{\alpha\alpha} & K_{\alpha\beta} \\
K_{\beta\sigma}^T & K_{\beta\Gamma}^T & K_{\alpha\beta}^T & K_{\beta\beta}
\end{bmatrix}, \\
f &= \begin{pmatrix}
f_\sigma \\
f_\Gamma \\
f_{I\alpha} + f_\alpha \\
f_{I\beta} + f_{I\beta}
\end{pmatrix}
\end{align*}
\tag{4.9}
\]

where the mass matrix follows the same partitionment and \( f_{I\alpha} \) and \( f_{I\beta} \) are internal cyclic interface forces. By introducing the dependent interface constraint \( x_\beta = e^{i\psi}x_\alpha \), where \( \psi = \frac{2\pi h}{N} \) is the inter-blade phase angle for harmonic \( h = 0, 1, \ldots, \text{int}[N/2] \) and \( i = \sqrt{-1} \), into the displacement vector \( \mathbf{x} \), the dependent interface DOF \( \beta \) are eliminated for a cyclically symmetric component. Note that \( h = N/2 \) is the largest attainable harmonic for even-valued \( N \), and is included throughout the subsequent formulations. Eliminating the dependent interface DOF, \( x_\beta \), is done with the following constraint matrix

\[
\begin{align*}
\mathbf{x} &= \begin{pmatrix}
x_\sigma \\
x_\Gamma \\
x_\alpha \\
x_\beta 
\end{pmatrix}, \\
0 & I & 0 \\
0 & 0 & I \\
I & 0 & 0 \\
e^{i\psi} & 0 & 0
\end{bmatrix}
\begin{pmatrix}
x_\alpha \\
x_\sigma \\
x_\Gamma \\
x_\beta 
\end{pmatrix} = T\tilde{\mathbf{x}}
\end{align*}
\tag{4.10}
\]

The dependent interface DOF are then eliminated from a cyclic component matrix by

\[
\tilde{K} = T^TKT
\tag{4.11}
\]

where the mass matrix undergoes the same transformation. Introducing Euler's formula \( e^{\pm i\psi} = \cos(\psi) \pm i\sin(\psi) \) and expressing the displacement vector in terms of cosine and sine terms, denoted by a right superscript \( c \) and \( s \), respectively, a real-valued matrix is obtained that is a function of the harmonic index, \( h \). This derivation is described in detail.
in Appendix A.

For harmonics $h = 0$, and if it exists, $N/2$, the sine component is eliminated so only a single sector description is required leaving the stiffness matrix

$$
\widetilde{K}^{(h)} = 
\begin{bmatrix}
\tilde{K}^{(h)}_{\tau\tau} & \tilde{K}^{(h)}_{\tau\Gamma} \\
\tilde{K}^{(h)}_{\tau\Gamma} & \tilde{K}^{(h)}_{\Gamma\Gamma}
\end{bmatrix}
$$

where the “tilde” overscript denotes a value that is in cyclic coordinates and the superscript $h$ denotes the harmonic index. The $\alpha$ and $\sigma$ DOF are combined into the sector interior DOF $\tau$

$$
\tilde{x}^c_{\tau} = \begin{cases}
\tilde{x}^c_{\alpha} \\
\tilde{x}^c_{\sigma}
\end{cases}
$$
\[ \tilde{K}_{\tau\tau}^{(h)} = \begin{bmatrix} K_{\alpha\alpha} + K_{\beta\beta} + (K_{\alpha\beta} + K_{\alpha\beta}^T) \cos (\psi) & K_{\sigma\alpha}^T + K_{\sigma\beta}^T \cos (\psi) \\ K_{\sigma\alpha} + K_{\sigma\beta} \cos (\psi) & K_{\sigma\sigma} \end{bmatrix} \]

\[ \tilde{K}_{\tau\Gamma}^{(h)} = \begin{bmatrix} K_{\Gamma\alpha}^T + K_{\Gamma\beta}^T \cos (\psi) \\ K_{\Gamma\sigma}^T \end{bmatrix}, \quad \tilde{K}_{\Gamma\Gamma}^{(h)} = K_{\Gamma\Gamma} \] (4.14)

For the remaining harmonics, \( h \neq 0 \) and \( h \neq N/2 \), the sine terms remain and a duplicate, or double, sector description is required and the displacement vector \( \tilde{x}^{(h)} \) is given by

\[ \tilde{x}^{(h)} = \begin{bmatrix} \tilde{x}^c_{\tau} \\ \tilde{x}^s_{\tau} \\ \tilde{x}^c_{\Gamma} \\ \tilde{x}^s_{\Gamma} \end{bmatrix} \] (4.15)

where

\[ \tilde{x}^c_{\tau} = \begin{bmatrix} \tilde{x}^c_{\alpha} \\ \tilde{x}^c_{\sigma} \end{bmatrix}, \quad \tilde{x}^s_{\tau} = \begin{bmatrix} \tilde{x}^s_{\alpha} \\ \tilde{x}^s_{\sigma} \end{bmatrix} \] (4.16)

and the sub-matrices to \( \tilde{K}^{(h)} \) in Eq. A.11 are computed by Equations 4.17 - 4.19. The interior \( \tau \) DOF partition of \( \tilde{K}^{(h)} \) is sub-partitioned according to

\[ \tilde{K}_{\tau\tau}^{(h)} = \begin{bmatrix} 1\tilde{K}_{\tau\tau}^{(h)} & 2\tilde{K}_{\tau\tau}^{(h)} \\ 2\tilde{K}_{\tau\tau}^{(h)} & 1\tilde{K}_{\tau\tau}^{(h)} \end{bmatrix} \] (4.17)
with

\[
\tilde{K}^{(h)}_{\Gamma\Gamma} = \begin{bmatrix}
K_{\alpha\alpha} + K_{\beta\beta} + (K_{\alpha\beta} + K_{\alpha\beta}^T) \cos(\psi) & K_{\sigma\alpha} + K_{\sigma\beta}^T \cos(\psi) \\
K_{\sigma\alpha} + K_{\sigma\beta}^T \cos(\psi) & K_{\sigma\sigma}
\end{bmatrix}
\]

(4.18)

\[
\tilde{K}^{(h)}_{\Gamma\Gamma} = \begin{bmatrix}
(K_{\alpha\beta}^T - K_{\alpha\beta}) \sin(\psi) & K_{\sigma\beta}^T \sin(\psi) \\
-K_{\sigma\beta}^T \sin(\psi) & 0
\end{bmatrix}
\]

(4.19)

The \(\Gamma\Gamma\) and \(\sigma\Gamma\) partitions of Eq. A.11 for harmonics, \(h \neq 0\) and \(h \neq N/2\), are given by

\[
\tilde{K}^{(h)}_{\Gamma\Gamma} = \begin{bmatrix}
K_{\Gamma\alpha}^T + K_{\Gamma\beta}^T \cos(\psi) & K_{\Gamma\beta}^T \sin(\psi) \\
K_{\Gamma\sigma}^T & 0 \\
-K_{\Gamma\beta}^T \sin(\psi) & K_{\Gamma\alpha}^T + K_{\Gamma\beta}^T \cos(\psi)
\end{bmatrix}
\]

(4.20)

The force vector is transformed to cyclic coordinates according to

\[
\tilde{f}^{(h)} = T^i f = \begin{bmatrix}
f_{\alpha} + e^{-i\psi} f_{\beta} \\
f_{\sigma} \\
f_{\Gamma}
\end{bmatrix}
\]

(4.21)

where the cyclic interface portions were eliminated since \(f_{I\beta} = -e^{i\psi} f_{I\alpha}\). This force vector can be cast into a real-value formulation just as the mass and stiffness matrices to yield

\[
\tilde{f}^{(h)} = \begin{bmatrix}
\tilde{f}_{\Gamma} \\
\tilde{f}_{\sigma}
\end{bmatrix}
\]
where for harmonics $h = 0$ or $h = N/2$

$$
\tilde{f}_r = \begin{cases} 
f_{\alpha} + f_{\beta} \cos(\psi) \\
f_{\sigma}
\end{cases}, \quad \tilde{f}_\Gamma = f_{\Gamma} \tag{4.22}
$$

and for the remaining harmonics

$$
\tilde{f}_r = \begin{cases} 
f_{\alpha} + f_{\beta} \cos(\psi) \\
f_{\sigma} \\
-f_{\sigma} \sin(\psi) \\
0
\end{cases}, \quad \tilde{f}_\Gamma = \begin{cases} 
f_{\Gamma} \\
0
\end{cases} \tag{4.23}
$$

### 4.3 IBR Test Case

The 16-blade Augmented Damping Low Aspect Ratio Fan (ADLARF) IBR [92], depicted in Fig. 4.2, used in this study offers geometries typically seen in modern IBRs. The FEM is meshed with eight-node linear solid (brick) elements with translations in the $x$-, $y$-, and $z$-directions at each node. The mesh density in Table 4.1 is used for both nominal and geometric mistuning prediction methods developed in subsequent chapters. The IBR was excited by a unit harmonic forcing function loaded at each blade’s leading edge tip node in the axial direction. While overly simplistic, more complex pressure loadings on the blade can be applied through the forcing vectors derived for each respective method. Regardless, a point load at the blade tip is sufficient for exciting the IBR and is often used in traveling wave experimental testing [93].
4.3.1 Free Response Characteristics

A nodal diameter (ND) plot illustrating frequency veering was constructed in Fig. 4.3 for the tuned ADLARF IBR by plotting the natural frequencies of the tuned system versus the number of nodal diameters. This plot characterizes the free vibration of the IBR, where nearly horizontal lines correspond to assembly modes dominated by blade motion, while the slanted lines are dominated by disk motion. These lines are slanted due to the stiffening of the disk as the circumferential wavelength decreases with a corresponding increase in nodal diameters. It is well established in the literature that the amount of interaction between blade and disk-dominated modes is indicative of inter-blade coupling, and ultimately, an IBR’s sensitivity to mistuning [94]. Regions of disk-blade mode interaction occur in veering regions of Fig. 4.3 where the connected natural frequency lines appear to approach and then diverge at a nodal diameter.

A tuned system is restricted to having natural modes at integer harmonics that lead to
Table 4.1: Basic FEM size data for the ADLARF rotor

<table>
<thead>
<tr>
<th>Component</th>
<th>Elements</th>
<th>Nodes</th>
<th>DOF</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cantilevered Blade</td>
<td>640</td>
<td>1020</td>
<td>3060</td>
</tr>
<tr>
<td>Disk Sector</td>
<td>600</td>
<td>723</td>
<td>2169</td>
</tr>
<tr>
<td>Disk-Blade Interface</td>
<td>n/a</td>
<td>51</td>
<td>153</td>
</tr>
<tr>
<td>Full Model</td>
<td>20384</td>
<td>28704</td>
<td>86112</td>
</tr>
</tbody>
</table>

Figure 4.3: Nodal diameter plot illustrating natural frequencies versus nodal diameters. The investigated frequency ranges at the specific EO excitations are circled.

discrete inter-blade phase angles that satisfy displacement continuity requirements at the sector interfaces. However, such discrete values leave it difficult to visualize the veering regions. A better visualization is achieved through computing modes at non-integer, continuous harmonics. Furthermore, it provides a means to calculate the curvature of the veering that can be related to the amount of inter-blade coupling [94]. Both integer and continuous harmonics are shown in Fig. 4.3, where the markers depicting natural frequencies are connected by lines illustrating mode families and the veering regions. The mode families are characterized by the type of occurring blade motion that strongly resembles cantilevered blade motion. Limited disk participation as the mode family lines become nearly horizontal
causes the blades to vibrate closely to that of a cantilevered blade.

The type of nominal, cantilevered blade motion belonging to each mode family highlighted in Fig. 4.3 are shown in Fig. 4.4. The low order modes have little complexity and can easily be characterized by the type of motion. However, higher order modes (HOM) illustrate the complexity of the nominal blade mode shapes that can be a combination of different mode shapes. It is with these complex displacements, as opposed to simple modes (e.g. first bend), that the mistuned blade modes are expected to differ from their nominal counterpart and have an impact forced response predictions calculated with a tuned mode expansion. At these frequency ranges the wavelength of the mode shape decreases and the blade response features higher amplitudes in smaller portions of the blade, e.g. leading edge scallops, that are expected to be sensitive to geometric perturbations.

4.3.2 External Forcing

A traveling wave force is considered for the blade DOF only. This allows for a more compact formulation, but it is not a requirement of the mistuning methods presented herein. This excitation force is constant in magnitude and differs only in phase from blade to blade by

\[
\varphi_{a,C} = \frac{2\pi C (a - 1)}{N} \quad a = 1, \ldots, N
\]  

(4.24)

where \( C \) is the EO excitation. A phase vector \( P_C \) between blades becomes

\[
P_C = \left\{ \exp(i\varphi_{1,C}), \quad \ldots, \quad \exp(i\varphi_{N,C}) \right\}
\]  

(4.25)
Figure 4.4: Cantilevered ADLARF blade modes shape types describing the blade motion for each mode family of Fig. 4.3 (HOM = Higher-Order Mode)
The forcing vector on all blades can then be expressed as a constant force $^A f_\tau$ and $^A f_\Gamma$ on the interior and interface DOF, respectively, as

$$ ^A V = \begin{cases} P_C \otimes ^A f_\tau \\ P_C \otimes ^A f_\Gamma \end{cases} $$

(4.26)

### 4.4 DFIBR Test Case

The DFIBR used in this study is a modification of the ADLARF IBR where the $I_N = 16$ inner-blades have been elongated and the $O_N = 32$ outer-blades are appropriately scaled, partial versions of the inner-blades. A total of $O_S = 2$ outer-blades are assigned to each cyclic sector. The outer-blades are circumferentially offset from inner-blades by a fourth of the inter blade phase angle of the DFIBR. This offset is arbitrary and can be changed to meet design requirements. The ring cross-section is has been arbitrarily defined and has not undergone any design optimization. While not a DFIBR from any specific engine, the geometries in this model are representative of those seen on modern DFIBRs and is shown in Fig. 4.5. The FEM is meshed with eight-node linear hexahedral elements with translations in the x-, y-, and z-directions at each node. The mesh density is listed in Table 4.2 is used for all mistuning prediction methods. Only the disk is constrained to zero displacements in all three directions at the hub of the disk.

#### 4.4.1 Free Response Characteristics

An ND plot illustrating the tuned system natural frequencies versus the harmonic index is shown in Fig. 4.6. This plot characterizes the free vibration of the dual flow-path rotor by highlighting regions consisting of system modes and those of mainly blade motion. These regions are evident by “connecting” system natural frequencies over harmonic in-
Table 4.2: Basic FEM size data for the DFIBR partitioning defined in Fig. 8.2

<table>
<thead>
<tr>
<th>Component</th>
<th>Nodes</th>
<th>DOF</th>
</tr>
</thead>
<tbody>
<tr>
<td>$D_T$</td>
<td>813</td>
<td>2439</td>
</tr>
<tr>
<td>$R_T$</td>
<td>1714</td>
<td>5142</td>
</tr>
<tr>
<td>$I_T$</td>
<td>1240</td>
<td>3720</td>
</tr>
<tr>
<td>$O_T$</td>
<td>252</td>
<td>756</td>
</tr>
<tr>
<td>$\Gamma_R + \Gamma_D$</td>
<td>124</td>
<td>373</td>
</tr>
<tr>
<td>$\Gamma_i$</td>
<td>42</td>
<td>126</td>
</tr>
<tr>
<td>Full Model</td>
<td>28704</td>
<td>86112</td>
</tr>
</tbody>
</table>

Figure 4.5: Full DFIBR test case
Figure 4.6: The nodal diameter plot of the DFIBR illustrating the tuned system natural frequencies versus the harmonic index

dices for a family of modes, as shown with the dashed and continuous lines through circle markers. The dashed lines denote integer harmonics, while the solid denotes fictitious continuous harmonics, but better illustrate veering regions. In traditional, single flow-path IBR designs, these veering regions have a tendency to be susceptible to mistuned response amplification. A veering region is evident where the continuous harmonic lines approach and then diverge, e.g. at harmonic one and approximately $1200\, Hz$.

The veering regions contain system modes that offer significant interaction between the motion of the blades, disk, and outer ring. The nearly horizontal portions of each mode family consist primarily of blade motion. In IBRs this mode family was characterized by the type of blade motion that took place, e.g. first bend or first torsion. In DFIBRs, there are two sets of blades of different geometries that will now characterize the mode family. In fact, it is shown that the primarily flat regions consist of either an inner- (outer-) blade mode type and then transition to an outer- (inner-) blade mode type through the slanted, or system mode regions of the ND plot. An example of this is shown in Fig. 4.7 for the fourth mode family on the ND plot. As the harmonic index increases from zero to four, a
relatively flat portion on the ND plot annotates primarily blade motion, which happens to be outer-blade 1\textsuperscript{st} bend motion (Fig. 4.7a). As harmonic four is approached the mode family line becomes slanted, denoting a system mode that has significant interaction between the inner- and outer-blades through the ring (Fig. 4.7b). Continuing to increase in harmonic then shows another relatively flat portion of primarily blade motion that is inner-blade 1\textsuperscript{st} fixed-fixed torsion motion (4.7c). Note there is still some outer-blade modal stresses at harmonic eight, but is due primarily from the motion of the ring induced by the inner-blade fixed-fixed first torsion mode instead of outer-blade modes. Since the ring is not as stiff as the disk, perfect fixed-fixed inner-blade boundary conditions are only approximate.

Furthermore, while the inner-blades are essentially subject to fixed-fixed boundary conditions at the blade root and tip, inner-blade motions will can still resemble a cantilevered-blade motion for certain mode families. An example of this phenomenon is illustrated in Fig. 4.8a that shows the first system mode shape that occurs at approximately 96\,Hz. In this mode, the inner-blade motion is described by the cantilevered first-bend since all inner-blades bend in-phase with each other, resulting in the ring and outer-blades oscillating about the axial direction while the disk remains essentially motionless. While the outer-blades undergo motion at this mode, the modal stresses of Fig. 4.8b show that this motion is due entirely to the motion of the inner-blades.

### 4.4.2 External Forcing

DFIBRs can be subjected to different EO excitations on the inner- and outer-blades. While the inner- or outer-blades are subject to a constant force that differs only in phase, the force resultant on the DFIBR will exhibit a non-constant force magnitude circumferentially around the rotor with a phase difference that may not pass from zero to 360\,deg before repeating.

The blade force vector is derived for a periodic EO forcing function that is constant in
Figure 4.7: Modal Stresses for Family 4 illustrating the transition from primarily outer-blade motion through a system mode to primarily inner-blade motion.
Figure 4.8: First system mode at Harmonic 0 illustrating the inner-blade cantilevered first bend mode under fixed-fixed blade root and tip conditions.
magnitude and differs only in phase from blade-to-blade by

$$\varphi_{a,C} = \frac{2\pi \bullet C (a - 1)}{N} \quad a = 1, \ldots, \bullet N$$  \hspace{1cm} (4.27)

where the bullet, $\bullet$, is a place holder for either an $I$ or $O$ to denote either, inner- or outer-blades, respectively, and $\bullet C$ is the EO excitation. This type of excitation is representative of engine forcing where stationary, non-uniform pressure distributions around the annulus is felt as a dynamic load as the DFIBR rotates through the flow field. The force is assumed be constant in magnitude and differ only in phase from blade-to-blade. The forcing phase vector between blades for a specific EO excitation becomes

$$\bullet P_C = \left\{ e^{i\varphi_1, C}, \ldots, e^{i\varphi_{\bullet N}, C} \right\}$$  \hspace{1cm} (4.28)

For example, consider Fig. 4.9 that plots the applied force magnitude and phase difference from Eq. 4.27 to the inner- and outer-blades as well as the resulting force magnitude and phase seen around the rotor. The inner-blades were subject to a $I C = 0$ EO excitation while the outer-blades were subject to a $O C = 2$ EO excitation, each of constant magnitude as shown in Fig. 4.9a. As a result, there is a non-constant force magnitude around the rotor that is characterized by a force magnitude and phase with a period of $P = |O C - I C| = 2$ around the rotor. While the forces on the DFIBR are not summed, but rather applied at different points circumferentially around the DFIBR, it illustrates how the DFIBR forced response response will be influenced with different EO excitations on the inner- and outer-blades.
Figure 4.9: Illustration how different EO excitations between the inner- and outer-blades causes non-constant force magnitudes around the DFIBR and the disruption in the phase of Eq. 4.27.
4.5 Blade Geometric Deviations

Mistuned response is highly sensitive to small geometric deviations and, as a result, gross-measurement quality control methods do not provide the quantitative details needed for mistuning studies. One requisite approach is the use of coordinate measurement machines (CMMs) that obtain a surface map of geometric locations through the use of a transversing probe. Consequently, thousands of data points are created, but the computational expense of assessing the sensitivity of mistuning to each geometric perturbation at every location is impracticable. The use of such a measurement device warrants the need of blade geometry ROMs that still retain geometric deviations. An attractive approach is Principal Component Analysis (PCA) because it reduces the dimensionality of interrelated data sets, where the user can choose the amount of variation that is retained. By transforming the original, correlated data set to a new set of uncorrelated Principal Components (PCs), the first few PCs bear the majority of the variation in the original data.

As applied to blade geometry, suppose that $\mathbf{x}$ is a vector of $p$ three-dimensional coordinate data points, where $\mathbf{x} \in \mathbb{R}^p$. If the variances of and the covariances between the $p$ data points are of interest, CMM measurement data will dictate that $p$ variances and $\frac{1}{2}p(p-1)$ covariances be reviewed, where $p$ measures in the thousands. Furthermore, a set of $N$ blades increases the original data size and results in a matrix $\mathbf{X} \in \mathbb{R}^{p,N}$. An alternative approach to processing thousands of pieces of geometric information is to find a few ($\ll p$) derived variables that maintain the majority of geometrical variances and covariances.

Implementation occurs by computing $\Delta \mathbf{X}$, an $(p \times N)$ matrix of measured deviations with the $(i,j)$th element $(x_{i,j} - \overline{x}_j)$, where

$$\overline{x}_j = \frac{1}{n} \sum_{i=1}^{n} x_{i,j} \quad j = 1, 2, \ldots, N$$  \hspace{1cm} (4.29)

This process of computing $\Delta \mathbf{X}$ measures the $j^{th}$ variable about its mean $\overline{x}_j$ for the $i^{th}$ observation, or in blade terminology, the variation around the average blade - which is
not necessarily the original design specification. The first-order covariance matrix of $\Delta X$, known as $\Sigma$, mandates that a total of $(p \times N)$ pieces of information be reviewed, which is unacceptable for CMM data. To provide the set of uncorrelated PCs, $\Sigma$ is cast into the standard eigen-problem formulation, where the PCs or weights are the eigenvectors $\Psi$ and $\Lambda$ is a diagonal matrix of eigenvalues

$$\Sigma \Psi = \Psi \Lambda \quad (4.30)$$

In general, the eigenvalues are the variances of the PCs that give an indication of the amount of variance of the original data captured by the PCs. Furthermore, due to the nature of the eigen-problem the PCs are also orthogonal, meaning they are statistically uncorrelated.

The measured deviations, $\Delta X$, is linearly transformed to Principal Component space by the following equation, where $Z$ is the score matrix and $Z \in \mathbb{R}^{m,N}$ with $m$ as the number of retained PCs

$$Z = [\Delta X] \Psi \quad (4.31)$$

The scores are fundamentally regression coefficients for the PC space and explain the participation of each PC in the CMM data. Thus, if $m < p$, PCA transforms a large set of interrelated data to a much smaller set of $m$ uncorrelated parameters.

If the PCs and scores are known, the measured deviations can be determined by

$$\Delta X = Z \Psi^T \quad (4.32)$$

Although this might seem unorthodox, it provides an opportunity to perturb each score $z_i$ in the score matrix and achieve new blade geometries while retaining the variation of the original blade geometries. This is done by

$$\tilde{z}_i = \xi_i z_i \quad i = 1, 2, \ldots, m \quad (4.33)$$
where $\xi_i$ is a randomly drawn scalar from the distribution of $z_i$. Eqs. 4.32 and 4.33 are the manner in which new IBR blade geometries are determined for geometric mistuning.

Principal Component Analysis of the 16 industrial fan blades used to generate blade geometry deviations generated 15 principal components (PCs), where Fig. 4.10 illustrates the variance explained by each principal component (PC) in the measurement data. The PCs are ordered such that each subsequent PC accounts for less spatial variation, until 100% of the variation is explained by all 15 PCs. This represents a significant reduction in model size, where if all physical nodal locations were retained, thousands of DOF would need to be perturbed. Figure 4.11 depicts a single blade surface with geometric deviations obtained with all 15 PCs. It is apparent that surface deviations correlate across the blade and that PCA accounts for this spatial correlation. A five mil (five thousandths of an inch) geometric tolerance limit was imposed on blade deviations to be consistent with modern blade manufacturing tolerance limits.
Figure 4.11: Random ADLARF blade (pressure side) x-direction surface deviations

4.6 Summary

This chapter provided the fundamental background of IBR and DFIBR dynamic response that will help formulate prediction methodologies in subsequent chapters. Furthermore, it described how as-manufactured airfoil geometry deviations were identified with Principal Component Analysis. These deviations are confined to the airfoils and is exploited in the next chapter by using the Craig-Bampton Component Mode Synthesis technique.
Craig-Bampton Component Mode

Synthesis

5.1 Introduction

Large, complex structural systems modeled with finite element techniques often result in exceptionally large models with total degrees of freedom (DOFs) easily extending into the millions. This presents a significant computational burden for both digital storage of system matrices and numerical methods. While modern digital computers become more and more powerful, the desire to have high-fidelity, multi-physics models\(^1\) embedded in optimization routines or probabilistic analysis greatly outweighs modern computational capabilities. This drives the need for development of approaches that reduce the order (e.g. model size, or total model DOF) of the modeled structural system.

Component mode synthesis (CMS) techniques have been proven to be very useful in solving large structural dynamics problems. These types of problems are based upon fundamental frequencies and associated mode shapes and do not require a large number of DOF. However, dynamic problems still require high finite element mesh densities to accurately map dynamic stresses and strains \textit{a posteriori}. The term \textit{component modes} refer to \textit{Ritz Vectors}, or \textit{assumed modes}, that are \textit{basis vectors} that describe nodal displacements

\(^{1}\text{Models that include many different types of physical phenomena, e.g. structural dynamics, thermal analysis, and mass flow}\)
within a substructure or component; e.g., eigenvectors are component normal modes that are just one category of component modes. Each specific CMS method is characterized by the types of modes retained in the solution. The CMS approach described herein is referred to as the Craig-Bampton (CB) method, which employs a combination of fixed-interface normal modes and interface constraint modes.

Three basic steps are performed in CB-CMS approach:

1. Division of a system FEM into components or substructures
2. Definition and calculation of Component Modes
3. Synthesis of the reduced-order model of the system by coupling the components

5.2 Formulation

For the substructuring step, the system is divided into components that share a common, redundant interface with one or all of the other components. DOF falling on this interface are referred to as boundary coordinates while the remaining DOF are referred to as interior coordinates. Figure 5.1 illustrates a partitioning of a disk-blade system model into subcomponents and interface DOF. The equation of motion (EOM) for a single, undamped component \( s \) is of the form

\[
M^{(s)} \ddot{x}^{(s)} + K^{(s)} x^{(s)} = f^{(s)} \tag{5.1}
\]

where \( M^{(s)}, K^{(s)}, \) and \( x^{(s)} \) are derived in the original physical coordinate system. The component’s physical displacement coordinates \( x \) are transformed to component generalized coordinates \( p \) by a Ritz coordinate transformation

\[
x^{(s)} = U^{(s)} p^{(s)} \tag{5.2}
\]
where $U^{(s)}$ is the CB component mode matrix composed of fixed-interface normal modes and interface constraint modes. The component modal model is then subject to the following EOM

$$\mathcal{M}^{(s)} \ddot{p}^{(s)} + \mathcal{K}^{(s)} p^{(s)} = \mathcal{F}^{(s)} \quad (5.3)$$

where the component mass matrix, stiffness matrix, and force vector are given by

$$\mathcal{M}^{(s)} = U^{\top(s)} M^{(s)} U^{(s)}, \quad \mathcal{K}^{(s)} = U^{\top(s)} K^{(s)} U^{(s)}, \quad \mathcal{F}^{(s)} = U^{\top(s)} f^{(s)} \quad (5.4)$$

To derive the component modes used in the CB method, Eq. 5.1 is partitioned according to

$$\begin{bmatrix} M_{\tau\tau} & M_{\tau\Gamma} \\ M_{\Gamma\tau} & M_{\Gamma\Gamma} \end{bmatrix} \begin{bmatrix} \ddot{x}_\tau \\ \ddot{x}_\Gamma \end{bmatrix} + \begin{bmatrix} K_{\tau\tau} & K_{\tau\Gamma} \\ K_{\Gamma\tau} & K_{\Gamma\Gamma} \end{bmatrix} \begin{bmatrix} x_\tau \\ x_\Gamma \end{bmatrix} = \begin{bmatrix} 0 \\ f_\Gamma \end{bmatrix} \quad (5.5)$$

where $\tau$ refers to interior coordinates and $\Gamma$ refers to boundary coordinates. Fixed-interface normal modes required for the CB method are obtained by restraining all boundary DOF.
and solving the classical eigenvalue problem (EVP)

\[
\begin{bmatrix}
K_{\tau\tau} - \omega_j^2 M_{\tau}\nend{bmatrix}\{\phi_i\}_j = 0 \quad j = 1, 2, \ldots, N_{\tau}
\] (5.6)

where \(N_{\tau}\) is the number of interior DOF. Combining the complete set of fixed-interface normal modes yields the matrix \(\Phi_{\tau\tau}\) and is assembled in the modal matrix

\[
\Phi_{\tau} = \begin{bmatrix}
\Phi_{\tau\tau} \\
0_{\Gamma\tau}
\end{bmatrix}
\] (5.7)

If the fixed-interface normal modes are normalized with respect to the interior partition of the mass matrix, \(M_{\tau\tau}\), they satisfy

\[
\Phi_{\tau}^T M_{\tau\tau} \Phi_{\tau\tau} = I_{\tau\tau} \quad \Phi_{\tau}^T K_{\tau\tau} \Phi_{\tau\tau} = \Lambda_{\tau\tau} = \text{diag} \left( \omega_j^2 \right)
\] (5.8)

Constraint modes are ascertained by statically deforming a structure by applying a unit displacement to one coordinate of an established set of constraint coordinates while the remaining coordinates of the set are restrained, and the remaining DOF of the component are force-free. Interface constraint modes are prescribed by using the boundary DOF as the established set of constraint coordinates, applying successive unit displacements on the boundary DOF, and leaving all of the interior DOF of the component force-free. This is given by

\[
\begin{bmatrix}
K_{\tau\tau} & K_{\tau\Gamma} \\
K_{\Gamma\tau} & K_{\Gamma\Gamma}
\end{bmatrix}
\begin{bmatrix}
\Psi_{\tau\Gamma} \\
I_{\Gamma\Gamma}
\end{bmatrix} =
\begin{bmatrix}
0_{\tau\Gamma} \\
R_{\Gamma\Gamma}
\end{bmatrix}
\] (5.9)
Solving Eq. 5.9 for $\Psi_{\tau\Gamma}$ allows for the interface constraint mode matrix to be given by

$$
\Psi_c = \begin{bmatrix} \Psi_{\tau\Gamma} \\ I_{\Gamma\Gamma} \end{bmatrix} = \begin{bmatrix} -K_{\tau\tau}^{-1}K_{\tau\Gamma} \\ I_{\Gamma\Gamma} \end{bmatrix} \tag{5.10}
$$

As a check, these constraint modes are stiffness-orthogonal to all of the fixed-interface normal modes.

The modes in $\Phi$ are linearly independent by definition, while the modes in $\Psi$ are ensured to be linearly independent through the successive unit displacements used in their generation. Furthermore, the two mode sets are linearly independent of each other. If full retention of the modes in $\Phi$ is prescribed, the number of modes will be equal to the number in interior DOF for the respective component. The size of $\Psi$ must equal the number of interface DOF for the component, as no reduction technique is applied to these DOF in this traditional formulation. As a result, if all the modes are retained in $\Phi$, the linear independence and completeness of the CB solution will span the same solution space and yield the exact solution of the system FEM.

From Eq. 5.3 and 5.4, the CB transformation matrix is a combination of Eq. 5.7 and 5.10 given by

$$
U^{(s)} = \begin{bmatrix} \Phi_{\tau} & \Psi_c \end{bmatrix}^{(s)} = \begin{bmatrix} \Phi_{\tau k} & \Psi_{\tau\Gamma} \\ 0_{\Gamma k} & I_{\Gamma\Gamma} \end{bmatrix}^{(s)} \tag{5.11}
$$

and Eq. 5.2 is expanded as

$$
x^{(s)} = \begin{bmatrix} x_{\tau} \\ x_{\Gamma} \end{bmatrix}^{(s)} = \begin{bmatrix} \Phi_{\tau k} \\ 0_{\Gamma k} \end{bmatrix}^{(s)} \begin{bmatrix} \Psi_{\tau\Gamma} \\ I_{\Gamma\Gamma} \end{bmatrix}^{(s)} \begin{bmatrix} p_k \\ p_{\Gamma} \end{bmatrix} \tag{5.12}
$$

where the subscript $k$ on $\Phi_{\tau k}$ represents the number of kept fixed-interface modes. When the component fixed-interface constraint modes are normalized according to Eq. 5.8, the
mass and stiffness matrices of the component modal EOM have the form

\[ M^{(s)}_{CB} = \begin{bmatrix} I_{kk} & M_{k\Gamma} \\ M_{k\Gamma}^T & M_{\Gamma\Gamma} \end{bmatrix}^{(s)} \quad K^{(s)}_{CB} = \begin{bmatrix} \Lambda_{kk} & 0_{k\Gamma} \\ 0_{k\Gamma}^T & K_{\Gamma\Gamma} \end{bmatrix}^{(s)} \] (5.13)

The coupling of components begins with the bottom row of Eq. 5.12, which implies that \( x^{(s)} = p^{(s)} \). Thus for a two component system, \( p^{(1)} = p^{(2)} = x \), and the separate components are coupled with the component coupling matrix \( S \) (for an example two-component system)

\[ \begin{bmatrix} p^{(1)}_{k_1} \\ p^{(1)}_{k_2} \\ p^{(2)}_{k_1} \\ p^{(2)}_{k_2} \end{bmatrix} = Tq = \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & I \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} p^{(1)}_{k_1} \\ p^{(2)}_{k_2} \\ x \end{bmatrix} \] (5.14)

that guarantees equality of DOF between components at the interface. Equation 5.14 can be expanded for a multiple interface system, such as the \( N \) blade-disk interfaces of a rotor. The reduced system *coupled* EOM is then given by

\[ C^{B}M_{\ddot{p}} + C^{B}K_p = C^{B}F \] (5.15)
where

\[
{CB\mathcal{M}} = T^T \sum_{s=1}^{N} \mathcal{B} \left[ \mathcal{M}^{(s)} \right] T \\
{CB\mathcal{K}} = T^T \sum_{s=1}^{N} \mathcal{K}^{(s)} T \\
{CB\mathcal{F}} = T^T \begin{cases} 
\mathcal{F}^{(1)} \\
\vdots \\
\mathcal{F}^{(N)} 
\end{cases}
\]

(5.16)

The reduced model eigenvalue problem can then be solved

\[
[{CB\mathcal{K}} - \lambda {CB\mathcal{M}}] \{\phi_i\}_j = 0 \quad j = 1, 2, \ldots, N_r
\]

(5.17)

where \(\lambda = \omega_j^2\). The mode shapes can be expanded to physical space, \(u\), with the following transformation

\[
x^{(s)} = U^{(s)} T^{(s)} p^{(s)}
\]

(5.18)

### 5.3 Summary

The CB-CMS approach accuracy and ease of implementation will be utilized in the remaining chapters for mistuned response prediction. However, there are certain attributes of the CB-CMS methodology that are uniquely adjusted in subsequent chapters to better suit turbomachinery components. Each of these unique modifications of the approach provide a novel contribution to the existing mistuning literature.
Geometric Mistuning ROM for IBRs
with Tuned Disk-Blade Connections

Abstract

Two methods that explicitly model blade geometry surface deviations for mistuning prediction in integrally bladed rotors are developed by performing a modal analysis on different degrees of freedom of a parent reduced-order model. The parent ROM is formulated with Craig-Bampton component mode synthesis in cyclic symmetry coordinates for an IBR with a tuned disk and small blade geometric deviations. The first method performs an eigen-analysis on the constraint-mode DOF that provides a truncated set of Interface modes while the second method includes the disk fixed-interface normal mode in the eigen-analysis to yield a truncated set of Ancillary modes. Both methods can utilize tuned or mistuned modes, where the tuned modes have the computational benefit of being computed in cyclic symmetry coordinates. Furthermore, the tuned modes only need to be calculated once which offers significant computational savings for subsequent mistuning studies. Each geometric mistuning method relies upon the use of geometrically mistuned blade modes in the component mode framework to provide a very accurate ROM.
forced response results are compared to both the full finite element model solutions and a
traditional frequency-based approach used widely in academia and the gas turbine industry.
It is shown that the developed methods provide highly accurate results with a significant
reduction in solution time compared to the full FEM and parent ROM.

6.1 Introduction

Integrally Bladed Rotors based on nominal design parameters are a rotationally periodic
structure. Unfortunately, there are small irregularities in the geometric and material char-
acteristics between individual blades, referred to as mistuning, causing the rotational peri-
odicity to break down. Even small mistuning that falls within strict manufacturing toler-
ances can have a dramatic effect on the IBR response. The ramifications of mistuning are
two-pronged. First, the system experiences a physical change in dynamic response where
individual blades can experience a localization of vibration energy causing forced response
levels greater than predicted in a tuned, cyclic symmetry analysis. Second, a computational
ramification is realized since the entire IBR must now be solved in an FEA application, as
opposed to a fundamental sector. This fact has driven the need for physics-based ROMs to
effectively and efficiently predict mistuned response, particularly for Monte Carlo simu-
lations seeking to characterize the full mistuned response distribution.

Significant research has been devoted to nominal based methods that are reviewed
in Chapter 2. However, Beck, et al. [51] and Brown [50] have shown that these nominal
methods can lead to significant prediction errors for even small mistuning. To account
for errors of the frequency-based approaches, geometric mistuning models were developed
to provide higher fidelity predictions (reviewed in Chapter 2). This chapter adds to the
existing body of research by addressing Major Contribution 1 of Section 3.2 on page 33.

This effort focuses on the development of two geometric mistuning ROMs synthe-
sized from an IBR composed of a tuned disk and geometrically perturbed blades that are
measured using a coordinate measurement machine (CMM). The containment of mistuning to the blades naturally partitions the IBR into mistuned blades and a tuned disk. This partitionment is exploited by component reduction techniques such as the Craig-Bampton component mode synthesis method (CB-CMS) [67]. A drawback of this CMS approach is the retention of all DOFs at the component interfaces. For IBRs, as the number interface DOFs and blades increase, the CB-CMS ROM is hardly reduced as the size is dominated by these DOF. Many works have sought to further reduce the CMS matrices by casting the interface DOF into a modal domain consisting of a truncated set of Interface modes [95–100]. The first mistuning method in this work further utilizes Interface modes for model reduction in a cyclic symmetry CB-CMS description. The second formulation follows suit and casts the the CB-CMS constraint and disk fixed-interface normal mode DOF into a modal domain consisting of a truncated set of Ancillary modes. Mistuning of the interface DOFs in the CB-CMS formulation are then projected onto the Interface or Ancillary modes.

This chapter is organized in the following manner: first, a description calculating the component modes and how to re-couple them is given in Sections 6.2 - 6.5; transformation of the excitation force is then described in Section 6.6; the CB-CMS ROM is then briefly described in Section 6.7; a description for calculating the Interface and Ancillary modes in Section 6.8. These modes are then shown how to reduce the CB-CMS system matrices for an even small ROM in Section 6.9. The results of each of the mistuning models is given in Section 6.10. Findings are then summarized in Section 6.11.

6.2 Disk Formulation

Calculation of the disk cyclic CB-CMS modes uses the the cyclic constraints and CB-CMS formulations of Section 4.2 on page 40 and Chapter 5, respectively. To calculate the fixed-interface normal modes, \( \tilde{\phi}^{(h)}_{ij} \), for a certain harmonic, \( h \), are obtained by restraining all
boundary DOF and solving the classical eigen-problem

\[
\begin{bmatrix}
D\tilde{K}_T^{(h)} - \lambda_j D\tilde{M}_T^{(h)}
\end{bmatrix}
D\tilde{\phi}_j^{(h)} = 0 \quad j = 1, 2, \ldots, D\tilde{N}_T.
\tag{6.1}
\]

where \(\lambda_j = \omega_j^2\) and \(\omega_j\) are the natural frequencies and \(D\tilde{N}_T\) are the total number of interior DOF. Usually, there is a frequency spectrum of interest that limits the required upper range of \(j\) to some cutoff, \(D\tilde{k}_n\), where \(D\tilde{k}_n \ll D\tilde{N}_T\). This subset of modes is then combined into the matrix \(D\tilde{\Phi}^{(h)}\) while the corresponding \(\lambda_j\) are combined into the diagonal spectral matrix \(D\Lambda\). The cyclic constraint modes, \(D\tilde{\Psi}^{(h)}\), are ascertained by statically deforming a component with a unit displacement to one coordinate of an established set of boundary coordinates while the remaining coordinates of the set are restrained, and the remaining interior DOF of the component are force-free. This is calculated by solving the first block of equations in the static problem with imposed unit deflections, \(I\), and reaction forces, \(R_T\).

\[
\begin{bmatrix}
D\tilde{K}_T^{(h)} & D\tilde{K}_{rT}^{(h)} \\
D\tilde{K}_{rT}^{(h)} & D\tilde{K}_{T\Gamma}^{(h)}
\end{bmatrix}
\begin{bmatrix}
D\tilde{\psi}^{(h)} \\
I
\end{bmatrix}
= \begin{bmatrix}
0 \\
R_T^{(h)}
\end{bmatrix}
\tag{6.2}
\]

which gives

\[
D\tilde{\psi}^{(h)} = -D\tilde{K}_{T\Gamma}^{(h)} -1 D\tilde{K}_T^{(h)}
\tag{6.3}
\]

The transformation from cyclic coordinates, \(D\tilde{x}^{(h)}\), to cyclic CB modal coordinates, \(D\tilde{p}^{(h)}\), is accomplished with the cyclic CB modal transformation matrix

\[
D\tilde{x}^{(h)} = \begin{bmatrix}
D\tilde{\Phi}^{(h)} & D\tilde{\Psi}^{(h)} \\
0 & I
\end{bmatrix}
\begin{bmatrix}
D\tilde{p}^{(h)}_n \\
D\tilde{p}^{(h)}_c
\end{bmatrix}
= D\tilde{U}^{(h)} D\tilde{p}^{(h)}
\tag{6.4}
\]

where the right subscript \(n\) denotes motion due to the fixed-interface normal modes and \(c\) is denotes motion due to the constraint modes. The CB modal transformation matrix can
be expanded for all harmonics (∀h) according to

\[ D\tilde{U} = \begin{bmatrix} B \forall h \left[D\tilde{\Phi}(h)\right] & B \forall h \left[D\tilde{\Psi}(h)\right] \\ 0 & I \end{bmatrix} \] (6.5)

Here \( B \) is the block diagonal operator that places the \( h^{th} \) argument in the \( h^{th} \) block on the diagonal, where the blocks are not required to be of equal size. Pre- and post-multiplying cyclic coordinate quantities by \( D\tilde{U}^\top \) and \( D\tilde{U} \), respectively, give the corresponding quantities in cyclic CB modal coordinates where

\[ D\tilde{\mathcal{M}} D\tilde{\bar{p}} + D\tilde{\mathcal{K}} D\tilde{\bar{p}} = 0 \] (6.6)

where

\[ D\tilde{\mathcal{M}} = \begin{bmatrix} I & D\tilde{\mathcal{M}}_{nc} \\ D\tilde{\mathcal{M}}_{nc}^\top & D\tilde{\mathcal{M}}_{cc} \end{bmatrix}, \quad D\tilde{\mathcal{K}} = \begin{bmatrix} D\Lambda & 0 \\ 0 & D\tilde{\mathcal{K}}_{cc} \end{bmatrix} \] (6.7)

with submatrices given by

\[ D\tilde{\mathcal{M}}_{nc} = B \forall h \left[D\tilde{\Phi}(h) \left(D\tilde{\mathcal{M}}_{\tau\tau}^{(h)} D\tilde{\Psi}(h) + D\tilde{\mathcal{M}}_{\tau\Gamma}^{(h)}\right)\right] \]

\[ D\tilde{\mathcal{M}}_{cc} = B \forall h \left[D\tilde{\Psi}(h) \left(D\tilde{\mathcal{M}}_{\tau\tau}^{(h)} D\tilde{\Psi}(h) + D\tilde{\mathcal{M}}_{\tau\Gamma}^{(h)}\right) + D\tilde{\mathcal{M}}_{\tau\Gamma}^{(h)} D\tilde{\Psi}(h) + D\tilde{\mathcal{M}}_{\Gamma\Gamma}^{(h)}\right] \] (6.8)

\[ D\tilde{\mathcal{K}}_{cc} = B \forall h \left[D\tilde{\Psi}(h) \left(D\tilde{K}_{\Gamma\Gamma}^{(h)} + D\tilde{K}_{\Gamma\Gamma}^{(h)} D\tilde{\Psi}(h)\right)\right] \]

\[ D\Lambda = B \forall h \left[D\Lambda^{(h)}\right] \]
The corresponding cyclic CB-CMS modal displacement vector is then given by

\[
\tilde{D}^{(\forall h)} = \begin{cases} 
D^{(0)}_p & \cdots \\
D^{(N/2)}_p & \cdots \\
D^{(0)}_c & \cdots \\
D^{(N/2)}_c 
\end{cases}
\]  

(6.9)

### 6.3 Blade Component Matrices

The CB-CMS formulation for the blades is simpler than the disk since the component FEM matrices do not need to be reduced to a cyclic format. This allows the CB formulation to begin from the partitioned matrices

\[
A_K = \begin{bmatrix} 
A_{K\tau\tau} & A_{K\tau\Gamma} \\
A_{K\tau\Gamma} & A_{K\Gamma\Gamma} 
\end{bmatrix}, \quad \begin{bmatrix} 
A_{X\tau} \\
A_{X\Gamma} 
\end{bmatrix}
\]  

(6.10)

Again, the mass matrix follows the same partitionment. The fixed-interface normal modes are computed from the eigen-problem for the interior, \(\tau\), DOF

\[
\left[ A_{K\tau\tau} - \lambda_j A_{M\tau\tau} \right] A_{\Phi_j} = 0 \quad j = 1, 2, \ldots, A_{N_{\tau}}
\]  

(6.11)

where \(\lambda_j = \omega_j^2\) and \(\omega_j\) are the cantilevered blade natural frequencies and \(A_{N_{\tau}}\) are the number of interior DOF per blade. Again, a frequency range of interest limits the required upper range of \(j\) to some cutoff, \(A_{k_{ni}}\), where \(A_{k_{ni}} \ll A_{N_{\tau}}\). This subset of modes is then combined into the matrix \(A_{\Phi}\) and the corresponding \(\lambda_j\) are placed in the diagonal spectral
matrix $^A\Lambda$. The constraint modes, $^A\Psi$, are then calculated by

$$^A\Psi = -^A\Lambda^{-1}^A\Gamma$$  \hfill (6.12)

The transformation from physical coordinates, $^A\mathbf{x}$, to CB modal coordinates, $^A\mathbf{p}$, is accomplished with the cyclic CB modal transformation matrix

$$^A\mathbf{x} = \begin{bmatrix} ^A\Phi & ^A\Psi \\ 0 & I \end{bmatrix} \begin{bmatrix} ^A\mathbf{p}_n \\ ^A\mathbf{p}_c \end{bmatrix} = ^A\mathbf{U}^A\mathbf{p}$$  \hfill (6.13)

where the right subscript $n$ denotes motion due to the fixed-interface normal modes and $c$ is denotes motion due to the constraint modes. Pre- and post-multiplying blade component matrices by $^A\mathbf{U}^\top$ and $^A\mathbf{U}$, respectively, results in an EOM in CB modal coordinates

$$^A\mathbf{M}^A\mathbf{\ddot{p}} + ^A\mathbf{K}^A\mathbf{p} = 0$$  \hfill (6.14)

where

$$^A\mathbf{M}^{(a)} = \begin{bmatrix} I & ^A\mathbf{M}_{nc} \\ ^A\mathbf{M}_{nc}^\top & ^A\mathbf{M}_{cc} \end{bmatrix}, \quad ^A\mathbf{K}^{(a)} = \begin{bmatrix} ^A\Lambda & 0 \\ 0 & ^A\mathbf{K}_{cc} \end{bmatrix}$$  \hfill (6.15)

with the superscript $a$ serving as a reminder that these matrices are for a single blade, $a = 1, \ldots, N$ and

$$^A\mathbf{M}_{nc}^{(a)} = ^A\Phi^\top (^A\mathbf{M}_{\tau\tau}^A\Psi + ^A\mathbf{M}_{\tau\Gamma})$$

$$^A\mathbf{M}_{cc}^{(a)} = ^A\Psi^\top (^A\mathbf{M}_{\tau\tau}^A\Psi + ^A\mathbf{M}_{\tau\Gamma}) + ^A\mathbf{M}_{\tau\Gamma}^\top ^A\Psi + ^A\mathbf{M}_{\Gamma\Gamma}$$

$$^A\mathbf{K}_{cc}^{(a)} = ^A\mathbf{K}_{\Gamma\Gamma} + ^A\mathbf{K}_{\tau\Gamma}^\top ^A\Psi$$

(6.16)

All the previous calculations in this subsection have been for a single blade. If all
blades are tuned, these calculations would only have to be done once since the component matrices would be the same for all \( N \) blades because

\[
A_U^{(\text{Tuned})} = A_U^{(1)} = A_U^{(2)} = \ldots = A_U^{(N)} \tag{6.17}
\]

The CB-CMS reduced mass and stiffness matrices containing all \( N \) blades are generated using a single blade, e.g. \( a = 1 \)

\[
\begin{bmatrix}
I & I \otimes A_M^{(1)} \\
I \otimes A_M^{(1)} & I \otimes A_M^{(1c)}
\end{bmatrix},
\begin{bmatrix}
I \otimes A_L^{(1)} & 0 \\
0 & I \otimes A_K^{(1c)}
\end{bmatrix} \tag{6.18}
\]

where the symbol \( \otimes \) is the Kronecker product. The corresponding CB-CMS modal displacement vector is given by

\[
A_p^{(\forall a)} = \begin{bmatrix}
A_p^{(1)} \\
\vdots \\
A_p^{(N)}
\end{bmatrix},
A_p^{(\forall a)} = \begin{bmatrix}
A_p^{(1)} \\
\vdots \\
A_p^{(N)}
\end{bmatrix} \tag{6.19}
\]

However, geometric mistuning perturbs both the mass and stiffness matrices of each blade by varying amounts. For example, the mistuned stiffness matrix of Eq. 6.10 can be represented by

\[
A_K = tK + \Delta K \tag{6.20}
\]

where \( tK \) represents the tuned blade stiffness matrix and \( \Delta K \) is a perturbation matrix of
full rank with small deviations. The mistuned blade mass matrix follows suit. This type of mistuning is referred to as large rank, small mistuning. Consequently, the CB-CMS component matrices must be recalculated for all \( N \) blades since

\[
A_U^{(\text{Tuned})} \neq A_U^{(1)} \neq A_U^{(2)} \neq \ldots \neq A_U^{(N)}
\]  

(6.21)

Considering that the blade matrices are mostly sparse, and that these calculations are done blade-by-blade, this additional computation is small compared to solving the full IBR FEM or even traditional CB-CMS ROMs. Furthermore, if probabilistic studies are required, a small population of blades can be generated and bootstrapping methods can be used to eliminate calculation of a larger population of blades. The CB-CMS reduced mass and stiffness matrices of Eq. 6.14 for each mistuned blade are then combined into the block diagonal matrices containing all blades

\[
A_M = \begin{bmatrix}
I & B_{\forall a}\begin{bmatrix}A_{M_{nc}}^{(a)} \end{bmatrix} \\
B_{\forall a}\begin{bmatrix}A_{M_{nc}}^{(a)} \end{bmatrix} & B_{\forall a}\begin{bmatrix}A_{M_{cc}}^{(a)} \end{bmatrix}
\end{bmatrix},
A_K = \begin{bmatrix}
B_{\forall a}\begin{bmatrix}A_{\Lambda(a)}^{(a)} \end{bmatrix} & 0 \\
0 & B_{\forall a}\begin{bmatrix}A_{K_{cc}}^{(a)} \end{bmatrix}
\end{bmatrix}
\]  

(6.22)

where the \( a^{th} \) block on the diagonal corresponds to the \( a^{th} \) blade for \( a = 1, \ldots, N \). The displacement vector follows that of Eq. 6.19.

### 6.4 Component Coupling for Tuned Blades

CB-CMS model assembly requires interface displacement compatibility \( A_{x_{\Gamma}}^{(\forall a)} = D_{x_{\Gamma}} \) in physical coordinates, where \( D_{x_{\Gamma}} \) is now the displacement vector for all \( N \) disk-blade
interfaces. Expanding this and utilizing the CB-CMS requirement that $x_{\Gamma} = p_c$ yields

$$A_{x_{\Gamma}^{(\gamma a)}} = \left\{ \begin{array}{c} A_{x_{\Gamma}^{(1)}} \\ \vdots \\ A_{x_{\Gamma}^{(N)}} \end{array} \right\} = \left\{ \begin{array}{c} A_{p_c^{(1)}} \\ \vdots \\ A_{p_c^{(N)}} \end{array} \right\} = A_{x_{\Gamma}^{(\gamma a)}} = (E \otimes I) D_{p_c^{(\gamma h)}} = \dot{E} D_{p_c^{(\gamma h)}} = D_{p_c} = D_{x_{\Gamma}}$$

(6.23)

where $E$ is the $N \times N$ real-valued Fourier matrix defined in Appendix B and $I$ is the Identity matrix of size $N_{\Gamma} \times N_{\Gamma}$, where $N_{\Gamma}$ is the length of $A_{x_{\Gamma}^{(a)}}$. To constrain the blades to the disk, $D_{p_c^{(\gamma h)}}$ are kept as active DOF when the IBR has tuned blades:

$$\left\{ \begin{array}{c} D_{p_n^{(\gamma h)}} \\ D_{p_c^{(\gamma h)}} \\ A_{p_n^{(\gamma a)}} \\ A_{p_c^{(\gamma a)}} \end{array} \right\} = \left[ \begin{array}{cccc} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \\ 0 & \hat{E} & 0 \end{array} \right] \left\{ \begin{array}{c} D_{p_n^{(\gamma h)}} \\ D_{p_c^{(\gamma h)}} \\ A_{p_n^{(\gamma a)}} \\ A_{p_c^{(\gamma a)}} \end{array} \right\} = T_{CB} p_{CB}$$

(6.24)

The synthesized mass and stiffness matrices for tuned blades can now be coupled with the disk by $T_{CB}$

$$C_{\hat{M}} = T_{CB}^{t} \left[ \begin{array}{cc} D_{\hat{M}} & 0 \\ 0 & A_M \end{array} \right] T_{CB} = \left[ \begin{array}{ccc} I & D_{\hat{M}_{nc}} & 0 \\ D_{\hat{M}_{nc}}^{t} & \hat{M}_{cc} & \dot{E}^{t} (I \otimes A_{M_{nc}}^{t}) \\ 0 & (I \otimes A_{M_{nc}}) \hat{E} & I \end{array} \right]$$

(6.25)

where the constraint mode partition is given by

$$\hat{M}_{cc} = D_{\hat{M}_{cc}} + I \otimes A_{M_{cc}}$$

(6.26)
and the stiffness matrix is

\[ CB \tilde{K} = T_{CB}^T \begin{bmatrix} D \tilde{K} & 0 \\ 0 & A \tilde{K} \end{bmatrix} T_{CB} = \begin{bmatrix} D \Lambda & 0 & 0 \\ 0 & \tilde{K}_{cc} & 0 \\ 0 & 0 & I \otimes A \Lambda \end{bmatrix} \] (6.27)

where the constraint mode partition is given by

\[ \tilde{K}_{cc} = D \tilde{K}_{cc} + I \otimes A \Lambda \] (6.28)

The choice to retain \( D \tilde{p}_{c}^{(\forall h)} \) as the active DOF in the constraints of Eq. 6.24 results in the desired effect of \( \tilde{M}_{cc} \) and \( \tilde{K}_{cc} \) being block-diagonal, with the blocks belonging to the decoupled harmonics. This block diagonal structure will be exploited in Section 6.8.

### 6.5 Component Coupling for Geometrically Mistuned Blades

CB-CMS model assembly for geometrically mistuned blades begins with the interface compatibility requirements defined in Eq. 6.23. To constrain the mistuned blades to the disks, the physical space \( D \tilde{p}_{c} \) are kept as active DOF, as opposed to the cyclic coordinates used in the previous tuned case of Eq. 6.24:

\[
\begin{bmatrix}
D \tilde{p}_{c}^{(\forall h)} \\
D \tilde{p}_{c}^{(\forall h)} \\
A \tilde{p}_{n}^{(\forall a)} \\
A \tilde{p}_{c}^{(\forall a)}
\end{bmatrix} = \begin{bmatrix} I & 0 & 0 \\ 0 & \hat{E}^T & 0 \\ 0 & 0 & I \\ 0 & I & 0 \end{bmatrix} \begin{bmatrix} D \tilde{p}_{n}^{(\forall h)} \\ D \tilde{p}_{c} \\ A \tilde{p}_{n}^{(\forall a)} \\ A \tilde{p}_{c}^{(\forall a)} \end{bmatrix} = T_{CB} \tilde{p}_{CB} \] (6.29)
The synthesized mass and stiffness matrices for mistuned blades can now be coupled to the disk by $T_{CB}$

$$
CB\mathcal{M} = T_{CB}^T \begin{bmatrix} D\tilde{\mathcal{M}} & 0 \\ 0 & A\mathcal{M} \end{bmatrix} T_{CB} = \begin{bmatrix} I & D\tilde{\mathcal{M}}_{nc} \hat{E}^T \\ \hat{E} D\tilde{\mathcal{M}}_{nc}^T & M_{cc} \mathbb{B} \left[ A\mathcal{M}_{nc}^{(a)} \right]^T \end{bmatrix}
$$

(6.30)

where the constraint mode partition is given by

$$
\mathcal{M}_{cc} = \hat{E} D\tilde{\mathcal{M}}_{cc} \hat{E}^T + \mathbb{B} \left[ A\mathcal{M}_{cc}^{(a)} \right]
$$

(6.31)

and the transformed stiffness matrix is

$$
CB\mathcal{K} = T_{CB}^T \begin{bmatrix} D\tilde{\mathcal{K}} & 0 \\ 0 & A\mathcal{K} \end{bmatrix} T_{CB} = \begin{bmatrix} D\Lambda & 0 & 0 \\ 0 & \mathcal{K}_{cc} & 0 \\ 0 & 0 & \mathbb{B} \left[ A\Lambda^{(a)} \right] \end{bmatrix}
$$

(6.32)

where the constraint mode partition is given by

$$
\mathcal{K}_{cc} = \hat{E} D\tilde{\mathcal{K}}_{cc} \hat{E}^T + \mathbb{B} \left[ A\mathcal{K}_{cc}^{(a)} \right]
$$

(6.33)

The choice to retain $^Dp_c$ as the active DOF in the constraints of Eq. 6.29 transforms the mistuned $\mathcal{M}_{cc}$ and $\mathcal{K}_{cc}$ of Eqs. 6.31 and 6.33, respectively, into physical space. This is required in order to add the disk and mistuned blade components in the CB-CMS modal space.
6.6 Excitation Force

This section derives the CB-CMS modal forces for both the tuned and mistuned blade configurations. Both tuned and mistuned IBRs are subject to the same external forcing conditions outlined in Section 4.3 on page 45.

6.6.1 Tuned CB-CMS Modal Force

The CB-CMS modal force for tuned blade components is obtained by projecting the tuned blade modes of Eq. 6.13 onto the forcing vector of Eq. 4.26 on page 50 by

\[
A\mathcal{F} = \begin{bmatrix} A\mathcal{F}_n \\ A\mathcal{F}_c \end{bmatrix} = \begin{bmatrix} I \otimes A\Phi^\top & 0 \\ I \otimes A\Psi^\top & I \end{bmatrix} A\mathcal{V} = \begin{Bmatrix} P_C \otimes A\Phi^\top A\mathbf{f}_r \\ P_C \otimes (A\Psi^\top A\mathbf{f}_r + A\mathbf{f}_\Gamma) \end{Bmatrix}
\]

(6.34)

where \( P_C \) is given in Eq. 4.25 on page 48. Enforcing the constraints of Eq. 6.24 results in the modal force vector for the CB-CMS reduced system

\[
^{CB}\mathcal{F} = T_{CB}^\top \begin{Bmatrix} 0 \\ 0 \\ A\mathcal{F}_n \\ A\mathcal{F}_c \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ ^{CB}\mathcal{F}_c \\ A\mathcal{F}_n \end{Bmatrix}
\]

(6.35)
where

\[
\tilde{F}_c = \begin{bmatrix}
0 \\
\vdots \\
0
\end{bmatrix} + \begin{bmatrix}
\hat{e}^{(h=C,c)^T} \left[ P_C \otimes (A \Psi^T A f_r + A f_r) \right] \\
\hat{e}^{(h=C,s)^T} \left[ P_C \otimes (A \Psi^T A f_r + A f_r) \right] \\
\vdots \\
0
\end{bmatrix}
\] (6.36)

where \( e^{(C,c)} \) and \( e^{(C,s)} \) are the column vectors of the real-valued Fourier matrix, \( E \), defined in Appendix B on page 236 corresponding to the cosine and sine terms

\[
E = \begin{bmatrix}
e^{(0)} & e^{(1,c)} & e^{(1,s)} & \ldots & e^{(h,c)} & e^{(h,s)} & \ldots & e^{(N/2)}
\end{bmatrix}
\] (6.37)

Note that orthogonality between \( \hat{E} \) and \( P_C \) leaves any rows in \( \tilde{F}_c \) zero where \( C \neq h \).

### 6.6.2 Mistuned CB-CMS Modal Force

The CB-CMS modal force for mistuned blade components is obtained by projecting the force vector of Eq. 4.26 on page 50 onto the mistuned blade modes by

\[
A \mathcal{F} = \begin{bmatrix}
A \mathcal{F}_n \\
A \mathcal{F}_c
\end{bmatrix} = \begin{bmatrix}
\mathbb{B}_{\forall a} \left[ A \Phi(a) \right]^T \quad 0 \\
\mathbb{B}_{\forall a} \left[ A \Psi(a) \right]^T \quad I
\end{bmatrix} A \mathbb{V} = \ldots
\]

\[
= \begin{bmatrix}
\mathbb{B}_{\forall a} \left[ A \Phi(a) \right]^T \left( P_C \otimes A f_r \right) \\
\mathbb{B}_{\forall a} \left[ A \Psi(a) \right]^T \left( P_C \otimes A f_r \right) + (P_C \otimes A f_r)
\end{bmatrix}
\] (6.38)
Imposing the constraints of Eq. 6.29 results in the modal force vector for the CB-CMS reduced system

\[
CB \mathbf{F} = T_{CB}^\top \begin{bmatrix}
0 \\
0 \\
A \mathbf{F}_n \\
A \mathbf{F}_c
\end{bmatrix} = \begin{bmatrix}
0 \\
\mathbf{F}_c \\
A \mathbf{F}_n
\end{bmatrix} = \cdots
\]

\[
= \begin{bmatrix}
B \left[ A \mathbf{\Phi}^{(a)} \right]^\top \left( \mathbf{P}_C \otimes A \mathbf{f}_r \right) + \left( \mathbf{P}_C \otimes A \mathbf{f}_r \Gamma \right) \\
B \left[ A \mathbf{\Phi}^{(a)} \right]^\top \left( \mathbf{P}_C \otimes A \mathbf{f}_r \right)
\end{bmatrix}
\]

(6.39)

6.7 CB-CMS Equations of Motion

In the previous sections a cyclic sector disk description was used that required two different component coupling mechanisms for tuned and mistuned blades. As a result, two CB-CMS EOM are developed: one with tuned blade modes and the second with mistuned modes. Furthermore, the first has constraint DOF in cyclic coordinates while the mistuned approach these DOF in physical coordinates. In each approach, a generic EOM neglecting any description of cyclic or physical space notations can be described by

\[
CB \mathbf{M} \ddot{\mathbf{p}} + CB \mathbf{C} \dot{\mathbf{p}} + (1 + G) CB \mathbf{K} \mathbf{p} = CB \mathbf{F}
\]

(6.40)

where the blade modal damping matrix and structural damping coefficient, \( CB \mathbf{C} \) and \( G \), respectively, are included to better model IBR dynamic response [39]. The mass, stiffness, and forcing coefficients to the above generic EOM for the tuned case are given by Equations
The tuned blade modal damping matrix is given by

\[
CBC = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & I \otimes \text{diag}(2\zeta_j) \sqrt{\Lambda}
\end{bmatrix}
\] (6.41)

where \(\zeta_j\) is the damping coefficient for the \(j^{th}\) cantilevered blade mode from Eq. 6.11. The mass, stiffness, and forcing coefficients for the mistuned case are given by Equations 6.30, 6.32, and 6.39. The mistuned blade modal damping matrix is given by

\[
CBC = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & B \forall a \left[ \text{diag}\left(2\zeta_j^{(a)}\right) \sqrt{\Lambda^{(a)}} \right]
\end{bmatrix}
\] (6.42)

where \(\zeta_j^{(a)}\) is the damping coefficient for the \(j^{th}\) cantilevered blade mode from Eq. 6.11 for blade \(a\). In both methods, the solution of Eq. 6.40 is dominated by the retention of all the interface DOF. The next section describes methods to further reduce the CB-CMS model size.

### 6.8 Secondary Modal Analysis

The CB-CMS methodology requires retention of all interface DOF of the ROM. For IBRs with large \(N\) and large \(N_G\), the ROM will be dominated by the \(N \cdot N_G\) interface DOF that prevent an ultimate reduction in model size. To reduce this burden, a secondary eigen-analysis can be performed on portions of the CB-CMS system matrices that will re-cast these portions into a new modal domain. Two approaches will be performed in this work. First, the secondary eigen-analysis will be carried out on the constraint DOF partitions of the CB-CMS system matrices. The resulting truncated set of eigenvectors will yield a set
of Interface modes, referred to as Characteristic Constraint modes in [100]. The second approach will perform an eigen-analysis on the constraint and disk fixed-interface normal mode DOF partitions. The resulting truncated set of eigenvectors are termed Ancillary modes. In each approach, there are a set of tuned and mistuned CB-CMS system matrices from which they are derived. The tuned approach offers computational savings by carrying out the analysis in cyclic coordinates which allows each harmonic index to be solved independently. The mistuned matrices require the calculation to be carried out in a non-cyclic, CB-CMS modal space.

6.8.1 Tuned Interface Modes

From Equations 6.26 and 6.28 it is seen that the constraint portions \( \tilde{M}_{cc} \) and \( \tilde{K}_{cc} \) for tuned blades are block diagonal, where each block corresponds to the constraint DOF symmetrical components at a specific harmonic. Computation of the Interface modes can be therefore be done one harmonic at a time by

\[
\begin{bmatrix}
\tilde{K}_{cc}^{(h)} - \omega_j^2 \tilde{M}_{cc}^{(h)}
\end{bmatrix}
\tilde{\phi}_j^{(h)} = 0 \quad j = 1, \ldots, N_T
\]  

(6.43)

Modal truncation can be used to limit the required upper range of \( j \) to some cutoff, \( k_{cc} \), where \( k_{cc} \ll N_T \), and the left superscript \( t \) denotes a tuned cutoff value. This subset of modes are then combined into the matrix \( \tilde{\Phi}^{(h)} = \begin{bmatrix} \tilde{\phi}_1, \ldots, \tilde{\phi}_k \end{bmatrix} \). Each \( \tilde{\Phi}^{(h)} \) for all harmonics are then assembled into the block diagonal modal matrix

\[
\tilde{\Phi}_{cc} = \mathbb{B} \left[ \begin{array}{c}
\tilde{\Phi}^{(h)}_{cc}
\end{array} \right]
\]  

(6.44)
6.8.2 Mistuned Interface Modes

The mistuned constraint partitions from Equations 6.31 and 6.33 do not have cyclic symmetry properties. Consequently, the Interface modes will be calculated for the full physical space by

\[
[K_{cc} - \omega_j^2 M_{cc}] \phi_j = 0 \quad j = 1, \ldots, N \cdot N_t. \tag{6.45}
\]

Again, a frequency range of interest can limit the required upper range of \( j \) to some cutoff, \( m_{kcc} \), where \( m_{kcc} \ll N \cdot N_t \) and the superscript \( m \) denotes the mistuned cutoff value. This subset of modes is then combined into the Interface modal matrix

\[
\Phi_{cc} = [\phi_1, \ldots, \phi_k]. \tag{6.46}
\]

6.8.3 Tuned Ancillary Modes

From Eq. 6.25 and 6.27 it is seen that the constraint and disk fixed-interface normal mode portions are block diagonal for tuned blades, where each block corresponds to a specific harmonic. Computation of the Ancillary modes can be done one harmonic at a time once the DOF are organized by harmonic with the following boolean matrix

\[
D\tilde{p}^{(\forall h)} = \begin{bmatrix}
B_{\forall h} & \left[T_{s1}^{(h)} \right] \\
B_{\forall h} & \left[T_{s2}^{(h)} \right]
\end{bmatrix}
\begin{bmatrix}
\tilde{d}\tilde{p}^{(\forall h)} \\
\tilde{d}s\tilde{p}^{(\forall h)}
\end{bmatrix} = T_s d\tilde{p}^{(\forall h)} \tag{6.47}
\]
where $D \mathbf{p}^{(v/h)}$ is from Eq. 6.9 and

$$
d \mathbf{p}^{(v/h)} = \begin{cases}
\begin{bmatrix} d_{\mathbf{p}}^{(0)} \\
\vdots \\
d_{\mathbf{p}}^{(N/2)}
\end{bmatrix}, & \text{if } h = 0, N/2 \\
\begin{bmatrix} d_{\mathbf{p}}^{(0)} \\
\vdots \\
d_{\mathbf{p}}^{(N/2)}
\end{bmatrix}, & \text{if } h \neq 0, N/2
\end{cases}
$$  \hspace{1cm} (6.48)

and

$$
T_s^{(h)} = \begin{cases}
\begin{bmatrix} I_{D_k \times D_k} & 0_{D_k \times N} \\
0 & I_{N \times N}
\end{bmatrix}, & \text{if } h = 0, N/2 \\
\begin{bmatrix} I_{2 \times D_k} & 0_{2 \times N} \\
0 & I_{2 \times N}
\end{bmatrix}, & \text{if } h \neq 0, N/2
\end{cases}
$$  \hspace{1cm} (6.49)

$$
T_s^{(h)} = \begin{cases}
\begin{bmatrix} 0 & I_N \\
0 & 0
\end{bmatrix}, & \text{if } h = 0, N/2 \\
\begin{bmatrix} 0 & I_N \\
0 & 0
\end{bmatrix}, & \text{if } h \neq 0, N/2
\end{cases}
$$  \hspace{1cm} (6.50)

The boolean matrix, $T_s$, can then be used to reorder the constraint and fixed-interface normal mode disk portions of the tuned CB-CMS mass and stiffness matrices by

$$
\widetilde{\mathcal{M}}_{ss} = \mathbb{B}_{\forall h} \left[ \widetilde{\mathcal{M}}^{(h)}_{ss} \right] = T_s^\top \begin{bmatrix} I & D \widetilde{\mathcal{M}}_{nc} \\
D \widetilde{\mathcal{M}}_{nc}^\top & \widetilde{\mathcal{M}}_{cc}
\end{bmatrix} T_s
$$  \hspace{1cm} (6.51)

$$
\tilde{\mathcal{K}}_{ss} = \mathbb{B}_{\forall h} \left[ \tilde{\mathcal{K}}^{(h)}_{ss} \right] = T_s^\top \begin{bmatrix} D \Lambda & 0 \\
0 & \tilde{\mathcal{K}}_{cc}
\end{bmatrix} T_s
$$  \hspace{1cm} (6.52)
where

\[
\tilde{\mathcal{M}}_{ss}^{(h)} = \begin{bmatrix}
I & D\tilde{\mathcal{M}}_{hc}^{(h)} \\
D\tilde{\mathcal{M}}_{hc}^{(h)} & \tilde{\mathcal{M}}_{cc}^{(h)}
\end{bmatrix}
\]

(6.53)

\[
\tilde{\mathcal{K}}_{ss}^{(h)} = \begin{bmatrix}
d\Lambda^{(h)} & 0 \\
0 & \tilde{\mathcal{K}}_{cc}^{(h)}
\end{bmatrix}
\]

(6.54)

The tuned Ancillary modes are then computed one harmonic at a time by

\[
\left[\tilde{\mathcal{K}}_{ss}^{(h)} - \omega_j^2 \tilde{\mathcal{M}}_{ss}^{(h)} \right] \tilde{\phi}_j^{(h)} = 0 \quad j = 1, \ldots, N_s
\]

(6.55)

where \(N_s = Dk_n + N_\Gamma\). Again, a frequency range of interest limits the required upper range of \(j\) to some cutoff, \(h_{ss}\), where \(h_{ss} \ll N_s\). This subset of modes are then combined into the matrix \(\tilde{\Phi}_{ss} = [\tilde{\phi}_1, \ldots, \tilde{\phi}_k]\). Each \(\tilde{\Phi}_{ss}^{(h)}\) for all harmonics are then assembled into the block diagonal modal matrix

\[
\tilde{\Phi}_{ss} = B_{\forall h} \left[ \tilde{\Phi}_{ss}^{(h)} \right]
\]

(6.56)

The ordering of DOF in \(\Phi_{ss}\) are currently organized according to Eq. 6.48, however, Eq. 6.47 can be used to reorder this vector back to the partitionment of Eq. 6.9 resulting in

\[
\tilde{\Phi}_{ss} = \left\{ \begin{array}{c}
\tilde{\Phi}_{ss,n}^{(\forall h)} \\
\tilde{\Phi}_{ss,c}^{(\forall h)}
\end{array} \right\}
\]

(6.57)

### 6.8.4 Mistuned Ancillary Modes

The mistuned constraint partitions from Equations 6.30 and 6.32 do not have cyclic symmetry properties. Consequently, the Ancillary modes must be calculated for the larger eigen-problem

\[
[K_{ss} - \omega_j^2 M_{ss}] \phi_j = 0 \quad j = 1, \ldots, N_s \cdot N_s
\]

(6.58)
where
\[ M_{ss} = \begin{bmatrix} I & D \tilde{M}_{nc} \tilde{E}^T \\ \dot{E} D \tilde{M}_{nc} \tilde{M}_{cc} \end{bmatrix}, \quad K_{ss} = \begin{bmatrix} D_A & 0 \\ 0 & K_{cc} \end{bmatrix} \] (6.59)

are the disk normal mode and constraint partitions of Equations 6.30 and 6.32 and
\[ p_s = \begin{Bmatrix} D_{\tilde{n}}(\forall h) \\ p_n \\ D_p_c \end{Bmatrix} \] (6.60)

The cutoff limit, \( m_{k_{ss}} \), is used to limit the upper range of \( j \), where \( m_{k_{ss}} \ll N \cdot N_s \). This subset of modes is then combined into the characteristic constraint modal matrix
\[ \Phi_{ss} = [\phi_1, \ldots, \phi_k] \] (6.61)

6.9 Mistuning Models

In the two subsections that follow, two mistuning approaches are presented that use the tuned and mistuned Interface and Ancillary modes described in the previous section to further reduce the CB-CMS ROM size of Section 6.7. Section 6.9.1 presents the traditional, industry standard tuned blade mode approximation with a tuned Interface mode reduction. Then, Section 6.9.2 presents four mistuning models utilizing mistuned blade modes with both tuned and mistuned Interface and Ancillary mode reductions. In all approaches, the generic EOM is given by
\[ M_r \ddot{q}_r + C_r \dot{q}_r + (1 + G_i) K_r q_r = F_r \] (6.62)

where subscript \( r \) refers to reduced and the EOM matrices are to be defined in the following subsections.
6.9.1 Traditional Tuned Mode Approximation with Tuned CC-Modes

This method uses a traditional tuned blade mode approximation with perturbations in cantilevered blade frequencies, or modal stiffnesses, to represent non-proportional mistuning resulting from blade geometry and elastic modulus perturbations. This traditional approach requires Eq. 6.17 to hold. The tuned CB-CMS system matrices, $\tilde{M}$, $\tilde{K}$, $\tilde{C}$, and $\tilde{F}$ of Equations 6.25, 6.27, 6.41, and 6.35 are utilized, except the tuned cantilevered blade frequencies in $\tilde{K}$ are replaced with $\tilde{B}_\forall^{A\Lambda(a)}$ of Eq. 6.22. This implies that blade material property and geometry perturbations result only in cantilevered blade frequency deviations and the mistuned IBR response can be estimated by a linear combination of the tuned modes.

The problem size of the CB-CMS system can be further reduced by limiting the number of modes retained in $\tilde{\Phi}_{cc}$ of Eq. 6.44. This reduction is carried out with the following transformation matrix

$$
CB \mathbf{p} = \begin{bmatrix}
D_{\mathbf{p}_n} \\
D_{\mathbf{p}_c} \\
A_{\mathbf{p}_n}
\end{bmatrix} = \begin{bmatrix}
I & 0 & 0 \\
0 & \tilde{\Phi}_{cc} & 0 \\
0 & 0 & I
\end{bmatrix} \begin{bmatrix}
D_{\mathbf{q}_n} \\
D_{\mathbf{q}_c} \\
A_{\mathbf{q}_n}
\end{bmatrix} = T_{CC} \mathbf{q}_r \tag{6.63}
$$

where the coefficients to the reduced EOM in Eq. 6.62 are

$$
\mathcal{M}_r = T_{CC}^T \tilde{M} T_{CC} = \cdots
$$

$$
= \begin{bmatrix}
I & \tilde{\Phi}_{cc}^T \tilde{M} \tilde{M}_{nc} \tilde{\Phi}_{cc} & 0 \\
\tilde{\Phi}_{cc}^T \tilde{M}_{nc} \tilde{\Phi}_{cc} & \tilde{\Phi}_{cc}^T (D_{\tilde{M}_{cc}} + I \otimes A_{\tilde{M}_{cc}}) \tilde{\Phi}_{cc} & \tilde{\Phi}_{cc}^T \tilde{F} (I \otimes A_{\tilde{M}_{nc}}) \\
0 & \left[ (I \otimes A_{\tilde{M}_{nc}}) \tilde{E} \right] \tilde{\Phi}_{cc} & I
\end{bmatrix} \tag{6.64}
$$
\[ \mathbf{K}_r = \mathbf{T}_{CC}^\top \mathbf{K} \tilde{\mathbf{C}} \mathbf{T}_{CC} = \begin{bmatrix} D\Lambda & 0 & 0 \\ 0 & \Phi_{cc}^\top (D\tilde{\mathbf{K}}_{cc} + I \otimes A\mathbf{K}_{cc}) \Phi_{cc} & 0 \\ 0 & 0 & \mathbb{R} \begin{bmatrix} A\Lambda^{(a)} \end{bmatrix} \end{bmatrix} \] (6.65)

\[ \tilde{\mathbf{F}}_r = \mathbf{T}_{CC}^\top \mathbf{K} \tilde{\mathbf{F}} = \begin{bmatrix} 0 \\ \tilde{\mathbf{F}}_{r-c} \\ A\mathbf{F}_n \end{bmatrix} \] (6.66)

where

\[ \tilde{\mathbf{F}}_{r-c} = \begin{bmatrix} \Phi_{cc}^{(c)\top} \left\{ \begin{array}{c} \mathbf{e}_{C,r}^\top \left[ \mathbf{P}_C \otimes (A\Psi \mathbf{A}_f + A\mathbf{f}_r) \right] \\ \mathbf{e}_{C,s}^\top \left[ \mathbf{P}_C \otimes (A\Psi \mathbf{A}_f + A\mathbf{f}_r) \right] \\ 0 \\ \vdots \\ 0 \end{array} \right\} \end{bmatrix} \] (6.67)

Note that \( \mathcal{C}_r \) is obtained with the same transformation as \( \mathcal{M}_r \) and \( \mathcal{K}_r \), but the non-zero components stay the same as listed in Eq. 6.41 and the zero components only change size, so it is not re-listed here.

### 6.9.2 Geometric Mistuning with Secondary Modal Projections

The following approaches are developed specifically for large rank, small deviation mistuning that do not make the same tuned mode assumptions as the traditional tuned mode approach. Beginning with the mistuned CB-CMS system matrices of Equations 6.30, 6.32, 6.42, and 6.39, four mistuning models are developed. The first method projects the mistuned constraint DOF portions of the system CB-CMS mass and stiffness matrices onto
a subset of tuned Interface modes. This has the added benefit of calculating the tuned Interface modes only once. Furthermore, they have the computational advantage of being calculated at decoupled harmonic indices. The second approach uses the mistuned Interface modes in place of the tuned modes for the reduction. This method requires recalculation of the Interface modes for different mistuned IBRs, but it does not use a tuned Interface mode reduction approximation. The third approach is similar to the first, except the method projects the mistuning in the constraint DOF onto the tuned Ancillary modes. This approach also has the added benefit of calculating the tuned Ancillary modes for decoupled harmonics. The fourth method is again similar except mistuned Ancillary modes are calculated and used for the reduction. Each method is described in detail below, with absence of the blade damping matrix of Eq. 6.42 since these reduction techniques only change the size of zero entities while non-zero entities remain the same.

6.9.2.1 Tuned Interface Mode Reduction

The CB-CMS system equations are further reduced by projecting the mistuning in constraint terms of Equations 6.31 and 6.33 onto a truncated set of tuned Interface modes from Eq. 6.44. First, this truncated set of modes is first transformed back to physical DOF by

$$\Phi_{cc} = (E \otimes I_{N_T \times N_T}) \tilde{\Phi}_{cc}$$ (6.68)

This reduction is then carried out with the following transformation matrix

$$CB_p = \begin{bmatrix} D_{p_n} \\ D_{p_c} \\ A_{p_n} \end{bmatrix} = \begin{bmatrix} I & 0 & 0 \\ 0 & \Phi_{cc} & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} D_{q_n} \\ D_{q_c} \\ A_{q_n} \end{bmatrix} = T_{CC} \mathbf{q}_r$$ (6.69)
where the reduced matrices to Eq. 6.62 are given by

\[ \mathcal{M}_r = T_{CC}^T \mathcal{M}_{CC} = \begin{bmatrix}
I & D \tilde{M}_{nc} \hat{E}^T \Phi_{cc} & 0 \\
\Phi_{cc}^T \hat{E} D \tilde{M}_{nc} & \Phi_{cc}^T \mathcal{M}_{cc} \Phi_{cc} & \Phi_{cc}^T \mathbb{B}_{\forall a} \left[ A \mathcal{M}_{nc}^{(a)} \right]^T \\
0 & \mathbb{B}_{\forall a} \left[ A \mathcal{M}_{nc}^{(a)} \right] \Phi_{cc} & I
\end{bmatrix} \tag{6.70} \]

\[ \mathcal{K}_r = T_{CC}^T \mathcal{K}_{CC} = \begin{bmatrix}
D \Lambda & 0 & 0 \\
0 & \Phi_{cc}^T \mathcal{K}_{cc} \Phi_{cc} & 0 \\
0 & 0 & \mathbb{B}_{\forall a} \left[ A \Lambda^{(a)} \right]
\end{bmatrix} \tag{6.71} \]

\[ \mathcal{F}_r = T_{CC}^T \mathcal{F} = \begin{bmatrix}
0 \\
\Phi_{cc}^T \mathcal{F}_{c} \\
A^T \mathcal{F}_{n}
\end{bmatrix} \tag{6.72} \]

### 6.9.2.2 Tuned Ancillary Mode Reduction

The CB-CMS system matrices matrices are further reduced by projecting the mistuning in the constraint terms of \( M_{ss} \) and \( K_{ss} \) of Eq. 6.59 onto a truncated set of tuned Ancillary modes of Eq. 6.56. However, since the constraint partitions are in physical coordinates for mistuned blades, the constraint DOF in the tuned Ancillary mode vector, \( \tilde{\Phi}_{ss} \), must be converted from cyclic to physical DOF by

\[ \Phi_{ss,c} = (E \otimes I_{N_f \times N_f}) \tilde{\Phi}_{ss,c} \tag{6.73} \]

The Ancillary mode vector the becomes

\[ \Phi_{ss} = \begin{bmatrix}
\tilde{\Phi}_{ss,n}^{(\forall h)} \\
\Phi_{ss,c}
\end{bmatrix} \tag{6.74} \]
Model reduction is then carried out with the following transformation matrix

\[
\begin{bmatrix}
D_p \\
A_p
\end{bmatrix} = \begin{bmatrix}
\Phi_{ss} & 0 \\
0 & I
\end{bmatrix} \begin{bmatrix}
D_q \\
A_q
\end{bmatrix} = T_{CA} q_r
\]

(6.75)

where the reduced matrices to Eq. 6.62 are given by

\[
\mathcal{M}_r = T_{CA}^T M_{CA} T_{CA} = \begin{bmatrix}
\mathcal{M}_{ss} & \mathcal{M}_{sn} \\
\mathcal{M}_{sn} & I
\end{bmatrix}
\]

(6.76)

\[
\mathcal{K}_r = T_{CA}^T K_{CA} T_{CA} = \begin{bmatrix}
\tilde{\Phi}_{ss,n}^T D\tilde{\Phi}_{ss,n} + \Phi_{ss,c}^T \mathcal{K}_{cc} \Phi_{ss,c} & 0 \\
0 & \mathbb{B}_{\forall a} [A\Lambda(a)]
\end{bmatrix}
\]

(6.77)

\[
\mathcal{F}_r = T_{CA}^T \mathcal{F}_{CA} = \begin{bmatrix}
0 \\
\Phi_{ss,c}^T \mathcal{F}_c \\
A^T \mathcal{F}_n
\end{bmatrix}
\]

(6.78)

with submatrices given by

\[
\mathcal{M}_{ss} = \tilde{\Phi}_{ss,n}^T D\tilde{\Phi}_{ss,n} + \Phi_{ss,c}^T \hat{E} \tilde{\mathcal{M}}_{nc}^T \tilde{\Phi}_{ss,n} + \tilde{\Phi}_{ss,n}^T D\tilde{\mathcal{M}}_{nc} \hat{E}^T \Phi_{ss,c} + \Phi_{ss,c}^T \mathcal{M}_{cc} \Phi_{ss,c}
\]

(6.79)

\[
\mathcal{M}_{sn} = \Phi_{ss,c}^T \mathbb{B}_{\forall a} [A\Lambda(a)]^T \mathcal{M}_{nc}
\]

(6.80)

### 6.9.2.3 Mistuned Interface and Ancillary Mode Reduction

The previous methods utilizing tuned Interface and Ancillary modes are approximations of the mistuned response. Accuracy can be gained by using the respective mistuned Interface and Ancillary modes from Eqs. 6.46 and 6.61 in place of the tuned modes in the transformation matrix of Eq. 6.69 and Eq. 6.75, respectively. As the number of retained mistuned modes approaches their respective limits, each method approaches the accuracy of the parent CB-CMS system. However, the increase in accuracy comes at the expense
Table 6.1: ROM names and associated EOM

<table>
<thead>
<tr>
<th>ROM Name</th>
<th>Mass</th>
<th>Stiffness</th>
<th>Damping</th>
<th>Force</th>
<th>Reduction Modes</th>
</tr>
</thead>
<tbody>
<tr>
<td>CB-CMS</td>
<td>Eq. 6.30</td>
<td>Eq. 6.32</td>
<td>Eq. 6.42</td>
<td>Eq. 6.39</td>
<td>n/a</td>
</tr>
<tr>
<td>CCN</td>
<td>Eq. 6.25</td>
<td>Eq. 6.27</td>
<td>Eq. 6.41</td>
<td>Eq. 6.35</td>
<td>Eq. 6.68</td>
</tr>
<tr>
<td>CCT</td>
<td>Eq. 6.70</td>
<td>Eq. 6.71</td>
<td>Eq. 6.42</td>
<td>Eq. 6.72</td>
<td>Eq. 6.68</td>
</tr>
<tr>
<td>CCM</td>
<td>Eq. 6.70</td>
<td>Eq. 6.71</td>
<td>Eq. 6.42</td>
<td>Eq. 6.72</td>
<td>Eq. 6.46</td>
</tr>
<tr>
<td>CAT</td>
<td>Eq. 6.76</td>
<td>Eq. 6.77</td>
<td>Eq. 6.42</td>
<td>Eq. 6.78</td>
<td>Eq. 6.74</td>
</tr>
<tr>
<td>CAM</td>
<td>Eq. 6.76</td>
<td>Eq. 6.77</td>
<td>Eq. 6.42</td>
<td>Eq. 6.78</td>
<td>Eq. 6.61</td>
</tr>
</tbody>
</table>

in additional computational requirements to recalculate the mistuned Interface modes for every new mistuned IBR configuration.

### 6.9.3 Method Comparison

The previously outlined methods and their corresponding system matrices are outlined in Table 6.1. The first method, CCN, is a tuned Interface mode (CC) reduction of a tuned CB-CMS matrix. The $N$ serves as a reminder that this method uses nominal, or tuned, blade modes in the reduction/expansion for mistuned rotors. This is ultimately a frequency-based approach used widely in academia and industry, that assumes blade geometric perturbations alter only the corresponding modal stiffnesses while its mode shapes remain unaffected.

The CB-CMS method is formulated from mistuned blade matrices and modes. CCT and CCM are formulated from the mistuned CB-CMS matrices by reducing the interface DOF through either a tuned (T) or mistuned (M) Interface mode reduction. CAT and CAM are also formulated from the mistuned CB-CMS matrices by reducing the constraint and disk fixed-interface normal mode DOF through either a tuned (T) or mistuned (M) Ancillary mode (CA) reduction.

Table 6.2 outlines the size governing equation of each ROM discussed as a function of the number of blades and truncated modes retained in each method’s formulation. As previously outlined, the traditional CB-CMS ROM is the largest of these approaches and
Table 6.2: Mistuning ROM sizes

<table>
<thead>
<tr>
<th>Models</th>
<th>Size Eq.</th>
</tr>
</thead>
<tbody>
<tr>
<td>CB-CMS</td>
<td>( N (D_{kn} + N_\Gamma + \alpha_{kn}) )</td>
</tr>
<tr>
<td>CC-Reduced</td>
<td>( N (D_{kn} + \alpha_{kn}) + k_{cc} )</td>
</tr>
<tr>
<td>CA-Reduced</td>
<td>( N \cdot \alpha_{kn} + k_{ca} )</td>
</tr>
</tbody>
</table>

its size can be prohibitively large as the number of interface DOF increases. The remaining CC- and CA-reduced methods outlined in Section 6.9 seek to further reduce the size of the CB-CMS method. The methods utilizing Interface mode reduction are prescribed to be the same size by requiring that \( k_{cc} = m_k_{cc} = N\alpha_{cc} \), where \( k_{cc} \) is the number of retained Interface modes. Furthermore, the truncated tuned and mistuned mode sets belong to the lowest frequency index. This same prescription is followed for Ancillary mode reduction with \( k_{ca} = m_k_{ca} = N\alpha_{ca} \). While it isn’t a requirement that these models using tuned and mistuned reduction matrices be the same size, it is done here to provide a fair comparison between the predictions. The CA-reduced model sizes are independent of \( d_k_{kn} \) since this reduction method includes the disk fixed-interface modes in the secondary modal analysis. Increasing \( d_k_{kn} \) has the benefit of increasing the accuracy of the Ancillary mode reduction method without increasing the ROM size. However, this increase comes at the computational cost of calculating the Ancillary modes from larger matrices.

As the number of fixed-interface normal modes retained for all \( N + 1 \) components approaches their respective maximum, the prediction converges to the full FEM as the number of retained mistuned Interface and Ancillary modes approach their respective maximum. However, as these limits are approached, the ROM can hardly be called reduced. Note that convergence to the full FEM solution is only true when using mistuned blade, Interface and Ancillary modes since the tuned mode approaches of Sections 6.9.1, 6.9.2.1, and 6.9.2.2 are approximations.
6.10 Results

Results are generated for two mistuned ADLARF IBRs, A and B, that are subject to EO excitations $C = 0$ and $C = 2$, respectively, resulting in a high and low forced mistuned response amplification. A complete description of the ADLARF IBR is given in Section 4.3 on page 45. These excitation conditions are shown by the circles on the Nodal Diameter plot of Fig. 4.3 on page 47. High and low responding rotors provide an opportunity to determine if the developed methods are capable of providing accurate results over the distribution of forced response levels. Accurate prediction of the full forced response distribution is critical for reducing design over-conservatism while still ensuring the peak response does not exceed a predetermined critical value.

In the sections that follow each ROM is compared against a full FEM solution obtained from commercially available ANSYS software. The best achievable ROM results will belong to those predicted by the mistuned formulation of the traditional M.CB-CMS approach, since it is formulated with mistuned component matrices and modes while retaining all constraint DOF. Here the prefix "M." is used to denote that it is a prediction for mistuned response. The mistuned Ancillary and Interface mode reduction methods, M.CCM and M.CAM, will approach the accuracy of M.CB-CMS since these methods are synthesized from this parent model using mistuned modes in the reduction process. The accuracy of the tuned Ancillary and Interface mode reduction methods, M.CCT and M.CAT, should follow suit, depending on the accuracy of the assumption that the tuned modes span the same space as the mistuned modes, which is investigated in Section 6.10.1. In the results that follow, M.CCT and M.CAT are always compared against their mistuned mode counterpart, M.CCM and M.CAM, respectively, since the latter methods make no approximations other than modal truncations. Mistuned free response results are first shown in Section 6.10.2 that compare system natural frequencies and select mode shapes. Section 6.10.3 then discusses forced response predictions. Finally, model size and computation time comparisons are made in Section 6.10.4.
6.10.1 CC and CA Tuned and Mistuned Mode Comparison

The Modal Assurance Criterion (MAC) is used to provide a measure of consistency between tuned and mistuned modes. This MAC is given by

\[
MAC_{jk} = \frac{\left| \{t\phi_j\}^T M_{xx} \{m\phi_k\} \right|^2}{\{t\phi_j\}^T M \{t\phi_j\} \{m\phi_k\}^T M \{m\phi_k\}} \tag{6.81}
\]

where \(MAC_{jk}\) takes on a value between zero and one and \(xx\) designates either mistuned mass matrix \(M_{cc}\) or \(M_{ss}\) from Eqs. 6.31 and 6.59, respectively. Furthermore, \(t\phi\) and \(m\phi\) are the corresponding \(j^{th}\) and \(k^{th}\) tuned and mistuned Interface or Ancillary modes, respectively. If the mistuned modes are mass normalized, a MAC value of one indicates that the modal vectors are consistent while a value of zero indicates the modal vectors are inconsistent. Since the modes are obtained from linear FEMs, the consistency can be interpreted as a degree of orthogonality between a tuned and mistuned mode. This provides a quantitative measure to determine if the tuned and mistuned modes span the same space. If the tuned modes closely span the same space as the mistuned modes, using tuned modes in the reduction process will still provide accurate results.

The Interface and Ancillary MAC values can be seen in Figs. 6.1 and 6.2, respectively. In both cases, the modes have a high indication of orthogonality, where a perfect diagonal would indicate perfect orthogonality. The decay into the off-diagonal terms is often at repeated modes, where mistuning causes mode splitting that destroys the orthogonality. These results provide a good indication that the tuned mode reduction should have favorable accuracy, which is highlighted in the following sections.
Figure 6.1: Modal Assurance Criterion for tuned and mistuned Interface modes
Figure 6.2: Modal Assurance Criterion for tuned and mistuned Ancillary modes
6.10.2 Mistuned Free Response Results

The free response data is composed of the system natural frequencies and the corresponding mode shapes of the IBR predicted by each ROM. The percent error in the natural frequencies are illustrated versus the frequency index in Fig. 6.3 and are determined by comparing each ROM prediction with the "true" full FEM solution. Positive error corresponds to a frequency prediction above the full solution while negative error corresponds to a prediction below the full solution. In general, the ROMs create a stiffer model resulting in predicted frequencies above the full solution. The three methods developed with Interface mode reduction (M.CCN, M.CCT, and M.CCM) perform the poorest, however, the maximum error is quite small at \( \approx 0.27\% \) for the high responding rotor and \( \approx 0.19\% \) for the low responding rotor. The Ancillary mode reduction methods, M.CAT and M.CAM, prove to be more accurate than the Interface mode reduction methods since the errors are, in general, smaller over the entire frequency spectrum. Furthermore, the tuned Ancillary mode approximation of M.CAT seemed to have little effect on the frequency predictions as they closely follow the mistuned Ancillary mode M.CAM prediction. Lastly, as expected, M.CB-CMS has the highest accuracy.

A subset of predicted system mode shapes are chosen for comparison in the free response results. For the EO excitation and frequency range of interest, as depicted in Fig. 4.3 on page 47, the modal participation factors are determined for a modal summation response and are shown in the Pareto plot of Fig. 6.4 on page 103. The first ten modes with the highest modal participation are plotted on the abscissa. The bars and stems corresponding to the left ordinate illustrate the modal contributions for M.CCN and M.CB-CMS, respectively. While M.CCN correctly identifies the modes with the largest participation factors for both test cases, the amount these modes participate have an associated error when compared to the more accurate ROM M.CB-CMS. Large errors for the first ten modes will have a negative impact on predicted forced response levels, since these modes contribute to
Figure 6.3: IBR natural frequency error for each ROM compared against full FEM predictions

(a) Rotor A: High Responding Rotor with $C = 2$

(b) Rotor B: Low Responding Rotor with $C = 0$

more than 90% of all the modes in the forced response levels, as illustrated by the line plot corresponding to the right ordinate.

Mistuned IBR mode 121 has the largest modal contribution for Rotor A and is shown in the stem plots of Fig. 6.5. Likewise, mistuned mode 124 is the second largest contributing mode for Rotor B and is shown in Fig. 6.6. In each figure, the modal response in the $z$-direction at the blade tip is plotted for each blade around the IBR for a respective ROM and the full FEM prediction. Figure 6.5a illustrates the M.CB-CMS and M.CCN predic-
Figure 6.4: Modal participation factors for the EO excitations and forcing frequency range on interest from Fig. 4.3 on page 47.
tions. Again, the M.CB-CMS method will have the highest accuracy and is shown to be in very good agreement with the full FEM. Accuracy diminishes for M.CCN for a majority of the blades, with large errors seen in Fig. 6.6a. It will be shown later that this error in conjunction with error on the modal participation factor will result in larger errors in forced response levels. Figures 6.5b and 6.6b on page 106 depicts the predicted mode shapes for M.CCT and M.CCM. There is good agreement between each ROM and the full FEM. Slight errors on select blades, e.g. seven and 16 in Fig. 6.5b, can be reduced by increasing the number of retained Interface modes. There is also good agreement between M.CCT and M.CCM, which provides an indicator that the tuned Interface mode reduction method is accurate. M.CAT and M.CAM predictions are shown in Figs. 6.5c and 6.6c, which show the highest accuracy of the ROMs utilizing Ancillary and Interface mode reductions.

6.10.3 Mistuned Forced Response Results

Calculated blade displacements corresponds to the Euclidean distance of blade tip displacements over the range of excitation frequencies of interest at the EO excitations $C = 2$ for Rotor A and $C = 0$ for Rotor B. This results in set of forced response levels for each blade that can be further organized by finding a single maximum response of all blades at each excitation frequency or by finding the maximum response of each blade over the entire spectrum of excitation frequencies of interest. The former results in the maximum mistuned IBR response and represents the worst case, and conservative, scenario that all blades will see this response level over the frequency range of interest. If all blades are tuned, then this assumption is true. The latter represents the predicted peak blade-to-blade responses and provides a better assessment of the responses (stresses) that each blade experiences.

The peak IBR response is shown in Fig. 6.7 for M.CCN and M.CBCMS for Rotors A and B compared against the tuned and mistuned full FEM peak IBR response. As shown,
Figure 6.5: Comparison of Rotor A IBR mistuned mode 121 against the full FEM
Figure 6.6: Comparison of Rotor B IBR mistuned mode 124 against the full FEM
Figure 6.7: IBR peak forced response levels over excitation frequency range
there is an \( \approx 45\% \) mistuned response amplification for the high responding rotor A, while the low responding rotor B experience little to no response amplification. Furthermore, there is \( \approx 11\% \) and \( \approx 25\% \) error in the peak IBR M.CCN prediction for rotors A and B, respectively. The blade-to-blade response predictions are illustrated in Figs. 6.8 and 6.9, where the response levels have been normalized by the tuned response levels. A forced response amplification will then appear greater than one, while a response level below the tuned prediction will fall below one. Figures 6.8a and 6.9a illustrates that while M.CCN has identified response pattern, there are errors in each blade prediction. M.CCT and M.CCM, seen in Figs. 6.8b and 6.9b, showed improved accuracy compared to the full FEM, while M.CAT and M.CAM present the best results in Figs. 6.8c and 6.9c.

The methods are compared against each other in Fig. 6.10 on page 111 where the percent error is calculated from the full FEM and plotted for each blade. Large errors are shown for M.CCN for both rotors A and B that manifest from the method’s inability to accurately identify the modal participation factors and system mode shapes. In other words, the assumption that a linear sum of tuned modes will predict the mistuned response can lead to inaccurate blade-to-blade responses. M.CCT and M.CCM had smaller errors, which can be further reduced by increasing the number of retained Interface modes. The Ancillary mode methods, M.CAT and M.CAM, produced the best results that are generally as good as the parent CB-CMS ROM. Furthermore, M.CAT had errors that are plotted almost directly on top of M.CAM which highlights the tuned Ancillary mode assumption has negligible effect on the accuracy of the method.

### 6.10.4 Model Size and Solution Times

A review of the solution times begin with the comparison between using a cyclic versus a full disk representation. While the benefits of using cyclic symmetry are nothing new, a
Figure 6.8: Comparison of Rotor A peak blade-to-blade mistuned forced response levels against the full FEM predictions
Figure 6.9: Comparison of Rotor B peak blade-to-blade mistuned forced response levels against the full FEM predictions
Figure 6.10: Peak blade-to-blade mistuned forced response error
Table 6.3: Secondary Modal Analysis Solution times

<table>
<thead>
<tr>
<th>Mode</th>
<th>Normalized Time</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Phi_{cc}$</td>
<td>1</td>
</tr>
<tr>
<td>$\tilde{\Phi}_{cc}$</td>
<td>0.15</td>
</tr>
<tr>
<td>$\Phi_{ss}$</td>
<td>1</td>
</tr>
<tr>
<td>$\tilde{\Phi}_{ss}$</td>
<td>0.56</td>
</tr>
<tr>
<td>$D\Phi$</td>
<td>1</td>
</tr>
<tr>
<td>$D\tilde{\Phi}$</td>
<td>0.22</td>
</tr>
<tr>
<td>$D\Psi$</td>
<td>1</td>
</tr>
<tr>
<td>$D\tilde{\Psi}$</td>
<td>0.016</td>
</tr>
</tbody>
</table>

A quantified comparison here provides a comparison between methods, as well as motivation for using a more rigorous cyclic ROM formulation. Solution times for solving all modes are computed using MATLAB’s tic/toc function and are shown in Table 6.3. The solution times for each cyclic mode have been normalized by the solution time of its full (non-cyclic) counterpart. In all cases using a cyclic representation resulting in significant computational savings. It should also be noted that there are further computational savings by using a cyclic representation for subsequent analyses of new IBRs because the tuned cyclic modes need only be calculated once. If using mistuned $\Phi_{cc}$ and $\Phi_{ss}$ in M.CCM and M.CAM, respectively, these need to be recalculated for every mistuned IBR.

The number of retained modes and resulting model size of each developed method for the previous free and forced response results can be viewed in Table 6.4. As shown, the CC- and CA-reduced systems offer a significant reduction in model size from the parent CB-CMS system. The benefits of the smaller ROM offer both reduced storage requirements and reduced solution times. A comparison of solution times for solving the ROM eigen-problem and are shown in Table 6.5. The solution times have been normalized by that of the CB-CMS method since this time is the worst case due to its larger size. The Interface and Ancillary mode reduced ROMs show a significant reduction in solution times from the traditional CB-CMS approach. The careful reader will note that the solution time
Table 6.4: Mistuning ROM sizes

<table>
<thead>
<tr>
<th>Models</th>
<th>$Dk_n$</th>
<th>$N_\Gamma$</th>
<th>$A_k_n$</th>
<th>$k_{cc}$</th>
<th>$k_{ca}$</th>
<th>Size</th>
</tr>
</thead>
<tbody>
<tr>
<td>CB-CMS</td>
<td>5</td>
<td>153</td>
<td>25</td>
<td>n/a</td>
<td>n/a</td>
<td>2928</td>
</tr>
<tr>
<td>CC-Reduced</td>
<td>5</td>
<td>n/a</td>
<td>25</td>
<td>80</td>
<td>n/a</td>
<td>555</td>
</tr>
<tr>
<td>CA- Reduced</td>
<td>5</td>
<td>n/a</td>
<td>25</td>
<td>n/a</td>
<td>130</td>
<td>530</td>
</tr>
</tbody>
</table>

Table 6.5: ROM Eigen-Problem Solution times

<table>
<thead>
<tr>
<th>ROM</th>
<th>Normalized Time</th>
</tr>
</thead>
<tbody>
<tr>
<td>CB-CMS</td>
<td>1</td>
</tr>
<tr>
<td>CC-Reduced</td>
<td>0.022</td>
</tr>
<tr>
<td>CA-Reduced</td>
<td>0.032</td>
</tr>
</tbody>
</table>

of the Ancillary reduced methods are larger than the Interface mode reduce ROMs, but the Ancillary methods have a smaller eigen-problem to solve. This results from how the projection methods affects the sparsity of the ROM matrices. The diagonal matrix, $D\Lambda$, remains intact for the Interface mode reduction methods, while the Ancillary mode methods destroy this diagonal and reduces the sparsity of the ROM matrices. This has a subsequent increase in algorithm floating point operations and solution times.

6.11 Conclusions

Two geometric mistuning approaches were developed by performing a secondary modal analysis on different submatrices of a parent CB-CMS ROM formulated in cyclic coordinates. The first method computed the Interface modes of the CB-CMS constraint DOF while the second method computed Ancillary modes of the constraint and disk fixed-interface normal modes. These modes could be either tuned or mistuned. The tuned modes were calculated in cyclic coordinates that offered significant computation savings, while the mistuned modes eliminated the approximation of using tuned modes in the reduction
process. Regardless, free and forced response results highlighted that this is an accurate approximation for both high and low responding rotors as compared to the full FEM. Furthermore, the geometric mistuning methods were shown to have much better accuracy for peak IBR response and blade-to-blade predictions than the traditional frequency-based approach. The geometric mistuning methods were also shown to have a significant reduction in solution time of the eigen-problem from the traditional CB-CMS ROM.
New geometric mistuning approaches for integrally bladed rotors are developed for incorporating geometric perturbations to a fundamental disk-blade sector, particularly the disk-blade boundary, or connection. The developed reduced-order models are formulated from a Craig-Bampton component mode synthesis framework that is further reduced by a truncated set of Interface modes that are obtained from an eigen-analysis of the CB-CMS constraint degrees of freedom. An investigation into using a set of tuned Interface modes and tuned constraint modes for model reduction is then performed. A tuned mode approximation has the added benefit of being only calculated once which offers significant computational savings for subsequent analyses. Two configurations of disk-blade connection mistuning are investigated: as-measured principal component deviations and random perturbations to the inter-blade spacing. Furthermore, the perturbation sizes are amplified to investigate the significance of incorporating mistuned disk-blade connection. Free and forced response results are obtained for each ROM and each disk-blade connection type.
and compared to full finite element model solutions. It is shown that the developed methods provide highly accurate results with a significant reduction in solution time compared to the full FEM. In addition, results indicate that the inclusion of a mistuned disk-blade connection becomes significant as the size of the geometric deviations at the connection become large.

7.1 Introduction

Integrally Bladed Rotors based on nominal design parameters are a rotationally periodic structure. Unfortunately, there are small irregularities in the geometric and material characteristics between individual blades, referred to as mistuning, causing the rotational periodicity to break down. Even small mistuning that falls within strict manufacturing tolerances can have a dramatic effect on the IBR response. The ramifications of mistuning are two-pronged. First, individual blades can experience a localization of vibration energy causing forced response levels greater than predicted in a tuned, cyclic symmetry analysis. Second, is an increase in model size and computational burden because the entire IBR must now be solved in an FEA application, as opposed to a fundamental sector. This fact has driven the need for physics-based ROMs to effectively and efficiently predict mistuned response, particularly for Monte Carlo simulations seeking to characterize the full mistuned response distribution.

Significant research has been devoted to nominal based methods that are reviewed in Chapter 2. However, Beck, et al. [51] and Brown [50] have shown that these nominal methods can lead to significant prediction errors for even small mistuning. To account for errors of the frequency-based approaches, geometric mistuning models were developed to provide higher fidelity predictions and are also reviewed in Chapter 2. This chapter adds to the existing body of research by addressing Major Contribution 2 of Section 3.2 on page 33.

This effort creates a new geometric mistuning model synthesized from an IBR by
partitioning the full system into mistuned sectors composed of a single geometrically mistuned blade and disk sector with a mistuned disk-blade connection. Avalos, et al. [101], also investigated mistuned interfaces and found that mistuned disk-blade interfaces have a significant effect on mistuned response. However, the method in [101] required the definition of an interface zone, where the disk DOFs are not necessarily collocated with those of the blades, and an adjustable dimensionless stiffness parameter. The method described in this current work eliminates the need for defining this zone and adjustable parameter by performing a sector partitionment. This partitionment is then exploited by the Craig-Bampton component mode synthesis method [67, 102].

A drawback of the traditional CB-CMS approach is the retention of all DOFs at the component interfaces. For IBRs, as the number of interface DOFs and sectors increase, the CB-CMS ROM is hardly reduced as the size is dominated by these DOF. Many works have sought to further reduce the CMS matrices by casting the interface DOF into a modal domain consisting of a truncated set of Interface modes [99, 100]. This work extends the use of Interface modes by utilizing a tuned mode reduction approximation in a cyclic symmetry CB-CMS description. Furthermore, this effort approximates mistuned response using tuned constraint modes that are used with mistuned fixed-interface modes to investigate further increases in computational efficiency.

This chapter is organized in the following manner: first, Section 7.2 outlines the ROM development as well as outline the approximations that will significantly decrease computation expenses; then, Section 7.3 outlines the three newly developed ROMs capable of incorporating disk-blade connection mistuning; Section 7.4 then outlines the configurations of disk-blade connection mistuning for the ADLARF IBR described in Section 4.3 on page 45; lastly, results and conclusions are outlined in Sections 7.5 and 7.6, respectively.
7.2 Reduced-Order Model Formulation

The full IBR is first divided into components that consist of a single blade and a corresponding disk sector, as shown in Fig. 7.1, that results in a total of $N$ sectors. By substructuring in this fashion, mistuning can be applied to the blade and disk-blade interface geometry without having to assume a tuned interface and without having to define an interface zone or arbitrary stiffness parameter as required in [101]. The tuned interface assumption is then utilized for sector interfaces where disk DOF are coupled.

The CB-CMS formulation begins with the partitioned equation of motion (EOM) according to Fig. 7.1 for sector $s$

\[ M^{(s)} \ddot{x}^{(s)} + K^{(s)} x^{(s)} = 0 \quad (7.1) \]
where

\[ K^{(s)} = \begin{bmatrix} K_{\tau\tau} & K_{\tau\Gamma} \\ K_{\tau\Gamma}^T & K_{\Gamma\Gamma} \end{bmatrix}, \quad x^{(s)} = \begin{bmatrix} x_\tau \\ x_\Gamma \end{bmatrix} \]  

(7.2)

and \( \tau \) denotes interior DOFs, \( \Gamma \) denotes interface DOFs, and submatrices given by

\[ K_{\tau\Gamma} = \begin{bmatrix} K_{\tau_1 \Gamma_1} & K_{\tau_1 \Gamma_2} \\ K_{\tau_2 \Gamma_1}^T & K_{\Gamma_2 \Gamma_2} \end{bmatrix}, \quad x_\Gamma = \begin{bmatrix} x_{\Gamma_1} \\ x_{\Gamma_2} \end{bmatrix} \]  

(7.3)

Note that the mass matrix follows the same partitionment. Furthermore, depending on the number of element divisions through the disk, \( K_{\Gamma_1 \Gamma_2} \) may be a null matrix. Calculation of the sector fixed-interface normal modes, \( \phi_j^{(s)} \), are obtained by restraining all boundary DOF and solving the classical eigen-problem

\[ [K_{\tau\tau} - \lambda_j M_{\tau\tau}] \phi_j = 0 \quad j = 1, 2, \ldots, N_\tau^{(s)}. \]  

(7.4)

where \( \lambda_j = \omega_j^2 \) and \( \omega_j \) are the natural frequencies and \( N_\tau^{(s)} \) are the total number of interior DOF. Usually, there is a frequency spectrum of interest that limits the required upper range of \( j \) to some cutoff, \( k_n^{(s)} \), where \( k_n^{(s)} \ll N_\tau^{(s)} \). This subset of modes is then combined into the matrix \( \Phi^{(s)} \) while the corresponding \( \lambda_j \) are combined into the diagonal spectral matrix \( \Lambda^{(s)} \). The constraint modes, \( \Psi^{(s)} \), are ascertained by statically deforming a sector with a unit displacement to one coordinate of an established set of boundary coordinates while the remaining coordinates of the set are restrained, and the remaining interior DOF of the component are force-free. This is done with the following equation

\[ \Psi^{(s)} = -K_{\tau\tau}^{-1}(s) K_{\tau\Gamma}^{(s)} \]  

(7.5)

The transformation from physical coordinates, \( x^{(s)} \), to CB modal coordinates, \( p^{(s)} \), is
accomplished with the cyclic CB modal transformation matrix

\[
\mathbf{x}^{(s)} = \begin{bmatrix}
\Phi^{(s)} & \Psi^{(s)} \\
0 & I
\end{bmatrix}
\begin{Bmatrix}
\mathbf{p}^{(s)}_n \\
\mathbf{p}^{(s)}_c
\end{Bmatrix} = U^{(s)}\mathbf{p}^{(s)}
\]  
\hspace{1cm} (7.6)

where the right subscript \( n \) denotes motion due to the fixed-interface normal modes and \( c \) is denotes motion due to the constraint modes. Substituting this into the sector EOM in Eq. 7.1 and pre-multiplying by \( U^\top \) results in an EOM in CB modal coordinates

\[
\mathcal{M}^{(s)}\ddot{\mathbf{p}}^{(s)} + \mathcal{K}^{(s)}\mathbf{p}^{(s)} = 0
\]  
\hspace{1cm} (7.7)

where

\[
\mathcal{M}^{(s)} = \begin{bmatrix}
I & \mathcal{M}_{nc} \\
\mathcal{M}^\top_{nc} & \mathcal{M}_{cc}
\end{bmatrix}, \quad \mathcal{K}^{(s)} = \begin{bmatrix}
\Lambda & 0 \\
0 & \mathcal{K}_{cc}
\end{bmatrix}
\]  
\hspace{1cm} (7.8)

and

\[
\mathcal{M}_{nc} = \Phi^\top(M_{\sigma \sigma} \Psi + M_{\sigma \Gamma})
\]
\[
\mathcal{M}_{cc} = \Psi^\top(M_{\sigma \sigma} \Psi + M_{\sigma \Gamma}) + M_{\sigma \Gamma}^\top \Psi + M_{\Gamma \Gamma}
\]  
\hspace{1cm} (7.9)
\[
\mathcal{K}_{cc} = K_{\Gamma \Gamma} + K_{\sigma \Gamma}^\top \Psi
\]

All the previous calculations in this subsection have been for a single sector, \( s \). If all sectors are tuned, these calculations will only have to be done once since the component matrices would be the same for all \( N \) sectors. However, geometric mistuning perturbs both the mass and stiffness matrices of each sector by varying amounts. Consequently, the CB-CMS component matrices must be recalculated for all \( N \) sectors since

\[
U^{(Tuned)} \neq U^{(1)} \neq U^{(2)} \neq \cdots \neq U^{(N)}
\]  
\hspace{1cm} (7.10)
The CB-CMS reduced mass and stiffness matrices of Eq. 7.7 for each mistuned sector are then combined into the block diagonal matrices containing all sectors \((\forall s)\)

\[
M = \begin{bmatrix}
I & B \begin{bmatrix} M^{(s)}_{nc} \end{bmatrix} \\
B \begin{bmatrix} M^{T(s)}_{nc} \end{bmatrix} & B \begin{bmatrix} M^{(s)}_{cc} \end{bmatrix}
\end{bmatrix}, \quad
K = \begin{bmatrix}
0 & B \begin{bmatrix} \Lambda^{(s)} \end{bmatrix} \\
B \begin{bmatrix} \Lambda^{T(s)} \end{bmatrix} & 0
\end{bmatrix}
\]

where \(B\) is the block-diagonal operator that places the \(s^{th}\) sector on the \(s^{th}\) block on the diagonal.

CB-CMS model assembly requires interface displacement compatibility \(x_{\Gamma_2}^{(s)} = x_{\Gamma_1}^{(s+1)}\) in physical coordinates and further requires that \(x_{\Gamma} = p_{c}\). Note that for sector \(s = N\), the \(N + 1\) sector is simply \(s = 1\). To constrain the sectors, \(x_{\Gamma_1}\) DOF are kept active by

\[
\begin{bmatrix}
\mathbf{x}_{\Gamma_1}^{(1)} \\
\mathbf{x}_{\Gamma_2}^{(1)} \\
\vdots \\
\mathbf{x}_{\Gamma_1}^{(N)} \\
\mathbf{x}_{\Gamma_2}^{(N)}
\end{bmatrix} = T_{\Gamma\Gamma} \begin{bmatrix}
\mathbf{x}_{\Gamma_1}^{(1)} \\
\vdots \\
\mathbf{x}_{\Gamma_1}^{(N)}
\end{bmatrix} = T_{\Gamma\Gamma} \begin{bmatrix}
p_{n}^{(\forall s)} \\
p_{c}^{(\forall s)}
\end{bmatrix} = T_{CB} p_{CB}
\]

where \(T_{\Gamma\Gamma}\) is a boolean matrix composed of zeros and ones that satisfies the interface displacement compatibility requirement. By introducing the transformation matrix \(T_{CB}\) in

\[
\begin{bmatrix}
p_{n}^{(\forall s)} \\
p_{c}^{(\forall s)}
\end{bmatrix} = \begin{bmatrix}
I & 0 \\
0 & T_{\Gamma\Gamma}
\end{bmatrix} \begin{bmatrix}
p_{n}^{(\forall s)} \\
p_{c}^{(\forall s)}
\end{bmatrix} = T_{CB} p_{CB}
\]

the assembled mass and stiffness matrices of the reduced system are obtained by

\[
CBM = T_{CB}^{T}MT_{CB} = \begin{bmatrix}
I & B \begin{bmatrix} M^{(s)}_{nc} \end{bmatrix} T_{\Gamma\Gamma} \\
T_{\Gamma\Gamma} B \begin{bmatrix} M^{T(s)}_{nc} \end{bmatrix} & T_{\Gamma\Gamma} B \begin{bmatrix} M^{(s)}_{cc} \end{bmatrix} T_{\Gamma\Gamma}
\end{bmatrix}
\]

121
\[ CBK = T_{CB}^T K T_{CB} = \begin{bmatrix} \mathbb{B} \Lambda^{(s)}_\forall & 0 \\ 0 & T_{\Gamma \Gamma}^T \mathbb{B} \kappa^{(s)}_\forall \end{bmatrix} \]  

(7.15)

The sector modal damping matrix is then given by.

\[ CB\mathcal{C} = \begin{bmatrix} \mathbb{B} \Lambda^{(s)}_\forall \left(2 \zeta_j^{(s)} \sqrt{\Lambda^{(s)}_\forall}\right) & 0 \\ 0 & 0 \end{bmatrix} \]  

(7.16)

where \( \zeta_j^{(s)} \) is the damping coefficient for the \( j^{th} \) sector mode from Eq. 7.4 for sector \( s \).

### 7.2.1 Excitation Force

The external forcing vector in physical coordinates for the system can be represented by

\[ F = \begin{bmatrix} F_r \\ F_\Gamma \end{bmatrix} = \begin{bmatrix} f_r^{(1)} \\ \vdots \\ f_r^{(N)} \\ f_\Gamma^{(1)} \\ \vdots \\ f_\Gamma^{(N)} \end{bmatrix} \]  

(7.17)

where

\[ f_r^{(s)} = \begin{bmatrix} D f_r \\ A f_r \end{bmatrix} \]  

(7.18)

where the left superscript \( A \) and \( D \) correspond to the blade and disk, respectively. A traveling wave force is considered for the blade DOF only resulting in \( F_\Gamma = 0 \) and \( D f_r^{(s)} = 0 \). This allows for a more compact formulation, but it is not a requirement of the mistuning methods presented herein. This excitation force is constant in magnitude and differs only
in phase from blade to blade by

$$\varphi_{a,C} = \frac{2\pi C (a - 1)}{N} \quad a = 1, \ldots, N$$  \hspace{1cm} (7.19)$$

where $C$ is the EO excitation. A phase vector $P_C$ between blades becomes

$$P_C = \left\{ \exp (i\varphi_{1,C}) \right\} \quad \vdots \quad \left\{ \exp (i\varphi_{N,C}) \right\}$$  \hspace{1cm} (7.20)

The forcing vector on all sector interior DOF from Eq. 7.17 can then be expressed as a constant magnitude force

$$F_\tau = P_C \otimes f^{(1)}_\tau$$  \hspace{1cm} (7.21)$$

where the symbol $\otimes$ is the Kronecker product.

The CB-CMS modal force for the sector components is obtained by projecting the force vector of Eq. 7.17 onto the component sector modes and then pre-multiplying by the transpose of the transformation matrix

$$\begin{align*}
C_B F_C &= \left\{ \mathcal{F}_n \mathcal{F}_c \right\} = T_{CB}^T \begin{bmatrix}
B_{\forall s} \left[ \Phi^{(s)} \right]^T & 0 \\
B_{\forall s} \left[ \Psi^{(s)} \right]^T & I
\end{bmatrix} F_\tau = \left\{ \begin{bmatrix}
B_{\forall s} \left[ \Phi^{(s)} \right]^T \\
T_{\Gamma \forall s} B_{\forall s} \left[ \Psi^{(s)} \right]^T
\end{bmatrix} F_\tau \right\}
\end{align*}$$  \hspace{1cm} (7.22)$$

### 7.2.2 Tuned Constraint Mode Approximation

As previously discussed and shown in Eq. 7.10, geometrically mistuned sectors require recalculation of a sector’s component modes, i.e. fixed-interface normal modes, $\Phi$, and constraint modes, $\Psi$. In this effort, inclusion of the disk in the whole sector component eliminates the need to specially model a disk-blade connection beyond the finite element mesh, however there is an added computational expense in calculating the component modes since
the required matrices are now much larger. Calculating the constraint modes for Eq. 7.6 is equivalent to solving \( N \cdot N_\Gamma \) linear systems of the form \( Ax = b \), where \( N_\Gamma \) is the length \( x_\Gamma \). Solving this equation for \( x \) by Cholesky’s method results in a computational cost of about \( O \left( N_\Gamma^3 \right) \) that dominates the ROM formulation time [103]. This expense is eliminated in this work by using a tuned constraint mode approximation in Eq. 7.6 for mistuned response prediction. The physical basis for this assumption rests upon the size of the geometric perturbations that mistune each blade. If these perturbations do not drastically alter the blade geometry or the blade location on the disk sector, e.g. no large geometric mistuning, then the static deformation induced by the unit displacements at the sector interfaces will differ little from a tuned blade, thus allowing the use of tuned constraint modes.

No approximations are used for the fixed-interface normal modes, so the component reduction requires recalculation of the eigen-problem for each mistuned sector. But, it will be shown that calculating the \( N \) sets of constraint modes is significantly more expensive than calculating the \( N \cdot k^{(s)}_n \) total sector Interface modes. Furthermore, considering that the sector matrices are mostly sparse, and that these calculations are done sector-by-sector, this additional computation is small compared to solving the full IBR FEM or even traditional CB-CMS ROMs. Furthermore, if probabilistic studies are required, these fixed-interface normal modes can be approximated or a small population of mistuned sectors can be generated and bootstrapping methods can be used to eliminate calculation of a larger population of sectors.

### 7.2.3 Interface Mode Reduction

The CB-CMS methodology requires retention of all sector interface DOF of the ROM. Defining the sector interfaces to be in the disk results in a large number of retained interface DOF that prevent an ultimate reduction in model size. To reduce this burden, a secondary eigen-analysis can be carried out on the constraint DOF partitions of the CB-CMS system matrices of Equations 7.14 and 7.15 that will re-cast these portions into a
new modal domain. The resulting truncated set of eigenvectors will yield a set of interface modes, referred to as Characteristic Constraint modes in [100]. There are a set of tuned and mistuned CB-CMS system matrices from which the Interface modes are derived. The tuned approach offers computational savings by carrying out the analysis in cyclic coordinates which allows each harmonic index to be solved independently. The mistuned matrices require the calculation to be carried out in a non-cyclic, CB-CMS modal space. Each method is outlined below.

7.2.3.1 Tuned Interface Modes

The first set is derived for a tuned IBR that results in the block-circulant constraint portions of Eqs. 7.14 and 7.15 and can be transformed to cyclic coordinates for faster computation of tuned modes by

$$\tilde{K}_{cc} = (E^\top \otimes I) K_{cc} (E \otimes I)$$

where $E$ is the $N \times N$ real-valued Fourier matrix defined in Appendix B and $I$ is the identity matrix of size $N_{\Gamma_1} \times N_{\Gamma_1}$, where $N_{\Gamma_1}$ is the length of $x_{\Gamma_1}$. The matrix $\tilde{K}_{cc}$ is block diagonal, where each block corresponds to the constraint DOF symmetrical components at a specific harmonic index. Computation of the Interface modes can then be computed one harmonic at a time by

$$[\tilde{K}_{cc}^{(h)} - \omega_j^2 \tilde{M}_{cc}^{(h)}] \tilde{\Phi}_j^{(h)} = 0 \quad j = 1, \ldots, N_{\Gamma}$$

Modal truncation can be used to limit the required upper range of $j$ to some cutoff, $t_{cc}$, where $t_{cc} \ll N_{\Gamma}$ and the left superscript $t$ denotes a tuned cutoff value. This subset of modes are then combined into the matrix $\tilde{\Phi}^{(h)} = [\tilde{\phi}_1, \ldots, \tilde{\phi}_{t_{cc}}]$. This truncated set of
tuned modes can then be transformed back to physical constraint DOF by

$$
\Phi_{cc} = (E \otimes I) B \left[ \Phi_{cc}^{(h)} \right]
$$

(7.25)

### 7.2.3.2 Mistuned Interface Modes

Mistuned constraint partitions from Eqs. 7.14 and 7.15 do not have cyclic symmetry properties. Consequently, the Interface modes will be calculated for the full physical space by

$$
[K_{cc} - \omega_j^2 M_{cc}] \phi_j = 0 \quad j = 1, \ldots, N \cdot N_{\Gamma}
$$

(7.26)

Again, a frequency range of interest can limit the required upper range of $j$ to some cutoff, $m_{k_{cc}}$, where $m_{k_{cc}} \ll N \cdot N_{\Gamma}$ and the superscript $m$ denotes the mistuned cutoff value. This subset of modes is then combined into the Interface modal matrix

$$
\Phi_{cc} = [\phi_1, \ldots, \phi_{m_{k_{cc}}}] 
$$

(7.27)

### 7.3 Mistuning Models

Three mistuning models are developed that rely upon different tuned and mistuned modal reductions and expansions. For the methods utilizing tuned modes, it is assumed that the tuned modes approximately span the same space as their mistuned counterparts, thereby resulting in accurate ROMs. This assumption will be tested in Section 7.5. In all three approaches, the generic EOM is given by

$$
M_r \ddot{q}_r + C_r \dot{q}_r + (1 + G_i) K_r q_r = F_r 
$$

(7.28)
where subscript $r$ refers to \textit{reduced} and the sector modal damping matrix and structural damping coefficient, $C_r$ and $G$, respectively, can be included to better model IBR dynamic response [39].

The reduced problem size of Eq. 7.28 is obtained by limiting the number of modes retained in the Interface modal matrix $\Phi_{cc}$, i.e. by limiting $\ell_{cc}$ or $m_{cc}$. This reduction is carried out with the following transformation matrix

\[
C_B p = \begin{pmatrix} p_n \\ p_c \end{pmatrix} = \begin{bmatrix} I & 0 \\ 0 & \Phi_{cc} \end{bmatrix} \begin{pmatrix} q_n \\ q_c \end{pmatrix} = T_{CC} q_r \tag{7.29}
\]

Substituting this into Eq. 7.28 and pre-multiplying by $T_{CC}^\top$ results in the following ROM matrices

\[
M_r = T_{CC}^\top C_B M T_{CC} = \begin{bmatrix} I & B_{\forall s} \left[ M_{nc}(s) \right] T_{IT} \Phi_{cc} \\ \Phi_{cc}^\top T_{IT} B_{\forall s} \left[ M_{nc}(s) \right] T_{IT} \Phi_{cc} \end{bmatrix} \tag{7.30}
\]

\[
K_r = T_{CC}^\top C_B K C_B T_{CC} = \begin{bmatrix} B_{\forall s} \left[ \Lambda(s) \right] & 0 \\ 0 & \Phi_{cc}^\top T_{IT} B_{\forall s} \left[ K_{cc}(s) \right] T_{IT} \Phi_{cc} \end{bmatrix} \tag{7.31}
\]

\[
F_r = T_{CC}^\top C_B F = \begin{bmatrix} B_{\forall s} \left[ \Phi(s) \right]^\top F_r \\ \Phi_{cc}^\top T_{IT} B_{\forall s} \left[ \Psi(s) \right]^\top F_r \end{bmatrix} \tag{7.32}
\]

Note that this reduction does not change the non-zero elements of the modal damping matrix of Eq. 7.16, so it is not re-listed here.

It remains to define the three models that compose the matrices of Eq. 7.28:

1. Tuned Interface and Tuned Constraint Modes (TITCM) reduction/expansion that uses tuned Interface modes, $\Phi_{cc}$, from Eq. 7.25 that are used in Eq. 7.29 and tuned constraint modes in Eq. 7.6 with mistuned mass, stiffness, and fixed-interface normal modes.
2. Tuned Interface and Mistuned Constraint Modes (TIMCM) reduction/expansion that uses tuned Interface modes, $\Phi_{cc}$, from Eq. 7.25 that are used in Eq. 7.29 and and mistuned constraint modes in Eq. 7.6 with mistuned mass, stiffness, and fixed-interface normal modes.

3. Mistuned Interface and Mistuned Constraint Modes (MIMCM) reduction/expansion that uses mistuned Interface modes of Eq. 7.27 and mistuned constraint modes in Eq. 7.6 with mistuned mass, stiffness, and fixed-interface normal modes.

Reduced-order models TITCM and TIMCM approximate the mistuned response by using tuned mode approximations. This reduces subsequent computational expenses since tuned modes only need to be calculated once and therefore increases mistuned model efficiency. Model MIMCM, however, makes no other approximations other than modal truncation by prescribing $m_{k_{cc}} \ll N \cdot N_1$ or $k^{(s)}_{n} \ll N^{(s)}_n$, but requires recalculation of all mistuned modes.

### 7.4 Disk-Blade Connection Modeling

A single geometrically mistuned ADLARF IBR is used in this study that has two types of mistuned disk-blade configurations. The first type has geometric deviations that are described by Principal Components (PCs) resulting from PCA. The second type has geometric deviations described by PCs in addition to random perturbations in the inter-blade spacing. Each mistuned configuration is explained in detail in the following subsections.

#### 7.4.1 Mistuned Principal Component Configuration

Three cases of this type of configuration are investigated. The first, A1, simply uses as-measured geometric deviations prescribed by the PCs. The next two cases, A2 and A3, amplify the deviations of A1 by $50x$ and $100x$, respectively to examine the sensitivity of mistuned response to the size of geometric deviations at the disk-blade connection.
A histogram of the geometric deviations for all 16 A1 disk-blade connections is shown in Fig. 7.2 and show that the deviations are quite small. In fact, the majority of the deviations are less than 2.5 mils (thousandths of an inch). The exact same histogram is obtained for A2 and A3, where only the deviations are larger, e.g. the maximum deviation of interface A3 is now 250 mils instead of 2.5 mils from A1. All 16 A3 disk-blade connections are superimposed on top of each other in Fig. 7.3, where it is seen that these deviations create perturbations along the upper and lower surfaces, however, they do not significantly alter the blade stagger angle or inter-blade spacing. Furthermore, the disk portion is mistuned as well since the disk side of the connection is also perturbed.

### 7.4.2 Mistuned Inter-blade Spacing

Configuration A4 incorporates geometric deviations of A1, but it also randomly perturbs the inter-blade spacing with a uniform distribution with limits of \( \pm 2 \text{ deg} \) that are listed in Table 7.1. Ideally, these spacing deviations will be captured by PCA for a full IBR after the blades are attached. However, if PCA is carried out for detached blades that are later attached to a disk, these spacing deviations will occur independent of the PCs. A histogram
Figure 7.3: Superposition of all 16 disk-blade connections for mistuned A3 configuration that illustrate upper and lower surface deviations
of the geometric deviations for all 16 blades is shown in Fig. 7.4. The maximum deviation is approximately that of A3, however, due to the physical location change of the entire blade the majority of the deviations are much greater than those in A3. All 16 A4 disk-blade connections are superimposed on top of each other in Fig. 7.5 where the variation in the inter-blade spacing is clearly visible.

7.5 Results

This section will serve two purposes: first, to determine the accuracy of each developed ROM and second, to assess the importance of considering a mistuned disk-blade connection for IBRs. With this in mind, results are generated for a mistuned IBR subject to mistuned configurations A1-A4 where only the connections vary, i.e. the elastic modulus and geometric perturbations (excluding the disk-blade connection) of each blade are not changed from the original mistuned IBR. Furthermore, the same model with perfectly tuned connections is generated that will aid in determining the impact of a mistuned connection. Each test case is subject to EO excitation $C = 2$ that is shown by the circle in the
Figure 7.5: Superposition of all 16 disk-blade connections for configuration A4 that illustrate inter-blade spacing deviations
Table 7.1: Inter-blade spacing deviations from the nominal placement

<table>
<thead>
<tr>
<th>Blade</th>
<th>Deviations deg</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.5944</td>
</tr>
<tr>
<td>2</td>
<td>-1.1383</td>
</tr>
<tr>
<td>3</td>
<td>1.5584</td>
</tr>
<tr>
<td>4</td>
<td>-0.9613</td>
</tr>
<tr>
<td>5</td>
<td>0.0951</td>
</tr>
<tr>
<td>6</td>
<td>-0.9809</td>
</tr>
<tr>
<td>7</td>
<td>0.8538</td>
</tr>
<tr>
<td>8</td>
<td>-0.5506</td>
</tr>
<tr>
<td>9</td>
<td>-0.5881</td>
</tr>
<tr>
<td>10</td>
<td>1.205</td>
</tr>
<tr>
<td>11</td>
<td>1.7388</td>
</tr>
<tr>
<td>12</td>
<td>0.4852</td>
</tr>
<tr>
<td>13</td>
<td>0.989</td>
</tr>
<tr>
<td>14</td>
<td>-1.3913</td>
</tr>
<tr>
<td>15</td>
<td>-0.2042</td>
</tr>
<tr>
<td>16</td>
<td>-1.3675</td>
</tr>
</tbody>
</table>

Nodal Diameter plot in Fig. 4.3 on page 47. In the subsections that follow, each developed ROM will be compared against a full FEM solution obtained from the commercially available ANSYS software for all mistuned configurations. However, ROM TITCM is not used for any predictions for interface A4 since the tuned constraint mode approximation is no longer valid, and is explained more in the next subsection.

7.5.1 Constraint Mode Comparison

The magnitude of the difference between a mistuned and tuned constraint mode is shown in Fig. 7.6. In each plot, the same nodal DOF on the disk sector interface is given a unit displacement that results in the largest difference for all \( N_f \) constraint modes, i.e. Fig. 7.6 represents the worst tuned constraint mode approximation. The magnitude is shown at each nodal location by the color designated to each node, where red represents the largest difference and blue the smallest. Figure 7.6a shows particularly great agreement between the
tuned and mistuned constraint mode for A1. As the magnitude of the interface deviations are increased 100x for configuration A3, the largest difference seen in Figure 7.6b is an order of magnitude larger. As the magnitude of the deviations at the connection increase, e.g. by adjusting the inter-blade spacing, these deviations alter the static shape of the mistuned constraint modes from their tuned counterparts and therefore leads to inaccurate results.

7.5.2 Interface Mode Comparison

The Modified Modal Assurance Criterion (MAC) is used to provide a measure of consistency between the tuned and mistuned Interface modes. If the mistuned modes are mass-normalized, a MAC value of one indicates that the modal vectors are consistent while a value of zero indicates that the modal vectors are inconsistent. Since the modes are obtained from linear FEMs, the consistency can be interpreted as a degree of orthogonality between a tuned and mistuned mode. This provides a quantitative measure to determine if the tuned and mistuned Interface modes span the same space. If they approximately span the same space, methods utilizing tuned modes can still provide accurate results.

The Interface MAC values can be seen in Fig. 7.7. The modes have a high indication of orthogonality for interface A1, where a perfect solid diagonal would indicate perfect orthogonality. The decay into the off-diagonal terms is often at repeated modes, where mistuning causes mode splitting that destroys perfect orthogonality. Interface A4 represents the worst case between all the mistuned interfaces were the orthogonality between the modes is not as clearly visible. However, the diagonal remains intact indicating that the tuned mode reduction should still provide a sound reduction basis.
Figure 7.6: Worst-case magnitude of nodal displacement deviations between a tuned and mistuned constraint mode
Figure 7.7: Modal Assurance Criterion for tuned and mistuned Interface modes
7.5.3 Natural Frequency Comparison

The percent error in predicted mistuned IBR natural frequencies is illustrated in Fig. 7.8 and are determined by comparing each ROM prediction with the full FEM solution for configurations A1, A3, and A4. Positive error corresponds to a frequency prediction above the full FEM, while negative error denotes a prediction below the FEM. Also shown in this figure is the CB-CMS prediction, since the ROMs can at best be as accurate as this parent model. Figure 7.8a illustrates the error for A1 and shows that there is generally good agreement since the maximum error seen is only $\approx 0.269\%$. Furthermore, the difference between the ROMs and the parent CB-CMS model are negligible, indicating that an increase in accuracy to the full FEM solution can be gained by retaining more fixed-interface normal modes for each sector. Figure 7.8b also shows great agreement and varies little from the error seen for configuration A3. However, TITCM has larger errors at lower frequencies than the other ROMs, but these errors are still less than the maximum. Accuracy of the ROMs for A2 (not shown) fell between that of A1 and A3. Figure 7.8c also shows great agreement for TIMCM and MIMCM predictions with the largest error measuring only 0.166\%. Furthermore, the ROMs compared very well to the parent CB-CMS model for all frequencies and configurations.

7.5.4 Forced Response Comparison

Calculated blade displacements correspond to the Euclidean distance of blade tip displacements over the range of excitation frequencies of interest at the EO excitation $C = 2$. This results in set of peak blade response vs. excitation frequency that can be further organized by finding a single maximum response of all blades at each excitation frequency or by finding the maximum response of each blade over the entire spectrum of excitation frequencies. The former results in the maximum mistuned IBR response and represents the conservative worst case scenario that all blades will see this response level over the frequency range.
Figure 7.8: IBR natural frequency error for each ROM compared against full FEM predictions for configurations A1, A3, and A4
Figure 7.9: Peak blade-to-blade response predictions for each ROM compared against full FEM predictions for configurations A1 and A4 of interest. If all blades are tuned, then this assumption is true. The latter represents the predicted peak blade-to-blade responses and provides a better assessment of the responses (stresses) that each blade experiences.

These peak blade-to-blade predictions are shown in Fig. 7.9 for configurations A1 and A4 where the responses have been normalized by the tuned response. A response amplifica-
tion will then appear larger than one, while a response attenuation will be less than one. All three ROMs accurately predicted the mistuned response pattern and blade-to-blade levels very well for A1, where the same results were obtained for A2 and A3. The same accuracy is also seen for the TIMCM and MIMCM ROMs for configuration A4. Furthermore, it is also interesting to note that, qualitatively, the different mistuned configurations did not significantly change the mistuning pattern of the blades. A4 only differs slightly, e.g. blades 8 and 9, while the peak responding blade didn’t change and remained at approximately 1.49. This illustrates that the mistuning present over the entire blade has a larger impact on forced response levels than mistuning at the disk-blade connection. This is different than what was observed in [101], but in that work the perturbations were applied directly to the stiffness matrices and did not manifest directly from perturbations to the physical geometry of an IBR. A quantitative view of the blade-to-blade prediction error is shown in Fig. 7.10. This confirms the agreement between the ROM predictions and the full FEM since all errors are less than 2%. However, TITCM shows the poorest performance for A1 and A3, while TIMCM and MIMCM errors are almost coincident for all configurations. This shows that using a tuned Interface mode reduction procedure is an accurate approximation, even for large disk-blade connection mistuning as seen in A4.

7.5.5 Mistuned and Tuned Disk-Blade Connection Comparison

The previous subsections compared mistuned responses to a full FEM prediction that shared the same mistuned connection. However, it is important to assess when a mistuned connection should be considered for IBRs. Figure 7.11 plots the peak blade-to-blade response error when a mistuned IBR with tuned disk-blade connections is compared against the same mistuned IBR except with mistuned disk-blade connections. Here, the most accurate ROM, MIMCM, is used for the comparison. Errors from ROMs with PC interface deviations shown in Fig. 7.11a illustrates that A1 has minimal difference than that of a tuned
Figure 7.10: Peak blade-to-blade response error for each ROM compared against full FEM predictions for configurations A1, A3, and A4
connection. Only when the geometric deviations are amplified by 50x and 100x for A2 and A3 does the error begin to grow. However, the maximum error seen was only $\approx 5\%$ which is still relatively good. Furthermore, the peak responding blade 4, has less than $\approx 0.25\%$ error for this mistuned IBR. Figure 7.11b shows the same error plot calculated for A4. Here it is shown that assuming a tuned connection is not an accurate approximation since large prediction errors are seen. Referring back to Figs. 7.2 and 7.4, it can be seen that the majority of the geometric perturbations for A4 are significantly larger and distributed over each disk-blade connection. Deviations of this size are considered large geometric mistuning. This provides evidence that approximating a mistuned connection with a tuned connection is only accurate for small mistuning or conditions where large perturbations are localized to a small areas of the disk-blade connection.

7.5.6 Model Size and Solution Times

The size of each developed ROM is given by $S_{\text{ROM}} = \sum_{s} N_{s}^{(s)} + m/k_{cc}$, where $N_{s}^{(s)} = 35$ and $m/k_{cc} = 300$. For the CB-CMS parent model, there is no reduction to the size of the interface DOF so this model size is given by $S_{\text{CB}} = \sum_{s} N_{s}^{(s)} + N \cdot N_{r} = 5840$. The use of Interface modes results in $\approx 85\%$ reduction in model size and a $95\%$ reduction in the computational time, computed by MATLAB’s tic/toc function, to find the same number of ROM system modes. While the resulting ROM size for all three methods is the same, the difference lies in the computations used in formulating the models. First, the expenses for calculating the tuned Interface modes and constraint modes are excluded since these costs are only experienced once in an initial, single model reduction. Therefore, the most computationally expensive model is MIMCM since this ROM requires calculation the mistuned Interface, constraint, and fixed-interface normal modes for each sector resulting in a normalized expense of 1. The next most expensive method is TIMCM since the interface modes are excluded and has a normalized (by MIMCM time) expense of 0.48. The least
Figure 7.11: MIMCM peak blade-to-blade response error compared against those assuming a tuned disk-blade connection
expensive model is the TITCM since this method only requires recalculation of the mistuned fixed-interface normal modes. The expense, again normalized by that of MIMCM, results in 0.075.

7.6 Conclusions

Three different mistuning prediction ROMs were developed that are capable of handling mistuned disk-blade connections. Each ROM was formulated from a parent CB-CMS model through the use of Interface and constraint mode reductions. ROMs utilizing tuned Interface mode reductions were shown to accurately approximate the mistuned free and forced response of a full FEM for both small and large disk-blade connection mistuning. ROMs utilizing tuned constraint mode approximations were shown to be accurate for both small disk-blade connection perturbations and cases where large deviations are localized. Each ROM was shown to offer significant computational savings compared to the full FEM. Results also indicated that inclusion of the mistuned interface does not drastically impact the forced response predictions for IBRs when the size of geometric perturbations are small. Furthermore, it was found that the blade-to-blade mistuned response pattern also varied little as the size of disk-blade connection mistuning increased. Only when a significant portion of the disk-blade connection was subject to large perturbations were changes seen in the mistuning pattern.
Abstract

Mistuning prediction reduced-order models are developed for dual flow-path integrally bladed rotors. These turbine engine components are similar to traditional integrally bladed rotors, but they have the ability to perform work on a secondary, or bypassed, flow-field. The ROMs are developed in a Craig-Bampton component mode synthesis framework that easily allows mistuning to be implemented through perturbations to blade geometries and natural frequencies. Furthermore, the tuned disk and outer ring components are formulated in a cyclic symmetry format that greatly reduces the computational burden in calculating their component matrices. The first methods, analogous to traditional nominal mode approaches for IBRs, introduce mistuning through perturbations to blade fixed-interface normal mode natural frequencies. The second methods explicitly account for blade geometry surface deviations. In either the nominal or geometric approach, DFIBRs are subject to a large retention of interface degrees of freedom. This is overcome by developing ROMs reduced by a set of Interface modes from a secondary modal analysis on the CB-CMS con-
straint DOF. The second approach calculates a set of Ancillary modes from a secondary modal analysis on the CB-CMS constraint and disk and ring normal DOF to further reduce the model size. Both approaches utilize tuned modes that carry the benefit of only needing to calculate blade, Interface, and Ancillary modes once in an up-front computational expense. DFIBR dynamics are investigated and free and forced response results are compared to full finite element model solutions. It is shown that the developed nominal methods have variable accuracy while the geometric approaches provide highly accurate results. A significant reduction in solution times were realized as compared to the full FEM and a traditional CB-CMS ROM without a secondary modal reduction.

8.1 Introduction

Advanced turbine engine designs require innovative technologies to push performance limits while simultaneously increasing engine efficiencies. This often requires the use and manipulation of secondary, or multiple bypassed flow-paths. Where simple bypasses only reduct airflow, advanced manipulation of the airflow relies on rotating engine components to further exchange mechanical energy with the bypassed flow stream. This may require special rotating components, termed Dual Flow-path Integrally Bladed Rotors (DFIBRs) and shown in Fig. 8.1, that are analogous to tradition IBRs except there are two sets of blades that are stacked radially. At the center of the DFIBR is a traditional IBR that has an integral ring attached to the inner-blade tips. A second set of integral blades are then attached to this outer-ring. As flow is bypassed around the inner-blades, the outer-blades can further energize the bypassed flow.

The closest current design to DFIBRs is a shrouded fan stage, but there are significant differences. First, shrouds are common to inserted blade designs and are uncommon to IBRs. These shrouds “lock” when the fan is at speed, but the connection is not rigid nor integral to the component as in DFIBRs. Furthermore, the shroud placement can vary
span-wise along the blades while the DFIBR ring is always placed at the inner-blade tip to seal off the bypassed flow. Another difference is the outer-blade count can differ from the inner-blade count. In addition, the inner- and outer-blades can be subject to different EO excitations since they operate in different flow-fields and can be affected by different up- and down-stream flow disturbances.

However, a commonality between DFIBRs and IBRs is their susceptibility to mistuning due the components’ integral nature and subsequent lack of both material and friction damping. Mistuning disrupts the rotational periodicity of a DFIBR, eliminating the possibility of cyclic symmetry analyses and can cause a localization of vibration energy [16]. Efficient and effective prediction of this phenomenon then requires advance ROMs since the entire DFIBR must now be modeled. These ROMs allow a greater exploration of design space as well provide efficient means to generate a full mistuned response distribution. However, modeling mistuned response of DFIBRs requires new approaches due to the many differences between DFIBRs and IBRs.
Existing research has been devoted to mistuned response prediction of traditional IBRs. This effort develops frequency and geometric mistuning ROMs for DFIBRs that contribute to Major Contribution 3 and Major Contribution 4 of Chapter 3. These ROMs are developed for DFIBRs composed of a tuned disk and ring with geometrically perturbed inner- and outer-blades. The containment of mistuning to the blades naturally partitions the DFIBR and is exploited by the Craig-Bampton component mode synthesis method (CB-CMS) [67]. The first mistuning method in this work further utilizes Interface modes for model reduction in a cyclic symmetry CB-CMS description. The second formulation follows suit and casts the CB-CMS constraint and disk fixed-interface normal mode DOF into a modal domain consisting of a truncated set of Ancillary modes. Mistuning of the constraint DOFs in the CB-CMS formulation are then projected onto a set of tuned Interface or Ancillary modes.

This study restricts the mistuning to the blades and assumes nominal, cyclically symmetric disk and ring geometries. Industrial DFIBR geometries still presents a computational burden since it can bear a majority of the DOFs of the entire DFIBR FEM. To accommodate this issue, the disk and ring are presented in a cyclic symmetry format that is partitioned accordingly for the CB-CMS framework. The single sector formulation begins by partitioning a DFIBR sector according to Fig. 8.2, where $\Gamma$ are interface DOF, $\alpha$ are independent interface DOF, $\beta$ are the dependent interface DOF, and $\sigma$ are non-interface DOF. The real-valued symmetrical components for a single sector have been calculated in Section 4.2 on page 40 and are carried out for cyclic disk and ring components that can be differentiated in the following sections with the notation: the $\tau$ DOF represent the $D\tau$ DOF for the Disk and $R\tau$ for the Ring in Fig. 8.2. The generic $\Gamma$ DOF represent the partition $\Gamma_D$, while the ring requires a different interface vector due to the additional outer-blades that is
Figure 8.2: Partitioned IBR index notation used in the mathematical formulation

given by

\[
\vec{x}_\Gamma^{(h)} = \begin{cases}
\vec{x}_{\Omega R} \\
\vec{x}_{\Omega 1} \\
\vdots \\
\vec{x}_{\Omega S}\end{cases}
\]  

(8.1)

where \( \Omega S \) is the number of outer-blades on each cyclic sector.
8.2 Cyclic Component Matrices

Calculation of the cyclic fixed-interface normal modes, $\tilde{\Phi}^{(h)}$, for a certain harmonic, $h$, are obtained by restraining all boundary DOF in Eq. A.11 and solving the classical eigenvalue problem (EVP)

$$
\left[ \tilde{K}^{(h)} - \lambda_j \tilde{M}^{(h)} \right] \tilde{\phi}^{(h)}_j = 0 \quad j = 1, 2, \ldots, k_n \ll N_\tau
$$

where $\lambda_j = \omega_j^2$ and $\omega_j$ are the natural frequencies, $k_n$ is some desired cutoff dictated by the frequency range of interest, and $N_\tau$ are the total number of interior DOF. This subset of modes is then combined into the matrix $\tilde{\Phi}^{(h)}$ while the corresponding $\lambda_j$ are combined into the diagonal spectral matrix $\Lambda^{(h)}$. The cyclic constraint modes, $\tilde{\Psi}^{(h)}$, are ascertained by statically deforming a component with a unit displacement to one coordinate of an established set of boundary coordinates while the remaining coordinates of the set are restrained, and the remaining interior DOF of the component are force-free by

$$
\tilde{\Psi}^{(h)} = -\tilde{K}^{-1(h)} \tilde{K}^{(h)}
$$

The transformation from cyclic coordinates, $\tilde{x}^{(h)}$, to cyclic CB modal coordinates, $\tilde{p}^{(h)}$, is accomplished with the cyclic CB modal transformation matrix

$$
\tilde{x}^{(h)} = \begin{bmatrix} \tilde{\Phi}^{(h)} & \tilde{\Psi}^{(h)} \\ 0 & I \end{bmatrix} \begin{bmatrix} \tilde{p}^{(h)}_n \\ \tilde{p}^{(h)}_c \end{bmatrix} = \tilde{U}^{(h)} \tilde{p}^{(h)}
$$

where the right subscript $n$ denotes motion due to the fixed-interface normal modes and $c$ is denotes motion due to the constraint modes. The CB modal transformation matrix can
be expanded for all harmonics \((∀h)\) according to

\[
\tilde{U} = \begin{bmatrix}
B_{∀h} \tilde{\Phi}^{(h)} & B_{∀h} \tilde{\Psi}^{(h)} \\
0 & I
\end{bmatrix}
\]  

(8.5)

Here \(B\) is the block diagonal operator that places the \(h^{th}\) argument in the \(h^{th}\) block on the diagonal, where the blocks are not required to be of equal size. Substituting Eq. 8.4 into the physical space equation of motion (EOM) and pre-multiplying by \(\tilde{U}^\top\) gives the cyclic CB space EOM as

\[
\tilde{M} \ddot{\tilde{p}} + \tilde{K} \tilde{p} = \tilde{F}
\]

(8.6)

where

\[
\tilde{M} = \begin{bmatrix}
I & \tilde{M}_{nc} \\
\tilde{M}_{nc}^\top & \tilde{M}_{cc}
\end{bmatrix}, \quad \tilde{K} = \begin{bmatrix}
\Lambda & 0 \\
0 & \tilde{K}_{cc}
\end{bmatrix}, \quad \tilde{F} = \begin{bmatrix}
\tilde{F}_n \\
\tilde{F}_c
\end{bmatrix}
\]

(8.7)

with submatrices given by

\[
\tilde{M}_{nc} = B_{∀h} \left[ \tilde{\Phi}^{(h)} \left( \tilde{M}_{rr}^{(h)} \tilde{\Psi}^{(h)} + \tilde{M}_{r\Gamma}^{(h)} \right) \right]
\]

\[
\tilde{M}_{cc} = B_{∀h} \left[ \tilde{\Psi}^{(h)} \left( \tilde{M}_{rr}^{(h)} \tilde{\Psi}^{(h)} + \tilde{M}_{r\Gamma}^{(h)} \right) + \tilde{M}_{r\Gamma}^{(h)} \tilde{\Psi}^{(h)} + \tilde{M}_{\Gamma\Gamma}^{(h)} \right]
\]

(8.8)

\[
\tilde{K}_{cc} = B_{∀h} \left[ \tilde{K}_{r\Gamma}^{(h)} + \tilde{K}_{r\Gamma}^{(h)} \tilde{\Psi}^{(h)} \right]
\]

\[
\Lambda = B_{∀h} \left[ \Lambda^{(h)} \right]
\]

and the cyclic force vector components are given by

\[
\tilde{F}_n^{(∀h)} = \begin{bmatrix}
\tilde{f}_n^{(1)} \\
\vdots \\
\tilde{f}_n^{(N/2)}
\end{bmatrix}, \quad \tilde{F}_c^{(∀h)} = \begin{bmatrix}
\tilde{f}_c^{(1)} \\
\vdots \\
\tilde{f}_c^{(N/2)}
\end{bmatrix}
\]

(8.9)
\[ \tilde{f}_n^{(h)} = \tilde{\Phi}^\top \tilde{f}_\tau^{(h)} \] \[ \tilde{f}_c^{(h)} = \tilde{\Psi}^\top \tilde{f}_\tau^{(h)} + \tilde{f}_\Gamma^{(h)} \] (8.10)

### 8.3 Blade Component Matrices

The CB-CMS formulation for the blades is simpler than the cyclic components since the blade component FEM matrices do not need to be reduced to a cyclic format. This allows the CB formulation to begin from the partitioned matrices for a single blade

\[ K = \begin{bmatrix} K_{\tau\tau} & K_{\tau \Gamma} \\ K_{\Gamma \tau} & K_{\Gamma \Gamma} \end{bmatrix}, \quad x = \begin{bmatrix} x_\tau \\ x_\Gamma \end{bmatrix}, \quad f = \begin{bmatrix} f_\tau \\ f_\Gamma \end{bmatrix} \] (8.11)

Again, the mass matrix follows the same partitionment. The inner-blade interface vector can be further partitioned by

\[ x_\Gamma = \begin{bmatrix} x_{\Gamma D} \\ x_{\Gamma R} \end{bmatrix} \] (8.12)

The fixed-interface normal modes, \( \Phi \), are computed from the eigen-problem for the interior, \( \tau \), DOF as done in Eq. 8.2 where \( \lambda_j = \omega_j^2 \) and \( \omega_j \) are the cantilevered blade natural frequencies for the outer-blades and the fixed-fixed natural frequencies of the inner-blades. The subset of modes is then combined into the matrix \( \Phi \) and the corresponding \( \lambda_j \) are placed in the diagonal spectral matrix \( \Lambda \). The constraint modes, \( \Psi \), are then calculated from Eq. 8.3 using blade matrices.

The transformation from physical coordinates, \( x \), to CB modal coordinates, \( p \), is accomplished with the cyclic CB modal transformation matrix

\[ x = \begin{bmatrix} \Phi & \Psi \\ 0 & I \end{bmatrix} \begin{bmatrix} p_n \\ p_c \end{bmatrix} = U p \] (8.13)
Using the above CB transformation matrix in the same manner as the cyclic components, the blade CB modal coordinates matrices are given by

\[
\mathcal{M}^{(a)} = \begin{bmatrix} I & \mathcal{M}_{nc} \\ \mathcal{M}_{nc}^T & \mathcal{M}_{cc} \end{bmatrix}, \quad \mathcal{K}^{(a)} = \begin{bmatrix} \Lambda & 0 \\ 0 & \mathcal{K}_{cc} \end{bmatrix}, \quad \mathcal{F}^{(a)} = \begin{bmatrix} f_n \\ f_c \end{bmatrix}
\]

(8.14)

with the superscript \((a)\) serving as a reminder that these matrices are for the single blade, \(a\), and

\[
\mathcal{M}_{nc} = \Phi^T (M_{\tau\tau} \Psi + M_{\tau\Gamma}) \\
\mathcal{M}_{cc} = \Psi^T (M_{\tau\tau} \Psi + M_{\tau\Gamma}) + M_{\tau\Gamma}^T \Psi + M_{\Gamma\Gamma} \\
\mathcal{K}_{cc} = K_{\Gamma\Gamma} + K_{\tau\Gamma}^T \Psi
\]

(8.15)

\[
f_n = \Phi^T f_\tau \\
f_c = \Psi^T f_\tau + f_\Gamma
\]

8.4 Nominal Mode Approaches

8.4.1 Formulation for all Blades

All the calculations in Section 8.3 have been for a single blade, \(a\). If all blades are tuned, these calculations would only have to be done once for the inner-blades and once for the outer-blades, since the component matrices would be the same for all \(IN\) or \(ON\) blades because

\[
U^{(Tuned)} = U^{(1)} = U^{(2)} = \ldots = U^{(*)N}
\]

(8.16)

where the bullet, \(\bullet\), is a placeholder for either \(I\) or \(O\). The CB-CMS reduced mass and stiffness matrices containing all \(IN\) or \(ON\) blades are generated using a single inner- and
outer-blade, say \( a = 1 \), so

\[
\mathcal{M} = \begin{bmatrix}
I & I \otimes \mathcal{M}_{nc}^{(1)} \\
I \otimes \mathcal{M}_{nc}^{(1)} & I \otimes \mathcal{M}_{cc}^{(1)}
\end{bmatrix},
\mathcal{K} = \begin{bmatrix}
I \otimes \Lambda^{(1)} & 0 \\
0 & I \otimes \mathcal{K}_{cc}^{(1)}
\end{bmatrix}
\] (8.17)

where the symbol \( \otimes \) is the Kronecker product and the dimensions of the identity matrix, \( I \), is either \( I_N \times I_N \) or \( O_N \times O_N \). The blade forcing on all blades can then be expressed as

\[
\mathcal{F} = P_C \otimes \begin{bmatrix}
f_n^{(1)} \\
f_c^{(1)}
\end{bmatrix}
\] (8.18)

The calculations of this section have been carried out for specific blade set, i.e. inner- or outer-blades. Therefore, the notation for all blades (\( \forall a \)) requires all blades be of the same set. The preceding section will then produce CB modal matrices for each blade type where a left superscript of \( I \) or \( O \) will be placed on the respective Inner- and Outer-blade component matrices to differentiate between each in the following sections.

### 8.4.2 Rearrange Blade Matrices

Once the formulation of the blade CB-CMS matrices is complete, the inner blade matrices must be re-ordered since these blades are coupled to both the disk and ring. Further inspection shows that the constraint portions of the inner-blade matrices for a single blade are organized according to

\[
\mathcal{I}^p_c = \begin{bmatrix}
\mathcal{I}^p_{ID} \\
\mathcal{I}^p_{IR}
\end{bmatrix}
\] (8.19)

where the subscript \( c \) has been replaced with \( \Gamma \) for convenience, but is still associated with the constraint DOF since the CB formulation requires that \( x_\Gamma = p_c \), where \( x_\Gamma \) is given in Eq. 8.12. The single component matrices of Eq. 8.15 inner-blade are further partitioned
according to

\[
[i \mathcal{M}_{nc} = 
\begin{bmatrix}
[i \mathcal{M}_{n\Gamma D} & i \mathcal{M}_{n\Gamma R}
\end{bmatrix},
\]

\[
i \mathcal{M}_{cc} = 
\begin{bmatrix}
[i \mathcal{M}_{\Gamma D\Gamma D} & i \mathcal{M}_{\Gamma D\Gamma R} \\
[i \mathcal{M}_{\Gamma R\Gamma D} & i \mathcal{M}_{\Gamma R\Gamma R}
\end{bmatrix}
\]

\[
(i f_{c} = \begin{bmatrix} i f_{\Gamma D} \\
i f_{\Gamma R}
\end{bmatrix}
\]

where \(i \mathcal{K}_{cc}\) follows the same partitionment. All inner- and outer-blade matrices are then rearranged for direct coupling to the disk and ring according to

\[
\begin{bmatrix}
I_p^{(\gamma a)} \\
\vdots \\
o_p^{(\gamma a)}
\end{bmatrix} = 
\begin{bmatrix}
I_p^{(\gamma a)} \\
o_p^{(\gamma a)}
\end{bmatrix} = 
\begin{bmatrix}
I_p^{(\gamma a)} \\
o_p^{(\gamma a)}
\end{bmatrix} = T
\begin{bmatrix}
A_p^{(\gamma a)} \\
o_p^{(\gamma a)}
\end{bmatrix}
\]

where the left superscript, \(A\), has been added to denote those matrices and vectors that have both inner- and outer-blade components. The boundary DOF partitions are then given by

\[
i p^{(\gamma a)}_{\Gamma D} = \begin{bmatrix} I_p^{(1)}_{\Gamma D} \\
\vdots \\
i p^{(N)}_{\Gamma D}
\end{bmatrix},
p^{(\gamma a)}_{\Gamma} = \begin{bmatrix} I_p^{(1)}_{\Gamma R} \\
\vdots \\
i p^{(N)}_{\Gamma R}
\end{bmatrix},
o p^{(\gamma a)}_{\Gamma O} = \begin{bmatrix} O_p^{(1)}_{\Gamma O} \\
\vdots \\
o p^{(N)}_{\Gamma O}
\end{bmatrix}
\]

(8.22)
The sub-partitions to the transformation matrix, $T$, are composed of matrices filled with ones and zeros that allow the re-ordering in Eq. 8.21 and can be represented for $OS = 2$ as

$$
\begin{align*}
T_1 &= I_{N \times N} \otimes \begin{bmatrix} I \\ 0 \end{bmatrix} \\
T_2 &= I_{N \times N} \otimes \begin{bmatrix} 0 & 0 \\ I & 0 \end{bmatrix} \\
T_3 &= I_{N \times N} \otimes \begin{bmatrix} 0 \\ I \end{bmatrix}
\end{align*}
$$

(8.23)

where the right subscripts $N \times N$ denote the size of the matrix. The remaining dimension of the Identity and Null matrices on the right of the equality are of dimension that allow the matrix multiplication of Eq. 8.21 to commute. The blade matrices are re-ordered with the transformation matrix, $T$ of Eq. 8.21, according to

$$
\begin{align*}
\mathbf{A}_K &= T^\top \begin{bmatrix} I_K & 0 \\ 0 & \mathbf{O}_K \end{bmatrix} \\
\mathbf{A}_M &= T^\top \begin{bmatrix} I_M & 0 \\ 0 & \mathbf{O}_M \end{bmatrix} \\
\mathbf{A}_F &= T^\top \begin{bmatrix} \mathbf{I}_F \\ \mathbf{O}_F \end{bmatrix} = \begin{bmatrix} \mathbf{A}_F_n \\ \mathbf{A}_F_c \end{bmatrix}
\end{align*}
$$

(8.24)

where the inner- and outer-blade component matrices, $K$ and $M$, are from Eq. 8.17 and the force vector, $F$, is from Eq. 8.18. The numerous sub-partitions to Eq. 8.24 can be further
described by

\[
\begin{bmatrix}
I_A & 0 \\
0 & O_A
\end{bmatrix}
\]  
(8.25)

\[
\begin{bmatrix}
M_{n_1\Gamma_D} & M_{n_1\Gamma} \\
0 & M_{n_0\Gamma}
\end{bmatrix},
\begin{bmatrix}
M_{\Gamma_D\Gamma_D} & M_{\Gamma_D\Gamma} \\
M_{\Gamma_D\Gamma}^T & A_M^{\Gamma_G}
\end{bmatrix}
\]  
(8.26)

\[
\begin{bmatrix}
I_P \otimes I_f_n \\
O_P \otimes O_f_n
\end{bmatrix},
\begin{bmatrix}
F^{\Gamma_D} \\
A^{\Gamma_G}
\end{bmatrix}
\]  
(8.27)

where \(A_K^{cc}\) follows the same partitionment as \(A_M^{cc}\). The previous equations are populated from a single inner- and single outer-blade through the following sub-partitions

\[
M_{\Gamma_D\Gamma_D} = I \otimes I^{M_{\Gamma_D\Gamma_D}}^{(1)}
\]

\[
M_{\Gamma_D\Gamma} = I \otimes \begin{bmatrix} I^{M_{\Gamma_D\Gamma_R}}^{(1)} & 0 \end{bmatrix}
\]

\[
A_M^{\Gamma_G} = I \otimes \begin{bmatrix} I^{M_{\Gamma_R\Gamma_R}}^{(1)} & 0 \\
0 & O \otimes O^{M^{(1)}}_{nc}
\end{bmatrix}
\]  
(8.28)

\[
M_{n_1\Gamma_D} = I \otimes I^{M_{n_1\Gamma_D}}^{(1)}
\]

\[
M_{n_1\Gamma} = I \otimes \begin{bmatrix} I^{M_{n_1\Gamma_R}}^{(1)} & 0 \end{bmatrix}
\]

\[
M_{n_0\Gamma} = I \otimes \begin{bmatrix} 0 & O \otimes O^{M_{nc}}^{(1)}
\end{bmatrix}
\]
where $I^I$ and $O^I$ are identity matrices of size $I^N \times I^N$ and $O^S \times O^S$, respectively. The traveling wave force components are given by:

$$\mathcal{F}_{\Gamma_D} = T_1 (I^I P_C \otimes I^f) = I^I P_C \otimes I^f_{\Gamma_D}$$

(8.29)

$$A\mathcal{F}_{\Gamma} = T_2 (I^I P_C \otimes I^f) + T_3 (O^S P_C \otimes O^f) = \begin{bmatrix} I^I e^{i\varphi_{1,C}} I^f_{\Gamma_R} \\
O^I e^{i\varphi_{1,C}} O^f_{\Gamma_1} \\
O^I e^{i\varphi_{2,C}} O^f_{\Gamma_2} \\
\vdots \\
I^I e^{i\varphi_{N,C}} I^f_{\Gamma_R} \\
O^I e^{i\varphi_{2N-1,C}} O^f_{\Gamma_1} \\
O^I e^{i\varphi_{2N,C}} O^f_{\Gamma_2} \end{bmatrix}$$

(8.30)

It is apparent from Eq. 8.29 that the inner-blade force component contains only inner-blade EO excitations. However, Eq. 8.30 illustrates that this force component has both inner- and outer-blade EO excitations.

### 8.4.3 Component Coupling

CB-CMS model assembly requires interface displacement compatibility $I^I x_{\Gamma_D} = D^I x_{\Gamma}$ and $A^A x_{\Gamma} = R^A x_{\Gamma}$ in physical coordinates. Expanding this and utilizing the CB-CMS requirement that $x_{\Gamma} = p_c$ yields

$$ \begin{bmatrix} I^I x_{\Gamma_D}^{(\varphi a)} \\ A^A x_{\Gamma}^{(\varphi a)} \end{bmatrix} = \begin{bmatrix} I^I P_{\Gamma_D}^{(\varphi a)} \\ A^A P_c^{(\varphi a)} \end{bmatrix} = \begin{bmatrix} I^I \hat{E} & 0 \\ 0 & O^I \hat{E} \end{bmatrix} \begin{bmatrix} D^I P_c^{(\varphi h)} \\ R^I P_c^{(\varphi h)} \end{bmatrix} = \begin{bmatrix} D^I x_{\Gamma} \\ R^A x_{\Gamma} \end{bmatrix}$$

(8.31)

where $I^I \hat{E} = E \otimes I^I$, $O^I \hat{E} = E \otimes O^I$, and $E$ is the $N \times N$ real-valued Fourier matrix defined in Appendix B. Identity matrices $I^I$ and $O^I$ are of size $N_{\Gamma_D} = length(I^I x_{\Gamma_D}^{(\varphi)})$ and $N_{\Gamma} = length(x_{\Gamma})$, respectively.
To constrain the blades to the disk and ring, cyclic DOF $D_{\mathbf{p}_n}^{(\gamma h)}$ and $R_{\mathbf{p}_n}^{(\gamma h)}$ are kept as active DOF by the following

$$
\begin{cases}
D_{\mathbf{p}_n}^{(\gamma h)} & I 0 0 0 0 \\
D_{\mathbf{p}_c}^{(\gamma h)} & 0 0 I 0 0 \\
R_{\mathbf{p}_n}^{(\gamma h)} & 0 I 0 0 0 \\
R_{\mathbf{p}_c}^{(\gamma h)} & 0 0 0 I 0 \\
I_{\mathbf{p}_n}^{(\gamma a)} & 0 0 0 0 I \\
O_{\mathbf{p}_n}^{(\gamma a)} & 0 0 0 0 I \\
I_{\mathbf{p}_D}^{(\gamma a)} & 0 0 I_{E} 0 0 \\
\mathbf{p}_\Gamma^{(\gamma a)} & 0 0 0_{E} 0 0 \\
\end{cases}
\begin{bmatrix}
I 0 0 0 0 \\
0 0 I 0 0 \\
0 I 0 0 0 \\
0 0 0 I 0 \\
0 0 0 0 I \\
0 0 0 0 I \\
0 0 I_{E} 0 0 \\
0 0 0_{E} 0 0 \\
\end{bmatrix}
\begin{cases}
D_{\mathbf{p}_n}^{(\gamma h)} \\
D_{\mathbf{p}_c}^{(\gamma h)} \\
R_{\mathbf{p}_n}^{(\gamma h)} \\
R_{\mathbf{p}_c}^{(\gamma h)} \\
I_{\mathbf{p}_n}^{(\gamma a)} \\
I_{\mathbf{p}_c}^{(\gamma a)} \\
\mathbf{p}_\Gamma^{(\gamma a)} \\
\end{cases}
= T\mathbf{p}
$$

The resulting coupled CB-CMS ROM matrices are obtained by

$$
\begin{align*}
C_{B}\mathbf{M} &= T^T 
\begin{bmatrix}
D\tilde{\mathbf{M}} & 0 & 0 \\
0 & R\tilde{\mathbf{M}} & 0 \\
0 & 0 & A\mathbf{M} \\
\end{bmatrix}
T = 
\begin{bmatrix}
I & \tilde{\mathbf{M}}_{nc} & 0 \\
\tilde{\mathbf{M}}_{nc} & \tilde{\mathbf{M}}_{cc} & A\mathbf{M}_{nc} \\
0 & A\mathbf{M}_{nc} & I \\
\end{bmatrix} \\
C_{B}\mathbf{K} &= T^T 
\begin{bmatrix}
D\tilde{\mathbf{K}} & 0 & 0 \\
0 & R\tilde{\mathbf{K}} & 0 \\
0 & 0 & A\mathbf{K} \\
\end{bmatrix}
T = 
\begin{bmatrix}
\Lambda & 0 & 0 \\
0 & \tilde{\mathbf{K}}_{cc} & 0 \\
0 & 0 & A\Lambda \\
\end{bmatrix} \\
C_{B}\mathbf{F} &= T^T 
\begin{bmatrix}
D\tilde{\mathbf{F}} \\
R\tilde{\mathbf{F}} \\
A\mathbf{F} \\
\end{bmatrix}
= 
\begin{bmatrix}
\tilde{\mathbf{F}}_n \\
\tilde{\mathbf{F}}_c \\
A\mathbf{F}_n \\
\end{bmatrix}
\end{align*}
$$

The sub-partitions $A\Lambda$, $A\mathbf{M}_{nc}$, and $A\mathbf{F}_n$ are given in Equations 8.25 - 8.27, respectively. The
remaining sub-partitions are given by

\[
\Lambda = \begin{bmatrix}
D_\Lambda & 0 \\
0 & R_\Lambda \\
\end{bmatrix}
\]

\[
\tilde{M}_{nc} = \begin{bmatrix}
D \tilde{M}_{nc} & 0 \\
0 & R \tilde{M}_{nc} \\
\end{bmatrix}
\]

\[
(8.34)
\]

\[
\tilde{M}_{cc} = \begin{bmatrix}
D \tilde{M}_{cc} + i \hat{E}^\dagger A \tilde{M}_{\Gamma D \Gamma D} i \hat{E} & i \hat{E}^\dagger A \tilde{M}_{\Gamma D \Gamma D} O \hat{E} \\
o \hat{E}^\dagger A \tilde{M}_{\Gamma D \Gamma D} i \hat{E} & R \tilde{M}_{cc} + o \hat{E}^\dagger A \tilde{M}_{\Gamma \Gamma} O \hat{E} \\
\end{bmatrix}
\]

(8.35)

where \( \tilde{K}_{cc} \) follows the same partitionment as \( \tilde{M}_{cc} \). The force components are given by

\[
\tilde{F}_n = \begin{bmatrix}
D \tilde{F}_n^{(\forall h)} \\
R \tilde{F}_n^{(\forall h)} \\
\end{bmatrix}
\]

\[
\tilde{F}_c = \begin{bmatrix}
D \tilde{F}_c^{(\forall h)} + \tilde{F}_\Gamma^{(\forall h)} \\
R \tilde{F}_c^{(\forall h)} + \tilde{F}_\Gamma^{(\forall h)} \\
\end{bmatrix}
\]

(8.36)

where

\[
\tilde{F}_\Gamma^{(\forall h)} = i \hat{E}^\dagger F_\Gamma 
\]

(8.37)

\[
\tilde{F}_\Gamma^{(\forall h)} = o \hat{E}^\dagger A F_\Gamma 
\]

(8.38)

are the force vectors in physical space transformed into cyclic space. The cyclic blade force vector \( \tilde{F}_\Gamma^{(\forall h)} \) will have rows of mostly zeros due to the orthogonality between the EO
excitation and the harmonics of the structure and can be represented by

\[
\tilde{\mathbf{F}}_{\Gamma D}^{(\forall h)} = \begin{bmatrix}
\tilde{f}_{\Gamma D}^{(0)} \\
\tilde{f}_{\Gamma D}^{(1,c)} \\
\tilde{f}_{\Gamma D}^{(1,s)} \\
\vdots \\
\tilde{f}_{\Gamma D}^{(h,c)} \\
\tilde{f}_{\Gamma D}^{(h,s)} \\
\vdots \\
\tilde{f}_{\Gamma D}^{(N/2)}
\end{bmatrix}, \quad \tilde{\mathbf{F}}_{\Gamma}^{(\forall h)} = \begin{bmatrix}
\tilde{f}_{\Gamma}^{(0)} \\
\tilde{f}_{\Gamma}^{(1,c)} \\
\tilde{f}_{\Gamma}^{(1,s)} \\
\vdots \\
\tilde{f}_{\Gamma}^{(h,c)} \\
\tilde{f}_{\Gamma}^{(h,s)} \\
\vdots \\
\tilde{f}_{\Gamma}^{(N/2)}
\end{bmatrix}
\]

(8.39)

where the rows \( \tilde{f}_{\Gamma D}^{(h)} = 0 \) where \( C \neq h \) due to orthogonality between natural harmonics of the DFIBR and the EO excitations and

\[
\tilde{f}_{\Gamma D}^{(h,c)} = e^{(h,c)} \mathbf{P}_C \tilde{\mathbf{f}}_{\Gamma D}^{(\forall a)}
\]

(8.40)

\[
\tilde{f}_{\Gamma D}^{(h,s)} = e^{(h,s)} \mathbf{P}_C \tilde{\mathbf{f}}_{\Gamma D}^{(\forall a)}
\]

(8.41)

and \( e^{(h)} \) are the columns of the Fourier matrix from Appendix B

\[
E = \begin{bmatrix}
e^{(0)} & e^{(1,c)} & e^{(1,s)} & \ldots & e^{(h,c)} & e^{(h,s)} & \ldots & e^{(N/2)}
\end{bmatrix}
\]

(8.42)

The cyclic force vector \( \tilde{\mathbf{F}}_{\Gamma}^{(\forall h)} \) will have rows, \( \tilde{f}_{\Gamma}^{(h)} = 0 \), when \( h \neq lC \) or \( O C \). The rows \( \tilde{f}_{\Gamma}^{(h)} \neq 0 \) can be decomposed into

\[
\tilde{f}_{\Gamma}^{(h)} = \begin{bmatrix}
I \tilde{f}_{\Gamma l}^{(h)} \\
O \tilde{f}_{\Gamma 1}^{(h)} \\
O \tilde{f}_{\Gamma 2}^{(h)}
\end{bmatrix}
\]

(8.43)
where the outer blade force vectors obey \( \bar{f}_{r_i}^{(h=IC)} = 0 \) and the inner-blade force vectors obey \( \bar{f}_{r_i}^{(h=OC)} = 0 \).

### 8.4.4 Nominal Method CB-CMS ROM

The previous formulations utilized a cyclic disk and ring description that kept the constraint DOF in cyclic coordinates. The CB-CMS EOM is then formulated as

\[
\begin{align*}
\bar{CB} \ddot{p} + \bar{CB} \dot{p} + (1 + G_i) \bar{CB} \kappa \dot{p} &= \bar{CB} \mathcal{F} \\
\end{align*}
\]  

(8.44)

where the blade modal damping matrix and structural damping coefficient, \( \bar{CB} \kappa_C \) and \( G \), respectively, can be included to better model dynamic response [39]. The mass and stiffness matrices and forcing vectors are given in Eq. 8.33 while the blade modal damping matrix is given by

\[
\bar{CB} \kappa_C = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & A_C
\end{bmatrix}
\]  

(8.45)

where

\[
A_C = \begin{bmatrix}
I_i \otimes \text{diag} \left( 2 \zeta_j \right) \sqrt{I_i} & 0 \\
0 & O_i \otimes \text{diag} \left( 2 \zeta_j \right) \sqrt{O_i}
\end{bmatrix}
\]  

(8.46)

where \( I_i \) and \( O_i \) are identify matrices of size \( I_N \times I_N \) and \( O_N \times O_N \), respectively, and \( \zeta_j \) is the damping coefficient for the \( j^{th} \) blade mode. This CB-CMS model is dominated by the unnecessary retention of all the interface DOF and, at times, the disk and ring normal DOF. The following sections seek to further reduce the size of this CB-CMS ROM by introducing a secondary modal analysis on this model.
8.5 Secondary Modal Reduction ROMs

The CB-CMS methodology requires retention of all constraint DOF. For DFIBRs with a large inner- and outer-blade count, the ROM will be dominated by the interface DOF that prevent an ultimate reduction in model size. To reduce this burden, a secondary eigen-analysis can be performed on portions of the CB-CMS system matrices that will re-cast these portions into a new modal domain. Two approaches will be performed in this work: first, the secondary eigen-analysis will be carried out on the constraint DOF partitions of the CB-CMS system matrices. The resulting truncated set of eigenvectors will yield a set of Interface modes. The second approach will perform an eigen-analysis on the constraint and disk and ring fixed-interface normal mode DOF partitions. The resulting truncated set of eigenvectors are termed Ancillary modes. The cyclic formulation offers computational savings by carrying out the analysis in cyclic coordinates which allows each harmonic index to be solved independently.

In the following subsections, two mistuning approaches are presented that use either the tuned Interface or Ancillary modes to further reduce the CB-CMS ROM size of Eq. 8.44. In both approaches, the generic EOM is given by

\[ M_r \ddot{q}_r + C_r \dot{q}_r + (1 + Gi) K_r q_r = F_r \] (8.47)

where subscript \( r \) refers to reduced and the EOM matrices are defined in the following subsections.

8.5.1 Interface Mode Reduction

Keeping \( D_p^{(vh)} \) and \( R_p^{(vh)} \) as active DOF in Eq. 8.32 keeps the constraint DOF in cyclic coordinates that allows the constraint modes to be calculated at decoupled harmonics. However, these constraint portions are ordered by component and need to be reordered by har-
monics according to

\[
\begin{pmatrix}
D_{\tilde{P}_c}^{(0)} \\
\vdots \\
D_{\tilde{P}_c}^{(N/2)}
\end{pmatrix}
= \begin{pmatrix}
D_{\tilde{P}_c}^{(0)} \\
\vdots \\
D_{\tilde{P}_c}^{(N/2)}
\end{pmatrix}
\begin{bmatrix}
T_4 \\
T_5
\end{bmatrix}
\]

(8.48)

where the above transformation matrices are filled with ones and zeros to allow the re-ordering to be done, or specifically

\[
T_4 = \begin{bmatrix}
I_{\Gamma_D} & 0 \\
0 & I_{(N/2 - 1)} \otimes \begin{bmatrix} I_{2\Gamma_D} & 0 \\
0 & I_{\Gamma_D}
\end{bmatrix}
\end{bmatrix}
\]

(8.49)

\[
T_5 = \begin{bmatrix}
0 & I_{\Gamma} \\
0 & I_{(N/2 - 1)} \otimes \begin{bmatrix} 0 & I_{2\Gamma} \\
0 & I_{\Gamma}
\end{bmatrix}
\end{bmatrix}
\]

(8.50)

Re-ordering the constraint portions of the CB-CMS matrices to ordered harmonics gives

\[
^{CB}_{\tilde{M}}_{cc}^{(h-ord)} = \begin{pmatrix}
T_4^T \\
T_5^T
\end{pmatrix}
^{CB}_{\tilde{M}}_{cc}
\begin{bmatrix}
T_4 \\
T_5
\end{bmatrix}
\]

(8.51)

where the superscript (h-ord) reminds that it is ordered by harmonic indices. This matrix has a block-diagonal structure, where the \(h^{th}\) block corresponds to the constraint DOF
symmetrical components of the $h^{th}$ harmonic. The stiffness matrix is ordered in such a manner as well. The Interface mode shapes can then calculated one harmonic at a time by the following EVP

$$
\begin{align*}
\left[ \tilde{K}_{cc}^{(h-ord)} - \lambda_j \tilde{M}_{cc}^{(h-ord)} \right] \tilde{\phi}_j^{(h)} &= 0, \quad j = 1, \ldots, k_{cc} \ll N_{\Gamma_I} + N_{\Gamma} \\
\end{align*}
$$

(8.52)

where the Interface modes are assembled into the modal matrix $\tilde{\Phi}_{cc}^{(h)} = [\tilde{\phi}_1^{(h)}, \ldots, \tilde{\phi}_{k_{cc}}^{(h)}]$ and then combined for all harmonics by

$$
\tilde{\Phi}_{cc} = \begin{bmatrix} T_4 & T_5 \end{bmatrix} \begin{bmatrix} \tilde{\Phi}^{(h)}_{cc} \end{bmatrix}
$$

(8.53)

Since a limited set, $k_{cc}$, of tuned Interface modes is retained, the CB-CMS system can be further reduced through the following transformation matrix

$$
\begin{align*}
\mathbf{p} &= \begin{bmatrix} \tilde{\mathbf{p}}_n \\ \tilde{\mathbf{p}}_c \\ A^{\mathbf{p}} \mathbf{n} \end{bmatrix} = \begin{bmatrix} I & 0 & 0 \\ 0 & \tilde{\Phi}_{cc} & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{q}}_n \\ \tilde{\mathbf{q}}_c \\ A^{\mathbf{q}} \mathbf{n} \end{bmatrix} = T_{CC} \mathbf{q}_r
\end{align*}
$$

(8.54)

Substituting this into the CB-CMS EOM of Eq. 8.44 and pre-multiplying by $T_{CC}^T$ yields the
following matrices for the reduced ROM of Eq. 8.47

\[
\mathcal{M}_r = \begin{bmatrix}
I & \tilde{\mathcal{M}}_{\text{nc}}\bar{\Phi}_{\text{cc}} & 0 \\
\bar{\Phi}_{\text{ec}}^\top \tilde{\mathcal{M}}_{\text{nc}} & \bar{\Phi}_{\text{ec}}^\top \tilde{\mathcal{M}}_{\text{nc}} \bar{\Phi}_{\text{cc}} & \bar{\Phi}_{\text{ec}}^\top \tilde{\mathcal{M}}_{\text{nc}} \bar{\Phi}_{\text{cc}} \\
0 & A\tilde{\mathcal{M}}_{\text{nc}} \bar{\Phi}_{\text{cc}} & I
\end{bmatrix}
\]

\[
\mathcal{K}_r = \begin{bmatrix}
\Lambda & 0 & 0 \\
0 & \bar{\Phi}_{\text{ec}}^\top \mathcal{K}_{\text{cc}} \bar{\Phi}_{\text{cc}} & 0 \\
0 & 0 & A^2\Lambda
\end{bmatrix}
\]

\[
\mathcal{F}_r = \begin{bmatrix}
\tilde{\mathcal{F}}_n \\
\bar{\Phi}_{\text{ec}}^\top \tilde{\mathcal{F}}_c \\
A\mathcal{F}_n
\end{bmatrix}
\]

Note that \( C_r \) is obtained in the same manner, except only the size of the null entries are changed, so it is not re-listed.

### 8.5.2 Ancillary Mode Reduction

The disk and ring components are modeled as tuned so the normal DOF pertaining to these substructures remain unchanged from one mistuned DFIBR to the next. Therefore, the cyclic constraint partitions and disk and ring normal partitions can be re-ordered by harmonics to yield a block-diagonal structure that shares the same computational benefits of cyclic symmetry. This re-ordering by harmonics is done according to

\[
\tilde{\mathbf{P}}^{(\text{vh})} = \begin{bmatrix}
D^{(\text{vh})} \mathbf{P}_n \\
R^{(\text{vh})} \mathbf{P}_n \\
D^{(\text{vh})} \mathbf{P}_e \\
R^{(\text{vh})} \mathbf{P}_e
\end{bmatrix} = \begin{bmatrix}
T_6 \\
T_7 \\
T_8 \\
T_9
\end{bmatrix} \begin{bmatrix}
\tilde{\mathbf{p}}^{(1)}_s \\
\vdots \\
\tilde{\mathbf{p}}^{(N/2)}_s
\end{bmatrix} = T_s \tilde{\mathbf{P}}^{(\text{h-ord})}_s
\]
\[ \tilde{p}^{(h)}_s = \begin{cases} \begin{pmatrix} D \tilde{p}^{(h)}_n \\ R \tilde{p}^{(h)}_n \\ D \tilde{p}^{(h)}_c \\ R \tilde{p}^{(h)}_c \end{pmatrix} \\ \end{cases} \quad (8.57) \]

and the re-ordering matrices are filled with ones and zeros given by

\[ T_6 = I_{(N^2+1)} \otimes \begin{bmatrix} I_{Dk_n} & 0 & 0 \\ 0 & I_{Bk_n} & 0 \end{bmatrix} \quad (8.58) \]

\[ T_7 = I_{(N^2+1)} \otimes \begin{bmatrix} 0 & I_{Bk_n} & 0 \end{bmatrix} \quad (8.59) \]

\[ T_8 = \begin{bmatrix} 0 & 0 & I_{\Gamma D} & 0 \\ 0 & I_{(N^2-1)} \otimes \begin{bmatrix} 0 & 0 & I_{2\Gamma D} & 0 \end{bmatrix} & 0 \\ 0 & 0 & 0 & I_{\Gamma D} \\ 0 & 0 & 0 & I_{(N^2-1)} \otimes \begin{bmatrix} 0 & 0 & 0 & I_{2\Gamma R} \end{bmatrix} \end{bmatrix} \quad (8.60) \]

\[ T_9 = \begin{bmatrix} 0 & 0 & I_{\Gamma R} \\ 0 & I_{(N^2-1)} \otimes \begin{bmatrix} 0 & 0 & 0 & I_{2\Gamma R} \end{bmatrix} & 0 \\ 0 & 0 & 0 & I_{\Gamma R} \end{bmatrix} \quad (8.61) \]

where \( T_6 \) and \( T_7 \) can vary depending on the number of fixed-interface normal modes retained at each harmonic. The corresponding matrices are obtained by

\[ \tilde{M}_{ss} = \mathbb{E}_{\forall \nu h} \left[ \tilde{M}_{ss}^{(h)} \right] = T_8^T \begin{bmatrix} I & \tilde{M}_{nc} \\ \tilde{M}_{nc}^T & \tilde{M}_{cc} \end{bmatrix} T_8 \quad (8.62) \]

\[ \tilde{K}_{ss} = \mathbb{E}_{\forall \nu h} \left[ \tilde{K}_{ss}^{(h)} \right] = T_8^T \begin{bmatrix} \Lambda & 0 \\ 0 & \tilde{K}_{cc} \end{bmatrix} T_8 \quad (8.63) \]
where the submatrices to the above equations are given in Eq. 8.34. Following this re-ordering, the harmonic ordered matrices are obtained

\[
\tilde{M}_{ss}^{(h)} = \begin{bmatrix}
D_I & 0 & D\tilde{M}_{nc}^{(h)} & 0 \\
0 & R_I & 0 & R\tilde{M}_{nc}^{(h)} \\
D\tilde{M}_{nc}^{(h)} & 0 & D\tilde{M}_{cc}^{(h)} + A\tilde{M}_{\Gamma_D \Gamma_D}^{(h)} & A\tilde{M}_{\Gamma_D \Gamma}^{(h)} \\
0 & R\tilde{M}_{nc}^{(h)} & A\tilde{M}_{\Gamma_D \Gamma}^{(h)} & R\tilde{M}_{cc}^{(h)} + A\tilde{M}_{\Gamma \Gamma}^{(h)}
\end{bmatrix}
\]  

\(8.64\)

\[
\tilde{K}_{ss}^{(h)} = \begin{bmatrix}
D\Lambda^{(h)} & 0 & 0 & 0 \\
0 & R\Lambda^{(h)} & 0 & 0 \\
0 & 0 & D\tilde{K}_{cc}^{(h)} + A\tilde{K}_{\Gamma_D \Gamma_D}^{(h)} & 0 \\
0 & 0 & 0 & R\tilde{K}_{cc}^{(h)} + A\tilde{K}_{\Gamma \Gamma}^{(h)}
\end{bmatrix}
\]  

\(8.65\)

and

\[
A\tilde{M}_{\Gamma_D \Gamma_D}^{(\gamma h)} = \hat{I}E^T A\tilde{M}_{\Gamma_D \Gamma_D} \hat{I}E
\]  

\(8.66\)

\[
A\tilde{M}_{\Gamma \Gamma}^{(\gamma h)} = \hat{O}E^T A\tilde{M}_{\Gamma \Gamma} \hat{O}E
\]  

\(8.67\)

\[
A\tilde{M}_{\Gamma_D \Gamma}^{(\gamma h)} = \hat{I}E^T A\tilde{M}_{\Gamma_D \Gamma} \hat{O}E
\]  

\(8.68\)

The stiffness terms \(A\tilde{K}_{\Gamma_D \Gamma_D}^{(\gamma h)}\) and \(A\tilde{K}_{\Gamma \Gamma}^{(\gamma h)}\) follow that of the corresponding mass terms.

The Ancillary mode shapes can then be calculated one harmonic at a time by the following EVP

\[
\left[\tilde{K}_{ss}^{(h)} - \lambda_j \tilde{M}_{ss}^{(h)}\right] \tilde{\phi}_j^{(h)} = 0, \quad j = 1, \ldots, k_{ss} \ll Dk_n + Rk_n + N_{\Gamma_D} + N_{\Gamma}
\]  

\(8.69\)

where the Ancillary modes are assembled into the modal matrix \(\tilde{\Phi}_{ss} = [\tilde{\phi}_1, \ldots, \tilde{\phi}_{k_{ss}}]\)
and then combined for all harmonics by

\[
\tilde{\Phi}_{ss} = \begin{bmatrix}
T_6 \\
T_7 \\
T_8 \\
T_9
\end{bmatrix}
\mathbb{B} \begin{bmatrix}
\tilde{\Phi}_{ss}^{(h)}
\end{bmatrix}
\quad (8.70)
\]

Since a limited set, \(k_{ss}\), of tuned Ancillary modes is retained, the CB-CMS system can be further reduced through the following transformation matrix

\[
p = \begin{bmatrix}
\tilde{p} \\
A_{p_n}
\end{bmatrix} = \begin{bmatrix}
\tilde{\Phi}_{ss} & 0 \\
0 & I
\end{bmatrix} \begin{bmatrix}
\tilde{q} \\
A_{q_n}
\end{bmatrix} = T_{CA}q_r
\quad (8.71)
\]

Substituting this into the CB-CMS EOM of Eq. 8.44 and pre-multiplying by \(T_{CA}^\top\) yields the following matrices for the reduced ROM of Eq. 8.47

\[
\mathcal{M}_r = \begin{bmatrix}
\tilde{\mathcal{M}}_{ss} & \tilde{\mathcal{M}}_{sn} \\
\tilde{\mathcal{M}}_{sn}^\top & I
\end{bmatrix}, \quad \mathcal{K}_r = \begin{bmatrix}
\tilde{\mathcal{K}}_{ss} & 0 \\
0 & A_{\Lambda}
\end{bmatrix}, \quad \mathcal{F}_r = \begin{bmatrix}
\tilde{\mathcal{F}}_s \\
A_{\mathcal{F}_n}
\end{bmatrix}
\quad (8.72)
\]

where the \(s\) DOF submatrices are given by

\[
\tilde{\mathcal{M}}_{ss} = \tilde{\Phi}_{as}^\top \begin{bmatrix}
I & \tilde{\mathcal{M}}_{nc} \\
\tilde{\mathcal{M}}_{nc}^\top & \tilde{\mathcal{M}}_{cc}
\end{bmatrix} \tilde{\Phi}_{ss}, \quad \tilde{\mathcal{M}}_{sn} = \tilde{\Phi}_{as}^\top \begin{bmatrix}
0 \\
A_{\mathcal{M}_{nc}^\top}
\end{bmatrix}
\]

\[
\tilde{\mathcal{K}}_{ss} = \tilde{\Phi}_{as}^\top \begin{bmatrix}
\Lambda & 0 \\
0 & \tilde{\mathcal{K}}_{cc}
\end{bmatrix} \tilde{\Phi}_{ss}
\quad (8.73)
\]

\[
\tilde{\mathcal{F}}_s = \tilde{\Phi}_{as}^\top \begin{bmatrix}
\tilde{\mathcal{F}}_n \\
\tilde{\mathcal{F}}_c
\end{bmatrix}
\]

169
Table 8.1: Mistuning ROM sizes

<table>
<thead>
<tr>
<th>Models</th>
<th>Size Eq.</th>
</tr>
</thead>
<tbody>
<tr>
<td>CB-CMS</td>
<td>$I_N \left( Dk_n + Rk_n + \Gamma_k + N_{TD} + N_F \right) + O N \Gamma k_n$</td>
</tr>
<tr>
<td>Interface-Reduced</td>
<td>$I_N \left( Dk_n + Rk_n + \Gamma_k \right) + O N \Gamma k_n + k_{cc}$</td>
</tr>
<tr>
<td>Ancillary-Reduced</td>
<td>$I_N \Gamma_k + O N \Gamma k_n + k_{ca}$</td>
</tr>
</tbody>
</table>

As in the case of Interface mode reduction, only the size of the null entries in $C_r$ are changed, so it is not re-listed.

### 8.5.3 Method Comparison

Table 8.1 outlines the size governing equation of each ROM discussed as a function of the number of blades, truncated modes retained, and boundary DOF in each method’s formulation. As previously outlined, the traditional CB-CMS ROM is the largest of these approaches and its size can be prohibitively large as the number of interface DOF increases. The remaining Interface and Ancillary mode reduction methods outlined in Section 8.5 seek to further reduce the size of the CB-CMS model. While the Interface method effectively reduces the size of the constraint DOF, it is still dependent on the number of disk and ring normal modes retained. The Ancillary mode ROM size is independent of $Dk_n$ and $Rk_n$ since this reduction method includes these DOF in the secondary modal analysis. Increasing $Dk_n$ and $Rk_n$ has the benefit of increasing the accuracy of the Ancillary mode reduction method without increasing the final ROM size. However, this increase comes at the computational cost of calculating the Ancillary modes from larger matrices.
8.6 Nominal Method Test Conditions

8.6.1 Mistuning Implementation

Mistuning is implemented through perturbation to the inner-blade fixed-fixed and outer-blade cantilevered natural frequencies through the following

\[
\text{•B}_{\forall a} \left[ \text{diag}_{j=1,\ldots,k_n} \left( 1 + \delta_j^{(a)} \right) \lambda_j \right]
\]

(8.74)

where \(\delta_j^{(a)}\) is a small perturbation to the \(j^{th}\) natural frequency of blade \(a\) and \(\bullet\) denotes either \(I\) or \(O\). Typically these values can be drawn randomly from a distribution to create new mistuning patterns and mistuned response results. Here, the mistuned blade frequencies are calculated explicitly from a geometrically perturbed blade with deviations that result in less than a 3% variation in frequency. The deviations are introduced through perturbations to a blade’s Principal Component features that quantify small geometrical variations in a blade’s geometry [57, 104, 105].

8.6.2 DFIBR External Forcing

As previously highlighted in Section 4.4 on page 50, DFIBRs can be subjected to different EO excitations on the inner- and outer-blades. While the inner- or outer-blades are subject to a constant force that differs only in phase, the force resultant on the DFIBR will exhibit a non-constant force magnitude circumferentially around the rotor with a phase difference that may not pass from zero to 360 deg before repeating. The inner-blades were subject to an \(I^C = 0\) EO excitation while the outer-blades were subject to an \(O^C = 2\) EO excitation, each of constant magnitude. As a result, there is a non-constant force magnitude around the rotor that is characterized by a force magnitude and phase with a period of \(P = |O^C - I^C| = 2\) around the rotor.
Actual displacement magnitudes and phases are illustrated in Fig. 8.3a and 8.3b for the inner-blades of the DFIBR when subject to the above forcing. Figure 8.3a clearly shows the non-constant blade response that has a period of $P$ (i.e. it repeats after eight blades) around the DFIBR due to the difference in EO excitation between the inner- and outer-blades. Furthermore, the phase difference between blades is no longer constant, but has a period of $P$ about the DFIBR.

### 8.7 Nominal Method Results

Results are generated for two different mistuned DFIBRs that are subject to different EO excitations as listed in Table 8.2 over the same excitation frequency range. For these forcing conditions, system modes $52 - 56$ are excited, where modes $52$ and $53$ on Family 4 are excited by $C = 2$, mode $54$ on Family 5 is excited by $C = 0$, and modes on Family $155 - 56$ are excited by $C = 1$. Both rotors exhibit response amplification and are good test cases for the developed methods since it will demonstrate the ability to predict the magnification. Capturing this phenomenon is critical for accurate life assessment by ensuring that peak response does not exceed some predetermined critical value.

For this study, a unit force is applied to a single location on the outer-blade trailing edge tips and inner-blades at mid-span. While not representative of in-flight loading, this type of forcing can demonstrate the mistuning phenomenon and is usually prescribed in bench-level testing. The response levels are “measured” at a single output location on the leading edge of outer-blade tips and inner-blades at mid-span. These input/output locations were determined by upfront studies that identified blade locations that had high transfer functions.

In the following sections, the developed Interface mode reduction (Int-Red) and Ancillary mode reduction (Anc-Red) ROMs from Equations 8.33 and 8.72, respectfully, are compared against full FEM solutions obtained from ANSYS. Furthermore, each developed
Figure 8.3: Illustration how different EO excitations between the inner- and outer-blades causes non-constant blade displacement magnitudes around the DFIBR and the disruption in the phase
Table 8.2: Engine order excitation conditions for the two mistuned DFIBR test cases

<table>
<thead>
<tr>
<th>DFIBR</th>
<th>IC</th>
<th>OC</th>
<th>Excitation Freq. Range</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>1</td>
<td>1</td>
<td>875 - 925 Hz</td>
</tr>
<tr>
<td>B</td>
<td>0</td>
<td>2</td>
<td>875 - 925 Hz</td>
</tr>
</tbody>
</table>

ROM is compared against the parent CB-CMS model from which they are derived. The Interface and Ancillary modal methods will converge to the CB-CMS solution as the number of Interface and Ancillary modes retained in the formulations approach their respective maximums, so the CB-CMS solution represents the best available nominal-mode prediction.

### 8.7.1 Free Response Results

The free response data is compose of the DFIBR natural frequencies and corresponding mode shapes predicted by each ROM. The percent error from the full FEM predicted frequencies for both mistuned DFIBRs, A and B, are plotted versus the frequency index in Fig. 8.4. Positive error corresponds to a frequency prediction above the full FEM solution, while negative error corresponds to predictions below the full FEM solution. The ROMs’ percent errors for both DFIBRs are quite small for the first 60 system modes. The ROM errors follow that of the CB-CMS approach quite accurately and show the secondary modal reduction does not introduce significant errors beyond the initial nominal mode assumption.

A stem plot of mistuned system modal $z$-direction displacements at each blade output location is illustrated in Figures 8.5 on page 176 and 8.6 on page 177 for the largest participating modes in the frequency range of interest. Mode shape predictions by each ROM are shown against the full FEM solutions. Sub-figures 8.5a and 8.5b depict system mode 55 with inner- and outer-blade displacement predictions for DFIBR A where it is shown that mistuning has not caused strong mode localization to a single blade since the single nodal
Figure 8.4: ROM natural frequency prediction error of the two mistuned DFIBR test cases when compared against full FEM solutions
Figure 8.5: Largest participating system mode plotted for the inner- and outer-blades for DFIBR A with the excitation conditions described in Table 8.2.

diameter is still evident. Sub-figures 8.6a and 8.6b illustrate system mode 53 for DFIBR B. Figure 8.6a shows the appearance of only a single nodal diameter for this mode, but mistuning has masked the expected second nodal diameter. However, both nodal diameters are still evident in Fig. 8.6b. For both DFIBRs the ROMs agree quite well with the full FEM mode shapes and capture the mode shape quite well. Furthermore, the developed ROMs are very accurate when compared to the CB-CMS prediction. This further highlights that the secondary modal reduction did not introduce significantly more error.
Figure 8.6: Largest participating system mode plotted for the inner- and outer-blades for DFIBR B with the excitation conditions described in Table 8.2
8.7.2 Forced Response Results

Modal participation factors that are a function of modal force were calculated and determined how much a particular mistuned mode contributed to the total forced response. Since the developed ROMs utilize a tuned mode assumption, the modal forces are calculated from tuned blade modes. The DFIBRs highest participating modes previously illustrated in Figs. 8.5 and 8.6 were accurately captured by the ROMs, so it remains to determine how much these modes participate in the forced response. Figure 8.7 depicts Pareto plots of DFIBRs A and B for the first ten largest participation factors. The bars and stems correspond to the left ordinate and show the individual values of the participation factors for each mode as calculated from each method.

Figure 8.7a highlights that the ROMs accurately determined the participation values, as well as the correct order of the largest contributing modes when compared to the full results particularly for the largest three system modes. This should contribute to accurate forced response levels since these three modes contribute to approximately 95% of the forced response levels as shown on the right ordinate. Of particular interest, is the splitting of repeated modes 55 and 56 due to mistuning, where the second repeated mode 56 is not the second largest contributor to forced response levels. In fact, mode 54 that corresponds to a zero nodal diameter is the second largest contributor to an EO excitation of $C = 1$ on both the inner- and outer-blades.

Accurate modal participation factors were also calculated by the developed ROMs for DFIBR B in Fig. 8.7b. Here the highest participating modes, 53 and 52, are split tuned modes that have two nodal diameters and are expected to be high responders since the outer-blades are subject to a second EO excitation. The zero nodal diameter mode 54 is the third largest, but contributes little to forced response levels as evident from the cumulative contribution curve. This is surprising considering a zero EO excitation is applied to the inner blades, but the excitation frequency range corresponds to system modes with little
Figure 8.7: Modal participation factors for DFIBRs A and B with the excitation conditions described in Table 8.2
inner-blade motion and large outer-blade motion.

Finally, the modal participation factors and mode shapes were used in a modal summation to calculate forced response levels. Displacements correspond to the Euclidean distance at blade output locations over the frequency range of interest. Two approaches are used to display the forced response values: first, the peak DFIBR response is the maximum responding blade at each excitation frequency step; second, the peak blade response over the entire excitation frequency range. The peak DFIBR response for both test cases is illustrated in Fig. 8.9 and represents the conservative assumption that all blades are responding at the maximum level. It is shown that in each case the developed ROMs accurately catch the peak response of the inner- and outer-blade levels. Furthermore, as expected from the low modal response levels of inner-blades from Fig. 8.6, the inner-blades are responding at a much lower level.

The peak blade responses from Fig. 8.11 represent a conservative case in assuming that blades experience this forced response level over the entire excitation frequency range. The forced response levels of the ordinate have been normalized by the blades’ corresponding tuned response levels, so mistuned response amplification will then appear larger than one. The inner-blades of DFIBR A in Fig. 8.10a experience an amplification of only approximately 25%, while the outer-blades from Fig. 8.10b have a large amplification of approximately 65%. DFIBR B has an inner-blade amplification of approximately 225% shown in Fig. 8.11a while only 58% in the outer-blades of Fig. 8.11b. This shows response amplification is not always restricted to the highest responding blade set in operating conditions that suggest that either the inner- or outer-blade set will be responding.
Figure 8.8: Peak DFIBR response for test case A with the excitation conditions described in Table 8.2
Figure 8.9: Peak DFIBR response for test case B with the excitation conditions described in Table 8.2
Figure 8.10: Peak blade response for test case A with the excitation conditions described in Table 8.2
Figure 8.11: Peak blade response for test case B with the excitation conditions described in Table 8.2
Table 8.3: Developed ROM sizes

<table>
<thead>
<tr>
<th></th>
<th>CB-CMS</th>
<th>Int-Red</th>
<th>Anc-Red</th>
</tr>
</thead>
<tbody>
<tr>
<td>$D_{kn}$</td>
<td>5</td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td>$R_{kn}$</td>
<td>5</td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td>$I_{kn}$</td>
<td>30</td>
<td>30</td>
<td>30</td>
</tr>
<tr>
<td>$O_{kn}$</td>
<td>30</td>
<td>30</td>
<td>30</td>
</tr>
<tr>
<td>$N_{\Gamma_D}$</td>
<td>186</td>
<td>n/a</td>
<td>n/a</td>
</tr>
<tr>
<td>$N_{\Gamma}$</td>
<td>438</td>
<td>n/a</td>
<td>n/a</td>
</tr>
<tr>
<td>$k_{cc}$</td>
<td>n/a</td>
<td>250</td>
<td>n/a</td>
</tr>
<tr>
<td>$k_{ca}$</td>
<td>n/a</td>
<td>n/a</td>
<td>250</td>
</tr>
</tbody>
</table>

8.7.3 Model Sizes and Solution Times

The values that determine the model sizes from Table 8.1 are listed in Table 8.3. The resulting model sizes in Table 8.4 have been normalized by the size of the full FEM and show that the developed Interface and Ancillary modal reduction methods can attain a smaller ROM as compared. This smaller size is beneficial to solution of the the EVP of each ROM. The solution times to solving the ROM EVP for the same number of modes calculated using MATLAB’s tic/toc function are also listed in Table 8.4 have again been normalized by the full FEM EVP solution. As shown, a drawback of the CB-CMS approach is the ROM EVP can take longer to solve than the full FEM. This is due to the fully-populated constraint DOF portion of the matrices that causes significant burden to sparse eigen-solvers. This sparsity is kept intact for the full FEM and allows faster computation. However, the CB-CMS approach still allows direct access to blade natural for intentional mistuning studies. The Interface and Ancillary reduction methods are necessary for obtaining a ROM that is both smaller with a computationally tractable EVP. While the Interface mode and Ancillary mode reduction methods result in similar accuracy, size, and solution time, the Ancillary mode method has the benefit of being of having better accuracy when increasing the number of disk and ring normal modes without increasing the overall ROM model size.
Table 8.4: Developed ROM Size Comparisons

<table>
<thead>
<tr>
<th>Model</th>
<th>Normalized Size</th>
<th>Normalized Solution Time</th>
</tr>
</thead>
<tbody>
<tr>
<td>CB-CMS</td>
<td>0.155</td>
<td>1.236</td>
</tr>
<tr>
<td>Int-Red</td>
<td>0.043</td>
<td>0.040</td>
</tr>
<tr>
<td>Anc-Red</td>
<td>0.037</td>
<td>0.038</td>
</tr>
</tbody>
</table>

### 8.8 Nominal Method Conclusions

The tuned and mistuned dynamic response of DFIBRs was investigated using two new mistuning ROMs that started with a CB-CMS formulation with cyclic components that were subsequently reduced with a secondary modal analysis on the parent CB-CMS model. A tuned disk and ring were assumed with perturbations to blade natural frequencies. The first mistuning ROM calculates a set of Interface modes from a secondary modal analysis on the CB-CMS constraint DOF to reduce the model size. The second approach calculates a set of Ancillary modes from a secondary modal analysis on the CB-CMS constraint and disk and ring normal DOF to further reduce the model size. Both approaches utilize tuned mode projections that carry the benefit of only needing to calculate blade, Interface, and Ancillary modes once in an up-front computational expense. This approach assumes that mistuned response can be approximated as a linear combination of tuned modes. The Ancillary mode reduced ROM has the benefit of being independent of the number of retained fixed-interface normal modes for the disk and ring, where increasing the retained amount increases the ROM accuracy without increasing the final ROM size.

Free and forced responses were investigated using the above methods and illustrated the many differences between traditional IBR designs and DFIBRs. DFIBR mode families are characterized by two sets of blade mode types that transition between each other through disk and ring interaction. This behavior is identified through a nodal diameter plot characterizing the DFIBR free vibration. Furthermore, having two sets of blades that can experience different EO excitations creates a non-constant displacement magnitude be-
tween blades of the same type, i.e. blades will experience different stress levels for constant forcing on each blade.

Both ROMs effectively and efficiently captured the mistuned response behavior of two different DFIBRs subject to different EO excitations. Both methods compared very well to the parent CB-CMS model from which they were derived. Using a tuned mode reduction method proved to be accurate in predicting DIFBR mode shapes, modal participation factors and peak rotor and blade-to-blade responses for the two DFIBRs considered. It was shown that large response amplification can result for a low responding blade set when the DFIBR is tuned. In other words, designing a DFIBR for operating conditions that suggest either the inner- or outer-blade set will be responding can be disastrous since mistuning can cause amplification in the non-responding blade set.

8.9 Geometric Methods

8.9.1 Formulation for all Blades

All the previous calculations in Section 8.3 have been for a single blade, \( a \). If all blades are tuned, these calculations would only have to be done once for the inner-blades and once for the outer-blades since the component matrices would be the same for all \( I_N \) or \( O_N \) blades. However, geometric mistuning perturbs both the mass and stiffness matrices of each inner- and outer-blade by varying amounts. For example, the mistuned stiffness matrix of Eq. 8.11 can be represented by

\[
K = \mathit{tK} + \Delta K
\]

where \( \mathit{tK} \) is a tuned stiffness matrix and \( \Delta K \) is a perturbation matrix of full rank with small deviations. The mistuned mass matrix follows suit. This type of mistuning is referred to as large rank, small mistuning. Consequently, the CB-CMS component matrices must be
recalculated for all blades since

\[ U^{(Tuned)} \neq U^{(1)} = U^{(2)} \neq \cdots \neq U^{(N)} \]  

(8.76)

where the bullet, •, is a placeholder for either I or O.

Considering that the blade matrices are mostly sparse, and that these calculations are done blade-by-blade, this additional computation is small compared to solving the full IBR FEM or even traditional CB-CMS ROMs. Furthermore, if probabilistic studies are required, a small population of blades can be generated and bootstrapping methods can be used to eliminate calculation of a larger population of blades. The CB-CMS reduced mass and stiffness matrices of Eq. 8.14 for each mistuned blade are then combined into the block diagonal matrices containing all blades

\[ \mathcal{M} = \begin{bmatrix} I & \mathcal{B} \left[ \mathcal{M}^{(a)}_{nc} \right] \\ \mathcal{B} \left[ \mathcal{M}^{(a)}_{nc} \right] & \mathcal{B} \left[ \mathcal{M}^{(a)}_{cc} \right] \end{bmatrix}, \quad \mathcal{K} = \begin{bmatrix} \mathcal{B} \left[ \Lambda^{(a)} \right] & 0 \\ 0 & \mathcal{B} \left[ \mathcal{K}^{(a)}_{cc} \right] \end{bmatrix} \]  

(8.77)

where the \( a^{th} \) block on the diagonal corresponds to the \( a^{th} \) blade for \( a = 1, \ldots, N \).

### 8.9.2 Blade Forcing

The CB-CMS modal force for mistuned blade components is obtained by projecting the force vector onto the mistuned blade modes

\[ \mathcal{F} = \begin{bmatrix} \mathcal{B} \left[ \Phi^{(a)} \right]^T & 0 \\ \mathcal{B} \left[ \Psi^{(a)} \right]^T & I \end{bmatrix} \mathcal{F} = \begin{bmatrix} \mathcal{B} \left[ \Phi^{(a)} \right]^T (\mathbf{P}_c \otimes \mathbf{f}_r) \\ \mathcal{B} \left[ \Psi^{(a)} \right]^T (\mathbf{P}_c \otimes \mathbf{f}_r) + (\mathbf{P}_c \otimes \mathbf{f}_r) \end{bmatrix} \]  

(8.78)

Theses calculations have been carried out for specific blade set, i.e. inner- or outer-blades. Therefore, the notation for all blades (\( \forall a \)) requires all blades be of the same set. The
preceding section will then produce CB modal matrices for each blade type where a left superscript of \( I \) or \( O \) will be placed on the respective Inner- and Outer-blade component matrices to differentiate between each in the following sections.

### 8.9.3 Rearrange Blade Matrices

Once the formulation of the blade CB-CMS matrices is complete, the inner blade matrices must be re-ordered just as the nominal methods. The blade matrices are re-ordered with \( T \) according to

\[
A_K^T = T^\top \begin{bmatrix} I_K & 0 \\ 0 & O_K \end{bmatrix} T = \begin{bmatrix} A_\Lambda & 0 \\ 0 & A_{Kcc} \end{bmatrix}
\]

\[
A_M^T = T^\top \begin{bmatrix} I_M & 0 \\ 0 & O_M \end{bmatrix} T = \begin{bmatrix} I & M_{nc} \\ M_{nc}^\top & A_{Mcc} \end{bmatrix}
\]

\[
A_F^T = T^\top \begin{bmatrix} I_F \\ O_F \end{bmatrix} = \begin{bmatrix} A_{Fn} \\ A_{Fc} \end{bmatrix}
\]

where the inner- and outer-blade component matrices, \( K \) and \( M \), are from Eq. 8.77 and the force vector, \( F \), is from Eq. 8.78. The numerous sub-partitions to Eq. 8.79 can be further described by

\[
A_\Lambda = \begin{bmatrix} I_\Lambda & 0 \\ 0 & O_\Lambda \end{bmatrix}
\]

\[
A_{M_{nc}} = \begin{bmatrix} M_{ni\Gamma D} & M_{ni\Gamma} \\ 0 & M_{no\Gamma} \end{bmatrix}, \quad A_{Mcc} = \begin{bmatrix} M_{\Gamma_D\Gamma_D} & M_{\Gamma_D\Gamma} \\ M_{\Gamma_D\Gamma}^\top & A_{M_{\Gamma\Gamma}} \end{bmatrix}
\]

\[
A_{F_{\text{n}}} = \begin{bmatrix} I_{F_{n}} \\ O_{F_{n}} \end{bmatrix}, \quad A_{F_{\text{c}}} = \begin{bmatrix} I_{F_{D}} \\ A_{F_{\Gamma}} \end{bmatrix}
\]
where $^A K_{cc}$ follows the same partitionment as $^A M_{cc}$. The previous equations are populated from each inner- and outer-blade through the following sub-partitions

\[
\begin{align*}
M_{\Gamma_D \Gamma_D} &= B \forall a \left[ I M_{\Gamma_D \Gamma_D}^{(a)} \right] \\
M_{\Gamma_D \Gamma} &= B \forall a \left[ I M_{\Gamma_D \Gamma_R}^{(a)} 0 \right] \\
^A M_{\Gamma} &= B \forall a \left[ I M_{\Gamma_D \Gamma_R}^{(a)} 0 \right] \tag{8.83} \\
M_{nI \Gamma_D} &= B \forall a \left[ I M_{nI \Gamma_D}^{(a)} \right] \\
M_{nI \Gamma} &= B \forall a \left[ I M_{nI \Gamma_R}^{(a)} 0 \right] \\
M_{nO \Gamma} &= B \forall a \left[ 0 B \forall a \left[ O M_{nO \Gamma}^{(a)} \right] \right]
\end{align*}
\]

The traveling wave CB-CMS force components are given by

\[
\begin{align*}
^A f_{\Gamma} = \begin{bmatrix}
I f_{\Gamma_R}^{(1)} \\
O f_{\Gamma_D}^{(1)} \\
\vdots \\
I f_{\Gamma_R}^{(N)} \\
O f_{\Gamma_D}^{(N)}
\end{bmatrix}, \quad
f_{\Gamma_D}^{(N)} = \begin{bmatrix}
O f_{\Gamma_1}^{(N)} \\
O f_{\Gamma_2}^{(N)} \\
\vdots \\
O f_{\Gamma_O}^{(N)}
\end{bmatrix} \tag{8.84}
\end{align*}
\]

The force component $^f_{\Gamma_D}$ in Eq. 8.82 consists of only the inner-blade force component and contains only inner-blade EO excitations. However, Eq. 8.84 illustrates that the $^A f_{\Gamma}$ force component has both inner- and outer-blade EO excitations.
8.9.4 Component Coupling for Mistuned Blades

To constrain the blades to the disk and ring, cyclic DOF $D_{p_c}^{(\nu h)}$ and $R_{p_c}^{(\nu h)}$ are kept as active DOF by the following

\[
\begin{pmatrix}
D_{p_c}^{(\nu h)} \\
D_{p_c}^{(\nu h)} \\
R_{p_c}^{(\nu h)} \\
R_{p_c}^{(\nu h)} \\
\Gamma_{c}^{(\nu a)} \\
\Gamma_{c}^{(\nu a)} \\
\end{pmatrix} =
\begin{pmatrix}
I & 0 & 0 & 0 & 0 \\
0 & 0 & \tilde{E}^T & 0 & 0 \\
0 & I & 0 & 0 & 0 \\
0 & 0 & 0 & O_{E}^T & 0 \\
0 & 0 & 0 & 0 & I \\
0 & 0 & 0 & 0 & I \\
\end{pmatrix}
\begin{pmatrix}
D_{p_c}^{(\nu h)} \\
R_{p_c}^{(\nu h)} \\
D_{c} \\
R_{c} \\
\Gamma_{p}^{(\nu a)} \\
\Gamma_{p}^{(\nu a)} \\
\end{pmatrix}
= T^{CB} \mathbf{p}
\quad (8.85)
\]

The resulting coupled CB-CMS ROM matrices are obtained by

\[
^{CB} \mathbf{M} = T^T \begin{pmatrix}
D_{\tilde{M}} & 0 & 0 \\
0 & R_{\tilde{M}} & 0 \\
0 & 0 & \Lambda_{M} \\
\end{pmatrix} T = \begin{pmatrix}
I & \mathcal{M}_{nc} & 0 \\
\mathcal{M}_{nc} & \mathcal{M}_{cc} & A_{M_{nc}}^T \\
0 & A_{M_{nc}} & I \\
\end{pmatrix}
\]

\[
^{CB} \mathbf{K} = T^T \begin{pmatrix}
D_{\tilde{K}} & 0 & 0 \\
0 & R_{\tilde{K}} & 0 \\
0 & 0 & \Lambda_{K} \\
\end{pmatrix} T = \begin{pmatrix}
\Lambda & 0 & 0 \\
0 & \mathcal{K}_{cc} & 0 \\
0 & 0 & \Lambda \\
\end{pmatrix}
\quad (8.86)
\]

\[
^{CB} \mathbf{F} = T^T \begin{pmatrix}
D_{\tilde{F}} \\
R_{\tilde{F}} \\
A_{F} \\
\end{pmatrix} = \begin{pmatrix}
\tilde{\mathbf{F}}_n \\
\mathbf{F}_c \\
A_{\mathbf{F}}_n \\
\end{pmatrix}
\]

The sub-partitions $A_{\Lambda}$, $A_{M_{nc}}$, and $A_{\mathbf{F}}_n$ are given in Equations 8.80 - 8.82, respectively. The
remaining sub-partitions are given by

\[
\Lambda = \begin{bmatrix}
D_A & 0 \\
0 & R_A 
\end{bmatrix}
\]  
\tag{8.87}

\[
\mathcal{M}_{nc} = \begin{bmatrix}
D \tilde{\mathcal{M}}_{nc} I \hat{E}^\top & 0 \\
0 & R \tilde{\mathcal{M}}_{nc} O \hat{E}^\top
\end{bmatrix}
\]  
\tag{8.88}

\[
\mathcal{M}_{cc} = \begin{bmatrix}
I \hat{E} D \tilde{\mathcal{M}}_{cc} I \hat{E}^\top + A \mathcal{M}_{\Gamma \Delta \Gamma} \\
A \mathcal{M}_{\Gamma \Delta \Gamma}^\top & O \hat{E} R \tilde{\mathcal{M}}_{cc} O \hat{E}^\top + A \mathcal{M}_{\Gamma \Gamma}
\end{bmatrix}
\]  
\tag{8.89}

where \( \mathcal{K}_{cc} \) follows the same partitionment as \( \mathcal{M}_{cc} \). The force components are given by

\[
\tilde{\mathcal{F}}_n = \begin{bmatrix}
D \tilde{\mathcal{F}}_n^{(\gamma h)} \\
R \tilde{\mathcal{F}}_n^{(\gamma h)}
\end{bmatrix}
\]  
\tag{8.90}

\[
\mathcal{F}_c = \begin{bmatrix}
I \hat{E} D \tilde{\mathcal{F}}_c^{(\gamma h)} + I \hat{f}_D \\
O \hat{E} R \tilde{\mathcal{F}}_c^{(\gamma h)} + A \hat{f}_\Gamma
\end{bmatrix}
\]  
\tag{8.91}

### 8.9.5 Geometric CB-CMS ROM

The previous formulations utilized a cyclic disk and ring description that kept the constraint DOF in physical coordinates. The CB-CMS EOM is then formulated as

\[
^{CB}M\ddot{\mathbf{p}} + ^{CB}C\dot{\mathbf{p}} + (1 + G_i)^{CB}\mathbf{p} = ^{CB}\mathbf{F}
\]  
\tag{8.92}

where the blade modal damping matrix and structural damping coefficient, \(^{CB}C\) and \(G\), respectively, have can be included to better model dynamic response [39]. The mass, stiffness, and forcing matrices and vectors are given in Eq. 8.86 while the blade modal damping
matrix is given by

\[
CBC = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & AC
\end{bmatrix}
\] (8.93)

where

\[
AC = \begin{bmatrix}
\mathbb{B} \begin{bmatrix}
\text{diag} \left( 2 I_{j}^{(a)} \right) \sqrt{I \Lambda^{(a)}} \\
\text{diag} \left( 2 O_{j}^{(a)} \right) \sqrt{O \Lambda^{(a)}}
\end{bmatrix} \\
0
\end{bmatrix}
\] (8.94)

where \( \zeta_j \) is the damping coefficient for the \( j^{th} \) blade mode. This CB-CMS model is dominated by the unnecessary retention of all the interface DOF and, at times, the disk and ring normal DOF. The following sections seek to further reduce the size of this CB-CMS ROM by introducing a secondary modal analysis on this model.

### 8.10 Secondary Modal Reduction ROMs

In the following subsections, two mistuning approaches are presented that use either the tuned Interface or Ancillary modes to further reduce the CB-CMS ROM size of Eq. 8.92. In both approaches, the generic EOM is given by

\[
M_r \ddot{\mathbf{q}}_r + C_r \dot{\mathbf{q}}_r + (1 + Gi) K_r \mathbf{q}_r = \mathbf{F}_r
\] (8.95)

where subscript \( r \) refers to reduced and the EOM matrices are defined in the following subsections. These two models are compared to a traditional tuned mode approximation that is formulated in Section 8.5.1 on page 163.
8.10.1 Interface Mode Reduction

This subsection describes the use of Interface modes for model reduction. First, the use of tuned Interface modes is discussed that utilizes the tuned modes derived in Section 8.5.1. This cyclic formulation greatly reduces the computation cost in calculating these modes, but they must be transformed from cyclic coordinates back to physical coordinates. Calculation of the mistuned modes are then discussed.

8.10.1.1 Tuned Interface Modes

The tuned Interface modes, $\tilde{\Phi}_{cc}$ of Eq. 8.53, are transformed back to physical space by

$$\Phi_{cc} = \hat{E} \tilde{\Phi}_{cc}$$

(8.96)

where $\hat{E} = \mathbb{B} \left[ E, \gamma \tilde{E} \right]$. These modes are then frequency ordered to provide the same lowest modes for use in the reduction process in the following Section 8.10.1.2. Since a limited set, $k_{cc}$, of tuned Interface modes is retained, the CB-CMS system of Eq. 8.92 can be further reduced through the following transformation matrix

$$C^B \mathbf{p} = \begin{bmatrix} I & 0 & 0 \\ 0 & \Phi_{cc} & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} \tilde{q}_n \\ q_c \\ Aq_m \end{bmatrix} = T_{CC} \mathbf{q}_r$$

(8.97)
Substituting this into the CB-CMS EOM of Eq. 8.92 and pre-multiplying by \( T_{cc}^T \) yields the following matrices for the reduced ROM of Eq. 8.95

\[
\mathcal{M}_r = \begin{bmatrix}
I & \mathcal{M}_{mc} \Phi_{cc}^\top & 0 \\
\Phi_{cc}^\top \mathcal{M}_{mc} & \Phi_{cc}^\top \mathcal{M}_{cc} \Phi_{cc} & \Phi_{cc}^\top \mathcal{A} \mathcal{M}_{mc}^\top \\
0 & \mathcal{A} \mathcal{M}_{mc} \Phi_{cc} & I
\end{bmatrix}
\]

\[
\mathcal{K}_r = \begin{bmatrix}
\Lambda & 0 & 0 \\
0 & \Phi_{cc}^\top \mathcal{K}_{cc} \Phi_{cc} & 0 \\
0 & 0 & \mathcal{A} \Lambda
\end{bmatrix}
\]

\[
\mathcal{F}_r = \begin{bmatrix}
\Phi_{cc}^\top \mathcal{F}_n \\
\Phi_{cc}^\top \mathcal{F}_c \\
\mathcal{A} \Phi_{cc}^\top \mathcal{F}_n
\end{bmatrix}
\]

\( (8.98) \)

Note that \( C_r \) is obtained in the same manner, except only the size of the null entries are changed, so it is not re-listed.

### 8.10.1.2 Mistuned Modes

The mistuned Interface modes in physical space can be calculated by the same EVP as in Eq. 8.52, except with matrices from Eq. 8.89. This subset of modes is then combined into the mistuned Interface modal matrix \( \Phi_{cc} = [\phi_1, \ldots, \phi_{kcc}] \). This set of modes is then substituted into Eq. 8.97 to provide a mistuned Interface mode reduction.

### 8.10.2 Ancillary Mode Reduction

This subsection first discusses the use tuned Ancillary modes derived in Section 8.5.2 on page 166 for model reduction. As for the tuned Interface mode formulation, this cyclic formulation greatly reduces the computation cost in calculating the modes. Then, calculation
of the mistuned modes are then discussed.

### 8.10.2.1 Tuned Ancillary Modes

The disk and ring components are modeled as tuned so the normal DOF pertaining to these substructures remain unchanged from one mistuned DFIBR to the next. Therefore, the cyclic constraint partitions discussed in Section 8.5.2 on page 166 and disk and ring normal partitions can be re-ordered by harmonics to yield a block-diagonal structure that shares the same computational benefits of cyclic symmetry. This yields tuned Interface modes, $\tilde{\Phi}_{ss}$ of Eq. 8.70, that are transformed back to physical space by

$$\Phi_{ss} = V\tilde{\Phi}_{ss}$$  \hspace{1cm} (8.99)

where $V = B \left( I, \hat{I}^E, \hat{O}^E \right)$. Since a limited set, $k_{ss}$, of tuned Ancillary modes is retained the CB-CMS system can be further reduced through the following transformation matrix

$$\begin{bmatrix} C^B \end{bmatrix} p = \begin{bmatrix} \Phi_{ss} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \mathbf{q}_s \\ A p_n \end{bmatrix} = T_{CA} q_r$$  \hspace{1cm} (8.100)

Substituting this into the CB-CMS EOM of Eq. 8.92 and pre-multiplying by $T_{CA}^T$ yields the following matrices for the reduced ROM of Eq. 8.95

$$\mathbf{M}_r = T_{CA}^T C^B \mathbf{M}_T \mathbf{C}_A, \quad \mathbf{K}_r = T_{CA}^T C^B \mathbf{K}_T \mathbf{C}_A, \quad \mathbf{F}_r = T_{CA}^T C^B \mathbf{F}$$  \hspace{1cm} (8.101)

As in the case of Interface mode reduction, only the size of the null entries in $C_r$ are changed, so it is not re-listed.
8.10.2.2 Mistuned Ancillary Modes

The mistuned Ancillary modes in physical space can be calculated by the same EVP as in Eq. 8.69, except with matrices from Eq. 8.87 - 8.89 that populate

\[
\mathbf{M}_{ss} = \begin{bmatrix}
I & \mathbf{M}_{nc} \\
\mathbf{M}_{nc}^\top & \mathbf{M}_{cc}
\end{bmatrix}, \quad \mathbf{K}_{ss} = \begin{bmatrix}
\Lambda & 0 \\
0 & \mathbf{K}_{cc}
\end{bmatrix}
\]

(8.102)

A subset of modes is combined into the mistuned Ancillary modal matrix \( \Phi_{ss} = [\phi_1, \ldots, \phi_{k_{ss}}] \). This set of modes is then substituted into Eq. 8.100 to provide a mistuned Ancillary mode reduction.

8.10.3 Traditional Tuned Mode Approximation

This formulation is the tuned Interface mode reduction outlined in Section 8.5.1 on page 163. As a reminder, this traditional tuned ROM is formulated from tuned blade, disk, and ring matrices that offers certain computational benefits that are outlined Section 8.5.1. Particularly, Eq. 8.13 is composed of tuned blade modes used in the model reduction process. This implies the mistuned response can be approximated as a linear combination of tuned modes and is analogous to the frequency mistuning methods developed for IBRs. Since these matrices are formulated from tuned blades, the component \( A\Lambda \) in \( \mathbf{K}_r \) of Eq. 8.33 will be compose of tuned blade frequencies. These are simply replaced by mistuned blade frequencies that are found in the mistuned blade formulation of Section 8.10.1.1.

8.10.4 Method Comparison

The CB-CMS method is formulated from mistuned blade matrices that yield mistuned modes that populate Eq. 8.13 for each blade. This CB-CMS model is then reduced using an Interface mode reduction (CC) with either a tuned (T) or mistuned (M) Interface
modes. These methods are referred to as CCT and CCM, respectively, for the results comparison. The CB-CMS model is also reduced using an Ancillary mode reduction (CA) with either a tuned (T) or mistuned (M) Ancillary modes. These methods are referred to as CAT and CAM, respectively. The last method, CCN, is a tuned Interface mode reduction of a CB-CMS matrix formulated with tuned components which is then mistuned by introducing the inner- and outer-blade frequencies from the geometrically perturbed DFIBR. The N serves as a reminder that this method uses nominal, or tuned, blade modes in the reduction/expansion for mistuned DFIBRs. This is a frequency-based approach developed in Section 8.5.1 and is analogous to frequency-based approaches for IBRs used widely in academia and industry. This CCN method assumes blade geometric perturbations alter only the corresponding modal stiffnesses while its mode shapes remain unaffected.

Table 8.1 outlines the size governing equation of each ROM discussed as a function of the number of blades and truncated modes retained in each method’s formulation. As previously outlined, the traditional CB-CMS ROM is the largest of these approaches and its size can be prohibitively large as the number of interface DOF increases. The remaining CC- and CA-reduced methods outlined in Sections 8.10.1 and 8.10.2 seek to further reduce the size of the CB-CMS method. The CA-reduced model sizes are independent of $D_{kn}$ and $R_{kn}$ since this reduction method includes the disk and ring fixed-interface modes in the secondary modal analysis. Increasing $D_{kn}$ and $R_{kn}$ has the benefit of increasing the accuracy of the Ancillary mode reduction method without increasing the ROM size. However, this increase comes at the computational cost of calculating the Ancillary modes in a larger EVP.

As the number of fixed-interface normal modes retained for all components approaches their respective maximum, the prediction converges to the full FEM as the number of retained mistuned Interface and Ancillary modes approach their respective maximum. However, as these limits are approached, the ROM can hardly be called reduced. Note that convergence to the full FEM solution is only true when using mistuned blade, Interface
and Ancillary modes since the tuned mode reduction methods are approximations.

### 8.11 Results

Results are generated for a geometrically mistuned DFIBR by perturbing the PCs that describe the geometry deviations of the blade surface. The DFIBR is subject to the EO excitations $iC = 0$ over an excitation frequency range of $1450-1750 \, Hz$. These conditions will excite tuned system modes 98, 110, and 122 of mode families 9, 10, and 11, respectively, at harmonic index $h = 0$ of Fig. 4.6. These modes are depicted in Fig. 8.12, where mode 98 in Fig. 8.12a is primarily inner-blade fixed-fixed blade torsion motion while modes 110 and 122 are primarily cantilevered blade torsion motion. This DFIBR exhibits response amplification and is a good test case for the developed methods since it will demonstrate the ability to predict the magnification. Capturing this phenomenon is critical for accurate life assessment by ensuring that peak response does not exceed some predetermined critical value.

For this study, a unit force is applied to a single location on the outer-blade trailing edge tips and inner-blades at mid-span. While not representative of in-flight loading, this type of forcing can demonstrate the mistuning phenomenon and is usually prescribed in bench-level testing. The response levels are “measured” at a single output location on the leading edge of outer-blade tips and inner-blades at mid-span. These input/output locations were determined by upfront studies that identified blade locations that had high transfer functions.

In the sections that follow each ROM is compared against a full FEM solution obtained from ANSYS. The best achievable ROM results will belong to those predicted by the mistuned formulation of the traditional CB-CMS approach (labeled CMS in the following figures), since it is formulated with mistuned component matrices and modes while retaining all constraint DOF. The mistuned Ancillary and Interface mode reduction meth-
Figure 8.12: Tuned DFIBR system modes in the frequency range of interest at harmonic index $h = 0$
ods (labeled CCM and CAM in the following figures) will approach the accuracy of CB-CMS since these methods are synthesized from this parent model using mistuned modes in the reduction process. The accuracy of the tuned Ancillary and Interface mode reduction methods (labeled CCT and CAT in the following figures), should follow suit, depending how accurate the assumption is that the tuned modes span the same space as the mistuned modes. In the results that follow, CCT and M.CAT are always compared against their mistuned mode counterpart, CCM and CAM, respectively, since the latter methods make no approximations other than modal truncations. Accuracy of calculated system modes and modal participation factors (MPF) are first discussed in Section 8.11.1. These values directly contribute to the accuracy of blade-to-blade and peak DFIBR forced responses, discussed in Sections 8.11.2 and 8.11.3, respectively. Finally, model size and computation time comparisons are made in Section 8.11.4.

8.11.1 Modal Participations

A subset of predicted system mode shapes is chosen for comparison in this section. For the excitation conditions of interest, the modal participation factors are determined for a modal summation response and are shown in the Pareto plot of Fig. 8.13. The first ten modes with the highest modal participation are plotted on the abscissa. The bars and stems corresponding to the left ordinate illustrate the modal contributions of each ROM. As illustrated, CCN incorrectly identifies the modes with the largest participation factors and the amount these modes participate, e.g. mode 107 is incorrectly identified by CNN as participating a significant amount to the forced response. Large errors for the first ten modes will have a negative impact on predicted forced response levels, since these modes contribute to more than 85% of all the modes in the forced response levels, as illustrated by the line plot corresponding to the right ordinate.

Mistuned IBR mode 109 is shown to have the second largest modal contribution to the
Figure 8.13: Modal participation factors for the EO excitations \( I_C = 0 \) over an excitation frequency range of 1450 – 1750 Hz.

forced response and is shown in the stem plots of Fig. 8.14 for the inner- and outer-blades. In each figure, the modal response in the \( z \)-direction at the blade output locations are plotted for each blade around the DFIBR for a respective ROM and the full FEM prediction. The CMS method has the highest accuracy, as expected, and is shown to be in very good agreement with the full FEM. CCT and CCM also show good agreement between each ROM and the full FEM. Slight errors on select blades can be reduced by increasing the number of retained Interface modes. There is also good agreement between CCT and CCM, which provides an indicator that the tuned Interface modes method provide an accurate reduction method. Similar results are obtained for CAT and CAM. The CCN ROM accuracy diminishes greatly for a majority of the inner- and outer-blades. It will be shown later that this error, in conjunction with error in the modal participation factor, will manifest in larger errors in the forced response levels.

### 8.11.2 Blade-to-Blade Responses

Calculated blade responses correspond to the Euclidean distance of displacements over the excitation frequency range. Blade-to-blade responses represent a conservative case in assuming that blades experience this forced response level over the entire excitation
Figure 8.14: Comparison of predicted mistuned mode 109 against the full FEM solution
frequency range, but provides a better assessment of the responses (stresses) that each blade experiences. Figure 8.15a illustrates a stem plot of CMS and CCN predicted response levels at the previously described output locations on each blade. Both the inner- and outer-blades are plotted circumferentially around the DFIBR on the abscissa, but only the inner blades ($I_1 - I_{16}$) are numbered. After each inner-blade, the two outer-blades that follow before the next inner-blade are shown with simple tick marks. The forced response levels of the ordinate have been normalized by the blades’ corresponding tuned response levels, so mistuned response amplification will then appear larger than one. This test case exhibits strong mistuned response amplification for the inner-blades since seven of the 16 inner-blades are responding above $2x$ the tuned prediction with the largest $3.7x$ belonging to inner-blade #2 ($I_2$).

The CB-CMS predictions are shown since they represent the best attainable since the only approximation in this formulation is a truncation of the number of retained component fixed-interface modes. Also depicted are the CCN predictions that have large errors on certain blades. This manifests from the method’s inability to accurately predict the system modes and modal participation factors previously discussed. Figure 8.15b plots the percent error between the ROM predictions and the full FEM results. It is clear from this figure that there are instances where the tuned mode approximation method, CCN, can have large errors.

The CCT and CCM blade-to-blade predictions are shown in Fig. 8.16a and show improvement over the CCN predictions. Similar results are also obtained for both CAT and CAM. Both approaches that use mistuned modes in the reduction process give an indication if enough Interface or Ancillary modes are retained since they will approach the CMS solution. The percent error of each blade response as compared to the full FEM solution is shown in Fig. 8.17. The errors of both CC- and CA- methods compare well to the errors of the CMS approach, with the largest belonging to the CA- ROMs. Of particular importance is how closely the error of tuned mode reductions methods, CCT and CAT, follows their
Figure 8.15: CCN and CMS blade-to-blade forced response predictions for the mistuned DFIBR compared against full FEM solutions
mistuned counterparts, CCM and CAM. In some cases, the errors are almost identical. This evidence suggests that using the tuned mode approximations for both methods introduces only minor errors beyond what is already inherent in the models. Furthermore, since CCM and CAM will approach the CMS solution, this closeness of errors suggests that simply increasing the number of retained Interface or Ancillary modes will provide even greater accuracy.
Figure 8.17: Interface (CC) and Ancillary (CA) reduced ROMs’ blade-to-blade forced response prediction percent error for the mistuned DFIBR compared against full FEM solutions

8.11.3 Peak DFIBR Responses

Peak DFIBR response is the maximum mistuned response seen on the DFIBR and represents the worst case, and conservative, scenario that all blades are experiencing this response level. Figure 8.18 illustrates the predicted peak DFIBR mistuned response for all methods for the outer-blades only. The CCN prediction in Fig. 8.18a over predicts that actual response levels with 17.8% error. On the higher responding inner-blades the CCN method under-predicts the peak response with an error of 8.3%. The remaining CC- and CA ROMs of Fig. 8.18b all capture the peak response quite well. For all cases the error was less than 1%, providing further evidence that the tuned Interface and Ancillary mode reduction methods do not introduce significant errors.

8.11.4 Model Sizes and Solution Times

The values that determine the model sizes from Table 8.1 are listed in Table 8.5. The resulting model sizes in Table 8.6 have been normalized by the size of the full FEM and show that the developed Interface and Ancillary modal reduction methods can attain a significantly smaller ROM. This smaller size is beneficial to solution of the the EVP of each
Figure 8.18: ROM peak DFIBR forced response predictions of the inner-blades as compared to the full FEM solution
Table 8.5: Developed ROM sizes

<table>
<thead>
<tr>
<th></th>
<th>CB-CMS</th>
<th>Int-Red</th>
<th>Anc-Red</th>
</tr>
</thead>
<tbody>
<tr>
<td>$D_H$</td>
<td>5</td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td>$R_H$</td>
<td>5</td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td>$I_{cn}$</td>
<td>30</td>
<td>30</td>
<td>30</td>
</tr>
<tr>
<td>$O_{cn}$</td>
<td>30</td>
<td>30</td>
<td>30</td>
</tr>
<tr>
<td>$N_{\Gamma_D}$</td>
<td>186</td>
<td>n/a</td>
<td>n/a</td>
</tr>
<tr>
<td>$N_{\Gamma}$</td>
<td>438</td>
<td>n/a</td>
<td>n/a</td>
</tr>
<tr>
<td>$k_{cc}$</td>
<td>n/a</td>
<td>250</td>
<td>n/a</td>
</tr>
<tr>
<td>$k_{ca}$</td>
<td>n/a</td>
<td>n/a</td>
<td>250</td>
</tr>
</tbody>
</table>

ROM. The ROM EVP solution times are calculated using MATLAB’s tic/toc function are also listed in Table 8.6 and have been normalized by the full FEM EVP solution time. As shown, a drawback of the CB-CMS approach is the ROM EVP can take longer to solve than the full FEM. This is due to the fully-populated constraint DOF portion of the matrices that causes significant burden to sparse eigen-solvers. Sparsity is kept intact for the full FEM and allows faster computation. However, the CB-CMS approach allows direct access to blade natural frequencies for intentional mistuning studies without regenerating a full FEM. The Interface and Ancillary reduction methods are necessary for obtaining a ROM that is smaller with a computationally tractable EVP. The formulation of the Interface and Ancillary ROMs require the additional computational expense of a secondary EVP, however, using tuned modes requires this expense only once as an upfront requirement. Furthermore, this secondary EVP is calculated in cyclic coordinates that greatly reduces the computational expenses. While the Interface mode and Ancillary mode reduction methods result in similar accuracy, size, and solution time, the Ancillary mode method has the benefit of being of having better accuracy when increasing the number of disk and ring normal modes without increasing the overall ROM model size.
Table 8.6: Developed ROM Size Comparisons

<table>
<thead>
<tr>
<th>Model</th>
<th>Normalized Size</th>
<th>Normalized Solution Time</th>
</tr>
</thead>
<tbody>
<tr>
<td>CB-CMS</td>
<td>0.155</td>
<td>1.202</td>
</tr>
<tr>
<td>Int-Red</td>
<td>0.043</td>
<td>0.039</td>
</tr>
<tr>
<td>Anc-Red</td>
<td>0.037</td>
<td>0.037</td>
</tr>
</tbody>
</table>

8.12 Conclusions

Two geometric mistuning approaches were developed by performing a secondary modal analysis on different submatrices of a parent CB-CMS ROM formulated in cyclic coordinates. A tuned disk and ring were assumed with geometric perturbations contained to inner- and outer-blades. The first method computed the Interface modes of the CB-CMS constraint DOF while the second method computed Ancillary modes of the constraint and disk and ring fixed-interface normal modes. These modes could be either tuned or mistuned. The tuned modes were calculated in cyclic coordinates that offered significant computation savings, while the mistuned modes eliminated the approximation of using tuned modes in the reduction process. Regardless, free and forced response results highlighted that this is an accurate approximation for the given test case. Furthermore, the geometric mistuning methods were shown to have higher accuracy for peak DFIBR response and blade-to-blade predictions than a frequency-based approach. The geometric mistuning methods were also shown to have a significant reduction in solution time of the eigen-problem from the traditional CB-CMS ROM. However, the developed methods require that a fundamental sector can be obtained from the full DFIBR. This is only possible if there is an integer ratio of inner-to-outer-blade counts.
Dissertation Closing

9.1 Conclusions

This effort increased reduced-order model fidelity for mistuned IBR and DFIBR response predictions by explicitly accounting for blade geometric and material property deviations. These methods were formulated in a component mode synthesis framework utilizing secondary modal reductions in a cyclic symmetry format. The resulting reduced-order models captured perturbations to both blade natural frequencies and mode shapes resulting from geometric deviations. Furthermore, the secondary modal reductions and cyclic symmetry format showed significant computational savings over traditional component mode synthesis methods. The first formulation for IBRs assumed a tuned disk-blade connection and presented two methods that explicitly model blade geometry surface deviations by performing a modal analysis on different degrees of freedom of a parent reduced-order model. The parent ROM was formulated with CB-CMS in cyclic symmetry coordinates for an IBR with a tuned disk and blade geometric deviations. The first method performed an eigen-analysis on the constraint-mode DOFs that provided a truncated set of Interface modes while the second method included the disk fixed-interface normal modes in the eigen-analysis to yield a truncated set of Ancillary modes. Both methods were able to utilize tuned or mistuned modes, where the tuned modes have the computational benefit of being computed in cyclic symmetry coordinates. Furthermore, the tuned modes only needed to be calculated once, which offered significant computational savings for subsequent mistuning
studies. Each geometric mistuning method relied upon the use of geometrically mistuned blade modes in the component mode framework to provide a very accurate ROM. Free and forced response results were compared to both the full FEM solutions and a traditional frequency-based approach used widely in academia and the gas turbine industry. It was shown that the developed methods provided highly accurate results with a significant reduction in solution time compared to the full FEM and parent ROM. An investigation into the assumed tuned disk-blade connection was then performed. Two types of disk-blade connection mistuning were investigated: as-measured principal component deviations and random perturbations to the inter-blade spacing. Finally, these methods were extended to ROM methodologies for DFIBRs to assess the susceptibility of these new designs to mistuning and to be able to efficiently and effectively predict response amplification. Two main approaches were presented: first, a frequency-based method that is analogous to traditional mistuning approaches for IBRs, and second, geometric approaches that explicitly model blade geometry surface deviations. These methods helped characterize DFIBR dynamic response and investigated the unique aspects that differentiate these advanced components from IBRs. In all methods, free and forced response results were compared to both the full FEM solutions and the traditional frequency-based approaches. It was shown that the developed methods provide highly accurate results with a significant reduction in solution time compared to the full FEM and parent ROM.

9.2 Future Work

This effort was restricted to single stage dynamic response predictions. Future work consists of adding to the basic models by adding levels of complexity. Particular interest includes:

- Multistage effects
- Inclusion of rotor dynamic response
• Inclusion of Computational Fluid Dynamic unsteady aero-loading

• Reduced-order aero modeling

• Damping effects: coatings and monolithic properties

While quite broad, inclusion of these topics play a critical role in life prediction of turbine engines and remain to be incorporated into a high-fidelity reduced order modeling approach.
Bibliography


Appendices
Cyclic Constraint Formulation

The single sector formulation begins by partitioning the disk according to Fig. 8.2, where \( \Gamma \) are interface DOF, \( \alpha \) are independent interface DOF, \( \beta \) are the dependent interface DOF, and \( \sigma \) are non-interface DOF. The disk DOF vector \( x \) and corresponding stiffness matrix \( K \) are given by

\[
x = \begin{pmatrix} x_\sigma \\ x_\Gamma \\ x_\alpha \\ x_\beta \end{pmatrix}, \quad K = \begin{bmatrix} K_{\sigma\sigma} & K_{\sigma\Gamma} & K_{\sigma\alpha} & K_{\sigma\beta} \\ K_{\sigma\Gamma}^T & K_{\Gamma\Gamma} & K_{\Gamma\alpha} & K_{\Gamma\beta} \\ K_{\sigma\alpha}^T & K_{\Gamma\alpha} & K_{\alpha\alpha} & K_{\alpha\beta} \\ K_{\sigma\beta}^T & K_{\Gamma\beta}^T & K_{\alpha\beta}^T & K_{\beta\beta} \end{bmatrix}
\]

(A.1)

Note that the mass matrix follows the same partitionment. By introducing the dependent interface constraint \( x_\beta = e^{i\psi} x_\alpha \), where \( \psi = \frac{2\pi h}{N} \) is the inter-blade phase angle for harmonic \( h = 0, 1, \ldots, \text{int}\left[\frac{N}{2}\right] \) and \( i = \sqrt{-1} \), into the displacement vector \( x \), the dependent interface DOF \( \beta \) are eliminated. Note that \( h = N/2 \) is the largest attainable harmonic for even \( N \), and is used throughout the subsequent formulations. Now solving the static equation with the dependent constraint

\[
Kx = \begin{bmatrix} K_{\sigma\sigma} & K_{\sigma\Gamma} & K_{\sigma\alpha} & K_{\sigma\beta} \\ K_{\sigma\Gamma}^T & K_{\Gamma\Gamma} & K_{\Gamma\alpha} & K_{\Gamma\beta} \\ K_{\sigma\alpha}^T & K_{\Gamma\alpha} & K_{\alpha\alpha} & K_{\alpha\beta} \\ K_{\sigma\beta}^T & K_{\Gamma\beta}^T & K_{\alpha\beta}^T & K_{\beta\beta} \end{bmatrix} \begin{pmatrix} x_\sigma \\ x_\Gamma \\ x_\alpha \\ e^{i\psi} x_\alpha \end{pmatrix} = 0
\]

(A.2)
yields

\[ K_{\sigma\sigma} x_\sigma + K_{\sigma\Gamma} x_\Gamma + (K_{\sigma\alpha} + e^{i\psi} K_{\sigma\beta}) x_\alpha = 0 \]  
(A.3)

\[ K_{\sigma\Gamma}^T x_\sigma + K_{\Gamma\Gamma} x_\Gamma + (K_{\Gamma\alpha} + e^{i\psi} K_{\Gamma\beta}) x_\alpha = 0 \]  
(A.4)

\[ K_{\alpha\sigma}^T x_\sigma + K_{\Gamma\alpha} x_\Gamma + (K_{\alpha\alpha} + e^{i\psi} K_{\alpha\beta}) x_\alpha = 0 \]  
(A.5)

\[ K_{\sigma\beta}^T x_\sigma + K_{\Gamma\beta} x_\Gamma + (K_{\alpha\beta}^T + e^{i\psi} K_{\beta\beta}) x_\alpha = 0 \]  
(A.6)

Multiplying Eq. A.6 by \( e^{-i\psi} \) and adding the result to Eq. A.5 yields

\[ (K_{\sigma\alpha}^T + e^{-i\psi} K_{\sigma\beta}) x_\sigma + (K_{\Gamma\alpha}^T + e^{-i\psi} K_{\Gamma\beta}) x_\Gamma + (K_{\alpha\alpha} + K_{\beta\beta} + e^{i\psi} K_{\alpha\beta} + e^{-i\psi} K_{\alpha\beta}^T) x_\alpha = 0 \]  
(A.7)

Eliminating the dependent interface and combining yields

\[
\begin{bmatrix}
K_{\alpha\alpha} + K_{\beta\beta} + e^{i\psi} K_{\alpha\beta} + e^{-i\psi} K_{\alpha\beta}^T & K_{\alpha\alpha}^T + e^{-i\psi} K_{\alpha\beta}^T & K_{\Gamma\alpha}^T + e^{-i\psi} K_{\Gamma\beta}^T \\
K_{\sigma\alpha} + e^{i\psi} K_{\sigma\beta} & K_{\sigma\sigma} & K_{\Gamma\sigma}^T \\
K_{\Gamma\alpha} + e^{i\psi} K_{\Gamma\beta} & K_{\Gamma\sigma} & K_{\Gamma\Gamma}
\end{bmatrix}
\begin{bmatrix}
x_\alpha \\
x_\sigma \\
x_\Gamma
\end{bmatrix} = 0
\]  
(A.8)

Now writing the displacements in the form

\[
x_\alpha = \bar{x}_\alpha^c + i\bar{x}_\sigma^s \]
\[
x_\sigma = \bar{x}_\sigma^c + i\bar{x}_\sigma^s \]  
(A.9)

\[
x_\Gamma = \bar{x}_\Gamma^c + i\bar{x}_\Gamma^s \]
and utilizing Euler’s formula $e^{\pm i\psi} = \cos \psi \pm i \sin \psi$, Eq. A.8 becomes

$$
\begin{bmatrix}
K_{\alpha\alpha} + K_{\beta\beta} + K_{\alpha\beta} [\cos \psi + i \sin \psi] + K_{\alpha\beta}^T [\cos \psi - i \sin \psi] \\
K_{\sigma\alpha} + K_{\sigma\beta} [\cos \psi + i \sin \psi] \\
K_{\Gamma\alpha} + K_{\Gamma\beta} [\cos \psi + i \sin \psi] \\
K_{\sigma\alpha}^T + K_{\sigma\beta}^T [\cos \psi - i \sin \psi] \\
K_{\Gamma\alpha}^T + K_{\Gamma\beta}^T [\cos \psi - i \sin \psi]
\end{bmatrix}
\begin{bmatrix}
\tilde{x}_c^c + i\tilde{x}_s^c \\
\tilde{x}_c^s + i\tilde{x}_s^s \\
\tilde{x}_c^\Gamma + i\tilde{x}_s^\Gamma
\end{bmatrix} = 0 \quad (A.10)
$$

where the “tilde” overscript denotes a value that is in cyclic coordinates. It is important to emphasize that for harmonics $h = 0$, and if it exists $N/2$, the sine component is eliminated and no sine terms exist in the displacement vector. For these harmonics, only a single sector description is required and Eq. A.10 becomes

$$
\tilde{K}^{(h)} \tilde{x}^{(h)} = \begin{bmatrix}
\tilde{K}_{\tau\tau}^c(h) & \tilde{K}_{\tau\Gamma}^c(h) \\
\tilde{K}_{\tau\Gamma}^c(h) & \tilde{K}_{\Gamma\Gamma}^c(h)
\end{bmatrix}
\begin{bmatrix}
\tilde{x}_c^c \\
\tilde{x}_c^\Gamma
\end{bmatrix} = 0 \quad (A.11)
$$

where superscript $h$ denotes the harmonic index and not an exponential. The $\alpha$ and $\sigma$ DOF are combined into the sector interior DOF $\tau$

$$
\tilde{x}_c^c = \begin{bmatrix}
\tilde{x}_c^c \\
\tilde{x}_c^\alpha \\
\tilde{x}_c^\sigma
\end{bmatrix} \quad (A.12)
$$
\[ \tilde{K}^{(h)}_{\tau \tau} = \begin{bmatrix} K_{\alpha \alpha} + K_{\beta \beta} + (K_{\alpha \beta} + K^T_{\alpha \beta}) \cos \psi & K^T_{\sigma \alpha} + K^T_{\sigma \beta} \cos \psi \\ K_{\sigma \alpha} + K_{\sigma \beta} \cos \psi & K_{\sigma \sigma} \end{bmatrix} \] (A.13)

\[ \tilde{K}^{(h)}_{\tau \Gamma} = \begin{bmatrix} K^T_{\Gamma \alpha} + K^T_{\Gamma \beta} \cos \psi \\ K^T_{\Gamma \sigma} \end{bmatrix} \] (A.14)

\[ \tilde{K}^{(h)}_{\Gamma \Gamma} = K_{\Gamma \Gamma} \] (A.15)

For the remaining harmonics, \( h \neq 0 \) and \( h \neq N/2 \), the sine terms of Eq. A.10 remain and a duplicate, or double, sector description is required. Expanding Eq. A.10 and recombining to keep the sine and cosine terms separate yields

\[
\begin{align*}
\left\{ \left[ K_{\alpha \alpha} + K_{\beta \beta} + (K_{\alpha \beta} + K^T_{\alpha \beta}) \cos \psi \right] \tilde{x}_c^c + \left[ - (K_{\alpha \beta} - K^T_{\alpha \beta}) \sin \psi \right] \tilde{x}_s^s + \cdots \\
+ \left[ K^T_{\sigma \alpha} + K^T_{\sigma \beta} \cos \psi \right] \tilde{x}_s^c + \left[ K^T_{\sigma \alpha} + K^T_{\sigma \beta} \cos \psi \right] \tilde{x}_c^s + \cdots \\
+ i \left\{ \left[ (K_{\alpha \beta} - K^T_{\alpha \beta}) \sin \psi \right] \tilde{x}_c^c + \left[ K_{\alpha \alpha} + K_{\beta \beta} + (K_{\alpha \beta} + K^T_{\alpha \beta}) \cos \psi \right] \tilde{x}_c^s - \left[ K^T_{\sigma \beta} \sin \psi \right] \tilde{x}_c^s + \cdots \\
+ \left[ K^T_{\sigma \alpha} + K^T_{\sigma \beta} \cos \psi \right] \tilde{x}_s^c - \left[ K^T_{\Gamma \beta} \sin \psi \right] \tilde{x}_s^c + \left[ K^T_{\Gamma \alpha} + K^T_{\Gamma \beta} \cos \psi \right] \tilde{x}_s^c \right\} = 0 \quad (A.16)
\end{align*}
\]

\[
\begin{align*}
\left\{ \left[ K_{\sigma \alpha} + K_{\sigma \beta} \cos \psi \right] \tilde{x}_c^c - \left[ - K_{\sigma \beta} \sin \psi \right] \tilde{x}_s^c + K_{\sigma \sigma} \tilde{x}_s^c + K^T_{\Gamma \sigma} \tilde{x}_s^c \right\} & + \cdots \\
+ i \left\{ \left[ \sigma_{\alpha} + K_{\sigma \beta} \cos \psi \right] \tilde{x}_c^s + \left[ K_{\sigma \alpha} + K_{\sigma \beta} \cos \psi \right] \tilde{x}_s^c + K_{\sigma \sigma} \tilde{x}_s^c + K^T_{\Gamma \sigma} \tilde{x}_s^c \right\} = 0 \quad (A.17)
\end{align*}
\]

\[
\begin{align*}
\left\{ \left[ K_{\Gamma \alpha} + K_{\Gamma \beta} \cos \psi \right] \tilde{x}_c^c + \left[ - K_{\Gamma \beta} \sin \psi \right] \tilde{x}_s^c + K_{\Gamma \sigma} \tilde{x}_s^c + K_{\Gamma \Gamma} \tilde{x}_s^c \right\} & + \cdots \\
+ i \left\{ \left[ K_{\Gamma \beta} \sin \psi \right] \tilde{x}_c^s + \left[ K_{\Gamma \alpha} + K_{\Gamma \beta} \cos \psi \right] \tilde{x}_s^c + K_{\Gamma \sigma} \tilde{x}_s^s + K_{\Gamma \Gamma} \tilde{x}_s^s \right\} = 0 \quad (A.18)
\end{align*}
\]
Now, Equations A.16 - A.18 can be placed into the following matrix format

\[
\begin{bmatrix}
\mathcal{R} \text{ (Eq. A.16)} & \mathcal{R} \text{ (Eq. A.17)} & \mathcal{I} \text{ (Eq. A.16)} & \mathcal{I} \text{ (Eq. A.17)} & \mathcal{R} \text{ (Eq. A.18)} & \mathcal{I} \text{ (Eq. A.18)} \\
\end{bmatrix}
\begin{bmatrix}
\tilde{x}_\alpha^c \\
\tilde{x}_\sigma^c \\
\tilde{x}_\alpha^s \\
\tilde{x}_\sigma^s \\
\tilde{x}_\Gamma^c \\
\tilde{x}_\Gamma^s \\
\end{bmatrix} = 0 \quad (A.19)
\]

which is expanded to

\[
\begin{bmatrix}
K_{\alpha\alpha} + K_{\beta\beta} + (K_{\alpha\beta} + K_{\alpha\beta}^T) \cos \psi & K_{\sigma\alpha}^T + K_{\sigma\beta}^T \cos \psi & - (K_{\alpha\beta} - K_{\alpha\beta}^T) \sin \psi \\
K_{\sigma\alpha} + K_{\sigma\beta} \cos \psi & K_{\sigma\sigma} & -K_{\sigma\beta} \sin \psi \\
(K_{\alpha\beta} - K_{\alpha\beta}^T) \sin \psi & -K_{\sigma\beta}^T \sin \psi & K_{\alpha\alpha} + K_{\beta\beta} + (K_{\alpha\beta} + K_{\alpha\beta}^T) \cos \psi \\
K_{\sigma\beta} \sin \psi & 0 & K_{\sigma\alpha} + K_{\sigma\beta} \cos \psi \\
K_{\Gamma\alpha} + K_{\Gamma\beta} \cos \psi & K_{\Gamma\sigma} & -K_{\Gamma\beta} \sin \psi \\
K_{\Gamma\beta} \sin \psi & 0 & K_{\Gamma\alpha} + K_{\Gamma\beta} \cos \psi \\
K_{\sigma\beta}^T \sin \psi & K_{\Gamma\alpha}^T + K_{\Gamma\beta}^T \cos \psi & K_{\Gamma\beta}^T \sin \psi \\
0 & K_{\Gamma\sigma}^T & 0 \\
K_{\sigma\alpha}^T + K_{\sigma\beta}^T \cos \psi & -K_{\Gamma\beta}^T \sin \psi & K_{\Gamma\alpha}^T + K_{\Gamma\beta}^T \cos \psi \\
0 & 0 & K_{\Gamma\sigma}^T \\
0 & K_{\Gamma\Gamma} & 0 \\
K_{\Gamma\sigma} & 0 & K_{\Gamma\Gamma} \\
\end{bmatrix}
\begin{bmatrix}
\tilde{x}_\alpha^c \\
\tilde{x}_\sigma^c \\
\tilde{x}_\alpha^s \\
\tilde{x}_\sigma^s \\
\tilde{x}_\Gamma^c \\
\tilde{x}_\Gamma^s \\
\end{bmatrix} = 0 \quad (A.20)
\]

To render the Craig-Bampton partitionment of interior and interface DOF, the \(\tilde{x}_\alpha\) and...
\( \ddot{x}_\sigma \) DOF are combined into CB interior DOF

\[
\ddot{x}_\tau = \begin{cases} 
\ddot{x}_\tau^c \\
\ddot{x}_\tau^s 
\end{cases} = \begin{cases} 
\ddot{x}_\alpha^c \\
\ddot{x}_\alpha^s 
\end{cases}
\] (A.21)

while the \( \ddot{x}_\Gamma \) interface DOF remain as previously defined. This new partitionment results in the following cyclic CB partitionment DOF vector

\[
\ddot{x}^{(h)} = \begin{cases} 
\ddot{x}_\tau^{(h)} \\
\ddot{x}_\Gamma^{(h)} 
\end{cases}
\] (A.22)

and the CB cyclic disk stiffness matrix

\[
\tilde{K}^{(h)} = \begin{bmatrix} 
\tilde{K}^{(h)}_{\tau\tau} & \tilde{K}^{(h)}_{\tau\Gamma} \\
\tilde{K}^{(h)}_{\tau\Gamma} & \tilde{K}^{(h)}_{\Gamma\Gamma} 
\end{bmatrix}
\] (A.23)

Equation A.19 is partitioned according to Eq. A.22 to give:

\[
\tilde{K}^{(h)}_{\tau\tau} = \begin{bmatrix} 
\tilde{K}^{(h)}_{\tau\tau} & \tilde{K}^{(h)}_{\tau\Gamma} \\
\tilde{K}^{(h)}_{\tau\Gamma} & \tilde{K}^{(h)}_{\Gamma\Gamma} 
\end{bmatrix}
\] (A.24)

where

\[
1\tilde{K}^{(h)}_{\tau\tau} = \begin{bmatrix} 
K_{\alpha\alpha} + K_{\beta\beta} + (K_{\alpha\beta} + K_{\alpha\beta}^T) \cos \psi & K_{\tau\alpha}^T + K_{\tau\beta}^T \cos \psi \\
K_{\tau\alpha} + K_{\tau\beta} \cos \psi & K_{\tau\tau}
\end{bmatrix}
\] (A.25)
\[ 2 \tilde{K}^{(h)}_{\tau \tau} = \begin{bmatrix} \left( K^\top_{\alpha \beta} - K_{\alpha \beta} \right) \sin \psi & K^\top_{\sigma \beta} \sin \psi \\ -K_{\sigma \beta} \sin \psi & 0 \end{bmatrix} \] (A.26)

Furthermore,
\[ \tilde{K}^{(h)}_{\tau \Gamma} = \begin{bmatrix} K^\top_{\Gamma \alpha} + K^\top_{\Gamma \beta} \cos \psi & K^\top_{\Gamma \beta} \sin \psi \\ K^\top_{\Gamma \sigma} & 0 \\ -K^\top_{\Gamma \beta} \sin \psi & K^\top_{\Gamma \alpha} + K^\top_{\Gamma \beta} \cos \psi \\ 0 & K^\top_{\Gamma \sigma} \end{bmatrix} \] (A.27)

\[ \tilde{K}^{(h)}_{\Gamma \Gamma} = \begin{bmatrix} K_{\Gamma \alpha} + K_{\Gamma \beta} \cos \psi & K_{\Gamma \sigma} & -K_{\Gamma \beta} \sin \psi & 0 \\ K_{\Gamma \beta} \sin \psi & 0 & K_{\Gamma \alpha} + K_{\Gamma \beta} \cos \psi & K_{\Gamma \sigma} \end{bmatrix} \] (A.28)

\[ \tilde{K}^{(h)}_{\Gamma \Gamma} = \begin{bmatrix} K_{\Gamma \Gamma} & 0 \\ 0 & K_{\Gamma \Gamma} \end{bmatrix} \] (A.29)

The final cyclic EOM then becomes
\[ \tilde{M}^{(h)} \ddot{\tilde{x}}^{(h)} + \tilde{K}^{(h)} \tilde{x}^{(h)} = 0 \] (A.30)

In summary, the symmetrical components of the disk substructure for each harmonic index can be calculated by Eq. A.11, with an analogous representation of the mass matrix. Particularly, the submatrices to \( \tilde{K}^{(h)} \) listed in Eq. A.15 are for harmonics \( h = 0 \), and if it exists \( \pi / 2 \), that give a single sector description.
Circulant Matrices

Matrices describing a linear system with cyclic symmetry properties have circulant matrix properties, beginning with a matrix structure described by

\[
K = \begin{bmatrix}
  c_1 & c_2 & \cdots & c_N \\
  c_N & c_1 & \cdots & c_{N-1} \\
  \vdots & \vdots & \ddots & \vdots \\
  c_2 & c_3 & \cdots & c_1
\end{bmatrix}
\]  

(B.1)

where for IBRs and DFIBRs

\[
c_j = \begin{cases}
  K_{c_1 c_1} & \text{if } j = 1, \\
  K_{c_1 c_2} & \text{if } j = 2, \\
  0 & \text{if } j = 3, \cdots, N - 1, \\
  K_{c_1 c_2}^T & \text{if } j = N.
\end{cases}
\]  

(B.2)

Furthermore, these circulant matrices of order \( N \) posses \( N \) independent eigenvectors that compose the complex Fourier matrix, \( F \)

\[
F = [f_{jk}]
\]  

(B.3)
where

\[ f_{j,k} = \frac{1}{\sqrt{N}} e^{i\alpha(j-1)(k-1)} \]  

(B.4)

where \( i = \sqrt{-1} \) and \( \alpha = \frac{2\pi}{N} \).

A real-valued formulation of the Fourier matrix is given by

\[
E = \begin{bmatrix}
e_0 & e_{1,c} & e_{1,s} & \cdots & e_{h,c} & e_{h,s} & \cdots & e_{N/2}
\end{bmatrix}
\]  

(B.5)

where \( e_{h,c} \) and \( e_{h,s} \) are the column vectors of \( E \) corresponding to the cosine and sine terms and

\[
e_0 = \left\{ \frac{1}{\sqrt{N}}, \ldots, \frac{1}{\sqrt{N}} \right\}^T
\]  

(B.6)

\[
e_{h,c} = \left\{ \sqrt{\frac{2}{N}}, \ldots, \sqrt{\frac{2}{N}} \cos [\alpha h (j-1)], \ldots, \sqrt{\frac{2}{N}} \cos [\alpha h (N-1)] \right\}^T
\]  

(B.7)

\[
e_{h,s} = \left\{ 0, \ldots, \sqrt{\frac{2}{N}} \sin [\alpha h (j-1)], \ldots, \sqrt{\frac{2}{N}} \sin [\alpha h (N-1)] \right\}^T
\]  

(B.8)

\[
e_{N/2} = \left\{ \frac{1}{\sqrt{N}}, \ldots, \frac{(-1)^{(j-1)}}{\sqrt{N}}, \ldots, \frac{(-1)^{(N-1)}}{\sqrt{N}} \right\}^T
\]  

(B.9)

where \( e_{N/2} \) only exists if \( N \) is even. Some useful properties of these matrices follow

\[
F^\dagger F = E^\dagger E = I
\]  

(B.10)

\[
F^{-1} = F^\dagger, \quad E^{-1} = E^\dagger
\]  

(B.11)
Kronecker Product

The Kronecker product for the matrices in this document are defined as

\[
I_{D \times D} \otimes E_{N \times N} = \begin{bmatrix}
E & 0 \\
& E \\
& & \ddots \\
0 & & & E
\end{bmatrix}_{D \times D \otimes E_{N \times N}} = B [E] \tag{C.1}
\]

\[
\hat{E} = E_{N \times N} \otimes I_{D \times D} = \begin{bmatrix}
I_{D \times D} \otimes E_{11} & I_{D \times D} \otimes E_{12} & \cdots & I_{D \times D} \otimes E_{1N} \\
I_{D \times D} \otimes E_{21} & I_{D \times D} \otimes E_{22} & \cdots & I_{D \times D} \otimes E_{2N} \\
\vdots & \vdots & \ddots & \vdots \\
I_{D \times D} \otimes E_{N1} & I_{D \times D} \otimes E_{N2} & \cdots & I_{D \times D} \otimes E_{NN}
\end{bmatrix} \tag{C.2}
\]

Other useful properties are

\[
(A \otimes B)(C \otimes D) \tag{C.3}
\]

\[
(A \otimes B)^{-1} = A^{-1} \otimes B^{-1} \tag{C.4}
\]

\[
(A \otimes B)^\top = A^\top \otimes B^\top \tag{C.5}
\]