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Effect of a Graded Layer on the Plastic Dissipation During Mixed-Mode Fatigue Crack Growth on Ductile Bimaterial Interfaces

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EFFECT OF A GRADED LAYER ON THE PLASTIC DISSIPATION DURING MIXED-MODE FATIGUE CRACK GROWTH ON DUCTILE BIMATERIAL INTERFACES

A dissertation submitted in partial fulfillment of the requirements for the degree of
Doctor of Philosophy

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Abstract


An energy-based theory for predicting mode I fatigue crack growth rates in ductile metals based on the total plastic dissipation ahead of a crack tip was proposed in 2003 [1]. Since then, this theory has been extended to layered material systems that typically include mixed-mode loading and elastic/plastic mismatch [2–4]. Previous research by the author first extended this theory to include a graded layer (i.e., no step change in material properties across the crack plane) with which to more accurately model a crack interface between two materials with a mismatch in plastic properties only [5]. In the current research, the graded layer model has been extended to include a mismatch in elastic properties as well. In so doing, the author has derived a beam-theory solution for the strain energy release rate for use in exploring nondimensional effects of graded layer height, elastic mismatch, and mode of loading. In addition, the graded layer model has led to a purely elastic method for determining an unambiguous definition of the mode-mix in the presence of an elastic mismatch and has been validated by elastic-plastic plane strain finite element results illustrating the resulting plastic zones. This has led to an independent validation of a physically based mode-mix definition for bimaterial crack tips based on the total plastic dissipation developed by Daily in [4]. In addition, this dissertation provides comprehensive numerical results for the effects of an elastic/plastic mismatch, mode-mix, and graded layer height on
the total plastic dissipation for steady-state fatigue cracks on ductile bimaterial interfaces following that of [4] and [6]. Finally, experimental results for a brazed specimen under four-point bending that include sustained fatigue cracking along an interface have been provided for use in validating the theory of [1] for mixed-mode loading.
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Dedication

All glory to God above. Any talents, successes, and achievements are not my own, but come from Him. This journey would also not have been possible if not for the unwavering support of my parents Jack and Sheryl and my sister Carly. I only hope that I can one day show the same love and support to a family of my own. And to that end, I’m extremely grateful for the love and support of my soon-to-be wife, Erika. She has been a cornerstone upon which I’ve come to rely and I truly look forward to spending the rest of my life with her at my side.
Chapter 1

Introduction

This research provides insight into the prediction of steady-state fatigue crack growth rates based on the total plastic dissipation ahead of a crack. Finite element models (FEM) have been modified to include a discretized grading of material properties along an interface crack that lies between two dissimilar, isotropic materials. By eliminating the abrupt step change commonly modeled in bimaterial interfaces, inclusion of this graded layer leads to three important implications for current layered material modeling:

- A graded layer more accurately represents the mixing of properties that occurs when two materials are joined together.
- A graded layer removes the effect of the stress oscillation present in bimaterial crack tip stress fields.
- A graded layer provides an unambiguous definition of the mode-mix in the presence of an elastic mismatch.

The results of this work could potentially lead to the advancement of fatigue life estimates based on monotonic fracture properties in conjunction with finite element results, without
the need for costly and lengthy crack growth measurements.

1.1 Motivation

In today’s modern world, the demand for new-age materials that support next generation technology is ever increasing. In order to maintain this current pace, new methods of material testing are needed that can give quick, accurate assessments of material behavior for a diverse set of applications. To this end, an energy-based theory for predicting steady-state fatigue crack growth rates based on the total plastic dissipation ahead of a crack was developed by Klingbeil in 2003 [1]. It was shown that for homogeneous materials, a correlation exists between the monotonic fracture toughness and steady-state fatigue crack growth rates under mode I loading. This correlation is based on the idea that the energy required for crack growth is independent upon the manner in which the crack is propagated, whether through monotonic or fatigue loading. It follows that fatigue behavior can be predicted with only monotonic fracture toughness data and cyclic elastic-plastic finite element results.

In a series of papers from 2004 to 2010, Daily and Klingbeil sought to expand the finite element models of [1] to include more load cases and material combinations for use in more practical applications in the field of layered manufacturing. It is well known that fatigue crack growth in homogeneous materials generally occurs in a mode I configuration, as it provides an energetically favorable situation for crack extension to occur. However, fatigue delamination of bimaterial interfaces (a potential mode of failure seen in layered manufacturing and other applications) is an inherently mixed-mode problem, involving both mode I and mode II loading. To this end, the first paper in 2004 introduced a two-layer double cantilever beam (DCB) mixed-mode finite element model with the ultimate goal of predicting delamination of layered material systems [2]. This model was slightly modified in
2006 and 2010 to include mismatches in plastic (i.e. yield strength) and elastic (i.e. Young’s modulus) properties between the top and bottom layers, culminating in plastic dissipation results for general bimaterial interfaces under a variety of loading conditions [3, 4]. Examples of these types of interfaces include welding, brazing, and soldering applications that inherently create bimetallic interfaces by high-temperature joining of two materials. Such an example is shown in Fig. 1.1(a) with the joining of aluminum to stainless steel [8–13]. Fig. 1.1(b) shows an additive manufacturing technique that builds parts layer-by-layer for applications where enhanced material properties are needed or complex geometries are required [14–18]. Applications in microelectronics, thin films, and flexible electronics exhibit layered geometries by adhering an elastic-plastic substrate to a relatively stiff film layer [19–22].

While Daily’s technique of handling material mismatches presented in [4] was quite novel¹, a recurring question provided the necessary motivation for the research presented herein. It is a common assumption of the previous models that a *perfectly sharp* crack exists between the top and bottom layers, so that the material properties exhibit a step change across the interface. This step change introduces added complexity in the fracture mechanics quantities needed to define the initial problem and is the source of some ambiguity in defining the mode-mix that a general layered system undergoes. In the current

¹Details presented in Section 1.2.4.
study, a graded layer model is introduced to allow a smooth transition in material properties between the two dissimilar layers, which provides a more realistic model for general bimaterial interfaces, allows for the use of classical fracture mechanics quantities, and provides an independent validation for Daily’s research in elastic mismatch problems. The effect of a graded layer is particularly relevant for additive manufacturing processes currently under development, which can potentially allow for intentional grading of material properties across a deposited interface (i.e., to influence mechanical behavior).

The following literature review outlines the previous research by Klingbeil and Daily that has been foundational to the current research.

1.2 A Total Dissipated Energy Theory

The first energy-based theory for crack growth was developed by Griffith in 1921 [23] and later modified by Irwin in 1958 [24]. While Griffith attributed the energy needed for crack growth to the difference in total potential energy and surface energy (experimentally valid for brittle materials), Irwin recognized that the total energy required for crack growth in ductile metals was largely due to crack tip plasticity\(^2\). Shortly thereafter, Rice proposed that there exists a critical amount of plastic dissipation (i.e., crack tip plasticity) that drives fatigue crack extension [25]. Since then, other researchers have explored the mechanism of crack tip plasticity as a driving force for fatigue crack growth through both analytical [25–37] and experimental [37–44] approaches. Furthermore, the dissipated energy theory presented by Klingbeil in 2003 [1] follows that of Bodner, Davidson, and Langford [45,46] where the total plastic dissipation contained in the reverse plastic zone ahead of a crack is proportional to the fatigue crack growth rate. It is this theory by Klingbeil [1] upon which all of the research presented herein is predicated.

\(^2\)Plasticity as defined by linear elastic fracture mechanics (LEFM), i.e. small scale yielding.
1.2.1 Plastic Dissipation Under Mode I Loading for Homogeneous Materials

In his 2003 paper, Klingbeil cited a need for advanced material modeling with which to enhance current methods for evaluating fatigue behavior of next generation materials. To this end, a link between monotonic fracture properties and fatigue crack growth was developed. It was hypothesized that the amount of energy needed to advance a crack a unit distance is independent of the manner in which the load is applied (i.e., monotonic or fatigue loading). The implications of this include being able to predict steady-state Paris-regime fatigue behavior with minimal experimental testing and modeling, thus saving time and money in new-age material development. Put in equation form (derived from the energy balance seen in Fig. 1.2),

\[
\frac{da}{dN} = \frac{1}{G_c} \frac{dW}{dN},
\]

where \(a\) is the crack length, \(N\) is the number of cycles, \(W\) is the plastic work per unit specimen width, and \(G_c\) is the monotonic fracture toughness of the material. The fatigue crack growth rate \(da/dN\) can be calculated using only \(G_c\) (i.e., monotonic fracture toughness) and the total plastic work per cycle \(dW/dN\). In ref. [1], this last term was extracted from an elastic-plastic finite element model of a C(T) specimen by integrating over the reversed

Figure 1.2: Energy balance for Klingbeil’s crack growth law [1].
Figure 1.3: Normalized data and “universal” crack growth law for several metals [1].

The validation of this theory was performed in [1] for homogeneous ductile metals under mode I loading. Fig. 1.3 shows crack growth data for several metals normalized with respect to finite element results and existing fracture toughness values. The plastic work per cycle \( dW/dN \) in eq. (1.1) can be nondimensionalized as

\[
\frac{dW}{dN} = \int \int_{r_p} \left\{ \sigma_{ij} d\epsilon_{ij}^p \right\} dA. \tag{1.2}
\]

The validation of this theory was performed in [1] for homogeneous ductile metals under mode I loading. Fig. 1.3 shows crack growth data for several metals normalized with respect to finite element results and existing fracture toughness values. The plastic work per cycle \( dW/dN \) in eq. (1.1) can be nondimensionalized as

\[
\frac{dW}{dN^*} = \frac{\sigma^2_y}{\overline{E} \Delta\mathcal{G}^2} \frac{dW}{dN^*}, \tag{1.3}
\]

where \( \sigma_y \) is the yield strength, \( \overline{E} = E \) for plane stress \( \overline{E} = E/(1 - \nu^2) \) for plane strain, and \( \Delta\mathcal{G} \) is the applied strain energy release rate. In the context of Fig. 1.3, substitution of
eq. (1.3) into eq. (1.1) results in a crack growth law of the form

$$\frac{da}{dN} = \frac{\Delta K^4}{\sigma^2_y K_{IC}} \frac{dW}{dN*}. \quad (1.4)$$

The plots in Fig. 1.3 show that Klingbeil’s crack growth theory is valid for homogeneous ductile metals under mode I loading. The next logical progression in this research was to apply this theory to a wider variety of loadings and materials (i.e., layered manufacturing) and to explore what other factors influence $dW/dN*$. 

### 1.2.2 Plastic Dissipation Under Mixed-Mode Loading for Homogeneous Materials

The dissipated energy approach was extended in 2004 by Daily and Klingbeil to cases of steady-state cyclic crack delamination of ductile layered material systems where the mode of loading is generally a combination of modes I and II (i.e., mixed-mode) [2]. This situation arises for a number of manufacturing techniques where a part is built up layer-by-layer (i.e., electron beam and laser based) where the overall material properties are identical. A potential failure mechanism observed for this type of homogeneous layered geometry is fatigue cracking between successive deposited layers.

The term “mixed-mode” refers to a load state that is neither pure mode I nor pure mode II as defined by the illustrations in Fig. 1.4(a) and (b), but is inherently a combination
of both opening and shearing at the same time. The research contained herein does not consider the out-of-plane tearing of mode III (Fig. 1.4(c)), which is a topic of ongoing research elsewhere. Elastic-plastic fracture mechanics concepts for bimaterial interfaces under both monotonic and cyclic loading are outlined in [47–49], and have been applied to ductile/brittle interfaces in [49–51]. Since that time, a small number of studies of soldered and welded joints have provided experimental results for mixed-mode fatigue crack growth along bimaterial interfaces [9, 52–57].

In order to extend the dissipated energy theory to layered material systems, Daily developed a double cantilever beam (DCB) model shown in Fig. 1.5 [2], that is a special case of the general bimaterial specimen geometry outlined by Suo and Hutchinson in [58]. This model has top and bottom layers of height \( h \) and overall length \( L \). The crack length \( a \) is sufficiently long to ensure steady-state conditions at the crack tip. Bending moments per unit width \( M_1 \) and \( M_2 \) are applied to the top and bottom layers, respectively, and are equilibrated by a symmetry condition on the right side. The DCB model is able to capture the full effect of mode mixity simply by varying the bending moments \( M_1 \) and \( M_2 \). This can be characterized by a mode-mix parameter in units of degrees defined as

\[
\psi = \tan^{-1} \left[ \frac{\sqrt{3} (M_2 - M_1)}{2 (M_2 + M_1)} \right].
\]

(1.5)

![Two Layer Model](image.png)

Figure 1.5: Two Layer Model [3].
Figure 1.6: (a) Mode I (b) Mode FPB (c) Mode II plastic zones [5].

The mode-mix is defined in the range $0^\circ \leq \psi \leq 90^\circ$ where $0^\circ$ corresponds to pure mode I loading and $90^\circ$ corresponds to pure mode II loading\(^3\). Results from this prior work not only include the nondimensional plastic dissipation $dW/dN^*$, but also depictions of the crack tip plastic zones that occur for various modes. These plastic zones can be thought of as visual representations of the plastic work that is occurring at different values of $\psi$. Mode I, mode four-point bend (FPB), and mode II forward plastic zones reproduced by the current author are shown in Figs. 1.6(a), (b), and (c) respectively.

For the model in Fig. 1.5, the fatigue crack growth law of eq. (1.1) can be written as

$$\frac{da}{dN} = \frac{E \Delta \psi^2}{\sigma^2 \gamma_c} \frac{dW}{dN^*} (\psi),$$

(1.6)

where $dW/dN^*$ represents dimensionless plastic dissipation results that are a function of the mode-mix $\psi$ as shown in Fig. 1.7. It can be seen that as the mode of loading increases from $0^\circ$ to $90^\circ$, the nondimensional plastic work per cycle increases by roughly an order of magnitude. This is due to the fact that the plastic zone in mode II is much larger than that in mode I as can be seen by length scale in Fig. 1.6.

\(^3\)A full treatment on the derivation and limits of $\psi$ are given in Section 2.1.2
1.2.3 Plastic Dissipation Under Mixed-Mode Loading for Plastically Mismatched Interfaces

In 2006, Daily expanded the DCB model in [2] to include a plastic mismatch as shown in Fig. 1.8. This mismatch refers to material combinations where the yield strength of the top layer can be different from the yield strength of the bottom layer. Examples of these can be seen in [49] where cases of ductile/brittle interfaces between metals and composites are
encountered. The magnitude of this mismatch in yield stresses is controlled by the yield strength mismatch parameter $\hat{\sigma}$, defined as

$$\hat{\sigma} = \frac{\sigma_1 - \sigma_2}{\sigma_1 + \sigma_2},$$

(1.7)

where the subscripts 1 and 2 indicate top and bottom layers, respectively. The mismatch parameter is defined in the range $-1 \leq \hat{\sigma} \leq 1$, where negative values indicate the top layer is weaker compared to the bottom layer, positive values indicate the top layer is stronger compared to the bottom layer, and $\hat{\sigma} = 0$ indicates both layers have equal yield strengths. In addition, eq. (1.7) can be rearranged as

$$\frac{\sigma_1}{\sigma_2} = \frac{1 + \hat{\sigma}}{1 - \hat{\sigma}},$$

(1.8)

where $\hat{\sigma} = 0.9$ corresponds to a top layer that is 19 times stronger than the bottom layer and $\hat{\sigma} = -0.9$ corresponds to a top layer that has about 5% of the strength of the bottom layer.

The effect of a plastic mismatch alters the shape of the plastic zones as well. Figs. 1.9 (a), (b), and (c) depict forward plastic zones in the presence of a plastic mismatch as reproduced by the current author in [5]. In sharp crack bimaterial models, this mismatch will decrease overall plasticity and its effect on $dW/dN^*$ has been shown to be asymptotic. In other words, for $\hat{\sigma} \geq 0.25$, there is negligible effect on the plastic dissipation.\(^4\) As an added conclusion to this prior work, the mode mixity of the problem has a more dominant effect on the overall plasticity than the plastic mismatch parameter. This dependence is seen in Fig. 1.10 where $dW/dN^*$ is plotted vs. $\psi$ with each data set representing a different $\hat{\sigma}$.

\(^4\)While valid for bimaterial models with a perfectly sharp crack, it has since been shown that the addition of a graded layer increases the overall plasticity and thus there is measurable response for $\hat{\sigma}$ values greater than 0.25.
Figure 1.9: (a) Mode I (b) Mode FPB (c) Mode II plastic zones in the presence of a plastic mismatch ($\hat{\sigma} = 0.25$) [5].

Figure 1.10: $dW/dN*$ vs. $\psi$ for $\hat{\sigma} \geq 0.0$ [3].
The fatigue crack growth law of eq. (1.1) can now be written as

\[
\frac{da}{dN} = \frac{E_2 \Delta \gamma^2}{\sigma^2_c \gamma_c} \frac{dW}{dN} \ast (\psi, \hat{\sigma}),
\]

where the subscript 2 indicates normalization with respect to the bottom layer and \( dW/dN \ast \) is a function of both mode-mix and yield strength mismatch. It is also worth noting here that this crack growth law has not been validated experimentally, and that a significant void in the literature exists for mixed-mode crack data in the presence of a plastic mismatch.

### 1.2.4 Plastic Dissipation Under Mixed-Mode Loading for Elastically Mismatched Interfaces

The next step in this research was again published by Daily and Klingbeil in 2010 [4] and included plastic dissipation results for not just mode-mixity and plastic mismatch, but for an elastic mismatch as well. The same model developed by Daily and shown in Fig. 1.11 is used to extract plastic dissipation results. The only physical difference between this and previous models is that a mismatch in elastic properties (Young’s modulus and Poisson’s ratio) can exist between the top and bottom layers.

This mismatch in elastic properties can be characterized by Dundurs’ parameters \( \alpha \) and
\( \beta \) as defined by

\[
\alpha = \frac{\mu_1 (\kappa_2 + 1) - \mu_2 (\kappa_1 + 1)}{\mu_1 (\kappa_2 + 1) + \mu_2 (\kappa_1 + 1)},
\]

(1.10)

and

\[
\beta = \frac{\mu_1 (\kappa_2 - 1) - \mu_2 (\kappa_1 - 1)}{\mu_1 (\kappa_2 + 1) + \mu_2 (\kappa_1 + 1)},
\]

(1.11)

where subscripts 1 and 2 refer to the top and bottom layers, respectively [59]. In the above equations, \( \mu \) is the shear modulus, \( \kappa = 3 - 4\nu \) in plane strain and \( \kappa = (3 - \nu)/(1 + \nu) \) in plane stress. A more useful form of \( \alpha \) is

\[
\alpha = \frac{E_1 - E_2}{E_1 + E_2},
\]

(1.12)

which indicates that \( \alpha \) measures the relative stiffness of the two materials. The elastic mismatch parameter \( \alpha \) is defined in the range \(-1 \leq \alpha \leq 1\) where positive values of \( \alpha \) indicate a stiffer top layer compared to the bottom layer, negative values of \( \alpha \) indicate a stiffer bottom layer compared to the top layer, and \( \alpha = 0 \) is the homogeneous case with identical elastic properties. Similar to eq. (1.8), eq. (1.12) can be rearranged as

\[
\frac{E_1}{E_2} = \frac{1 + \alpha}{1 - \alpha},
\]

(1.13)

where \( \alpha = 0.9 \) corresponds to the top layer being 19 times stiffer than the bottom layer and \( \alpha = -0.9 \) corresponds to the top layer being about 5\% of the stiffness of the bottom layer.

The parameter \( \beta \) is defined in the range \(-0.5 \leq \beta \leq 0.5\) for real solids.

While including an elastic mismatch seems to be a relatively small change to the model, the implications of such are quite complex. In the presence of an elastic mismatch, the applied mode as defined by eq. (1.5) no longer matches the crack tip mode defined by the local K-fields. This can be further understood by examining the plastic zones that occur in the presence of an elastic mismatch. For symmetric loading of a homogeneous material, the plastic zone of Fig. 1.12(a) occurs at the crack tip. For the same symmetric loading and an
elastic mismatch of $\alpha = 0.8$, the plastic zone of Fig. 1.12(b) occurs at the crack tip. Simply by changing the elastic properties of the top layer, the symmetric mode I plastic zone of Fig. 1.12(a) becomes asymmetric in Fig. 1.12(b) for the exact same physical loading case! Furthermore, it is well known that the stress fields at a crack tip in a homogeneous material exhibit an inverse square root singularity. However, in the presence of a bimaterial elastic mismatch, they also include an oscillatory singularity as given by

$$\left(\sigma_{yy} + i\sigma_{xy}\right) |_{y=0} = \left(\frac{K_1 + iK_2}{\sqrt{2\pi r_x}}\right)^{ie}.$$  \hspace{1cm} (1.14)

Because of this, the mode-mix parameter is defined as

$$\psi = \tan^{-1}\frac{Im(K^{ie})}{Re(K^{ie})},$$  \hspace{1cm} (1.15)

where $l$ is an arbitrary characteristic length and $\epsilon$ is the oscillation index or bimaterial constant defined as

$$\epsilon = \frac{1}{2\pi}\ln\left(\frac{1 - \beta}{1 + \beta}\right).$$  \hspace{1cm} (1.16)

Because $l$ is arbitrary, the definition of what mode of loading the crack tip actually sees in the presence of an elastic mismatch is also arbitrary. This definition was used by Suo and Hutchinson in [60] where $l$ was set equal to the layer height $h$. Any subsequent reporting
of mode-mix values must have corresponding characteristic lengths reported as well. For comparison of results involving different characteristic lengths, the required transformation equation is given by

$$\psi_2 = \psi_1 + \varepsilon \ln \frac{l_2}{l_1}.$$  \hspace{1cm} (1.17)

Not only does this ambiguous mode definition complicate the bimaterial mismatch problem, but the stress field for an elastically mismatched problem become oscillatory as the crack tip is approached, which is generally accepted as a non-physical result.

One of the main focal points of Daily and Klingbeil’s latest research was to provide a more physically based definition of the characteristic length \(l\). To this end, plastic dissipation was used to “calibrate” the mode-mix for any combination of loading or material mismatch. Instead of using one characteristic length in defining the mode-mix, Daily proposed using various characteristic lengths based on the minimum plasticity that occurs in mode I and the maximum plasticity that occurs in mode II. Using an interpolation scheme for mixed-mode cases, Daily was able to map out the plastic dissipation for the full range of mode-mix in the presence of an elastic and plastic mismatch. These results led to eq. (1.4) being modified to

$$\frac{da}{dN} = \frac{E_2 \Delta G^2}{\sigma^2 G_c} \frac{dW}{dN*}(\psi, \hat{\sigma}, \alpha, \beta),$$  \hspace{1cm} (1.18)

where \(dW/dN*\) is a function of mode-mix, plastic mismatch, and elastic mismatch. These results are depicted in Fig. 1.13(a) and (b), where \(dW/dN*\) is plotted vs. \(\psi\) for \(\alpha \geq 0.0\) in Fig. 1.13(a) and corresponding characteristic lengths used for each of \(\psi\) are plotted in Fig. 1.13(b).

The motivation for the proposed research largely stems from the complications presented in this section for bimaterial interfaces. Ambiguous definitions for the mode-mix and non-physical oscillatory stresses at the crack can be avoided by eliminating the perfect crack interface. This is accomplished by using a graded layer.
In modeling bimaterial interfaces (as seen in the previous research), it is a common assumption that a perfectly sharp crack exists between the top and bottom layers, so that the material properties exhibit a step change across the interface. In the current study, a graded layer model is introduced to allow a smooth transition in material properties between the two dissimilar layers, which provides a more realistic model for deposited metal interfaces. The effect of a graded layer is particularly relevant for additive manufacturing processes currently under development, which can potentially allow for intentional grading of material properties across a deposited interface. While numerous studies have considered the behavior of cracks in the vicinity of a graded interface [61–68], none has provided numerical results for the total plastic dissipation ahead of the crack.

Not only does a graded layer model more accurately represent the physical problem, it makes the fracture mechanics quantities easier to deal with as well. The oscillatory stress singularity is no longer present and the actual mode-mixity of the problem can be determined using conventional methods. This will be outlined in detail in Chapter 4.

The previous models of Daily [2–4] have been modified to include a graded layer in
between the top and bottom layers with which to smoothly transition material properties from bottom to top. This new geometry is shown in Fig. 1.14. It consists of overall layer heights \( h \) and length \( L \), however, a middle layer now exists of height \( H \) where the material properties vary linearly from the bottom to the top.\(^5\) Moments \( M_1 \) and \( M_2 \) are still applied to the top and bottom layers in order to control the applied mode of loading.

### 1.3.1 Plastic Dissipation Under Mixed-Mode Loading for Plastically Mismatched Interfaces with a Graded Layer

The effect that a graded layer has on plastic dissipation when in the presence of only a plastic mismatch has been previously explored by the author in [6]. It was found that the addition of a graded layer in the presence of a plastic mismatch increases the total plastic dissipation in all cases where the crack lies on the interface of the weaker material. The rise in \( dW/dN^* \) with increasing graded layer height can be attributed to a corresponding increase in crack tip plasticity. This is best illustrated by the evolution of the plastic zones with increasing graded layer height \( h^* \).\(^6\) To this end, mode I, mode FPB, and mode II plastic zones are plotted in Figs. 1.15, 1.16, and 1.17. As the graded layer height increases, more

\(^5\) A linear function was used for simplicity, but any mathematical expression can be used (i.e., exponential or hyperbolic).

\(^6\) Where \( h^* \) is the nondimensional graded layer height described in Section 5.2
Figure 1.15: Mode I plastic zones for $\hat{\sigma} = 0.25$ [6].

Figure 1.16: Mode FPB plastic zones for $\hat{\sigma} = 0.25$ [6].

Figure 1.17: Mode II plastic zones for $\hat{\sigma} = 0.25$ [6].
plasticity is allowed to occur in the graded layer itself due to the smooth transition in yield strength. For each value of mode-mix, a graded layer which is large relative to the plastic zone (i.e., $h^* \to \infty$) would result in homogeneous plastic zones as shown in Fig. 1.6.

For the purpose of illustrating the relative effects of strength mismatch, graded layer thickness, and mode-mix, the total plastic dissipation is plotted in Fig. 1.18. The smallest $dW/dN^*$ values occur in mode I (the lowest set of lines in Fig. 1.18) while the largest values occur in mode II (the topmost lines in Fig. 1.18). Mode FPB lies in the middle of these two sets of data. Overall, the maximum plasticity that can occur is in a pure mode II case while the minimum plasticity occurs in a pure mode I case. For each value of mode-mix, all numerical results are bounded between the extremes in plastic mismatch for a perfect crack interface ($\hat{\sigma} = 0.0$ and $\hat{\sigma} = 1.0$), for which a graded layer has no effect.

For comparison with the results reported in [3], the dimensionless plastic dissipation is plotted over the full range of mode-mix and for different values of $\hat{\sigma}$ and $h^*$ in Fig. 1.19. Each plot represents a different strength mismatch, while the family of curves in each plot
Figure 1.19: $dW/dN^*$ vs. mode-mix for (a) $\tilde{\sigma} = 0.0$ (b) $\tilde{\sigma} = 0.05$ (c) $\tilde{\sigma} = 0.1$ (d) $\tilde{\sigma} = 0.25$ [6].
represents a variation in graded layer thickness. The plot of Fig. 1.19(a) corresponds to the homogeneous case of no mismatch ($\hat{\sigma} = 0.0$), and is in keeping with the results of [3]. As can be seen in Figs. 1.19(b)-(d), the sensitivity of the plastic dissipation to the addition of the graded layer increases with both strength mismatch and applied mode-mix ratio. Thus, the graded layer has the greatest effect on the plastic dissipation in mode II and when there is a higher mismatch in plastic properties between the layers. Although the introduction of a graded layer has a measurable effect, the applied mode-mix ratio clearly has the greatest impact on the plastic dissipation per cycle.

1.4 Overview & Contributions

The true motivation of modeling a graded layer lies in more accurately representing a physical system. As such, previous research by the author has explored its effect on the plastic dissipation in the presence of a plastic mismatch only. In extending the graded layer approach to elastic mismatches as well, the benefits of including a graded layer increase dramatically. The research presented herein provides new contributions in the field of fracture mechanics involving layered material systems that include:

1. A previously unpublished beam theory solution for the strain energy release rate in the presence of a graded layer.

2. Previously unpublished nondimensional results of the effect of a graded layer, elastic mismatch, and mode of loading on the strain energy release rate.

3. A previously unpublished elastic technique for numerically determining an unambiguous definition of the mode in the presence of an elastic mismatch.

4. An independent validation of Daily’s physically based definition of the mode in the presence of an elastic mismatch using total plastic dissipation.
5. Previously unpublished results for the effect of a graded layer, elastic mismatch, plastic mismatch, and mode of loading on the plastic dissipation during fatigue crack growth on bimaterial interfaces.

6. Experimental steady-state fatigue crack growth rates for a four-point bend specimen under mixed-mode loading.

Chapter 2 provides relevant background information on fracture and fatigue modeling parameters and inherent assumptions of the included research. The contents of the first two contributions are presented in Chapter 3, along with a discussion of the implications of the nondimensional results. Chapter 4 discusses the numerical technique involved in contribution 3 including examples and validation. In so doing, the method of contribution 3 is used to support contributions 4 and 5, which are included in Chapter 5 along with a full discussion of the FEA models employed. The last contribution involving the experimental studies is the subject of Chapter 6, followed by conclusions and future work in Chapter 7.
Chapter 2

Background

2.1 Fracture Mechanics

In general, the field of fracture mechanics describes the propagation of cracks within a body and has been explored since World War I. In contrast, the more general field of solid mechanics has its origins beginning in the early 1800’s. Since that time, engineers have become proficient in designing components that resist failure on the first cycle. It is the process of repeated loading (often below that of yield) over time that drives failure. More specifically, initiation and propagation of micro-flaws into cracks ultimately lead to failure of engineering components. Thus, the field of fracture mechanics exists to characterize a material’s resistance to fracture, and to describe the driving forces governing crack extension under both monotonic and fatigue loading.

2.1.1 Crack Tip Fields and Plastic Zones

In most fracture mechanics texts, it is common to find a full asymptotic analysis of the stress fields near a crack tip, the result of which is an infinite series solution with a dominant first
term. The corresponding crack-tip fields are often defined in tensor notation as

\[ \sigma_{ij}^{I} = \frac{K_I}{\sqrt{2\pi r}} f_{ij}^{I}(\theta), \]  
\[ \sigma_{ij}^{II} = \frac{K_{II}}{\sqrt{2\pi r}} f_{ij}^{II}(\theta), \]  
\[ \sigma_{ij}^{III} = \frac{K_{III}}{\sqrt{2\pi r}} f_{ij}^{III}(\theta), \]

where the roman numerals I, II, and III indicate the different modes of loading shown in Fig. 1.4. In all three equations, there exists a $1/\sqrt{r}$ singularity whereby the stresses approach infinity as $r \to 0$. While the equations predict this in theory, practicality dictates that before the stresses reach infinity, yielding (and ultimately failure) of the material occurs. In most cases, material defects or inclusions serve as stress risers leading to micro-cracks and ultimately full fatigue cracks. Fracture occurs when these crack lengths reach a critical value and extend suddenly or even catastrophically.

It is of interest to note here that even in the presence of plasticity at the crack tip (due to stresses above yield near the crack tip), the assumptions of linear elastic fracture mechanics (LEFM) are still valid. Plasticity in these cases tend to be very localized and driven by the elastic K-fields that surround the crack tip. The area around the crack tip can be broken into three regions as depicted in Fig. 2.1. The size of the plastic zone of Region 1 is the actual yielded material nearest the crack tip defined by

\[ r_p = \gamma \left( \frac{K_I}{\sigma_y} \right)^2, \]

that scales with the mode I fracture toughness $K_I$ and the yield stress $\sigma_y$ of the material through the proportionality constant $\gamma$. Region 2 is the area directly surrounding Region 1 where the stresses are altered by a redistribution of the load from the plastic zone. Region 3 is considered the elastic “K-field” or “K-dominant” zone where the elastic solution is dominated by the stress singularities in eqs. (2.1)-(2.3) [5]. This last region serves as the
boundary condition for the nonlinear material behavior occurring at the crack tip. The larger elastic fields that surround the area of plasticity still govern crack growth as long as that plasticity is kept to a minimum. There exists a quantitative “check” that can ensure small scale yielding and thus give confidence to the elastic solution. This check is called a $J$-integral and is described in more detail in a following section.

In 3D applications, the plastic zone size not only varies in the $x-y$ plane but also in the $z$-direction through the thickness of a cracked specimen. For sufficiently thick specimens, out-of-plane contraction due to the high crack tip stresses is prevented by the bulk of the surrounding material. While plane strain conditions can be assumed for these thick specimens, a region of plane stress occurs near the edges. Even for globally plane stress problems, however, the conditions near the crack tip are closer to plane strain. Typically, plane strain fracture toughness values are reported in the literature (as opposed to plane stress) due to the decreased plastic zone sizes and thus decreased ductility and toughness in the material. To this end, plane strain fracture toughness values are more conservative than plane stress values in design. For mode I plane strain conditions to occur, the rule of
thumb is

\[ B \geq 2.5 \left( \frac{K_I}{\sigma_y} \right)^2, \]  

(2.4)

where \( B \) is the specimen thickness in the \( z \)-direction of Fig. 2.1. As such, a minimum thickness can be calculated to ensure a plane strain condition. [5]

### 2.1.2 Mode-Mix & Stress Intensity Factors

Eq. (2.5) shows a classical formula for the stress intensity factor for a variety of plates and beams [69] under mode I loading:

\[ K = \lambda \sigma \sqrt{\pi a} \]  

(2.5)

In the above equation, \( \lambda \) is a geometry factor, \( \sigma \) is the far field normal stress, and \( a \) is the crack length. Similar formulas exist for mixed-mode loading. In particular, a semi-analytical solution for the mixed-mode stress intensity factors has been developed by Suo and Hutchinson [60] that accounts for mismatches in layer thickness and elastic properties during steady-state cracking along an interface of a bimaterial layered specimen. The stress intensity factors derived for the geometry of Fig. 1.11 are

\[ K_I = \sqrt{3} \left( -C_2 \frac{M_1 - M_2}{h} \right) h^{-1/2} + 2 \sqrt{3} \{M_1 - C_3 (M_1 - M_2)\} h^{-3/2}, \]  

(2.6)

and

\[ K_{II} = 2 \left( -C_2 \frac{M_1 - M_2}{h} \right) h^{-1/2}, \]  

(2.7)

where \( M_1 \) is the applied moment to the top layer, \( M_2 \) is the applied moment to the bottom layer, and \( h \) is the height of the top layer and of the bottom layer. The parameters \( C_2 \) and \( C_3 \) are dimensionless constants that account for mismatches in elastic properties and layer thickness.

The geometry used in this research is depicted in Fig. 1.14, where \( H \) is the height of
the graded layer with $C_2 = 3/4$ and $C_3 = 1/8$ for identical layer thicknesses and elastic properties. Substituting these values for $C_2$ and $C_3$ into eqs. (2.6) and (2.7) gives simplified expressions for the closed form mode I and mode II stress intensity factors. For mode I,

$$K_I = \frac{\sqrt{3} (M_1 + M_2)}{h^{3/2}} \quad (2.8)$$

and for mode II,

$$K_{II} = -\frac{3 (M_1 - M_2)}{2h^{3/2}}. \quad (2.9)$$

In the current study, the height of the top and bottom layers remain constant, while the height of the graded layer $H$ is increased from the crack plane up into the top layer.

The mode-mix parameter described previously is defined as

$$\psi = \tan^{-1}\left(\frac{K_{II}}{K_I}\right), \quad (2.10)$$

and characterizes the amount of mode I and mode II (i.e., tension and shear) at the crack tip. Substitution of Eqs. (2.8) and (2.9) into eq. (2.10) results in eq. (1.5) shown in a previous section. By holding $M_1$ constant and varying the magnitude of $M_2$ in the range $-M_1 \leq M_2 \leq M_1$, $\psi$ is defined in the range $-90^\circ \leq \psi \leq 0^\circ$. Likewise, by holding $M_2$ constant and varying the magnitude of $M_1$ in the range $-M_2 \leq M_1 \leq M_2$, $\psi$ is defined in the range $0^\circ \leq \psi \leq 90^\circ$. In this way, the mode-mix parameter is defined for the full range of mode-mix $-90^\circ \leq \psi \leq 90^\circ$ simply by varying the combination of the applied moments. For the case of equal moments $M_1 = M_2$, the specimen experiences a pure mode I loading condition where $\psi = 0^\circ$ and the moments in Fig. 2.2(a) are symmetrically pulling the top and bottom layers apart. Conversely, if $M_2 = -M_1$ then $\psi = -90^\circ$ and both moments are applied in the clockwise direction resulting in a pure mode II state of loading as shown in Fig. 2.2(c). Similarly, if $M_1 = -M_2$ then $\psi = 90^\circ$ and both moments are applied in a counterclockwise direction, also resulting in a pure mode II state of loading. It is also worth noting that if
$M_1$ or $M_2$ equals zero then $\psi = \pm 41^\circ$ as shown in Fig. 2.2(b), which is a special case of the four-point bend test specimen geometry used in interfacial fracture toughness testing of layered materials [70–73].

2.1.3 Strain Energy Release Rate

While the mode-mix provides a way to describe the type of applied loading\(^1\), the strain energy release rate serves to characterize the magnitude of a given mode-mix. The mode I and mode II stress intensity factors in eqs. (2.8) and (2.9) are directly related to the strain energy release rate by the fracture mechanics relation

$$G = \frac{|K|^2}{E}, \quad (2.11)$$

where $|K| = \sqrt{K_I^2 + K_{II}^2}$ and $E = E$ for plane stress and $E = E/(1 - \nu^2)$ for plane strain. The strain energy release rate can be described as the amount of energy per unit fracture area created during crack extension and can be used to quantify the magnitude of loading since it contains contributions from both tensile and shear stresses (i.e., $K_I$ and $K_{II}$).

As described in [2], eqs. (2.8) and (2.9) can be substituted into eq. (2.11) to provide a relationship between the applied loading and the strain energy release rate as

$$G = \frac{3(7M_1^2 + 2M_1M_2 + 7M_2^2)}{4Eh^3}, \quad (2.12)$$

where $G$ is a function of the applied moments, elastic modulus, and layer height $h$. In the current work, $G$ is held constant and the moments are allowed to vary in order to map out the effect of the mode-mix.

The above equation can also be derived by simple beam theory. Strain energy is defined

\(^1\)For cases where no elastic mismatch is present.
Figure 2.2: Full range of mode-mix depiction.
as the stored work done by external forces that cause deformation and takes the form

\[ U = \int \frac{1}{2} \sigma_{ij} \varepsilon_{ij} dV. \]  \hspace{1cm} (2.13)

This is also equal to the area under the elastic stress-strain curve resulting from loading and unloading of the model. When dealing with beams in pure bending, eq. (2.13) reduces to

\[ U = \int \frac{M^2}{2EI} dx. \]  \hspace{1cm} (2.14)

From [1],

\[ \mathcal{G} = - \frac{d\Pi}{da} = - \frac{dU}{da} \]  \hspace{1cm} (2.15)

and substitution of eq. (2.13) into eq. (2.15) yields

\[ \mathcal{G} = \frac{dU}{da} = \frac{M^2}{2EI}. \]  \hspace{1cm} (2.16)

The change in strain energy for steady-state crack extension is \( dU_{\text{ahead}} - dU_{\text{behind}} \) and substitution of eq. (2.16) yields

\[ \mathcal{G} = - \left( \frac{6(M_1 - M_2)^2}{E(h + H)^3} - \left( \frac{6M_1^2}{Eh^3} + \frac{6M_2^2}{EH^3} \right) \right). \]  \hspace{1cm} (2.17)

As discussed in [60], this can be further simplified to that of eq. (2.12), assuming that the beam width is unity or that the moments are per unit width.

2.1.4 \textit{J-Integral}

The \textit{J}-integral is a path independent line integral that represents the strain energy release rate of both linear and nonlinear elastic materials. Rice [25] showed this in 1968 with the equation

\[ J = - \frac{d\Pi}{da}. \]  \hspace{1cm} (2.18)
where $\Pi = U - W$. Here, $U$ is the stored energy in the body and $W$ is the work done by external forces on the body. The J-integral can also be represented in tensor notation as

$$J = \oint_{\Gamma} \left( U dy - T_i \frac{\partial u_i}{\partial x} ds \right), \quad (2.19)$$

where $T_i$ is the traction vector [74].

Within Region 1 of Fig. 2.1, $J$-integrals lose path independence [75]. Outside of Region 1, the $J$-integral is a measure of small scale yielding and if $J$ equals $G$, then linear elastic fracture mechanics governs the behavior of the plastic zone. This is used as a check for small scale yielding of all models presented herein. Recent studies have shown that this is still true for a grading of layers perpendicular to the crack face, but not so with properties graded parallel to the crack face [76].

### 2.2 Elastic-Plastic Response

Although the plastic zones are governed by the elastic stresses surrounding the crack tip, the material in the plastic zone still exhibits elastic-plastic material response. To this end, the typical stress-strain curve of Fig. 2.3(a) is simplified herein using the elastic-plastic models of Fig. 2.3(b) and Fig. 2.3(c). A material hardening rate is defined using the ratio $E_t/E$ in Fig. 2.3(b). The case $E_t = 0$ corresponds to an elastic-perfectly plastic response.
as defined in Fig. 2.3(c). The case $E_t = 1.0$ corresponds to a perfectly elastic response (no plasticity at the crack tip). An exploration of the effects of material hardening by varying $E_t/E$ was performed by Daily in [7]. For simplicity, the models presented herein assume the elastic-perfectly plastic material response of Fig. 2.3(c). More detail on how to define such a material response is outlined in the following chapter.

### 2.3 Hardening Model

Material hardening is the strengthening of the material due to plastic deformation. This is known to occur because microscopic defects called dislocations tend to “pile up” or impede each other when strained. When these dislocations stop moving, the material’s yield strength is increased and ductility is decreased. ABAQUS contains ways to model the hardening of materials and is described below.

#### 2.3.1 Isotropic Hardening

Isotropic hardening means that the yield surface changes size uniformly in all directions such that the yield stress increases (or decreases) in all stress directions as plastic straining occurs. In other words, upon tensile plastic deformation, the yield strength would increase in another loading direction (compression, torsion). More cycles of plastic deformation would lead to the yield strength increasing after each cycle (Fig. 2.4(a)). While in theory this trend would continue indefinitely, real materials only exhibit this behavior transiently.

#### 2.3.2 Kinematic Hardening

Kinematic hardening is a translation of the yield surface in the direction of plastic strain, which results in a reduction of the yield strength upon load reversal. The effect of this decrease in yield strength is commonly known as the Bauschinger effect. Under cyclic loading, materials actually exhibit both isotropic and kinematic hardening. The material
will either harden (increase in yield strength), or soften (decrease in yield strength) from cycle to cycle, and finally settle into a steady-state for any added cycles. This transient behavior and the time it takes to reach steady-state is referred to as plastic shakedown. The model used herein uses a bi-linear kinematic hardening model as shown in Fig. 2.4(b) and predicts plastic shakedown after one cycle. This can be considered a first order approximation to a steady-state cyclic response, in keeping with the assumptions of Klingbeil in [1].

2.4 Paris-Regime Fatigue Crack Growth

Fig. 2.5 shows a typical crack growth curve. Crack initiation occurs within Region I while Region II is commonly referred to as the Paris-regime. This region tends to be exponential and can be described by

$$\frac{da}{dN} = C(\Delta K)^m.$$  \hspace{1cm} (2.20)
It is important to point out that the scope of this research deals with Paris-regime crack growth data only. The models presented do not account for crack initiation and moreover, the plastic work and surface energy contributions associated with crack extension are assumed to be negligible compared with the total plastic dissipation occurring throughout the reversed plastic zone ahead of the crack tip [1, 4, 6].

### 2.4.1 Stationary Crack Modeling

As outlined in [1, 3, 4, 6, 77], the elastic-plastic finite element models are for stationary (i.e., not growing) cracks in a general bimaterial specimen. As such, the results of this work should be considered a first approximation to the stabilized cyclic response under constant amplitude loading. This type of modeling does not account for the transient evolution of the crack tip plastic zone or the effects of plasticity induced crack closure. Modeling of a growing crack is not necessary as the energy contributions of plastic work and surface energy associated with Paris-regime crack extension in a single cycle are negligible compared to the total plastic dissipation ahead of the crack [1, 3, 4, 6, 77].

Figure 2.5: Typical Crack Growth Curve
Chapter 3

Effect of a Graded Layer: Derivation of the Strain Energy Release Rate

3.1 Overview

This chapter describes in detail the derivation of a beam theory solution for the strain energy release rate in the presence of a graded layer and provides elastic finite element results (i.e., $J$-integral) that confirm its validity. In addition, nondimensional results for the effect of graded layer height, applied load, and elastic material mismatch on the strain energy release rate are discussed.

3.2 Strain Energy Release Rate in the Presence of a Graded Layer

In general, the strain energy release rate $G$ is the amount of energy dissipated per unit new crack area (with units of $J/m^2$) and describes the overall magnitude of loading for any given mode-mix. For a unit area of crack extension, $G$ is the difference in potential energy
behind and ahead of the crack tip. For the two layer DCB model developed by Daily [2],

\[ G = \left( \frac{M_1^2}{2E_1I_1} + \frac{M_2^2}{2E_2I_2} \right)_{\text{behind}} - \left( \frac{(M_1 - M_2)^2}{2E_1I_3} \right)_{\text{ahead}}, \]  

(3.1)

where the subscripts 1 and 2 refer to the elastic properties and geometry of the top and bottom layers, respectively. The “behind” quantity consists of two terms, each corresponding to the layers that have separated. The “ahead” quantity consists of one term that characterizes the uncracked portion of the beam, where the quantity \( \frac{1}{E} \) is defined as

\[ \frac{1}{E} = \frac{1}{E_1} + \frac{1}{E_2}, \]  

(3.2)

and \( I_3 \) is a composite moment of inertia of the full beam ahead of the crack. In the case of equal layer heights and matching elastic properties, eq. (3.1) simplifies to that of eq. (2.12) from the previous chapter and given again below for clarity:

\[ G = \frac{3(7M_1^2 + 2M_1M_2 + 7M_2^2)}{4Eh^3}. \]  

(3.3)

For the case of equal layer heights and an elastic mismatch (i.e., nonzero \( \alpha \)), eq. (3.1) was simplified in [4] to

\[ G = \frac{(18\alpha^3 - 15\alpha^2 - 4\alpha + 21)M_1^2 - 6(\alpha^2 - 1)M_1M_2 - (18\alpha^3 + 15\alpha^2 - 24\alpha - 21)M_2^2}{E_2h^3(-3\alpha^3 - 3\alpha^2 + 4\alpha + 4)}. \]  

(3.4)

It is of interest to obtain a similar equation for the graded layer model.

### 3.2.1 A Composite Beam Theory Solution

Instead of trying to account for a continuous grading of Young’s modulus by adding terms to eq. (3.1) for each layer, the effect of a property gradient can be captured by manipulating the moment of inertia for each sublayer. Consider the elastically mismatched two-layer
beam in Fig. 3.1(a). It is a common strength of materials method to transform one dimension of the cross-sectional area in the horizontal direction by a factor

\[ n = \frac{E_1}{E_2}, \]  

resulting in a transformed beam made of all the same material\(^1\). Thus for a two-layer model, the geometry of Fig. 3.1(a) composed of two materials with different elastic moduli is transformed to the geometry of Fig. 3.1(b) now composed entirely of material #2. The same transformation can be applied in the presence of a graded layer. Consider the cross-sectional area of the graded layer model in Fig. 3.2(a). Because of the linear grading of material properties through the graded layer, the transformed cross-sectional area contracts the top layer by a factor \(n\) and connects to the bottom layer with a trapezoidal shape. By employing this strength of materials approach, the effect of having a linear property gradient can be captured with the moment of inertia of the newly transformed composite beam.

\(^1\)Note that the strain and deformation of the transformed section is the same, while the bending stress above the interface must be multiplied by \(n\).
Equation (3.1) can now be written as

$$\mathcal{G} = \left(\frac{M_1^2}{2E_1I_{c2}} + \frac{M_2^2}{2E_2I_{c2}}\right)_{\text{behind}} - \left(\frac{(M_1 - M_2)^2}{2E_2I_{c3}}\right)_{\text{ahead}},$$  \hspace{1cm} (3.6)$$

where $I_{c2}$ and $I_{c3}$ are the composite moments of inertia for the shapes that include the trapezoid (shown in Figs. 3.3(a) and 3.3(c)) and $I_2$ is the moment of inertia of the bottom layer (shown in Fig. 3.3(b)). The composite centroidal moment of inertia for $I_{c2}$ is given as

$$I_{c2} = \frac{h^3n}{3} - \frac{H^3n}{12} - \frac{(3h^2n - H^2n + H^2)^2}{18(H - Hn + 2hn)} + \frac{H^3}{12},$$  \hspace{1cm} (3.7)$$
where \( n = \frac{E_1}{E_2} \), \( H \) is the height of the graded layer, and \( h \) is the height of the bottom layer. Also, the composite centroidal moment of inertia for \( I_{c3} \) is given as

\[
I_{c3} = \frac{(-H^4 + 6H^3h - 12H^2h^2 + 12Hh^3 - 6h^4)n^2 + (2H^4 + 24H^2h^2 - 84h^4)n}{(36H - 72h)n - (36H + 72h)}
\]

\[
- \frac{(H^4 + 6H^3h + 12H^2h^2 + 12Hh^3 + 6h^4)}{(36H - 72h)n - (36H + 72h)}.
\]  

A validation of eq. (3.6) is provided in the following section with an elastic graded layer FEA model.

### 3.3 Elastic Validation of \( G \): The Graded Layer Model

An elastic plane strain finite element model was developed using the commercial finite element package ABAQUS for the purpose of validating the beam theory solution of eq. (3.6). The overall macro mesh of the graded layer model shown in Fig. 1.14 is shown in Fig. 3.4. The graded layer spans the total length \( L \) and varies linearly in the \( y \)-direction through the graded layer height \( H \). The discretization of the graded layer will be discussed in the upcoming sections. For the following results, the parameters chosen were \( L = 50\text{mm} \) and...
$a = 25\, mm$, in keeping with that of [2–4, 6].

### 3.3.1 Boundary Conditions

The boundary conditions for this half plane model include a symmetry condition on the right side in the $x$-direction and a pinned connection at the right hand corner in the $y$-direction. A seam is defined from $(-L, 0)$ to $(0, 0)$ to allow the layers to pull apart.

### 3.3.2 Loading

As illustrated in Fig. 3.5, the moments $M_1$ and $M_2$ are applied with equal and opposite pressures $P_1$ and $P_2$ applied to each layer

$$P_1 = \frac{2M_1}{d^2},$$  \hspace{1cm} (3.9)

and

$$P_2 = \frac{2M_2}{d^2},$$  \hspace{1cm} (3.10)

where $d$ is equal to the overall height of one layer $h$. The overall magnitude of loading for the model is controlled by the strain energy release rate $G$, which is held constant at $200\, J/m^2$. It becomes necessary here to emphasize again that for the following results of this chapter, the physical mode of loading (i.e., $M_1$ and $M_2$) is not equivalent to the actual
mode of loading (i.e., \( K_I \) and \( K_{II} \) definition) at the crack tip, due to the elastic mismatch between the layers. For this reason, the plots in this chapter contain a parameter called the moment ratio \( M_r \) to describe the physical loading conditions rather than the mode-mix parameter \( \psi \) which describes the mode of loading at the crack tip.\(^2\) Regardless, the validation of the beam theory solution for the strain energy release rate in the presence of a graded layer is unhindered by having to know the mode mixity at the crack tip.

Two definitions exist for the moment ratio to physically span fully symmetric loading conditions. These definitions are

\[
M_r = \frac{M_1}{M_2},
\]

(3.11)

and

\[
M_r = \frac{M_2}{M_1}.
\]

(3.12)

The reason for having two definitions and not just one can more easily be explained using Tables 3.1 and 3.2, which consider the actual moments applied to the model. The value of \( M_1 \) is found by rearranging eq. (3.11) to give

\[
M_1 = M_r M_2,
\]

(3.13)

and substituting into eq. (3.6) which yields

\[
\mathcal{G} = \frac{M_2^2}{2E_2} \left( \frac{M_2^2}{I_{c2}} + \frac{1}{I_2} - \frac{(M_r - 1)^2}{I_{c3}} \right).
\]

(3.14)

Solving this equation explicitly for \( M_2 \) results in

\[
M_2 = \sqrt[4]{2\mathcal{G}E_2 \left( \frac{M_2^2}{I_{c2}} + \frac{1}{I_2} - \frac{(M_r - 1)^2}{I_{c3}} \right)}.
\]

(3.15)

The value of \( M_1 \) is then simply calculated using eq. (3.13). For \( M_r \) in the range \(-1 \leq M_r \leq\)

\(^2\)This disparity is resolved in the following chapter.
\[ H = 0, \alpha = 0 \quad M_1 / M_2 \]

<table>
<thead>
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<th>( M_r )</th>
<th>( M_1 )</th>
<th>( M_2 )</th>
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<td>-1.0</td>
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<tr>
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<td>594.73</td>
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<tr>
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<td>0</td>
<td>625.79</td>
</tr>
<tr>
<td>0.5</td>
<td>265.12</td>
<td>530.24</td>
</tr>
<tr>
<td>1.0</td>
<td>413.19</td>
<td>413.19</td>
</tr>
</tbody>
</table>

Table 3.1: Applied moments according to \( M_r = M_1 / M_2 \) for \( H = 0 \) and \( \alpha = 0 \).

\[ H = 0, \alpha = 0 \quad M_2 / M_1 \]

<table>
<thead>
<tr>
<th>( M_r )</th>
<th>( M_1 )</th>
<th>( M_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1.0</td>
<td>477.95</td>
<td>-477.95</td>
</tr>
<tr>
<td>-0.5</td>
<td>594.73</td>
<td>-297.37</td>
</tr>
<tr>
<td>0.0</td>
<td>625.79</td>
<td>0</td>
</tr>
<tr>
<td>0.5</td>
<td>530.24</td>
<td>265.12</td>
</tr>
<tr>
<td>1.0</td>
<td>413.19</td>
<td>413.92</td>
</tr>
</tbody>
</table>

Table 3.2: Applied moments according to \( M_r = M_2 / M_1 \) for \( H = 0 \) and \( \alpha = 0 \).

1, the values of \( M_1 \) and \( M_2 \) are given in Table 3.1 for no graded layer and \( \alpha = 0 \). What this table shows is that using \( M_r = M_1 / M_2 \) by itself only gets half of the physical modes of loading! For \( M_r = -1.0 \), \( M_1 \) and \( M_2 \) are both applied in a counterclockwise direction to produce a positive mode 2 loading condition.\(^3\) For \( M_r = 1.0 \), a pure mode 1 loading condition is achieved. The negative mode 2 loading conditions must come from the second definition of the moment ratio \( M_r = M_2 / M_1 \). These values are shown in Table 3.2, and are found using similar substitutions outlined previously. Rearranging eq. (3.12) results in

\[ M_2 = M_r M_1, \]  \( \text{(3.16)} \)

and substitution into eq. (3.6) results in

\[ \mathcal{G} = \frac{M_1^2}{2E_2} \left( \frac{1}{I_{c2}} + \frac{M_r^2}{I_2} - \frac{(1 - M_r)^2}{I_{c3}} \right). \]  \( \text{(3.17)} \)

\(^3\)Defined using arabic numerals as this is not the actual mode at the crack tip.
Solving for \( M_1 \) yields

\[
M_1 = \frac{2\mathcal{G}E_2}{\frac{1}{I_2} + \frac{M_2^2}{I_2} + \frac{(1-M_r)^2}{I_3}},
\]

(3.18)

either \( M_2 \) given by eq. (3.16). Now for \( M_r = -1.0 \), the magnitudes of the moments \( M_1 \) and \( M_2 \) remain the same, but the sign has switched, thus allowing the moments to be applied in opposite directions than that of the moments in Table 3.1. In this way, the full span of symmetric (mode 1) and asymmetric (mode 2) physical loading conditions are achieved. This is not to say that the full span of mode-mix is covered by this definition. In fact, it will be shown in the following chapter that the range of \( M_r \) must be increased in order to achieve a full range of mode mixity. That being said, for the following results of this chapter, \( M_r \) is defined in the range \(-1 \leq M_r \leq 1\) in order to explore trends and validate the beam theory solution for \( \mathcal{G} \).

### 3.3.3 Meshing and Material Properties

The graded layer model is meshed using 8-node bi-quadratic reduced integration elements, which are standard for elastic-plastic analyses. The mesh is highly biased towards the crack plane in order to accurately resolve the material grading and plastic dissipation results presented in the next chapter. The graded layer height \( H \) is discretized into sublayers, and to provide a continuous grading, must contain enough of these sublayers to not have a significant change in material properties from one layer to the next. The mesh was constructed to have each successive layer of elements above the crack plane assigned a slightly different material property than the one below it. In the presence of an elastic mismatch (i.e. non-zero \( \alpha \)) the elastic modulus of each \( i^\text{th} \) sublayer above the crack plane can be calculated by

\[
E_{i+1} = \left( \frac{E_1 - E_2}{n + 1} \right) i + E_2,
\]

(3.19)
or in the presence of a plastic mismatch (i.e. non-zero $\sigma^\theta$) the yield strength of each $i^{th}$ sublayer above the crack plane can be calculated by

$$\sigma_y(i+1) = \left( \frac{\sigma_y1 - \sigma_y2}{n+1} \right) i + \sigma_y2,$$

(3.20)

where $n$ is the number of sublayers in the graded layer. A zoomed-in view of the crack tip is shown in Fig. 3.6, where the first three sublayers of the graded layer are labeled. Located within the first sublayer and the bottom layer is a “micro spider-web” crack tip mesh biased toward the crack tip itself. This spider-web mesh is where the $J$-integral and contour integral information will be extracted. Even in the presence of an elastic or plastic mismatch, the first sublayer and bottom layer are effectively the same material as the properties differ by only a small percentage. The importance of this will be discussed in the next chapter.

The graded layer resolution $n$ (i.e. number of sublayers) can be calculated based on the percentage change in material properties required per sublayer. For $\alpha > 0.0$,

$$n > \frac{1}{x} \left( \frac{2\alpha}{1-\alpha} \right) - 1,$$

(3.21)
and for $\alpha < 0$,

$$n > \frac{1}{x} \left( \frac{2\alpha}{\alpha - 1} \right) - 1,$$

(3.22)

where $x$ is the percentage difference between any two sublayers. Thus, for a given $\alpha$ value, a minimum number of sublayers are needed to keep the resolution $x$. A plot of $n$ vs. $\alpha$ is given in Fig. 3.7 in the range $-0.8 \leq \alpha \leq 0.8$, which is the same as that considered by Daily in [4]. Clearly, the limiting case in resolving the graded layer property gradient is $\alpha = 0.8$. While the graded layer resolution could have changed with each $\alpha$ value used in the forthcoming models, a constant resolution was necessary to ensure adequate resolution of the plastic zones. Of the two, plastic zone resolution governs over the number of sublayers and to this end, a value of $n = 300$ was used throughout this study. As it stands, the worst case scenario is that when $\alpha = 0.8$, the largest percentage difference from sublayer to sublayer is $x = 3\%$. For all other $\alpha$ values, the percentage difference per sublayer is substantially smaller.

The elastic modulus of the bottom layer is held constant at $E_2 = 73.1 \text{ GPa}$. Table 3.3 shows the overall elastic mismatch for different values of $\alpha$. 

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<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$E_1$ (GPa)</th>
<th>$E_2$ (GPa)</th>
</tr>
</thead>
<tbody>
<tr>
<td>-0.8</td>
<td>8.12</td>
<td>73.1</td>
</tr>
<tr>
<td>-0.6</td>
<td>18.28</td>
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</tr>
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</tr>
<tr>
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</tr>
<tr>
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</tr>
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</tr>
<tr>
<td>0.8</td>
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</tr>
</tbody>
</table>

Table 3.3: Elastic moduli for different $\alpha$.

### 3.3.4 Material Response

For the purpose of validating the beam theory solution for $G$ and mapping out the effect of the applied load, elastic mismatch, and graded layer height on the nondimensional strain energy release rate, the material response is simply elastic and the analysis entails only one linear static load step.

### 3.4 Validation of $G$

The validation of the beam theory solution for $G$ of eq. (3.6) requires calculating the $J$-integral for a range of applied loads (derived for a fixed $G$ using the method of the previous section) and a range of elastic mismatches and graded layer heights. Because $G$ is fixed, the extracted $J$-integral values for all models should equal $200 J/m^2$. This proved true for all symmetric and asymmetric loading conditions (controlled by $M_r$) with $\alpha$ in the range $-0.8 \leq \alpha \leq 0.8$ and $H$ in the range $0 \leq H \leq 2.5$. Provision of figures in this case would be quite trivial in that the $J$-integrals for all cases were within less than a percent of $G$ and would thus show a family of horizontal surfaces at $G = 200 J/m^2$ ($z$-axis) for all values of $\alpha$ with $H$ plotted on the $x$-axis and $M_r$ plotted on the $y$-axis.
3.5 Nondimensional Analysis of \( \mathcal{G} \)

While the previous model and validation were performed for a fixed \( \mathcal{G} \), an exploration of the effects of the applied loading, elastic mismatch, and graded layer height on the magnitude of \( \mathcal{G} \) is of interest. These effects can be seen by forming nondimensional quantities for each parameter. The applied loading parameter \( M_r \) will be used to describe the ratio of moments applied to the model, the bounds of which were discussed in Section 3.3.2. If \( M_r = 1.0 \), then the applied moments are symmetric, equal in magnitude, and applied in opposite directions. If \( M_r = -1.0 \), the moments are perfectly asymmetric, equal in magnitude, and applied in the same direction. As discussed previously, the elastic mismatch parameter \( \alpha \) will be used to describe the elastic property difference between the top and bottom layers. A nondimensional graded layer height \( h_r \) will specify the height of the graded layer relative to the overall layer height and will be defined in the next section.

3.5.1 Effect of the Moment Ratio \( M_r \): The Homogeneous Case

Because the homogenous case contains no material mismatch, substitution of either definition for \( M_r \) (i.e. \( M_1/M_2 \) or \( M_2/M_1 \)) results in the same equation. A nondimensional definition for \( \mathcal{G} \) can be derived by substituting \( M_r \) into eq. (2.11) as

\[
\mathcal{G} = \frac{3M_{1.2}^2}{4Eh^3}(7M_r^2 + 2M_r + 7),
\tag{3.23}
\]

where

\[
\mathcal{G} = \frac{3M_{1.2}^2}{4Eh^3} \mathcal{G}^*.
\tag{3.24}
\]

Thus for the homogenous case, the nondimensional strain energy release rate is given by

\[
\mathcal{G}^* = 7M_r^2 + 2M_r + 7
\tag{3.25}
\]
and is normalized by either of the applied moments, elastic modulus, and overall layer height. A plot of $G^*$ as a function of $M_r$ is shown in Fig. 3.8. It can be noted that a minimum $G^*$ occurs for a moment ratio of $-1/7$.

### 3.5.2 Effect of the Moment Ratio $M_r$ and Elastic Mismatch $\alpha$: The Bimaterial Case

Substitution of $M_r = M_1/M_2$ into eq. (3.4) results in

$$G = \frac{M_2^2}{E_2 h^3} \frac{[6(3\alpha^2 - 4)(\alpha - 1) + 3\alpha^2 - 3]M_2^2 + 6(1 - \alpha^2)M_r - [6(3\alpha^2 - 4)(\alpha - 1) + 33\alpha^2 - 45]}{(4 - 3\alpha^2)(\alpha + 1)},$$

(3.26)

leading to

$$G = \frac{M_2^2}{E_2 h^3} G^*, \quad \text{(3.27)}$$
where the nondimensional strain energy release rate, now a function of $M_r$ and $\alpha$, is

$$\mathcal{G}^* = \frac{[6(3\alpha^2 - 4)(\alpha - 1) + 3\alpha^2 - 3]M_r^2 + 6(1 - \alpha^2)M_r - [6(3\alpha^2 - 4)(\alpha - 1) + 33\alpha^2 - 45]}{(4 - 3\alpha^2)(\alpha + 1)}.$$  

(3.28)

Similarly, substituting $M_r = M_2/M_1$ into eq. (3.4) gives

$$\mathcal{G} = \frac{M_1^2}{E_2h^3} \frac{[6(3\alpha^2 - 4)(\alpha - 1) + 3\alpha^2 - 3] + 6(1 - \alpha^2)M_r - [6(3\alpha^2 - 4)(\alpha - 1) + 33\alpha^2 - 45]M_r^2}{(4 - 3\alpha^2)(\alpha + 1)},$$

(3.29)

leading to

$$\mathcal{G} = \frac{M_1^2}{E_2h^3} \mathcal{G}^*,$$

(3.30)

where the nondimensional strain energy release rate, now a function of $M_r$ and $\alpha$, is

$$\mathcal{G}^* = \frac{[6(3\alpha^2 - 4)(\alpha - 1) + 3\alpha^2 - 3] + 6(1 - \alpha^2)M_r - [6(3\alpha^2 - 4)(\alpha - 1) + 33\alpha^2 - 45]M_r^2}{(4 - 3\alpha^2)(\alpha + 1)}.$$  

(3.31)

The effect of $M_r$ and $\alpha$ on $\mathcal{G}^*$ is plotted for $M_r = M_1/M_2$ and $M_r = M_2/M_1$ in Figs. 3.9(a) and (b), respectively. In both cases, $\mathcal{G}^*$ increases for increasingly negative values of $\alpha$. This is due to the overall lower stiffness of the model for $\alpha \leq 0.0$. In addition, the overall
effect of the moment ratio is less when normalizing with respect to $M_1$ rather than $M_2$.

### 3.5.3 Effect of the Moment Ratio $M_r$, $\alpha$, and $h_r$: The Graded Layer Case

The graded layer geometry consists of a top and bottom layer height $h$ and the graded layer height $H$ from Fig. 1.14. Because the bottom layer height remains constant for any graded layer height, $\mathcal{G}$ will be normalized with respect to $h$ and the graded layer height will be controlled by the height ratio, defined as

$$h_r = \frac{H}{h-H}. \tag{3.32}$$

The height ratio is defined in the range $0 \leq h_r \leq 100$, where a value of 0 corresponds to the bimaterial case and a value of 100 corresponds to the graded layer being 99% of the total top layer. In the context of this work, $h_r$ will be explored in the range $0 \leq h_r \leq 1$. A value of 1 means that the graded layer height $H$ is half of the top layer. As many applications will never contain graded layers this large, focus will be limited to a maximum value of 1. In addition, there is an asymptotic effect as the graded layer becomes larger, which will be seen in the upcoming sections. Rearranging eq. (3.32) yields the graded layer height in terms of the nondimensional graded layer height $h_r$ as

$$H = \frac{h_r h}{1+h_r}. \tag{3.33}$$

Substitution of this equation into $I_{c2}$ and $I_{c3}$ results in an equation for $\mathcal{G}$ as

$$\mathcal{G} = \frac{M_r^2}{2E_2h^3} \left[ \frac{M_r^2}{f_c2(\alpha, h_r)} + 12 - \frac{(M_r-1)^2}{f_c3(\alpha, h_r)} \right], \tag{3.34}$$
where the quantities $f_{c2}$ and $f_{c3}$ are functions of $\alpha$ and $h_r$. The nondimensional strain energy release rate normalized by $M_2$ can now be defined as

$$G^* = \frac{M_r^2}{f_{c2}(\alpha, h_r)} + 12 - \frac{(M_r - 1)^2}{f_{c3}(\alpha, h_r)}.$$  \hspace{1cm} (3.35)

Similarly, if $M_r = M_2/M_1$, the strain energy release rate changes slightly to

$$G = \frac{M_1^2}{2E_2h^3} \left[ \frac{1}{f_{c2}(\alpha, h_r)} + 12M_r^2 - \frac{(1 - M_r)^2}{f_{c3}(\alpha, h_r)} \right],$$  \hspace{1cm} (3.36)

and the nondimensional strain energy release rate normalized by $M_1$ becomes

$$G^* = \frac{1}{f_{c2}(\alpha, h_r)} + 12M_r^2 - \frac{(1 - M_r)^2}{f_{c3}(\alpha, h_r)}.$$  \hspace{1cm} (3.37)

For the plots of Fig. 3.10, the moment ratio is plotted on the x-axis, the relative graded layer height $h_r$ on the y-axis and $G^*$ on the z-axis. Each figure contains five separate plots, each for a different $\alpha$ value. Similar conclusions can be drawn about the effect of $M_r$ on $G^*$ in the presence of a graded layer. For $\alpha \leq 0.0$, $M_r$ has more of an overall effect. Also, negative $\alpha$ values lead to higher $G^*$ values in all cases. The effect of the graded layer is slight and $G^*$ tends to increase for higher relative graded layer heights in all cases. This effect seems to also be asymptotic after $h_r \approx 0.2$.

For the plots of Fig. 3.11, the same quantities are plotted along the axes. The effect of $M_r$ on $G^*$ is opposite that of the plots in Fig. 3.10. The quantity $G^*$ seems to be more sensitive to $M_r$ for positive $\alpha$ values, however negative $\alpha$ values lead to higher $G^*$ values overall. The effect of the relative graded layer height also has the same impact as before. A slight increase of $G^*$ occurs for $0 \leq h_r \leq 0.2$ but is asymptotic for larger values.
Figure 3.10: Effect of $M_r$ and $h_r$ on $J^*$ for (a) $\alpha = 0.0$ (b) $\alpha = -0.8$ (c) $\alpha = 0.8$ (d) $\alpha = -0.5$ (e) $\alpha = 0.5$
Figure 3.11: Effect of $M_r$ and $h_r$ on $\mathcal{G}$ for (a) $\alpha = 0.0$ (b) $\alpha = -0.8$ (c) $\alpha = 0.8$ (d) $\alpha = -0.5$ (e) $\alpha = 0.5$
3.5.4 Conclusions

A previously unpublished beam theory solution for the strain energy release rate in the presence of a graded layer has been presented. This is integral to determining an unambiguous definition of mode-mix for problems containing an elastic mismatch, as presented in the following chapter. In addition, previously unpublished nondimensional results for the strain energy release rate as a function of the applied load $M_r$, the elastic mismatch $\alpha$, and the relative graded layer height $h_r$ have been presented. An understanding of how material properties and geometry affect the overall loading intensity parameter $\mathcal{G}$ is key in designing debond resistant interfaces, and ultimately predicting fatigue behavior of layered material systems.
Chapter 4

Effect of a Graded Layer: Determination of the Mode-mix in the Presence of an Elastic Mismatch

4.1 Overview

A major contribution of this work includes a technique whereby the actual mode at the crack tip can be determined in a unique manner, which eliminates the need for an arbitrary characteristic length. The graded layer model facilitates this by allowing for a smooth change in material properties between the top and bottom layers. This method involves extracting stress intensity factors from a “micro” homogeneous stress field right at the crack tip. This micro field is capable of defining the overall mode of loading at the crack tip even in the presence of material mismatches. The following sections outline this process and present plastic zones and plastic dissipation results to support this method.
4.2 Micro-crack Tip Modeling Approach for Calculating Stress Intensity Factors

The classical stress intensity factors $K_I$ and $K_{II}$ (for 2D problems) characterize the influence of the load or deformation on the magnitude of the crack tip stress and strain fields [78]. Extraction of these quantities using contour integrals similar to the $J$-integral are performed using ABAQUS. What makes this extraction unique is the use of the “micro spider-web” crack tip mesh from Fig. 3.6. In particular, this part of the mesh, contained within the first sublayer and bottom layer, allows for the use of the classical definition of the mode-mix given by eq. (2.10) because the material properties within the micro spider-web are effectively homogeneous (quantitatively within a few percent across the interface). Analyses performed without the micro spider-web mesh involve extracting contour integral values that pass through all the sublayers contained within the graded layer. As a result of crossing all of these boundaries, the contour integrals used to capture the stress intensity factors are not path independent. In addition, contour integrals for just a bimaterial interface exhibiting a perfect interface provide stress intensity factors $K_1$ and $K_2$, where the arabic subscripts indicate the real and imaginary parts of the complex stress intensity factor defined in eq. (1.14). The K-factors as extracted from within the micro spider-web mesh at the crack tip, where the material is effectively homogeneous ($\alpha = 0.0$), are path independent and remained constant even in the presence of a global elastic mismatch. Moreover, these $K$ factors represent the classical $K_I$ and $K_{II}$ that are used to calculate the mode and the $J$-integrals extracted from this region match the beam theory solution for the strain energy release rate.
### 4.3 Elastic Technique to Find Actual Mode

In the presence of an elastic mismatch (nonzero $\alpha$), it is unknown *a priori* what physical loading combination of $M_1$ and $M_2$ would achieve a particular mode-mix at the crack tip. For a fixed $\mathcal{G}$, there exists some combination of moments that will produce mode I (or any other mode of interest) at the crack tip. To this end, an elastic sweep of relevant $M$-ratios (i.e., $M_r$ from previous chapter) are performed for each $\alpha$ value in order to locate particular modes (mode I when $K_{II} = 0$ and mode II when $K_I = 0$). A plastic analysis is then performed in ABAQUS for each elastic mismatch $\alpha$ and its corresponding moment ratio for any mode of interest.

#### 4.3.1 Elastic Sweep Results in General

For each of the six graded layer heights (1.0, 0.5, 0.25, 0.1, 0.06, 0.03 mm), an elastic sweep of moment ratios is performed using an automated script in Matlab coupled with the python interface in ABAQUS. The range of $M_r$ is chosen on a trial and error basis and Table 4.1 shows the ranges needed for each graded layer height. This should be regarded as a “brute-force” method, as more elegant algorithms employing sensitivity and automatic expansion of the range could have been utilized. This would have decreased the number of elastic runs needed to locate the loads corresponding to all the different modes for each $\alpha$. As it stands, computation time for these elastic runs was generally trivial compared with the plasticity calculations carried out in later chapters.

<table>
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</thead>
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</tr>
<tr>
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<td>$-4.0 \leq M_r \leq 10.0$</td>
</tr>
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<td>$-4.0 \leq M_r \leq 14.0$</td>
</tr>
<tr>
<td>0.1</td>
<td>$-4.0 \leq M_r \leq 16.0$</td>
</tr>
<tr>
<td>0.06</td>
<td>$-2.0 \leq M_r \leq 16.0$</td>
</tr>
<tr>
<td>0.03</td>
<td>$-2.0 \leq M_r \leq 16.0$</td>
</tr>
</tbody>
</table>

Table 4.1: Range of $M_r$ needed for different graded layer heights.
The resulting output of these elastic runs is the actual mode of loading at the crack tip calculated by eq. (2.10), where the $K$-factors are extracted from Abaqus contour integrals. In this way, plots of mode-mix vs. $M_r$ for all $\alpha$ can be constructed for each graded layer height. As discussed in section 3.3.2, a single definition for the moment ratio fails to capture the full effect of mode-mix. As before, both definitions ($M_1/M_2$ and $M_2/M_1$) of $M_r$ are needed to fully define the mode-mix. This is supported by the plots in Figs. 4.1 and 4.2. By looking at Fig. 4.1(a), it is seen that a mode of positive $90^\circ$ is not even achievable! In fact, the results become asymptotic and fail to exceed $60^\circ$ for that case. The second definition of $M_r$ is used for Fig. 4.1(b) and it is seen that $-90^\circ$ is reached, but $90^\circ$ is not. Therefore, $M_r = M_1/M_2$ is able to produce results for the effect of mode-mix in the range $0^\circ \leq \psi \leq 90^\circ$ and $M_r = M_2/M_1$ provides results in the range $-90^\circ \leq \psi \leq 0^\circ$. In this way, the full range of actual mode-mix at the crack is explored for different graded layer heights and shown in both Figs. 4.1 and 4.2.

### 4.3.2 Elastic Sweep Results for Specific Modes of Interest

While the results of the previous section show the general effect of the moment ratio on the mode-mix, it is of more interest to know what moment ratio produces a particular mode of loading. This is possible by plotting the moment ratio vs. $\alpha$ for a family of modes. In other words, a single curve on an $M_r$ vs. $\alpha$ plot is the intersection of points of a horizontal slice from Figs. 4.1 and 4.2 (seen as the bold horizontal lines at $\psi = 0^\circ$ and $\psi = \pm90^\circ$). This is performed for all the previous plots and shown in Figs. 4.3 and 4.4. These plots are integral to providing the correct loading for each non-zero value of $\alpha$. In order to validate that this method does in fact produce the correct mode at the crack tip, plasticity calculations are performed and plastic zones are examined, as described in the following section.
Figure 4.1: $\psi$ vs. $M_r$ for all $\alpha$ for (a)-(b) $H = 1.0 \text{mm}$ (c)-(d) $H = 0.5 \text{mm}$ (e)-(f) $H = 0.25 \text{mm}$. 
Figure 4.2: $\psi$ vs. $M_r$ for all $\alpha$ for (a)-(b) $H = 0.1 \text{mm}$ (c)-(d) $H = 0.05 \text{mm}$ (e)-(f) $H = 0.03 \text{mm}$.
Figure 4.3: $M_r$ vs. $\alpha$ for all $-90^\circ \leq \psi \leq 90^\circ$ for (a)-(b) $H = 1.0\,mm$ (c)-(d) $H = 0.5\,mm$ (e)-(f) $H = 0.25\,mm$. 
Figure 4.4: $M_r$ vs. $\alpha$ for all $-90^\circ \leq \psi \leq 90^\circ$ for (a)-(b) $H = 0.1 \text{ mm}$ (c)-(d) $H = 0.06 \text{ mm}$ (e)-(f) $H = 0.03 \text{ mm}$.
4.3.3 Model Plasticity

The current work employs elastic-perfectly plastic material response, which represents an upper bound on plastic dissipation for ductile metals. The load step simply consists of a linear static load and unload as shown in Fig. 4.5. A forward plastic zone at the crack tip is formed upon completion of the Load step 1 while during the Unload step 2, a reversed plastic zone occurs. The reversed plastic zone is roughly four times smaller than the forward plastic zone, as shown in the mode I example of Fig. 4.6. The plastic zones shown in the following section are reversed plastic zones for three different modes and positive
values of $\alpha$.

4.3.4 Mode I

The plot in Fig. 4.7 shows what moment ratio is needed for every $\alpha$ value to achieve a mode I condition at the crack tip (i.e. $K_{II} = 0$), as determined by the sweep of elastic analyses previously described. The moment ratios are plotted on the $y$-axis, while the $\alpha$ values are plotted on the $x$-axis. Notice that as $\alpha$ increases (i.e. the top layer gets stiffer) the moment ratio becomes greater than one. This is because a larger moment is needed on the stiffer layer to produce a symmetric stress field. The dotted lines in this figure correspond to the homogeneous case where symmetric loading is applied.

The results here can be validated by looking at the plastic zones that occur for each $\alpha$ value. It can be seen in Fig. 4.8 that for every $\alpha$ value, the plastic zones look very similar and symmetric in shape. A slight counterclockwise “twisting” of the plastic zones can be seen for very high values of $\alpha$, but similar behavior is seen in [4]. In addition, a clockwise “twisting” of the plastic zones is seen in Fig. 4.9 for high negative values of $\alpha$. 

Figure 4.7: Moment ratios needed for mode I conditions at the crack tip for all $\alpha$ values.
Figure 4.8: Mode I Reversed Plastic Zones for (a) \( \alpha = 0.0 \) (b) \( \alpha = 0.2 \) (c) \( \alpha = 0.4 \) (d) \( \alpha = 0.6 \) (e) \( \alpha = 0.8 \).
Figure 4.9: Mode I Plastic Zones for (a) $\alpha = -0.2$ (b) $\alpha = -0.4$ (c) $\alpha = -0.6$ (d) $\alpha = -0.8$. 
Figure 4.10: Moment ratios needed for mode FPB conditions at the crack tip for all $\alpha$ values.

### 4.3.5 Mode FPB

A similar plot for $\psi = 41^\circ$ is shown in Fig. 4.10. The moment ratio is plotted on the $y$-axis and $\alpha$ on the $x$-axis. The shape of this plot is similar to that of Fig. 4.7 but is less in overall magnitude. The $\alpha = 0.8$ case for mode I requires a moment ratio of about 13 whereas the same case for FPB is only about 1.75. The dotted lines intersect for the homogeneous case and a moment ratio of zero. It should also be noted that for $\alpha < 0.0$, the moment ratio becomes negative. The mode FPB plastic zones for all positive $\alpha$ are shown in Fig. 4.11.

### 4.3.6 Mode II

And lastly, Fig. 4.12 shows the moment ratios needed to produce a mode II state of stress at the crack tip for a range of $\alpha$ values. Due to the amount of shear in the mode II case, all of the moment ratios are negative. The dotted lines intersect for the homogeneous case at a moment ratio of $-1$. The mode II plastic zones are shown in Fig. 4.13. In this chapter, an elastic method for determining the mode-mix at a crack tip in the presence of an elastic
Figure 4.11: Mode FPB Reversed Plastic Zones for (a) $\alpha = 0.0$ (b) $\alpha = 0.2$ (c) $\alpha = 0.4$ (d) $\alpha = 0.6$ (e) $\alpha = 0.8$
mismatch using a graded layer was presented. It was validated using elastic-plastic FEA results which show the correct orientation of the crack tip plastic zones for the given modes.

Figure 4.12: Moment ratios needed for mode II ($\psi = +90^\circ$) conditions at the crack tip for all $\alpha$ values.
Figure 4.13: Mode II Reversed Plastic Zones ($\psi = +90$) for (a) $\alpha = 0.0$ (b) $\alpha = 0.2$ (c) $\alpha = 0.4$ (d) $\alpha = 0.6$ (e) $\alpha = 0.8$
Chapter 5

Total Plastic Dissipation in the Presence of a Graded Layer

The total plastic dissipation $dW/dN*$ is calculated for all cases in the presence of an elastic mismatch.

5.1 Calculation of $dW/dN*$

The calculation of the total plastic dissipation ahead of the crack is performed in the same way following the methods of [1–4]. As the dissipated energy theory is based on a stationary crack, four load cycles are required for plastic shakedown to occur. So far, the plastic runs from the previous section included two load steps: a load and an unload. To calculate $dW/dN$, two more load steps are required as shown in Fig. 5.1. It has been shown previously in Fig. 4.6 that a forward plastic zone occurs at Step 1 and a reversed plastic zone in Step 2. For stationary cracks, the plastic zones that occur at Step 3 and Step 4 are identical in size to the plastic zone at Step 2. Thus, determining a steady-state accumulation of $dW/dN$ for a unit crack advance requires subtraction of $dW/dN$ at Step 4 and Step 2. The implications of modeling a steady-state crack are outlined in [1] where it is discussed that stationary crack modeling does not account for the energy associated with the actual crack
advance. In general, for Paris regime crack growth in ductile metals, plastic energy dissipation throughout the reversed plastic zone is much larger than that associated with the crack advancement in a single cycle, so that modeling the actual crack extension is unnecessary.

Nondimensional results for \( \frac{dW}{dN} \) are calculated following that of [4] where \( \frac{dW}{dN}^{*} \) is normalized by the elastic/plastic material properties (i.e. Young’s modulus/yield strength), elastic mismatch, and applied load given as

\[
\frac{dW}{dN^{*}} = \frac{\sigma_{y2}^{2}}{\Delta \gamma^{2} \Delta \alpha^{2} (1 + \alpha)} \frac{dW}{dN},
\]

The subscript 2 refers to the bottom layer where the material and geometry parameters are held constant.

### 5.2 Calculation of \( h^{*} \)

In order to present the effect of a graded layer on the plastic dissipation, a different normalization is required than that presented in section 3.5.3. For real materials, the effect on the total plastic dissipation of the graded layer is most seen for graded layers on the order of the plastic zone size. For this reason (and in keeping with the author’s previous research), the graded layer height is nondimensionalized with respect to the plastic zone size, which
scales with the applied stress intensity factor as

\[ r_p \approx \left( \frac{|\Delta K|}{\sigma_y} \right)^2. \]

Knowing that \( \Delta G = \frac{|\Delta K|^2}{E} \) and substituting into the previous equation, the nondimensional graded layer height is defined as

\[ h^* = \frac{H\sigma_y^2}{\Delta G E}. \]

This quantity is valid in the range \( 0 \leq h^* \leq \infty \) (although the authors previous research showed that the effect of the graded layer is predominantly seen in the range \( 0 \leq h^* \leq 1 \)). As \( h^* \) increases, the graded layer becomes large relative to the plastic zone size.

### 5.3 Results & Discussion

#### 5.3.1 Validation of a Physically Based Definition of the Mode-Mix in the Presence of an Elastic Mismatch by [4]

In 2010, Daily presented a method with which to define the mode-mix at a crack tip in the presence of an elastic mismatch. This method used maximum and minimum plasticity to calibrate characteristic lengths in such a way that minimum plasticity corresponded to a mode I configuration and maximum plasticity corresponded to a mode II configuration. The graded layer research contained herein can provide an independent validation for Daily’s approach.

This is accomplished by first combining the plots of Figs. 4.3 and 4.4 in such a way as to directly see the effect of the graded layer for specific modes. A plot of \( M_r \) vs. \( h^* \) for the full range of \( \alpha \) and \(-90^\circ \leq \psi \leq 0^\circ\) is shown in Fig. 5.2, while \( M_r \) vs. \( h^* \) for the full range of \( \alpha \) and \( 0^\circ \leq \psi \leq 90^\circ\) is plotted in Fig. 5.3. Using a Piecewise Cubic Hermite
Figure 5.2: $M_r$ vs. $h^*$ for the full range of $\alpha$ for (a) $\psi = -90^\circ$ (b) $\psi = -60^\circ$ (c) $\psi = -30^\circ$ (d) $\psi = 0^\circ$. 

---

\[ \alpha = 0.8 \]
\[ \alpha = 0.6 \]
\[ \alpha = 0.4 \]
\[ \alpha = 0.2 \]
\[ \alpha = 0.1 \]
\[ \alpha = 0.0 \]
\[ \alpha = 0.2 \]
\[ \alpha = 0.4 \]
\[ \alpha = 0.6 \]
\[ \alpha = 0.8 \]
Figure 5.3: $M_r$ vs. $h^*$ for the full range of $\alpha$ for (a) $\psi = 0^\circ$ (b) $\psi = 30^\circ$ (c) $\psi = 60^\circ$ (d) $\psi = 90^\circ$. 
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Table 5.1: \( M_1 \) values for corresponding \( \alpha \) and \( \psi \) for a bimaterial interface.

Interpolating Polynomial (pchip function) in Matlab, the trend-lines were extrapolated to the left until they crossed \( h^* = 0 \). These y-intercepts correspond to the ratio of moments needed for the bimaterial model with only two layers (i.e., Daily [4]). Once these moment ratios are known, it becomes trivial to back out the corresponding \( M_1 \) and \( M_2 \) needed for each value of \( \alpha \) and \( \psi \). As discussed in Section 3.3.2, eqs. (3.15) and (3.12) can be used to find \( M_2 \) and \( M_1 \) for \( \psi \geq 0^\circ \), respectively. Likewise, eqs. (3.18) and (3.16) can be used to find \( M_1 \) and \( M_2 \) for \( \psi \leq 0^\circ \). A listing of these moments \( M_1 \) and \( M_2 \) is found in Tables 5.1 and 5.2.

A recreation of Daily’s bimaterial model was developed in Abaqus for the purpose of applying the moments derived from the graded layer models. For verification, the same plastic zones (progression of increasing \( \alpha \) for mode 0° and 30°) resulting from this solution
technique as that of Daily in [4] are shown in Fig. 5.4. While the length scale is different due to differing $G$ values (200 vs. 1000 $J/m^2$) and elastic moduli, the overall shape of the plastic zones is almost identical. The same progression can be seen for modes $60^\circ$ and $90^\circ$ in Fig. 5.5 and for completeness, modes $0^\circ$ and $-30^\circ$ in Fig. 5.6 and modes $-60^\circ$ and $-90^\circ$ in Fig. 5.7. These plastic zones correspond directly to the results in [4] and as such provide an independent validation of the method used therein.

5.3.2 Effect of Mode-mix and Elastic Mismatch on the Total Plastic Dissipation

The section presents comprehensive numerical results for the full range of $h^*$, $\alpha$, and $\psi$ for the elastic mismatch case.

A series of plots for the nondimensional plastic dissipation $dW/dN^*$ vs. mode-mix $\psi$ are shown in Fig. 5.8 for different graded layer heights and $\alpha \geq 0.0$. Following the work presented in [6], the most notable effect of the graded layer is increased plasticity due to the smooth change in material properties. By not having a step change across the interface, more plasticity can occur in the upper layer as the first several sub-layers within the graded layer are closer in material stiffness. Another conclusion to be drawn from these results is that for positive $\alpha$, the only visible symmetry is for a large graded layer. As the graded layer gets smaller (i.e., moving from (f) to (a)), more asymmetry presents itself, especially for negative modes. The interaction between $\alpha$, mode, and graded layer heights cause this behavior. In general, for $\alpha \geq 0.0$, the stiffer material on top tends to push plasticity into the bottom layer. Positive mixed-modes, by definition, produce plastic zones that protrude into the bottom layer versus negative modes that produce plastic zones mostly in the top layer. In this way, the plastic dissipation for negative modes is greater than that of the positive modes due to the graded layer allowing more plasticity into the top layer. However, in the limit as $h^* \rightarrow 0$, the behavior of the plastic zones mimic that of a bimaterial interface and the curves collapse as shown by the spread of the lines becoming smaller for smaller $h^*$. 78
Figure 5.4: Reversed plastic zones for mode I and mixed mode I/II.
Figure 5.5: Reversed plastic zones for mode II and mixed mode I/II.
Figure 5.6: Reversed plastic zones for mode I and mixed mode I/II.
Figure 5.7: Reversed plastic zones for mode I and mixed mode I/II.
Figure 5.8: $dW/dN*$ vs. $\psi$ for positive $\alpha$ and (a) $h* = 5.472$ (b) $h* = 2.736$ (c) $h* = 1.368$ (d) $h* = 0.547$ (e) $h* = 0.365$ (f) $h* = 0.182$
Representative plastic zones for this behavior are shown in Figs. 5.9-5.12.

Likewise, a series of plots for the nondimensional plastic dissipation $dW/dN^*$ vs. mode-mix $\psi$ are shown in Fig. 5.17 for different graded layer heights and $\alpha \leq 0.0$. 
Figure 5.9: Representative plastic zones for mixed mode with $h^* = 0.547$. 
Figure 5.10: Representative plastic zones for mixed mode with $h^* = 0.547$. 

(a) $\alpha = 0.0$ and $\psi = -90^\circ$

(b) $\alpha = 0.0$ and $\psi = -60^\circ$

(c) $\alpha = 0.4$ and $\psi = -90^\circ$

(d) $\alpha = 0.4$ and $\psi = -60^\circ$

(e) $\alpha = 0.8$ and $\psi = -90^\circ$

(f) $\alpha = 0.8$ and $\psi = -60^\circ$
Figure 5.11: Representative plastic zones for mixed mode with $h^* = 0.547$. 
Figure 5.12: Representative plastic zones for mixed mode with $h^* = 0.547$. 
Figure 5.13: Representative plastic zones for mixed mode with $h^* = 0.547$. 
Figure 5.14: Representative plastic zones for mixed mode with $h^* = 0.547$. 
Figure 5.15: Representative plastic zones for mixed mode with $h^* = 0.547$. 
Figure 5.16: Representative plastic zones for mixed mode with $h^* = 0.547$. 
Figure 5.17: $dW/dN*$ vs. $\psi$ for negative $\alpha$ and (a) $h* = 5.472$ (b) $h* = 2.736$ (c) $h* = 1.368$ (d) $h* = 0.547$ (e) $h* = 0.365$ (f) $h* = 0.182$
The plots in this figure are more symmetrical across the modes as negative $\alpha$ generally tends to limit the total plastic dissipation in the bottom layer. This effect pushes plasticity into the top layer only, and because the graded layer allows more plasticity, an increase in the normalized plastic dissipation is seen for decreasing graded layer height. This is not to say that plasticity increases without bound as the graded layer height decreases, since once the graded layer is small enough, the plastic zones will return to the shapes as seen in the bimaterial case presented by Daily. As the graded layer height increases (moving from (a)-(f) in Fig. 5.17) the curves tend to collapse together as the effect of the graded layer decreases. Representative plastic zones for this behavior are shown in Fig. 5.13-5.16.

Overall, the effect of the graded layer is to increase plasticity, as a smooth change in material properties allows for the plastic zones to expand into the upper layer particularly for negative values of $\alpha$.

5.3.3 Effect of Strength Mismatch on the Total Plastic Dissipation

An exploration of the effect of strength mismatch on the total plastic dissipation was covered for a bimaterial interface in [3] and again for models in including a graded layer in [6]. Following that of [4], the previous section explored the effect of a graded layer in the presence of an elastic mismatch. The last set of results are for the effect of a graded layer on the general interface problem containing a graded layer with an elastic and plastic interface. Again, in keeping with the author’s previous research, the results of including a strength mismatch will be for a crack that lies on the boundary of the graded layer and the *weaker* material, thus limiting the range of strength mismatches to $0 \leq \hat{\sigma} \leq 0.25$.

Similar plots as those in Figs. 5.17 and 5.8 are shown in Figs. 5.18 and 5.19, where $dW/dN*$ is plotted vs. $\psi$ for $\alpha \geq 0.0$ and $\alpha \leq 0.0$, respectively. Positive values of $\hat{\sigma}$ indicate that the top layer is stronger relative to the bottom layer. As such, increasing $\hat{\sigma}$ decreases the amount of plasticity in every case. This effect increases with decreasing graded layer height.
Figure 5.18: Effect of strength mismatch on $dW/dN^*$ with solid lines $\dot{\sigma} = 0$ and dotted lines $\dot{\sigma} = 0.25$. 
Figure 5.19: Effect of strength mismatch on $dW/dN^*$ with solid lines $\hat{\sigma} = 0$ and dotted lines $\hat{\sigma} = 0.25$. 
5.4 Conclusions

In the context of predicting steady-state fatigue crack growth rates, eq. (1.1) can now be written as

\[
\frac{da}{dN} = \frac{E_2 \Delta \gamma^2}{\sigma^2 y_c} \frac{dW}{dN^*} (\psi, \hat{\sigma}, \alpha, h^*) ,
\]

where \( dW/dN^* \) is now a function of the mode-mix, elastic and plastic mismatch, and also the graded layer height. In all cases, the crack is assumed to occur on the interface with the weaker material (\( \hat{\sigma} \geq 0.0 \)), in which case consideration of the full range of elastic mismatch \((-0.8 \leq \alpha \leq 0.8)\) and mode-mix \((-90^\circ \leq \psi \leq 90^\circ)\) yields comprehensive results. The overall effect of the graded layer is an increase in normalized plastic dissipation, as the smooth change in material properties allows more plasticity to occur in the upper layer. As the strength of the upper layer increases, the resulting normalized plastic dissipation decreases as higher yield strengths decrease plastic zone sizes. The interaction between elastic and plastic mismatches affects the normalized plastic dissipation as well. For cases where the upper layer is stiffer due to increased \( \alpha \), less effect of strength mismatch occurs. More effect of strength mismatch is seen for \( \alpha \leq 0.0 \), as most of the plasticity is already pushed into the graded layer due the bottom layer being much stiffer. This increases the influence of the strength mismatch in the top layer.

Overall, the effect of mode-mix is the dominant factor in controlling plastic dissipation. Both elastic and plastic mismatches produce secondary effects that influence plasticity in a quantifiable way. In many cases, the elastic mismatch controls the overall shape of the plastic zone, while the strength mismatch tends to decrease overall plasticity.
Chapter 6

An Experimental Validation of the Dissipated Energy Theory for Mixed-Mode Loading

6.1 Overview

This chapter presents experimental results for sustained mixed-mode fatigue crack growth data with the goal of validating the total dissipated energy theory for fatigue crack growth rates under mixed-mode loading. As a point of clarification, this chapter is not meant to be a validation for the graded layer research, although future iterations of experimental testing could be coupled to the previous results. This chapter simply provides sustained mixed-mode fatigue crack growth rates for a steady-state crack between two layers of the same material. It is the intent of the author to link these fatigue crack growth rates to the monotonic fracture toughness of the same specimen using the method presented in [1].
6.2 Testing Equipment

6.2.1 Bending Fixture

Courtesy of Carnegie Mellon University, the four-point bending fixture shown in Fig. 6.1 was used for the experiments. This fixture was designed by Klingbeil for interfacial fracture testing of deposited metal layers [72]. Wright State University’s machine shop was able to replicate the design and produce WSU’s own in-house bending fixture. The fixture is equipped with in-plane and out-of-plane degrees of freedom that ensure the specimen is in true four-point bending. The top part of the fixture shown in Fig. 6.2(a) rotates in-plane to account for any asymmetry in specimen geometry. In addition, Fig. 6.2(b) shows the top two load points which rotate out-of-plane in order to account for asymmetry in the out-of-plane direction.

As previously noted, four-point bending of a bimaterial specimen geometry with identical layers of the same material corresponds to a steady-state loading configuration with a mode-mix of $\psi = 41^\circ$. The steady-state behavior (independent of crack length) is due to the constant moment (thus constant $G$) occurring between the two inside load points.
A free body diagram is shown in Fig. 6.3(a), and shear and bending moment diagrams in Fig. 6.3(b) and Fig. 6.3(c) respectively.

Movement of the load points is allowed by sliding the trapezoidal blocks along a track cut into the large rectangular base. This allows for different length specimens to be tested. In addition, the magnitude of the applied moments can be adjusted by changing the distance between the load points, as shown in Fig. 6.4.

### 6.2.2 MTS Machine

The MTS servo-hydraulic machine shown in Fig. 6.5 provided by the AFRL/PRTZ TEFF lab was used to fatigue the specimens. This machine consists of a load cell located at the top and a displacement control located at the bottom. The machine operates in a combination of two regimes, displacement and force. Each regime has two ranges, a low and a high. The large displacement range allows the stroke of the machine to operate from $-5 \leq d_{stroke} \leq 5\,\text{in}$ and the small displacement range from $-0.5 \leq d_{stroke} \leq 0.5\,\text{in}$. The large force range allows the machine to apply forces in the range $-20000 \leq F_{applied} \leq 20000\,\text{lbf}$ while the small force range is $-2000 \leq F_{applied} \leq 2000\,\text{lbf}$. Each range in both regimes has associated errors and for the application of this four-point bend test, the small scale displacement range was implemented. In addition, the machine employs a proportional integral derivative (PID) controller that must be tuned with test setup to ensure test stability. The University of Dayton Research Institute (UDRI) provided insight and guidance to properly tune the machine.
Figure 6.3: Four-point bend (a) Free body diagram (b) Shear diagram (c) Moment diagram

Figure 6.4: Distance between the load points
6.2.3 Hydraulic Grip Adapters

The MTS machine is equipped with hydraulic grips that serve to hold a variety of specimens in different testing configurations. These grips are occasionally pulled off for calibration and maintenance. Due to this and machine availability, adapters for the four-point bending fixture needed to be made. The threaded adapters shown in Fig. 6.6 were used without the hydraulic grips installed. Another set was also made without threads to be used when the hydraulic grips were in place.
6.3 Layered Specimen Design

In order to successfully validate the theory, two experimental tests need to be performed on the same specimen. The first is obtaining a Paris-regime fatigue crack growth curve for a particular specimen. The second is the measurement of the monotonic fracture toughness (critical energy release rate, $G_c$) of that specimen. Obtaining fatigue crack growth is the easier of the two, although potentially time consuming if the interface is extremely tough. This is because it is generally possible to drive the crack along the interface through continued cycling of the specimen. The fracture toughness, on the other hand, is the quantity where specimen design becomes very important. In theory, a load is applied to a pre-cracked, crack initiated specimen (again in four-point bend) and slowly increased. At some load, the crack will start propagating along the interface and the load will plane off, or remain constant. This critical load can be used to obtain the fracture toughness of the layered specimen with

$$K_c = \sqrt{K_I^2 + K_{II}^2},$$

where $K_I$ and $K_{II}$ are defined by eqs. (2.6) and (2.7) and are functions of the applied moments and specimen geometry. The critical energy release rate for crack extension is then defined by

$$G_c = \frac{|K_c|^2}{E}.$$

By definition of LEFM, large-scale yielding must not occur and as such, this load cannot yield the lower layer in bending. A problem exists if the critical energy release rate to cause a steady-state fracture corresponds to a stress above the yield strength of the lower layer. This means that a monotonic crack cannot be grown along the interface without plastic deformation of the whole specimen. If that state is reached, then that particular geometry of the specimen does not yield a brittle enough interface with which to validate the theory. That is not to say that the applicability of the theory is now void in regards to the manufacturing process used to make the specimen. That is because the specimen itself
could always be scaled up so as to put the critical energy release rate below the yield stress of the bottom layer. The limiting factor then becomes the equipment with which the test is performed. The goal here is to find a specimen geometry to validate the theory, not to test the functionality of layered manufacturing processes for particular real world applications. To this end, not only must the geometry be large enough to accommodate large loads, the process to bond the layers together needs to produce an interface brittle enough to obtain monotonic delamination prior to yielding. As such, many different bonding processes and geometries were attempted.

All specimens consisted of two layers bonded together. The different manufacturing processes are discussed in the following sections. Once the layers were joined together, a notch was cut in the center all the way through the top layer down to the interface, as shown in Fig. 6.7. This pre-crack provided a crack initiation site due to the stress concentration. A significant point of interest with the four point bend setup is the fact that both fatigue and monotonic fracture data can come from the same specimen. After crack initiation, a fatigue crack growth rate can be recorded by cycling the specimen for $N$ cycles. Knowing the crack length at the beginning and the end of the cycle, the crack extension is obtained by optical examination. For a range of loads, a fatigue crack growth rate curve can be mapped. With the same specimen, a constant increasing load is applied as described above to find the critical energy release rate thus yielding the fracture toughness. This can be repeated as many times as needed as long as the crack remains within the inner loading points.
6.4 General Procedure

6.4.1 Specimens

6.4.1.1 Epoxy

Material specimens involving bonded aluminum interfaces were the subject of some previous in-house studies at WSU. There were, however, no crack growth results from this type of specimen. Bonded with Loctite U-05FL Hysol, the interface proved too brittle when placed in the MTS machine. Brittle fracture occurred with the layers becoming separated rather abruptly. Fig. 6.8 shows one such aluminum bonded specimen that was used during testing.

6.4.1.2 Laser Deposition

The University of Missouri Rolla provided a laser deposited layered specimen with which to conduct fracture tests. This specimen was manufactured by applying laser melted powder to a substrate. The substrate was made of Ti-6Al-4V, as was the powder. This particular specimen was 6 inches long by 1 inch wide. The upper layer was \( \approx 0.25 \) inches thick while the bottom layer was \( \approx 0.75 \) inches thick. The interface between the laser melted powder and the substrate proved to be overly tough, and therefore monotonic failure could not be achieved. To this end, extensive fatigue data was not pursued. Fig. 6.9 shows the laser deposited specimen provided for testing.

Figure 6.8: Aluminum layered specimen.
6.4.1.3 Brazed

A brazed specimen manufactured by Wal Colmonoy has provided the best configuration to allow steady-state crack growth measurements. The specimens were prepared by brazing two pieces of bar stock together using a tin/gold alloying foil in a hydrogen vacuum, the thickness of which was negligible when compared with the overall specimen thickness\(^1\). The brazing surfaces were polished to ensure maximum contact area for the alloying foil. The bulk thickness of the specimen was 1 1/8 inches. Following brazing, each longitudinal side was machined until the overall thickness was 1 inch. Upon completion of the brazing/machining, the pre-crack notch was cut using a wire EDM process performed by Republic EDM. The resulting specimen is shown in Fig. 6.10.

Other brazed specimens were tested unsuccessfully due to the nature of the interface. Without the interface polishing, the brazing of the copper bonded to copper proved to be non-uniform. For the steel bonded to steel specimens, the interface proved too tough and

---

\(^1\)Any effects of such a small thickness would contribute to the overall \(G\) of the interface.
mode I failure occurred in the bottom layer. In both cases, even if successful steady-state cracking had occurred, measurement of the actual crack length would have proved difficult without polishing. These specimens are shown in Fig. 6.11.

6.4.2 Applied Fatigue Loading

The successful specimen was placed in the four point bend apparatus of Fig. 6.1, which was mounted in the axial tension/compression machine of Fig. 6.5. The load ratio $R = P_{low}/P_{high}$ was kept at 0.1 to ensure that the specimen remained in stable compression. Due to the machine setup, a load ratio of zero would have been potentially unstable as the specimen was not constrained in the tensile direction. It has also been shown in [1] that the load ratio $R$ has little effect on the total plastic dissipation. The value of $P_{high}$ was determined based on the yield strength of the bottom layer, and $P_{low}$ was computed from the load ratio $R = 0.1$. In terms of cyclic loading, the mean load was determined as

$$P_{mean} = \frac{P_{high} + P_{low}}{2},$$

(6.1)
while the alternating load was determined as

\[ P_{\text{alternating}} = \frac{P_{\text{high}} - P_{\text{low}}}{2}. \] (6.2)

These quantities were entered into the MTS software along with the desired frequency of cyclic loading. This particular MTS machine had a maximum frequency of about 40 Hz. A frequency of 32 Hz was initially used for all experiments, although it became necessary to decrease this value based on the resonance of the hydraulic system. For certain cyclic frequencies, fluid vibrations became excessively noisy and it became necessary to choose lower frequencies. As such, due to the steady-state nature of this study, changes in load frequency were neglected. The number of cycles was then entered and the machine allowed to run until a pre-crack formed along the interface.
6.5 Crack Measurements (\(da/dN\) Curve)

An inherent difficulty in measuring cracks along the interface between two layers (as in the four-point bend specimen) is trying to accurately identify exactly where the crack tip is at any point in time. It became necessary to polish each side of the bar stock after the brazing process in order to create a flat surface on which to measure the crack. After the pre-crack had propagated to the interface and then extended to at least one beam width away from the center-point, measurements were able to be taken. These involved cycling the specimen for a pre-defined number of cycles and then removing it from the machine. A penetrant dye was painted on the surface along the interface and allowed to permeate the crack face. After approximately 3-5 minutes, the leftover dye was wiped off and a developer then sprayed on the surface. The developer wicks the dye that is present “in” the crack up onto the surface, thus revealing a line in the dried developer showing the length of the crack. Further clarification was provided by a black light, as the dye was sensitive under this type of exposure and as such caused the crack line to glow. A micrometer was attached to the specimen just beneath the crack face, which allowed for accurate measurement of the total crack length. Independent measurements were taken for each side of the crack and each side of the specimen providing four separate crack length measurements. These crack lengths are shown for one side of the specimen in Fig. 6.12. If the interface toughness is uniform, the right-side crack should grow approximately the same amount on both sides of the specimen as the left-side crack, although this was not always the case.

For the specimen discussed here, the crack grew away from the pre-crack notch on both sides for approximately 1/2” and arrested on one side only. Thus, the data in the following section is for the side that continued to grow. This crack arrest was likely a result of non-uniform bonding during the brazing process. Regardless, one side of the specimen is sufficient in providing crack growth measurements, as either side can grow independently of the other without altering the applied range of energy release rate. This is one of the advantages of the chosen specimen geometry.
Figure 6.12: Crack length measurements.
6.5.1 Calculation of $\Delta K$

To construct the $da/dN$ curve, it was necessary to measure fatigue crack growth rates at different load ($\Delta K$) levels. These levels were limited by the yield stress of the bottom layer, as large-scale plasticity must be avoided. Based on classical bending stress the maximum load for a given percentage of yield strength was given by

$$P_{\text{high}} = \frac{x\sigma_y bh^2}{3d},$$

where $x$ is a percent, $b$ and $h$ are the width and height of the beam, and $d$ is the distance between the load points on the fixture. Based on the load ratio $r = 0.1$, $P_{\text{low}} = rP_{\text{high}}$, then eqs. (6.1) and (6.2) were used to calculated the mean and alternating loads. The different load levels $\Delta K$ were calculated as

$$\Delta K = \frac{\Delta Pd}{4bh} \sqrt{\frac{21}{h}},$$

where $\Delta P = P_{\text{high}} - P_{\text{low}}$. To this end, Table 6.1 shows the $\Delta K$ values and the corresponding load values for a yield strength of $\sigma_y = 45000$ psi and specimen dimensions $b = 1"$, $h = 1.1875\"$, and $d = 2.5\"$. Although the tests were conducted in English units, the table has been converted to SI units.

<table>
<thead>
<tr>
<th>%$\sigma_y$</th>
<th>$P_{\text{high}}$ (N)</th>
<th>$P_{\text{low}}$ (N)</th>
<th>$\Delta K$ MPa$\sqrt{m}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>30</td>
<td>11291</td>
<td>1129.1</td>
<td>5.556</td>
</tr>
<tr>
<td>35</td>
<td>13173</td>
<td>1317.3</td>
<td>6.482</td>
</tr>
<tr>
<td>40</td>
<td>15054</td>
<td>1505.4</td>
<td>7.408</td>
</tr>
<tr>
<td>42</td>
<td>15807</td>
<td>1580.7</td>
<td>7.778</td>
</tr>
<tr>
<td>54</td>
<td>20324</td>
<td>2032.4</td>
<td>10.001</td>
</tr>
<tr>
<td>64</td>
<td>24087</td>
<td>2408.7</td>
<td>11.853</td>
</tr>
</tbody>
</table>

Table 6.1: Four-point bend specimen loading
6.5.2 Calculation of $da/dN$

For one particular load level, the specimen was fatigued for a set number of cycles after which the crack length was measured and recorded using the process previously described. This process was repeated 4-5 times for each load level. By plotting crack length $a$ vs. cycle number $N$ and using a least squares linear fit, the fatigue crack growth rate for a particular $\Delta K$ level was given as the slope of the resulting line. Each of these plots for every load level is shown in Fig. 6.13. Combining the $\Delta K$ loads in Table 6.1 and the slopes from Fig. 6.13, a $da/dN$ curve was determined for both sides of the specimen as shown in Figs. 6.14(a) and (b) plotted on a log-log scale. These represent previously unpublished results for fatigue crack growth under sustained mixed-mode loading in a layered specimen geometry. In order to assess the validity of the dissipated energy theory for mixed-mode loading, the author has attempted to determine the monotonic fracture toughness of the same specimen. An additional specimen was tested that was theoretically identical in manufacturing process to the first specimen. Unlike the first, no crack arrest occurred on a single side and $a$ vs. $N$ plots for all four sides are shown in Fig. 6.15 and 6.16. The slopes of a least squares fit of each set of data were averaged and plotted on a log-log scale in Fig. 6.17. With both specimens, monotonic fracture proved elusive. After loading both specimens up to near yield, the crack did not extend in a steady-state manner, however, the fatigue data is still valid. The brazed interface proved too tough for this particular specimen geometry, but future work could look into scaling up the specimen size until monotonic delamination is possible.

6.5.3 Conclusions

The experimental results presented here represent previously unpublished results for fatigue crack growth under sustained mixed-mode loading in a layered specimen geometry. The intent of these experiments is to assess the validity of the theory presented in [1] for mixed-mode loading, and not necessarily to validate the graded layer models presented in
Figure 6.13: $a$ vs. $N$ for various loads (a) 30% (b) 35% (c) 40% (d) 42% (e) 54% (f) 64%
Figure 6.14: log – log plot of $da/dN$ vs. $\Delta K$
Figure 6.15: $a$ vs. $N$ for various loads (a) 30% (b) 35% (c) 40% (d) 45%
Figure 6.16: $a$ vs. $N$ for various loads (a) 50% (b) 55% (c) 60% (d) 65%

Figure 6.17: log – log plot of $da/dN$ vs. $\Delta K$
the previous section. That said, the results of this work may help form the basis for application of the dissipated energy theory in the design of fatigue-resistant interfaces in additive manufacturing of other applications including layered material systems.
Chapter 7

Conclusions & Future Work

7.1 Conclusions

This dissertation has presented a full treatment of the effect of a graded layer on the plastic dissipation at a bimaterial crack tip under mixed-mode loading. These types of layered material systems are common in many manufacturing techniques that involve material deposition on a layer-by-layer basis. The understanding of fatigue cracks that propagate in this type of media has provided the motivation for this research to extend a dissipated energy theory of fatigue crack growth to bimaterial systems with a graded interface. Previous research assumed a perfect crack interface, where a step change in material properties exists, while the current research sought to eliminate this non-physical scenario by inserting a graded layer along the crack plane.

Chapters 1 and 2 provided the necessary background and literature framework to support the extension of previous work by Daily and Klingbeil in the context of graded layers. In Chapter 3, the author presented a previously unpublished beam theory solution for the strain energy release rate for models with a graded layer. This led to a full set of previously unpublished results exploring the effect of three nondimensional quantities (loading, material mismatch, and graded layer height) on the nondimensional strain energy release rate.
These results could ultimately lead to a quicker advancement of new age material adoption with the goal of creating fatigue resistant layered geometry. Chapter 4 is arguably the most notable contribution to the field of bimaterial crack modeling. By using a graded layer and unique meshing technique, the author presented a previously unpublished method with which to determine the actual mode-mix at a crack tip in the presence of an elastic mismatch. Previous methods include an ambiguous definition of the mode-mix that involved an arbitrary characteristic length, while the graded layer model provides an *unambiguous* definition of the mode for bimaterial models and bypasses the need for a characteristic length. In so doing, this research validates a previous method by Daily that used extrema in plastic dissipation to calibrate traditional characteristic lengths corresponding to crack tip modes of interest. This validation was presented in Chapter 5, along with full plastic dissipation results following that of Daily and Klingbeil for a bimaterial model with a graded layer. This directly supports the method Klingbeil used to predict steady-state Paris-regime fatigue crack growth rates using only monotonic fracture toughness values and more realistic FEA models. Finally, Chapter 6 presented some mixed-mode experimental fatigue data for a four-point-bend specimen geometry. While direct validation of Klingbeil’s theory for mixed-mode loading has proved difficult, the resulting fatigue crack growth data provides previously unpublished $da/dN$ curves for brazed interfaces between steel layers.

Overall, the results of this work show that while the quantifiable effect on the plastic dissipation of a graded layer is small compared to that of the actual mode-mix, there is in fact an effect of an imperfect interface. The implications of this have become more important with the advent of manufacturing techniques that are potentially able to tailor material properties in local regions to a high degree of resolution. These opportunities for intentional grading serve as direct applications for the research contained herein, and provide motivation for future work in the area of graded layer modeling.

The research presented herein provides new contributions in the field of fracture mechanics involving layered material systems that include:
1. A previously unpublished beam theory solution for the strain energy release rate in the presence of a graded layer.

2. Previously unpublished nondimensional results of the effect of a graded layer, elastic mismatch, and mode of loading on the strain energy release rate.

3. A previously unpublished elastic technique for numerically determining an unambiguous definition of the mode in the presence of an elastic mismatch.

4. An independent validation of Daily’s physically based definition of the mode in the presence of an elastic mismatch using total plastic dissipation.

5. Previously unpublished results for the effect of a graded layer, elastic mismatch, plastic mismatch, and mode of loading on the plastic dissipation during fatigue crack growth on bimaterial interfaces.

6. Experimental steady-state fatigue crack growth rates for a four-point bend specimen under mixed-mode loading.

### 7.2 Future Work

First and foremost, there still exists a large void in the literature for comprehensive mixed-mode crack growth data. The results of this dissertation provide a small step in the right direction, but more is needed to fully validate the total dissipated energy theory as applied to specimens under mixed-mode loading. As such, future work could include more four point bend specimen testing or even other configurations for different modes of loading. In addition, more analytical work could be done on the effect of crack location relative to the graded layer. It was assumed in this work that the crack stayed on the boundary of the weaker material yet not necessarily the less stiff material. As such, exploration of the effect of moving the crack to different material stiffness boundaries could be explored.
Appendix A

Matlab Scripts for Composite Beam Analysis

tclc
clear all

%% This script calculates the formula for Ic2 (top rect and trapezoid)
syms h1 h2 b n x y HR h3 alpha h H E1 E2

% Material is All E2
% x(y) for Ic2
x1 = b/h2*(n-1)*y + b;  % trapezoid
x2 = n*b;  % top rectangle

% Area
A = n*b*h1 + b*h2/2*(n+1);  % total area

% Material is All E1
% x1 = b/h2*(1-n)*y + n*b;
% x2 = b;
% A = b*h1 + b*h2/2*(n+1);

% Neutral axis location
19 \ ybar = (\text{int}(\text{y} \cdot \text{x1}, \text{y}, 0, \text{h}2) + \text{int}(\text{y} \cdot \text{x}2, \text{y}, \text{h}2, \text{h}1+\text{h}2))/\text{A};

20 \ \text{Moment of Inertia}

21 \ I = \text{int}(\text{y}^2 \cdot \text{x1}, \text{y}, 0, \text{h}2) + \text{int}(\text{y}^2 \cdot \text{x}2, \text{y}, \text{h}2, \text{h}1+\text{h}2);

22

23 \ \text{Centroidal moment of inertia}

24 \ Iprime = I - \text{A} \cdot \text{ybar}^2; \ Iprime = \text{subs}(Iprime, [b], [1]);

25 \ \% Iprime = \text{subs}(Iprime, [\text{h}1, \text{h}2, \text{b}], [\text{h}3/(\text{HR}+1), \text{h}3+3\text{HR}/(\text{HR}+1), 1]);

26 \ Iprime = \text{subs}(Iprime, [\text{h}1, \text{h}2, \text{h}3, \text{b}], [\text{h}1-\text{H}1]);

27 \ Iprime = \text{subs}(Iprime, [\text{H}], [\text{HR}+\text{h}/(1+\text{HR})]);

28

29 \ X = \text{simplify}(Iprime)

30 \ Y = \text{simple}(X)

31 \ \text{pretty}(Y)

cle

clear all

\text{This script calculates the formula for Ic3 (top rect, trapezoid, and bottom rect together)}

4 \ \text{syms} \ \text{h1} \ \text{h2} \ \text{h3} \ \text{b} \ \text{n} \ \text{x} \ \text{y} \ \text{HR} \ \text{h} \ \text{H}

5

6 \ \% \text{Material is All E2}

7 \ \% x(y) for Ic3

8 \ \text{x1} = \text{b}; \ \% \text{Bottom Rect}

9 \ \text{x2} = \text{b}/\text{h}2*(\text{n}-1)*(\text{y}-\text{h}3) + \text{b}; \ \% \text{Trapezoid}

10 \ \text{x3} = \text{n} \cdot \text{b}; \ \% \text{Top Rect}

11 \ \% \text{Area}

12 \ \text{A} = \text{b} \cdot \text{h}3 + \text{n} \cdot \text{b} \cdot \text{h}1 + \text{b} \cdot \text{h}2/2*(\text{n}+1); \ \% \text{Total Area}

13

14 \ \% \text{Material is All E1}

15 \ \% \text{x1} = \text{n} \cdot \text{b};

16 \ \% \text{x2} = \text{b}/\text{h}2*(1-\text{n})*(\text{y}-\text{h}3) + \text{n} \cdot \text{b};
\% x3 = b;
\% A = n*b*h3 + b*h1 + b*h2/2*(n+1);

\% Neutral axis location
\[ y_{\text{bar}} = \frac{\int (y \times x1, y, 0, h3) + \int (y \times x2, y, h3, h2+h3) + \int (y \times x3, y, h2 + h3, h1+h2+h3)}{A}; \]

\% Moment of inertia
\[ I = \int (y^2 \times x1, y, 0, h3) + \int (y^2 \times x2, y, h3, h2+h3) + \int (y^2 \times x3, y, h2 + h3, h1+h2+h3); \]

\% Centroidal moment of inertia
\[ I_{\text{prime}} = I - A \times y_{\text{bar}}^2; \]

\% I_{\text{prime}} = \text{subs}(I_{\text{prime}}, b, 1);
\% I_{\text{prime}} = \text{subs}(I_{\text{prime}}, \{b, h1, h2\}, \{1, h3/(HR+1), h3*HR/(HR+1)\});
\% I_{\text{prime}} = \text{subs}(I_{\text{prime}}, \{h1, h2, h3\}, \{h*(1-\text{HR}), h*\text{HR} \times h \times 1\});
[1, 2, 3, 4]
Appendix B

Matlab Script Invoking Abaqus Python Script

1 %% Variables
2 Modes = [−90:30:0 0:30:90];
3 Alpha = −0.8:0.2:0.8;
4 %AllMratios = −0.087172009972292;
5
6 AllMratios = [INSERT MATRIX OF LOADS HERE];
7 %% Write Variable . py File
8 MeshLayerHeight = 0.1;
9 sendmail(address , 'Job Started', 'Plastic')
10 for i = 1:length(Modes);
11     Mode = Modes(i);
12     if i<=4
13         M_status = ‘‘M1_Constant’’;
14     else
15         M_status = ‘‘M2_Constant’’;
16     end
17  end
\%M_status = ' ' ' M2_Constant ' ' ' ;

Mratios = AllMratios (: , i ); \%i

for j = 1: length ( Alpha )
    alpha = Alpha ( j );
    Mratio = Mratios ( j ); \%j

\%
Make python file with variables

delete( ' Variables . py' , ' * . rpy' , ' * . log' );

fid = fopen ( ' Variables . py' , ' w' );
fprintf ( fid , ' mode = %2.0 f \n ', Mode );
fprintf ( fid , ' alpha = %3.1 f \n ', alpha );
fprintf ( fid , ' M_ratio = %12.6 f \n ', Mratio );
fprintf ( fid , ' meshLayerHeight = %4.2 f \n ', MeshLayerHeight );
fprintf ( fid , strcat ( ' M_status = ' , M_status ));

fclose ( fid );

\%Make part ( run Abaqus)

system ( ' abaqus cae noGUI = ElasticPlasticAnalysis_matlab . py ' ); \%Windows system?
sendmail ( address , ' h2 = 0.1 ' , ' Plastic ' , ' sigmahat = 0.1 ' )

end

end

\% Send the email

sendmail ( address , ' Job is Complete ' , ' Plastic ' )
# Do not delete the following import lines
from abaqus import *
from abaqusConstants import *
from odbAccess import *
#from smtplib import SMTP #for emailing

import section
import regionToolset
import displayGroupMdbToolset as dgm
import part
import material
import assembly
import step
import interaction
import load
import mesh
import job
import sketch
import visualization
import xyPlot
import displayGroupOdbToolset as dgo
import connectorBehavior

import sys
import os
# Load variables
execfile('Variables.py')

#M_status = 'M1_Constant'
mode = float(mode)

def main():
    session.viewports['Viewport: 1'].setValues(displayedObject=None)

    # Change from "hex" format to "coordinate" format or "index" format (used for naming geometrical features)
    cliCommand("""session.journalOptions.setValue(replayGeometry=COORDINATE)"")

    # Change working directory
    zzz = str(mode)

    #workingDirectory = " C:/Users/Craig Baudendistel/Desktop/Abaqus/Completed_Jobs/" # Windows Bootcamp
    workingDirectory = "C:/Users/Craig Baudendistel/Desktop/Abaqus/Completed_Jobs/"

    if not os.path.exists(workingDirectory+zzz):
        os.makedirs(workingDirectory+zzz)
        os.chdir(workingDirectory+zzz)

    job = 'submit' # submit or wait
    modelType = 'plastic' # elastic or plastic
    # If the modelType is plastic, a submodel is created to run the plasticity

    Mdb()

    # Make sure Length/Height is an integer for meshing purposes
    Length = 50. # mm
    topHeight = 5. # mm
    bottomHeight = 5. # mm
    Height = 10. # mm

    G = 0.2 # N mm/mm^2
    sigmaHat = 0.1
    # hStar = h*sigma2**2/(G+Ebar)

    ElementType = CPE8R # or CPE4R
contourNumber = 16

#meshLayerHeight = 1.0
if meshLayerHeight == 0.01:
    propertyLayers = 25
    meshLayers = 25
else:
    propertyLayers = 100
    meshLayers = 100
m=1. #Controls smallest element size which is 2.5e-5 meshResolution/m
meshResolution = meshLayerHeight/meshLayers
gradedLayerHeight = float(propertyLayers)/float(meshLayers)*meshLayerHeight

#Setup File Names based on Alpha, Moment Ratio, and GradedLayerHeight
a = str(alpha)
b = str(M_ratio)
c = str(gradedLayerHeight)
AID = a.replace('.', 'o')
MID = b.replace('.', 'o')
CID = c.replace('.', 'o')
jobname = 'Alpha_' + AID + '_status_' + MID + '_h2_' + CID
print jobname

nu_1 = 1./3.
nu_2 = 1./3.
E_2 = 73100.
E_1 = (1.+alpha)*E_2/(1.-alpha)
sigma_2 = 300.0
sigma_1 = (1.+sigmaHat)/(1.-sigmaHat)*sigma_2
littleBoxSize = meshResolution/100.
h1 = topHeight - gradedLayerHeight
h2 = gradedLayerHeight
h3 = bottomHeight

#h_2 = hStar*G*E_2/(sigma_2**2.*(1.-nu_2**2.))
n = E_1/E_2
Ibot = h3**3./12.

I_c2_calc = (h3**3.*n)/3. - (h2**3.*n)/12. - (3.*h3**2.*n - h2**2.*n + h2**2.)
  **2./((h2 - h2*n + 2.*h3*n)) + h2**3./12.

#I_c2_calc = (n*(h1 + h2))**3./3. - (h2**3.*n)/12. - (h2**2*(n/3. + 1./6.)) + (h1*n*(h1
  + 2.*h2))/2.)**2./((h1*n + (h2*(n + 1.))/2.) + h2**3./12.

#I_c3_calc = (h2*(4.*h2+h3 + 3.*h2**2.*n + 6.*h3**2.*n + h2**2. + 6.*h3**2. + 8.*h2+h3
  *n))/12. - (n*(h2 + h3))**3./3. - ((h2*(h2 + 3.*h3 + 2.*h2*n + 3.*h3*n))/6. + h3
  **2./2. + (h1*n*(h1 + 2.*h2 + 2.*h3))/2.)**2./((h3 + h1*n + (h2*(n + 1.))/2.) + (n
  *h1 + h2 + h3))**3./3. + h3**3./3.

I_c3_calc = ((-h2**4. + 6.*h2**3.*h3 - 12.*h2**2.*h3**2. + 12.*h2**2.*h3**2. - 6.*h3**4.)*
  n**2. + (2.*h2**4. + 24.*h2**2.*h3**2. - 84.*h3**4.)*n - h3**4. - 6.*h2**3.*h3 -
  12.*h2**2.*h3**2. - 12.*h2**2.*h3**3. - 6.*h3**4.)/((36.*h2 - 72.*h3)*n - 36.*h2 -
  72.*h3)

if M_status == 'M1.Constant':
    M1 = ((2.*G*E/2/(1. - nu_2**2.2.))/((1./1_c2_calc+(M_ratio**2.2.)/Ibot - (1. - M_ratio)**2.2./
    I_c3_calc))**(1./2.)

    M2 = M_ratio*M1

elif M_status == 'M2.Constant':
    M2 = ((2.*G*E/2/(1. - nu_2**2.2.))/((M_ratio**2.2.1_c2_calc + 1.)*Ibot - (M_ratio - 1.)**2.2./
    I_c3_calc))**(1./2.)

    M1 = M_ratio*M2

# CREATE THE GLOBAL MODEL

# Rename Model
mdb.models.changeKey (fromName='Model-1', toName='Global_Model')
del mdb.models['Global Model'].sketches['__profile__']

# PARTITION the part

# Horizontal Crack Plane

p = mdb.models['Global Model'].parts['Global Part']
f, e, dl = p.faces, p.edges, p.datums
t = p.MakeSketchTransform(sketchPlane=f.findAt(coordinates=(0.0, 0.0, 0.0), normal=(0.0, 0.0, 1.0)), sketchPlaneSide=SIDE1, origin=(0.0, 0.0, 0.0))
s1 = mdb.models['Global Model'].ConstrainedSketch(name='__profile__', sheetSize=Length, gridSpacing=Length/10.0, transform=t)
g, v, d, c = s1.geometry, s1.vertices, s1.dimensions, s1.constraints
s1.sketchOptions.setValues(decimalPlaces=6)
s1.setPrimaryObject(option=SUPERIMPOSE)
p.projectReferencesOntoSketch(sketch=s1, filter=COPLANAR_EDGES)
s1.Line(point1=(-Length/2., 0.0), point2=(Length/2., 0.0))

if meshLayerHeight != topHeight/2.:
s1.Line(point1=(-Length/2., -bottomHeight/2.), point2=(Length/2., -bottomHeight/2.)) # Upper Horiz Line (if needed)
s1.Line(point1=(-Length/2., topHeight/2.), point2=(Length/2., topHeight/2.)) # Lower Horiz Line (if needed)
s1.Line(point1=(0.0, topHeight), point2=(0.0, -bottomHeight))

s1.Line(point1=(meshResolution, -bottomHeight), point2=(meshResolution, topHeight)) # Vertical Middle Line
s1.Line(point1=(-meshResolution, -bottomHeight), point2=(-meshResolution, topHeight)) # Vertical Right Line
s1.Line(point1=(meshResolution, -bottomHeight), point2=(meshResolution, topHeight)) # Vertical Left Line
s1.rectangle(point1=(-littleBoxSize, -littleBoxSize), point2=(littleBoxSize, littleBoxSize)) # Crack Tip Box
s1.Line(point1=(meshResolution, meshResolution), point2=(littleBoxSize, littleBoxSize)) # Right Top Diag Line
s1.Line(point1=(-meshResolution, -meshResolution), point2=(-littleBoxSize, -littleBoxSize)) # Left Bottom Diag Line
s1.Line(point1=(-meshResolution, meshResolution), point2=(-littleBoxSize, littleBoxSize)) # Left Top Diag Line
s1.Line(point1=(meshResolution, -meshResolution), point2=(littleBoxSize, -littleBoxSize)) # Right Bottom Diag Line

for i in range(1,meshLayers+1):
s1.Line(point1=(-Length/2., meshResolution*i), point2=(Length/2., meshResolution*i)) # Top Horiz Lines
157    s1.Line(point1=(Length/2., -meshResolution*i), point2=(Length/2., -meshResolution *i)) #Bottom Horiz Lines
158    s1.Line(point1=(0.15*Height, topHeight), point2=(0.15*Height, -bottomHeight))
159    s1.Line(point1=(-0.15*Height, topHeight), point2=(-0.15*Height, -bottomHeight))
160    pickedFaces = f.findAt(((0.0, 0.0, 0.0), ))
161    #e1, d2 = p.edges, p.datums
162    p.PartitionFaceBySketch(faces=pickedFaces, sketch=s1)
163    s1.unsetPrimaryObject()
164    del mdb.models["Global\Model"].sketches["__profile__"]
165
166    # INSTANCE Global Part into Assembly
167    a = mdb.models["Global\Model"].rootAssembly
168    a.DatumCsysByDefault(CARTESIAN)
169    p = mdb.models["Global\Model"].parts["Global\Part"]
170    a.Instance(name='Global\Instance', part=p, dependent=OFF)
171
172    # DEFINE ELASTIC MATERIALS AND ASSIGN TO SECTIONS
173    # Top Section
174    mdb.models["Global\Model"].Material(name='Material-1')
175    mdb.models["Global\Model"].materials['Material-1'].Elastic(table=((E_1, nu_1), ))
176    mdb.models["Global\Model"].HomogeneousSolidSection(name='topSection', material='Material-1', thickness=1.0)
177    # Bottom Section
178    mdb.models["Global\Model"].Material(name='Material-2')
179    mdb.models["Global\Model"].materials['Material-2'].Elastic(table=((E_2, nu_2), ))
180    mdb.models["Global\Model"].HomogeneousSolidSection(name='bottomSection', material='Material-2', thickness=1.0)
181    # Graded layer Section
182    for i in range(1, propertyLayers+1):
183        mdb.models["Global\Model"].Material(name="MiddleLayer\+\'i\'")
184        mdb.models["Global\Model"].materials['MiddleLayer\+\'i\'].Elastic(table=((E_2 + (E_1
185        -E_2)/(propertyLayers+1)\*(i), (nu_2 + (nu_1-nu_2)/(propertyLayers+1)\*(i))), ))
186        mdb.models["Global\Model"].HomogeneousSolidSection(name="MiddleSection\+\'i\',
187        material='MiddleLayer\+\'i\'', thickness=1.0)
188
189    #Add in the plasticity properties
190    #\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\n181
mdb.models['Global Model'].materials['Material -1'].Plastic(hardening=KINEMATIC, table=
=((sigma_1, 0.0), (sigma_1+sigma_1*.0002, 300))

mdb.models['Global Model'].materials['Material -2'].Plastic(hardening=KINEMATIC, table=
=((sigma_2, 0.0), (sigma_2+sigma_2*.0002, 300))

for i in range(1,propertyLayers+1):#propertyLayers+1):
  mdb.models['Global Model'].materials['MiddleLayer'+str(i)].Plastic(hardening=
                        KINEMATIC, table=((sigma_2 + (sigma_1−sigma_2)/(propertyLayers+1)*i, 0.0),

# ASSIGN SECTIONS TO PARTITIONS

# This block of code assigns just the very first gradedLayer (because of crack tip mesh box)

middlefaces = f.findAt(((meshResolution*2./3.,meshResolution*1./3.,0.0), ),
  ((meshResolution*1./3.,meshResolution*2./3.,0.0), ),
  ((−meshResolution*2./3.,meshResolution*1./3.,0.0), ),
  ((−meshResolution*1./3.,meshResolution*2./3.,0.0), ),
  ((littleBoxSize/2.,littleBoxSize/2.,0.0), ),

middleregion = regionToolset.Region(faces=middlefaces)
p.SectionAssignment(region=middleregion, sectionName='MiddleSection1', offset=0.0)

#This block of code assigns the remaining grading of the properties
for i in range(2, propertyLayers+1):
    middlefaces = f.findAt(((Length/4., (i-0.5)*meshResolution, 0.0), ),
                           ((Length/4., (i-0.5)*meshResolution, 0.0), ),
                           ((-0.5*0.15*Height, (i-0.5)*meshResolution, 0.0), ),
                           ((0.5*0.15*Height, (i-0.5)*meshResolution, 0.0), ))
    middleregion = regionToolset.Region(faces=middlefaces)
p.SectionAssignment(region=middleregion, sectionName='MiddleSection '+str(i), offset=0.0)

#This block of code assigns the very top sections
topfaces = f.findAt(((Length/4., 3./4.*topHeight, 0.0), ),
                    ((-meshResolution/2., 3./4.*topHeight, 0.0), ),
                    ((meshResolution/2., 3./4.*topHeight, 0.0), ),
                    ((Length/4., 3./4.*topHeight, 0.0), ),
                    ((-0.5*0.15*Height, 3./4.*topHeight, 0.0), ),
                    ((0.5*0.15*Height, 3./4.*topHeight, 0.0), ))

#Extra Space if the meshLayerHeight isn't equal to half the topHeight
if meshLayerHeight != topHeight/2.:
    topfaces = topfaces + f.findAt(((Length/4., meshLayerHeight+meshResolution, 0.0), ),
                                   ((Length/4., meshLayerHeight+meshResolution, 0.0), ),
                                   ((-meshResolution/2., meshLayerHeight+meshResolution, 0.0), ),
                                   ((meshResolution/2., meshLayerHeight+meshResolution, 0.0), ),
\((-0.5 \times 0.15 \times \text{Height}, \text{meshLayerHeight} + \text{meshResolution}, 0.0),\)
\((-0.5 \times 0.15 \times \text{Height}, \text{meshLayerHeight} + \text{meshResolution}, 0.0),\))

topregion = regionToolset.Region(faces=topfaces)

# Extra layers if property layers aren't equal to the mesh layers
if propertyLayers != meshLayers:
    for i in range(propertyLayers + 1, meshLayers + 1):
        topfaces = topfaces + f.findAt(((
            -Length / 4., (i - 0.5) * meshResolution, 0.0), ),
            (Length / 4., (i - 0.5) * meshResolution, 0.0), ),
            (-meshResolution / 2., (i - 0.5) * meshResolution, 0.0), ),
            (meshResolution / 2., (i - 0.5) * meshResolution, 0.0), ),
            (-0.5 * 0.15 * Height, (i - 0.5) * meshResolution, 0.0), ),
            ((0.5 * 0.15 * Height, (i - 0.5) * meshResolution, 0.0), ),
            ((meshResolution * 2./3., -meshResolution * 1./3., 0.0), ),
            ((meshResolution * 1./3., -meshResolution * 2./3., 0.0), ),
            ((-meshResolution * 2./3., -meshResolution * 1./3., 0.0), ),
            ((-meshResolution * 1./3., -meshResolution * 2./3., 0.0), ),
            ((littleBoxSize / 2., -littleBoxSize / 2., 0.0), ),
            ((-littleBoxSize / 2., -littleBoxSize / 2., 0.0), ),
            ((-Length / 4., -3./4.*bottomHeight, 0.0), ),
            ((-meshResolution / 2., -3./4.*bottomHeight, 0.0), ),
            ((meshResolution / 2., -3./4.*bottomHeight, 0.0), ),
            ((Length / 4., -3./4.*bottomHeight, 0.0), ),
            ((-0.5 * 0.15 * Height, -3./4.*bottomHeight, 0.0), ),
            ((0.5 * 0.15 * Height, -3./4.*bottomHeight, 0.0), ))
	opregion = regionToolset.Region(faces=topfaces)

p.SectionAssignment(region=topregion, sectionName='topSection', offset=0.0)

# Assign all the bottom sections
# The very first layer below the crack plane
bottomfaces = f.findAt(((
    -Length / 4., -(1.-0.5) * meshResolution, 0.0), ),
    ((Length / 4., -(1.-0.5) * meshResolution, 0.0), ),
    ((-0.5 * 0.15 * Height, -(1.-0.5) * meshResolution, 0.0), ),
    ((0.5 * 0.15 * Height, -(1.-0.5) * meshResolution, 0.0), ),
    ((meshResolution / 2., -meshResolution * 1./3., 0.0), ),
    ((meshResolution * 1./3., -meshResolution / 2., 0.0), ),
    ((-meshResolution / 2., -meshResolution * 1./3., 0.0), ),
    ((-meshResolution * 1./3., -meshResolution / 2., 0.0), ),
    ((littleBoxSize / 2., -littleBoxSize / 2., 0.0), ),
    ((-littleBoxSize / 2., -littleBoxSize / 2., 0.0), ),
    ((-Length / 4., -3./4.*bottomHeight, 0.0), ),
    ((-meshResolution / 2., -3./4.*bottomHeight, 0.0), ),
    ((meshResolution / 2., -3./4.*bottomHeight, 0.0), ),
    ((Length / 4., -3./4.*bottomHeight, 0.0), ),
    ((-0.5 * 0.15 * Height, -3./4.*bottomHeight, 0.0), ),
    ((0.5 * 0.15 * Height, -3./4.*bottomHeight, 0.0), )))
# The rest of the layers

```python
for i in range(2, meshLayers + 1):
    bottomfaces = bottomfaces + f.findAt(((−meshResolution/2., (i−0.5)∗meshResolution, 0.0), ),
            ((meshResolution/2., (i−0.5)∗meshResolution, 0.0), ),
            ((−Length/4., (i−0.5)∗meshResolution, 0.0), ),
            ((Length/4., (i−0.5)∗meshResolution, 0.0), ),
            ((−0.5∗0.15∗Height, (i−0.5)∗meshResolution, 0.0), ),
            ((0.5∗0.15∗Height, (i−0.5)∗meshResolution, 0.0), ))
```

# Extra space if needed

```python
if meshLayerHeight != topHeight/2.:
    bottomfaces = bottomfaces + f.findAt(((−Length/4., −meshLayerHeight+meshResolution, 0.0), ),
            ((Length/4., −(meshLayerHeight+meshResolution), 0.0), ),
            ((−meshResolution/2., −(meshLayerHeight+meshResolution), 0.0), ),
            ((meshResolution/2., −(meshLayerHeight+meshResolution), 0.0), ),
            ((−0.5∗0.15∗Height, −(meshLayerHeight+meshResolution), 0.0), ),
            ((0.5∗0.15∗Height, −(meshLayerHeight+meshResolution), 0.0), ))
```

bottomregion = regionToolset.Region(faces=bottomfaces)
p. SectionAssignment(region=bottomregion, sectionName='bottomSection', offset=0.0)

# MESH CONTROLS AND ELEMENT TYPE

`# All the TOP regions`

```python
a = mdb.models['Global Model'].rootAssembly
f1 = a.instances['Global Instance'].faces
topRegions = f1.findAt(((−Length/4., 3.∗topHeight/4., 0.0), ),
            # Top Left
```
if meshLayerHeight != topHeight/2.:
    topRegions = topRegions + f1.findAt(((Length/4., meshLayerHeight+meshResolution, 0.0), ),
    ((Length/4., meshLayerHeight+meshResolution, 0.0), ),
    ((-0.5*0.15*Height, meshLayerHeight+meshResolution, 0.0), ),
    ((0.5*0.15*Height, meshLayerHeight+meshResolution, 0.0), ),
    ((-meshResolution/2., meshLayerHeight+meshResolution, 0.0), ),
    ((meshResolution/2., meshLayerHeight+meshResolution, 0.0), ))

#All the BOTTOM regions
bottomRegions = f1.findAt(((Length/4., -3.*bottomHeight/4., 0.0), ),
    ((Length/4., -3.*bottomHeight/4., 0.0), ),
    ((-0.5*0.15*Height, -3.*bottomHeight/4., 0.0), ),
    ((0.5*0.15*Height, -3.*bottomHeight/4., 0.0), ),
    ((-meshResolution/2., -3./4.*bottomHeight, 0.0), ),
    ((meshResolution/2., -3./4.*bottomHeight, 0.0), ),
    ((meshResolution/2., meshLayerHeight, 0.0), ),
    ((-meshResolution/2., meshLayerHeight, 0.0), ),
    ((meshLayerHeight+meshResolution, meshLayerHeight, 0.0), ),
    ((meshLayerHeight+meshResolution, -3.*bottomHeight/4., 0.0), ),
    ((-meshResolution/2., meshLayerHeight, meshLayerHeight+meshResolution, 0.0), ),
    ((meshResolution/2., meshLayerHeight, meshLayerHeight+meshResolution, 0.0), ))
if meshLayerHeight != topHeight/2.:
    bottomRegions = bottomRegions + f1.findAt(((Length/4., -(meshLayerHeight+meshResolution), 0.0), ),
        ((Length/4., -(meshLayerHeight+meshResolution), 0.0), ),
        ((0.5*0.15*Height, -(meshLayerHeight+meshResolution), 0.0), ),
        ((meshResolution/2., -(meshLayerHeight+meshResolution), 0.0), ))

    #All the MIDDLE regions
    middleRegions = f1.findAt(((Length/4., (1. - 0.5)*meshResolution, 0.0), ),
        ((Length/4., (1. - 0.5)*meshResolution, 0.0), ),
        ((Length/4., -(1. - 0.5)*meshResolution, 0.0), ),
        ((0.5*0.15*Height, -(1. - 0.5)*meshResolution, 0.0), ),
        ((0.5*0.15*Height, (1. - 0.5)*meshResolution, 0.0), ),
        ((0.5*0.15*Height, (1. - 0.5)*meshResolution, 0.0), ))

    for i in range(2, meshLayers+1):
        middleRegions = middleRegions + f1.findAt(((Length/4., (i - 0.5)*meshResolution, 0.0), ),
            ((Length/4., (i - 0.5)*meshResolution, 0.0), ),
            ((Length/4., -(i - 0.5)*meshResolution, 0.0), ),
            ((Length/4., -(i - 0.5)*meshResolution, 0.0), )
            ((-meshResolution/2., (i - 0.5)*meshResolution, 0.0), )

            for j in range(2, meshLayers+1):
                middleRegions = middleRegions + f1.findAt(((Length/4., (i - 0.5)*meshResolution, 0.0), ),
                    ((Length/4., (i - 0.5)*meshResolution, 0.0), ),
                    ((Length/4., -(i - 0.5)*meshResolution, 0.0), ),
                    ((Length/4., -(i - 0.5)*meshResolution, 0.0), )
                    ((-meshResolution/2., (i - 0.5)*meshResolution, 0.0), )

                        for k in range(2, meshLayers+1):
                            middleRegions = middleRegions + f1.findAt(((Length/4., (i - 0.5)*meshResolution, 0.0), ),
                                ((Length/4., (i - 0.5)*meshResolution, 0.0), ),
                                ((Length/4., -(i - 0.5)*meshResolution, 0.0), ),
                                ((Length/4., -(i - 0.5)*meshResolution, 0.0), )
                                ((-meshResolution/2., (i - 0.5)*meshResolution, 0.0), )
allRegions = topRegions + bottomRegions + middleRegions

a.setMeshControls(regions=allRegions, elemShape=QUAD, technique=STRUCTURED)

elemType1 = mesh.ElemType(elemCode=ElementType, elemLibrary=STANDARD,
  secondOrderAccuracy=OFF, hourglassControl=DEFAULT, distortionControl=DEFAULT)

a.setElementType(regions=(allRegions,), elemTypes=(elemType1,))

# SEEDING

a = mdb.models['Global_Model'].rootAssembly
el1 = a.instances['Global_Instance'].edges

# Seed the horizontal lines biased towards the crack tip
pickedEdges1 = el1.findAt((((0.5*0.15*Height, topHeight, 0.0),), ((0.5*0.15*Height,
  0.0, 0.0),), ((-0.5*0.15*Height, -bottomHeight, 0.0),))
pickedEdges2 = el1.findAt(((0.5*0.15*Height, topHeight, 0.0),), ((-0.5*0.15*Height, 0.0, 0.0),), ((0.5*0.15*Height, 0.0, 0.0),), ((-0.5*0.15*Height, -bottomHeight, 0.0),))

if meshLayerHeight != topHeight/2.:
pickedEdges1 = pickedEdges1 + el1.findAt(((0.5*0.15*Height, topHeight/2., 0.0),),
  ((0.5*0.15*Height, -bottomHeight/2., 0.0),))
pickedEdges2 = pickedEdges2 + el1.findAt(((0.5*0.15*Height, -bottomHeight/2., 0.0)
  ,), ((-0.5*0.15*Height, topHeight/2., 0.0),))
for i in range(1, meshLayers+1):
pickedEdges1 = pickedEdges1 + el1.findAt(((0.5*0.15*Height, (i)*meshResolution,
  0.0),), ((0.5*0.15*Height, -(i)*meshResolution, 0.0),))
pickedEdges2 = pickedEdges2 + e1.findAt(((−0.5×0.15×Height, (i)×meshResolution, 0.0), ),((−0.5×0.15×Height, (−i)×meshResolution, 0.0), ))

# a. seedEdgeByBias(end1Edges=pickedEdges1, end2Edges=pickedEdges2, minSize=meshResolution/m, maxSize=1000.×meshResolution/m, constraint=FIXED)# ratio=RATIO, number=100)

# a. seedEdgeByBias(end1Edges=pickedEdges1, end2Edges=pickedEdges2, ratio=50, number=100, constraint=FIXED)# 10, 100 for .25 and up

# Seed the outer horizontal lines biased towards the crack tip
pickedEdges1 = el.findAt(((Length/4., topHeight, 0.0), ),((Length/4., 0.0, 0.0), )
,((−Length/4., −bottomHeight, 0.0), )
pickedEdges2 = el.findAt(((−Length/4., topHeight, 0.0), ),((−Length/4., 0.0, 0.0), )
,((Length/4., −bottomHeight, 0.0), ))

if meshLayerHeight != topHeight/2.:  
pickedEdges1 = pickedEdges1 + e1.findAt(((Length/4., topHeight/2., 0.0), ),((
Length/4., −bottomHeight/2., 0.0), ))
pickedEdges2 = pickedEdges2 + e1.findAt(((−Length/4., −bottomHeight/2., 0.0), )
,((−Length/4., topHeight/2., 0.0), ))

for i in range(1, meshLayers+1):
  pickedEdges1 = pickedEdges1 + e1.findAt(((Length/4., (i)×meshResolution, 0.0), )
,((Length/4., (−i)×meshResolution, 0.0), ))
pickedEdges2 = pickedEdges2 + e1.findAt(((−Length/4., (i)×meshResolution, 0.0), )
,((−Length/4., (−i)×meshResolution, 0.0), ))
# a. seedEdgeByBias(end1Edges=pickedEdges1, end2Edges=pickedEdges2, minSize=1000.×meshResolution/m, maxSize=10000.×meshResolution/m, constraint=FINER)# ratio=RATIO, number=100) # 100

pickedEdges = pickedEdges1 + pickedEdges2

# a. seedEdgeByNumber(edges=pickedEdges, number=int((Length/2.−0.25×Height)/(10.×meshResolution/m)), constraint=FIXED)
a.seedEdgeByNumber(edges=pickedEdges, number=25, constraint=FIXED)# 25

# Seed the vertical lines along the perimeter (biased if meshLayerHeight=topHeight/2 else fixed)
pickedEdges1 = el.findAt(((−Length/2., .75×topHeight, 0.0), ),
((Length/2., −.75×bottomHeight, 0.0), ),
((0.0, −.75×bottomHeight, 0.0), ),
((-meshResolution, .75×topHeight, 0.0), ),
((meshResolution, .75×topHeight, 0.0), ),
((−0.15×Height, −.75×bottomHeight, 0.0), ),
((0.15×Height, −.75×bottomHeight, 0.0), ),

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pickedEdges2 = e1.findAt(((−Length/2. , −0.75∗bottomHeight , 0.0) , ) ,
((Length/2. , .75∗topHeight , 0.0) , ) ,
((0.0 , .75∗topHeight , 0.0) , ) ,
((−meshResolution , −0.75∗bottomHeight , 0.0) , ) ,
((meshResolution , −0.75∗bottomHeight , 0.0) , ))
pickedEdges = pickedEdges1 + pickedEdges2
a.seedEdgeByNumber(edges=pickedEdges , number=4, constraint=FIXED)

#ratio=50, number=10) #10

#Seed the lines around the tiny crack tip box and all lines coming from that region
pickedEdges = e1.findAt(((littleBoxSize ,0.5∗littleBoxSize , 0.0) , ) ,
((littleBoxSize ,−0.5∗littleBoxSize , 0.0) , ) ,
(−littleBoxSize ,0.5∗littleBoxSize , 0.0) , ) ,
(−littleBoxSize ,−0.5∗littleBoxSize , 0.0) , ) ,
(0.5∗littleBoxSize ,littleBoxSize , 0.0) , ) ,
(−0.5∗littleBoxSize ,littleBoxSize , 0.0) , ) ,
(0.5∗littleBoxSize ,−littleBoxSize , 0.0) , ) ,
(−0.5∗littleBoxSize ,−littleBoxSize , 0.0) , ) )
for i in range(1,meshLayers+1):
pickedEdges = pickedEdges + e1.findAt(((meshResolution , (i−0.5)∗meshResolution , 0.0) , ) ,
(−meshResolution , (i−0.5)∗meshResolution , 0.0) , ) ,
(meshResolution , −(i−0.5)∗meshResolution , 0.0) , ) ,
(−meshResolution , −(i−0.5)∗meshResolution , 0.0) , ) ,
(0.5∗meshResolution , i∗meshResolution ,0.0) , ),
(−0.5∗meshResolution , i∗meshResolution ,0.0) , ),
(0.5∗meshResolution , −i∗meshResolution ,0.0) , ),
(−0.5∗meshResolution , −i∗meshResolution ,0.0) , ),
(−meshResolution , (i−0.5)∗meshResolution , 0.0) , ) ,
((0.0 , (i−0.5)∗meshResolution , 0.0) , ),
((0.0 , −(i−0.5)∗meshResolution ,0.0) , )
a.seedEdgeByNumber(edges=pickedEdges , number=2, constraint=FIXED)
# Seed the spiderweb

pickedEdges1 = e1.findAt(((meshResolution/2., 0.0, 0.0), ),
((0.0, -meshResolution/2., 0.0), ))
pickedEdges2 = e1.findAt(((meshResolution/2., -meshResolution/2., 0.0), ),
((meshResolution/2., meshResolution/2., 0.0), ),
((-meshResolution/2., meshResolution/2., 0.0), ),
((0.0, meshResolution/2., 0.0), ),
((-meshResolution/2., 0.0, 0.0), ),
((meshResolution/2., -meshResolution/2., 0.0), ))
a.seedEdgeByBias(end1Edges=pickedEdges1, end2Edges=pickedEdges2, ratio=100, number=16, constraint=FIXED)

if meshLayerHeight != topHeight/2.:
pickedEdges1 = e1.findAt(((Length/2., meshLayerHeight+meshResolution, 0.0), ),
((-meshResolution, meshLayerHeight+meshResolution, 0.0), ),
((0.0, -meshLayerHeight*1.01, 0.0), ),
((meshResolution, meshLayerHeight+meshResolution, 0.0), ),
((Length/2., -(meshLayerHeight+meshResolution), 0.0), ),
((-0.15*Height, -(meshLayerHeight+meshResolution), 0.0), ),
((0.15*Height, -(meshLayerHeight+meshResolution), 0.0), ))
pickedEdges2 = e1.findAt(((Length/2., -(meshLayerHeight+meshResolution), 0.0), ),
((-meshResolution, -(meshLayerHeight+meshResolution), 0.0), ),
((0.0, (meshLayerHeight+meshResolution), 0.0), ),
((meshResolution, -(meshLayerHeight+meshResolution), 0.0), ),
((Length/2., meshLayerHeight+meshResolution, 0.0), ),
((-0.15*Height, (meshLayerHeight+meshResolution), 0.0), ),
((0.15*Height, (meshLayerHeight+meshResolution), 0.0), ))

pickedEdges1 = e1.findAt(((meshResolution, meshLayerHeight+meshResolution, 0.0), ),
((-meshResolution, meshLayerHeight+meshResolution, 0.0), ))
pickedEdges2 = e1.findAt(((0.0, .49*topHeight, 0.0), ))
a.seedEdgeByBias(end1Edges=pickedEdges1, minSize=1.*meshResolution/m, maxSize=100.*meshResolution/m)#1 and 100

# a.seedEdgeByBias(end2Edges=pickedEdges1, minSize=meshResolution/m, maxSize=99.666666*meshResolution/m)
a.seedEdgeByBias(end2Edges=pickedEdges2, minSize=1.*meshResolution/m, maxSize=100.*meshResolution/m)#1 and 100

# MESH

partInstances=(a.instances['Global Instance'],)
a.generateMesh(regions=partInstances)

# ASSIGN CRACK DEFINITION TO SEAM FOR HISTORY OUTPUTS

v1 = a.instances['Global Instance'].vertices
verts1 = v1.findAt(((0.0, 0.0, 0.0),))
crackFront = regionToolset.Region(verts=verts1)
crackTip = regionToolset.Region(verts=verts1)
v11 = a.instances['Global Instance'].vertices
a.engineeringFeatures.ContourIntegral(name='Crack−1', symmetric=OFF, crackFront=crackFront, crackTip=crackTip, extensionDirectionMethod=Q_VECTORS, qVectors=((v1.findAt(coordinates=(0.0, 0.0, 0.0)), v11.findAt(coordinates=(Length/2., 0.0, 0.0))),), midNodePosition=0.5, collapsedElementAtTip=NONE)

# CREATE STEPS

mdb.models['Global Model'].StaticStep(name='Load', previous='Initial', description='Load', maxNumInc=100000, initialInc=0.01, minInc=1e−14, maxInc=0.5)
if modelType == 'plastic':
    mdb.models['Global Model'].StaticStep(name='Unload', previous='Load', description='Unload', maxNumInc=100000, initialInc=0.01, minInc=1e−14, maxInc=0.5)
    mdb.models['Global Model'].StaticStep(name='Reload', previous='Unload', description='Reload', maxNumInc=100000, initialInc=0.01, minInc=1e−14, maxInc=0.5)
    mdb.models['Global Model'].StaticStep(name='ReUnload', previous='Reload', description='ReUnload', maxNumInc=100000, initialInc=0.01, minInc=1e−14, maxInc=0.5)

# CREATE BOUNDARY CONDITIONS

------------------------------
# Symmetry along right side

```python
draw1 = e1.findAt(((Length/2., -0.75*bottomHeight, 0.0), ((Length/2., 0.75*topHeight, 0.0)),)
if meshLayerHeight != topHeight/2.:
    draw1 = draw1 + e1.findAt(((Length/2., -(meshLayerHeight+meshResolution), 0.0),
        ((Length/2., meshLayerHeight+meshResolution, 0.0)),)
for i in range(1, meshLayers+1):
    draw1 = draw1 + e1.findAt(((Length/2., (i-0.5)*meshResolution, 0.0), ((Length
        /2., -(i-0.5)*meshResolution, 0.0)),))
region = regionToolset.Region(edges=draw1)
```

# Pin the bottom right corner

```python
v1 = a.instances['Global Instance'].vertices
verts1 = v1.findAt(((Length/2., -bottomHeight, 0.0), ))
region = regionToolset.Region(vertices=verts1)
```

```python
drawModels['Global'].DisplacementBC(name='BC-2', createStepName='Initial',
    region=region, u1=UNSET, u2=SET, ur3=UNSET, amplitude=UNSET, distributionType=
    UNIFORM, fieldName='', localCsys=None)
```

# APPLY LOADS

```python
pbottom = abs(4.*M2/(bottomHeight)**2.)
ptop = abs(4.*M1/(topHeight)**2.)
```

#TopHeight Loads

```python
side1Edges1 = e1.findAt(((Length/2., 0.75*topHeight, 0.0), ))
side1Edges2 = e1.findAt(((Length/2., 0.5*meshResolution, 0.0), ))
if meshLayerHeight != topHeight/2.:
    side1Edges2 = side1Edges2 + e1.findAt(((Length/2., meshLayerHeight+meshResolution
        , 0.0), ))
for i in range(2, meshLayers+1):
    side1Edges2 = side1Edges2 + e1.findAt(((Length/2., (i-0.5)*meshResolution, 0.0),
        ))
region1 = regionToolset.Region(side1Edges=side1Edges1)
region2 = regionToolset.Region(side1Edges=side1Edges2)
```

```python
if Mi > 0.0:
```
mdb.models['GlobalModel'].Pressure(name='Load−1', createStepName='Load', region=region1, distributionType=UNIFORM, field='', magnitude=ptop, amplitude=UNSET)
mdb.models['GlobalModel'].Pressure(name='Load−2', createStepName='Load', region=region2, distributionType=UNIFORM, field='', magnitude=−ptop, amplitude=UNSET)

else:
    mdb.models['GlobalModel'].Pressure(name='Load−1', createStepName='Load', region=region1, distributionType=UNIFORM, field='', magnitude=−ptop, amplitude=UNSET)
    mdb.models['GlobalModel'].Pressure(name='Load−2', createStepName='Load', region=region2, distributionType=UNIFORM, field='', magnitude=ptop, amplitude=UNSET)

#Bottom height loads
side1Edges1 = el.findAt(((−Length/2., −0.75*bottomHeight, 0.0),))
side1Edges2 = el.findAt(((−Length/2., −0.5*meshResolution, 0.0),))
if meshLayerHeight != topHeight/2.:
    side1Edges2 = side1Edges2 + el.findAt(((−Length/2., −(meshLayerHeight+meshResolution), 0.0),))
for i in range(2,meshLayers+1):
    side1Edges2 = side1Edges2 + el.findAt(((−Length/2., −(i−0.5)*meshResolution, 0.0),))
region1 = regionToolset.Region(side1Edges=side1Edges1)
region2 = regionToolset.Region(side1Edges=side1Edges2)
if M2 > 0.0:
    mdb.models['GlobalModel'].Pressure(name='Load−3', createStepName='Load', region=region1, distributionType=UNIFORM, field='', magnitude=pbottom, amplitude=UNSET)
    mdb.models['GlobalModel'].Pressure(name='Load−4', createStepName='Load', region=region2, distributionType=UNIFORM, field='', magnitude=pbottom, amplitude=UNSET)
else:
    mdb.models['GlobalModel'].Pressure(name='Load−3', createStepName='Load', region=region1, distributionType=UNIFORM, field='', magnitude=−pbottom, amplitude=UNSET)
    mdb.models['GlobalModel'].Pressure(name='Load−4', createStepName='Load', region=region2, distributionType=UNIFORM, field='', magnitude=−pbottom, amplitude=UNSET)

# Only add in all the extra load steps if the run is plastic
if modelType == 'plastic':
    #Deactivate them all during the unload step
    mdb.models['GlobalModel'].loads['Load−1'].deactivate('Unload')
    mdb.models['GlobalModel'].loads['Load−2'].deactivate('Unload')

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mdb.models['GlobalModel'].loads['Load−3'].deactivate('Unload')
mdb.models['GlobalModel'].loads['Load−4'].deactivate('Unload')

#Topheight Loads

a = mdb.models['GlobalModel'].rootAssembly
s = a.instances['GlobalInstance'].edges
side1Edges1 = s.findAt(((−Length/2., 0.75*topHeight, 0.0),))
side1Edges2 = s.findAt(((−Length/2., 0.5*meshResolution, 0.0),))
if meshLayerHeight != topHeight /2.:
    side1Edges2 = side1Edges2 + s.findAt(((−Length/2., meshLayerHeight+meshResolution, 0.0),))
for i in range(2,meshLayers+1):
    side1Edges2 = side1Edges2 + s.findAt(((−Length/2., (i−0.5)*meshResolution, 0.0),))
region1 = regionToolset.Region(side1Edges=side1Edges1)
region2 = regionToolset.Region(side1Edges=side1Edges2)
if M1 > 0.0:
    mdb.models['GlobalModel'].Pressure(name='Load−5', createStepName='Reload',
        region=region1, distributionType=UNIFORM, field='', magnitude=ptop,
        amplitude=UNSET)
    mdb.models['GlobalModel'].Pressure(name='Load−6', createStepName='Reload',
        region=region2, distributionType=UNIFORM, field='', magnitude=−ptop,
        amplitude=UNSET)
else:
    mdb.models['GlobalModel'].Pressure(name='Load−5', createStepName='Reload',
        region=region1, distributionType=UNIFORM, field='', magnitude=−ptop,
        amplitude=UNSET)
    mdb.models['GlobalModel'].Pressure(name='Load−6', createStepName='Reload',
        region=region2, distributionType=UNIFORM, field='', magnitude=ptop,
        amplitude=UNSET)

#Bottomheight loads

a = mdb.models['GlobalModel'].rootAssembly
s = a.instances['GlobalInstance'].edges
side1Edges1 = s.findAt(((−Length/2., −0.75*bottomHeight, 0.0),))
side1Edges2 = s.findAt(((−Length/2., −0.5*meshResolution, 0.0),))
if meshLayerHeight != topHeight /2.:
    side1Edges2 = side1Edges2 + s.findAt(((−Length/2., −(meshLayerHeight+meshResolution), 0.0),))
for i in range(2,meshLayers+1):
sidelEdges2 = sidelEdges2 + s.findAt(((−Length/2., −(i−0.5)*meshResolution, 0.0), ))

region1 = regionToolset.Region(sidelEdges=sidelEdges1)
region2 = regionToolset.Region(sidelEdges=sidelEdges2)

if M2 > 0.0:
    mdb.models['GlobalModel'].Pressure(name='Load−7', createStepName='Reload', region=region1, distributionType=UNIFORM, field='', magnitude=pbottom, amplitude=UNSET)
    mdb.models['GlobalModel'].Pressure(name='Load−8', createStepName='Reload', region=region2, distributionType=UNIFORM, field='', magnitude=−pbottom, amplitude=UNSET)
else:
    mdb.models['GlobalModel'].Pressure(name='Load−7', createStepName='Reload', region=region1, distributionType=UNIFORM, field='', magnitude=−pbottom, amplitude=UNSET)
    mdb.models['GlobalModel'].Pressure(name='Load−8', createStepName='Reload', region=region2, distributionType=UNIFORM, field='', magnitude=pbottom, amplitude=UNSET)

mdb.models['GlobalModel'].loads['Load−5'].deactivate('ReUnload')
mdb.models['GlobalModel'].loads['Load−6'].deactivate('ReUnload')
mdb.models['GlobalModel'].loads['Load−7'].deactivate('ReUnload')
mdb.models['GlobalModel'].loads['Load−8'].deactivate('ReUnload')

# CREATE SEAM

edges1 = e1.findAt(((−Length/4., 0.0, 0.0), ),((-0.5*0.15*Height, 0.0, 0.0), ), ((−meshResolution/2., 0.0, 0.0), ), ((−littleBoxSize/2., 0.0, 0.0), ))
pickedRegions = regionToolset.Region(edges=edges1)
mdb.models['GlobalModel'].rootAssembly.engineeringFeatures.assignSeam(regions=pickedRegions)

# Request only last increment info

mdb.models['GlobalModel'].fieldOutputRequests['F−Output−1'].setValues(frequency=LAST_INCREMENT)

# mdb.models['GlobalModel'].HistoryOutputRequest(name='H−Output−2', createStepName='Load', frequency=LAST_INCREMENT, contourIntegral='Crack−1', sectionPoints=DEFAULT, rebar=EXCLUDE,
# numberOfContours=contourNumber)
# mdb.models['Global Model'].HistoryOutputRequest(name='H-Output-3', createStepName='Load', frequency=LAST_INCREMENT, contourIntegral='Crack-1', sectionPoints=DEFAULT, rebar=EXCLUDE,
  # numberOfContours=contourNumber, contourType=K_FACTORS, kFactorDirection=MERR)

# mdb.models['Global Model'].HistoryOutputRequest(name='H-Output-4', createStepName='Load', frequency=LAST_INCREMENT, contourIntegral='Crack-1', sectionPoints=DEFAULT, rebar=EXCLUDE,
  # numberOfContours=contourNumber, contourType=K_FACTORS)

# Added for full model plasticity run
mdb.models['Global Model'].HistoryOutputRequest(name='H-Output-1', createStepName='Load', variables=('ALLPD', ), frequency=LAST_INCREMENT)

# Turn on LINE SEARCH for plasticity

mdb.models['Global Model'].steps['Load'].control.setValue(allowPropagation=OFF, resetDefaultValues=OFF, lineSearch=(10.0, 1.0, 0.0001, 0.25, 0.01))

mdb.models['Global Model'].steps['Unload'].control.setValue(allowPropagation=OFF, resetDefaultValues=OFF, lineSearch=(10.0, 1.0, 0.0001, 0.25, 0.01))

mdb.models['Global Model'].steps['Reload'].control.setValue(allowPropagation=OFF, resetDefaultValues=OFF, lineSearch=(10.0, 1.0, 0.0001, 0.25, 0.01))

mdb.models['Global Model'].steps['ReUnload'].control.setValue(allowPropagation=OFF, resetDefaultValues=OFF, lineSearch=(10.0, 1.0, 0.0001, 0.25, 0.01))

# CREATE JOB

globalJob = mdb.Jobs(name=jobname, model='Global Model', description='', type=ANALYSIS, atTime=None, waitMinutes=0, waitHours=0, queue=None, memory=95, memoryUnits=PERCENTAGE, getMemoryFromAnalysis=True, explicitPrecision=SINGLE, nodalOutputPrecision=FULL, echoPrint=OFF, modelPrint=OFF, contactPrint=OFF, historyPrint=OFF, userSubroutine='', scratch='', parallelizationMethodExplicit=DOMAIN, multiprocessingMode=DEFAULT, numDomains=1, numCpus=1)
# Create the SUBMODEL if plasticity is requested
#
# if modelType == 'plastic':

# plasticZoneViewSize = 0.105  #mm  #Half of what you actually want. Coded weird ...
#(0.03 for mode 0)
# plasticZoneViewSize = 6.4156e−6*abs(mode)+0.0012004*abs(mode)+0.03
# print plasticZoneViewSize
# layersToKeep = meshLayers
# layersToKeep = int(plasticZoneViewSize / (meshResolution)) + 1

# if layersToKeep >= meshLayers:
# layersToKeep = meshLayers

# # Copy the global model and call it Sub Model
#
# /////////////////////////////////////////////////////////////////////////////////
# mdb.Model(name='Sub Model', objectToCopy=mdb.models['Global Model'])
# # Delete the Instance in the Sub Model to prevent warning/error when I rename the part
# a = mdb.models['Sub Model'].rootAssembly
# a.recalculate()
# del a.features['Global Instance ']
# # Rename the part "Sub Part"
# mdb.models['Sub Model'].parts.changeKey(fromName='Global Part ', toName='Sub Part ')
# # Re Instance the newly named part
# a1 = mdb.models['Sub Model'].rootAssembly
# p = mdb.models['Sub Model'].parts['Sub Part ']
# a1.Instance(name='Sub Instance ', part=p, dependent=OFF)

# ##Add in the plasticity properties
#
# ///////////////////////////////////////////////////////////////////////////////////

# mdb.models['Sub Model'].materials['Material − 1'].Plastic(hardening=KINEMATIC,
table = ((sigma_1, 0.0), (sigma_1+sigma_1*0.0002, 300.0)))
# mdb.models['Sub Model'].materials['Material − 2'].Plastic(hardening=KINEMATIC,
table = ((sigma_2, 0.0), (sigma_2+sigma_2*0.0002, 300.0)))
# for i in range(1,propertyLayers+1):#propertyLayers+1):
Plastic

# mdb.models["Sub Model"].materials["Middle Layer"+"i"].Plastic(hardening=
    KINEMATIC, table=((sigma_2 + (sigma_1-sigma_2)/(propertyLayers+1)*i, 0.0),

# #Cut out the view box

# s1 = mdb.models['Sub Model'].ConstrainedSketch(name='_profile_', sheetSize=
    plasticZoneViewSize, gridSpacing=plasticZoneViewSize/10.)
# g, v, d, c = s1.geometry, s1.vertices, s1.dimensions, s1.constraints
# s1.sketchOptions.setValue(decimalPlaces=6)
# s1.setPrimaryObject(option=SUPERIMPOSE)
# p = mdb.models['Sub Model'].parts['Sub Part']
# s1.rectangle(point1=(-plasticZoneViewSize, -plasticZoneViewSize), point2=(
    plasticZoneViewSize, plasticZoneViewSize))
# s1.rectangle(point1=(-Length/2., -bottomHeight), point2=(Length/2., topHeight))
# p = mdb.models['Sub Model'].parts['Sub Part']
# p. Cut ( sketch=s1 )
# s1. unsetPrimaryObject ()
# del mdb.models['Sub Model'].sketches['__profile__']

# #Delete the loads and BCs from Global Model

# p1 = mdb.models['Sub Model'].parts['Sub Part']
# mdb.models['Sub Model'].setValues(globalJob=jobname)
# a = mdb.models['Sub Model'].rootAssembly
# a.regenerate()
# mdb.models['Sub Model'].loads.delete(('Load−1', 'Load−2', 'Load−3', 'Load−4',
# Load−5', 'Load−6', 'Load−7', 'Load−8'))
# mdb.models['Sub Model'].boundaryConditions.delete(('BC−1', 'BC−2', ))

# #Pick the perimeter and name it as a set for BC application

# a = mdb.models['Sub Model'].rootAssembly
# el = a.instances['Sub Instance'].edges
# edges1 = el.findAt(((−plasticZoneViewSize /2., plasticZoneViewSize , 0.0) , ),
# ((−0.5*meshResolution , plasticZoneViewSize , 0.0) , ), ((0.5*meshResolution ,
# plasticZoneViewSize , 0.0) , ), ((plasticZoneViewSize /2., plasticZoneViewSize ,
# 0.0) , ),
# ((−plasticZoneViewSize /2., −plasticZoneViewSize , 0.0) , ),
# ((−0.5*meshResolution , −plasticZoneViewSize , 0.0) , ),
# ((0.5*meshResolution , −plasticZoneViewSize , 0.0) , ),
# ((plasticZoneViewSize /2., −plasticZoneViewSize , 0.0) , ),
# ((plasticZoneViewSize , 0.5*meshResolution , 0.0) , ),
# ((plasticZoneViewSize , −0.5*meshResolution , 0.0) , ))

# if plasticZoneViewSize > layersToKeep*meshResolution :
#     # loopSize = layersToKeep+2
# else :
#     # loopSize = layersToKeep+1
# for i in range(2,loopSize):
#     # start at 2, not 1, to avoid BC problems ( overlapping nodes )
#     edges1 = edges1 + el.findAt((−plasticZoneViewSize , (i−0.5)*meshResolution ,
# 0.0) , ),
#     # ((plasticZoneViewSize , (i−0.5)*meshResolution ,
# 0.0) , ),
#     # ((plasticZoneViewSize , −(i−0.5)*meshResolution ,
# 0.0) , ))
for i in range(1, loopSize):
    # edges1 = edges1 + el.findAt(((−plasticZoneViewSize, −(i−0.5)∗meshResolution, 0.0), ))
    # a.Set(edges=edges1, name='SubmodelPerimeter')

    # Assign the BC

    # region = a.sets['SubmodelPerimeter']
    # mdb.models['Sub Model'].SubmodelBC(name='BC−1', createStepName='Load', region=region, globalStep='1', globalIncrement=0, timeScale=OFF, dof=(1, 2), globalDrivingRegion='', absoluteExteriorTolerance=0.0, exteriorTolerance=0.05)
    # mdb.models['Sub Model'].boundaryConditions['BC−1'].setValuesInStep(stepName='Unload', fixed=OFF, globalStep='2')
    # mdb.models['Sub Model'].boundaryConditions['BC−1'].setValuesInStep(stepName='Reload', fixed=OFF, globalStep='3')
    # mdb.models['Sub Model'].boundaryConditions['BC−1'].setValuesInStep(stepName='ReUnload', fixed=OFF, globalStep='4')

    # Reassign the seam

    # a = mdb.models['Sub Model'].rootAssembly
    # el = a.instances['Sub Instance'].edges
    # edges1 = el.findAt(((−1.0∗meshResolution, 0.0, 0.0), ), ((−0.99∗meshResolution, 0.0, 0.0), ), ((−0.5∗littleBoxSize, 0.0, 0.0), ))
    # pickedRegions = regionTools.Region(edges=edges1)
    # mdb.models['Sub Model'].rootAssembly.engineeringFeatures.assignSeam(regions=pickedRegions)

    # Delete the history requests from the Global Model

delete mdb.models['Sub Model'].historyOutputRequests['H−Output−1']
delete mdb.models['Sub Model'].historyOutputRequests['H−Output−2']
delete mdb.models['Sub Model'].historyOutputRequests['H−Output−3']
delete mdb.models['Sub Model'].historyOutputRequests['H−Output−4']

    # Set Mesh Controls and Element Type

    # a = a.instances['Sub Instance'].edges
# if fl = a.instances['Sub Instance'].faces

# allRegions = fl.findAt(((−1.01*meshResolution, (0.5)*meshResolution, 0.0), ),
# ((1.01*meshResolution, (0.5)*meshResolution, 0.0), ),
# ((1.01*meshResolution, (0.5)*meshResolution, 0.0), ),
# ((−2./3.*meshResolution, 1./3.*meshResolution, 0.0), ),
# ((−1./3.*meshResolution, 2./3.*meshResolution, 0.0), ),
# ((1./3.*meshResolution, 1./3.*meshResolution, 0.0), ),
# ((−1./3.*meshResolution, −1./3.*meshResolution, 0.0), ),
# ((2./3.*meshResolution, −1./3.*meshResolution, 0.0), ),
# ((1./3.*meshResolution, −2./3.*meshResolution, 0.0), ),
# ((−0.5*littleBoxSize, −0.5*littleBoxSize, 0.0), ),
# ((−0.5*littleBoxSize, 0.5*littleBoxSize, 0.0), ),
# ((0.5*littleBoxSize, −0.5*littleBoxSize, 0.0), ),
# ((0.5*littleBoxSize, 0.5*littleBoxSize, 0.0), ))

# for i in range(2, layersToKeep+1): #meshLayers+1):

# allRegions = allRegions + fl.findAt(((−1.01*meshResolution, (i−0.5)*
# meshResolution, 0.0), ),
# ((1.01*meshResolution, (i−0.5)*
# meshResolution, 0.0), ),
# ((1.01*meshResolution, (i−0.5)*
# meshResolution, 0.0), ),
# ((−0.99*meshResolution, (i−0.5)*
# meshResolution, 0.0), ),
# ((−0.99*meshResolution, (i−0.5)*
# meshResolution, 0.0), ),
# ((0.99*meshResolution, (i−0.5)*
# meshResolution, 0.0), ),
# ((0.99*meshResolution, (i−0.5)*
# meshResolution, 0.0), ),
# ((0.99*meshResolution, (i−0.5)*
# meshResolution, 0.0), ))

# if plasticZoneViewSize > layersToKeep*meshResolution:

# allRegions = allRegions + fl.findAt((−plasticZoneViewSize/2., 0.99*
# plasticZoneViewSize, 0.0), ),
# ((plasticZoneViewSize/2., 0.99*
# plasticZoneViewSize, 0.0), ),
# Perform Bounding Zone

# 

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# a.setMeshControls (regions=allRegions, elemShape=QUAD, technique=STRUCTURED)

type1 = mesh.ElemType (elemCode=ElementType, elemLibrary=STANDARD,
    secondOrderAccuracy=OFF, hourglassControl=DEFAULT, distortionControl=DEFAULT)

a.setElementType (regions=(allRegions, ), elemTypes=(type1, ))

# Seeding

------------------------------

a = mdb.models [ 'Sub Model' ] . rootAssembly

partInstances = (ainstances [ 'Sub Instance' ], )

# a.seedInstance (regions=partInstances, size=2.7778e−8+abs (mode)+6.3888889 e−6+
    abs (mode)+.0002, deviationFactor=0.1)

# a.seedInstance (regions=partInstances, size=0.0008, deviationFactor=0.1)

# MESH

# a = mdb.models [ 'Sub Model' ] . rootAssembly

# partInstances = (a.instances [ 'Sub Instance' ], )

# a.generateMesh (regions=partInstances)

# Adjust Steps

------------------------------

# mdb.models [ 'Sub Model' ] . steps[ 'Load' ] . setValues (maxNumInc=60000, initialInc
    =0.01, minInc=1e−08, maxInc=0.5)
# mdba.models["Sub Model "].steps["Unload "].setValues(maxNumInc=60000, initialInc=0.01, minInc=1e-08, maxInc=0.5)
# mdba.models["Sub Model "].steps["Reload "].setValues(maxNumInc=60000, initialInc=0.01, minInc=1e-08, maxInc=0.5)
# mdba.models["Sub Model "].steps["ReUnload "].setValues(maxNumInc=60000, initialInc=0.01, minInc=1e-08, maxInc=0.5)

# #Redefine crack tip for J integrals during plastic analysis

# # a = mdba.models["Sub Model "].rootAssembly
# v1 = a.instances["Sub Instance "].vertices
# verts1 = v1.findAt(((0.0, 0.0, 0.0),))
# crackFront = regionToolset.Region(vertices=verts1)
# a = mdba.models["Sub Model "].rootAssembly
# v1 = a.instances["Sub Instance "].vertices
# verts1 = v1.findAt(((0.0, 0.0, 0.0),))
# crackTip = regionToolset.Region(vertices=verts1)
# a.engineeringFeatures.cracks["Crack -1"].setValues(crackFront=crackFront, crackTip=crackTip, extensionDirectionMethod=Q_VECTORS, qVectors=((0.0, 0.0, 0.0), (25.0, 0.0, 0.0)),)

# #Request output at every step and get Energy
# mdba.models["Sub Model "].fieldOutputRequests["F-Output -1"].setValues(variables=('S', 'PE', 'PEEQ', 'PEMAG', 'LE', 'U', 'RF', 'CF', 'CSTRESS', 'CDISP', 'ENER', 'ELEN', 'ELEDEN'), frequency=LAST_INCREMENT)

# #Ask for history output

# mdba.models["Sub Model "].HistoryOutputRequest(name='H-Output -1', createStepName='Load', variables=('ALLPD', ), frequency=LAST_INCREMENT)
# mdba.models["Sub Model "].HistoryOutputRequest(name='H-Output -2', createStepName='Load', frequency=LAST_INCREMENT,
# contourIntegral='Crack -1', sectionPoints=DEFAULT, rebar=EXCLUDE, numberOfContours=176)

# #Turn on LINE SEARCH for plasticity

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# Create JOB

"""
Create JOB
"""

# subJob = mdb.Job(name='Sub_' + jobname, model='Sub Model', description='', type=ANALYSIS,
               # atTime=None, waitMinutes=0, waitHours=0, queue=None, memory=95,
               # memoryUnits=PERCENTAGE, getMemoryFromAnalysis=True,
               # explicitPrecision=SINGLE, nodalOutputPrecision=FULL, echoPrint=OFF,
               # modelPrint=OFF, contactPrint=OFF, historyPrint=OFF, userSubroutine='',
               # scratch='', parallelizationMethodExplicit=DOMAIN,
               # multiprocessingMode=DEFAULT, numDomains=1, numCpus=1)

# Submit JOBS

if job == 'submit':
    globalJob.submit()
    globalJob.waitForCompletion()

    # if modelType == 'plastic':
    #    subJob.submit()
    #    subJob.waitForCompletion()

# Post processing consists of extracting relevant data. This includes:
# If its just an elastic sweep to find the modes, I need rpt files that contain
# the Ks/Js. A Matlab script then finds the modes.
# Once I know the modes, I want to run the plasticity and I only want the rpts
# with ALLPD. A Matlab script then finds \( \frac{dW}{dN} \).

if modelType == 'elastic':
    # Open ODB
odb = session.openOdb(name=workingDirectory+'/'+zzz+'/'+jobname+'.odb')

# Extract J integrals using 16 contours around the crack tip. Convergence is seen within 3 or so.

i=0

labelList = ['Stress intensity factor, K1 : K1 at H-OUTPUT-3 CRACK-1
__PICKEDSET17 Contour', 'Stress intensity factor, K1 : K1 at H-OUTPUT-4 CRACK-1
__PICKEDSET17 Contour',
'Stress intensity factor, K2 : K2 at H-OUTPUT-3 CRACK-1
__PICKEDSET17 Contour', 'Stress intensity factor, K2 : K2 at H-OUTPUT-4 CRACK-1
__PICKEDSET17 Contour',
'J-integral estimated from Ks, JKS at H-OUTPUT-3 CRACK-1
__PICKEDSET17 Contour', 'J-integral estimated from Ks, JKS at H-OUTPUT-4 CRACK-1
__PICKEDSET17 Contour',
]

for label in labelList:

IDList = ['01', '02', '03', '04', '05', '06', '07', '09', '10', '11',
'12', '13', '14', '15', '16']

for ID in IDList:

session.XYDataFromHistory(name='XYData-'+ID, odb=odb,
outputVariableName=label+ID, steps=('Load', ))

xy1 = session.xyDataObjects['XYData-01']
xy2 = session.xyDataObjects['XYData-02']
xy3 = session.xyDataObjects['XYData-03']
xy4 = session.xyDataObjects['XYData-04']
xy5 = session.xyDataObjects['XYData-05']
xy6 = session.xyDataObjects['XYData-06']
xy7 = session.xyDataObjects['XYData-07']
xy8 = session.xyDataObjects['XYData-08']
xy9 = session.xyDataObjects['XYData-09']
xy10 = session.xyDataObjects['XYData-10']
xy11 = session.xyDataObjects['XYData-11']
xy12 = session.xyDataObjects['XYData-12']
xy13 = session.xyDataObjects['XYData-13']
xy14 = session.xyDataObjects['XYData-14']
xy15 = session.xyDataObjects['XYData-15']
xy16 = session.xyDataObjects['XYData-16']

xy33 = append((xy1, xy2, xy3, xy4, xy5, xy6, xy7, xy8, xy9, xy10, xy11,
xy12, xy13, xy14, xy15, xy16))

if i == 0:

name = 'K1_MERR'
elif i == 1:
    name = 'K1_Normal'
elif i == 2:
    name = 'K2_MERR'
elif i == 3:
    name = 'K2_Normal'
elif i == 4:
    name = 'JfromKs_MERR'
else:
    name = 'JfromKs_Normal'

xy33.setValues(sourceDescription=name)
tmpName = xy33.name
session.xyDataObjects.changeKey(tmpName, name)
i=i+1

for ID in IDList:
    del session.xyDataObjects['XYData-'+ID]

x0 = session.xyDataObjects['JfromKs_MERR']
x1 = session.xyDataObjects['JfromKs_Normal']
x2 = session.xyDataObjects['K1_MERR']
x3 = session.xyDataObjects['K1_Normal']
x4 = session.xyDataObjects['K2_MERR']
x5 = session.xyDataObjects['K2_Normal']

session.xyReportOptions.setValues(numDigits=9)
session.writeXYReport(fileName=jobname+'.rpt', xyData=(x0, x1, x2, x3, x4, x5))

del session.xyDataObjects['JfromKs_MERR']
del session.xyDataObjects['JfromKs_Normal']
del session.xyDataObjects['K1_MERR']
del session.xyDataObjects['K1_Normal']
del session.xyDataObjects['K2_MERR']
del session.xyDataObjects['K2_Normal']

odb.close()

# Remove everything but the dat
fileTypes = ['.com','.inp','.prt','.sta','.sim','.msg','.odb']

for extension in fileTypes:
os.remove(jobname+extension)

# This block opens the Sub Model odb to extract ALLPD and write to an rpt file
if modelType == 'plastic':
    # Open the odb
    odb = session.openOdb(name=workingDirectory+zzz+'/'+jobname+'.odb')
    # Write ALLPD to XY-Data and then export as an rpt file
    session.XYDataFromHistory(name='XYData-1', odb=odb, outputVariableName='Plastic dissipation: ALLPD for Whole Model', steps=('Load', 'Unload', 'Reload', 'ReUnload', ), )
    x0 = session.xyDataObjects['XYData-1']
    session.xyReportOptions.setValues(numDigits=9)
    session.writeXYReport(fileName='ALLPD_'+jobname+'.rpt', xyData=(x0, ))
    odb.close()

    # Remove everything but the Sub model odb and dat
    fileTypes = ['.com', '.inp', '.prt', '.sta', '.sim', '.msg', '.dat']
    for extension in fileTypes:
        os.remove(jobname+extension)
        #os.remove('Sub_'+jobname+extension)
        #os.remove(jobname+'.odb')
main()
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