2008

A Novel Accurate Approximation Method of Lognormal Sum Random Variables

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A Novel Accurate Approximation Method of Lognormal Sum Random Variables

A thesis submitted in partial fulfillment of the requirements for
the degree of Master of Science in Engineering

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November 7, 2008

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ABSTRACT

Xue Li. M.S. Egr Department of Electrical Engineering, Wright State University, 2008. A Novel Accurate Approximation Method of Lognormal Sum Random Variables

Sums of lognormal random variables occur in many problems in wireless communications due to large-scale signal shadowing from multiple transmitters. With the emerging of cognitive radio technology, accurate analysis of interferences from multiple primary users and multiple secondary users is required. Such interferences are well modeled by the lognormal sum distribution. The lognormal sum distribution is known to have no close-form and is difficult to compute numerically. Several approximations to the lognormal sum distribution have been proposed and employed in literature. However, these approximation methods are not without their drawbacks. Some widely used approximation methods are not accurate at the lower region, while some other approximation methods require the CDF (cumulative distribution function) curve from the Monte Carlo Simulation which is very computational demanding. In this master's thesis, we propose a novel approximation method, namely the Log Skew Normal (LSN) approximation, to accurately model the sum of M lognormal distributed random variables. The LSN approximation has good accuracy in the entire PDF (probability density function) region, especially in the lower PDF region. Furthermore, the proposed LSN approximation does not require the CDF curve. The close-form PDF of the resulting LSN random variable (RV) is presented and its parameters derived from those of the M individual lognormal RVs by using the moment matching technique. Simulation results on the CDF of sum of M lognormal random variables in different conditions are used as reference curves to compare various approximation techniques. Simulation results confirm that the proposed LSN approximation provides better accuracy over a wide CDF range with no computational complexity increase.

Contents

List of Tables

1

Introduction

1.1 Cognitive Radio

With an ever increasing demand for higher data rates, coupled with an increase in new applications and the number of users, spectrum crowding and congestion continue to increase. At first glance by looking at the FCC's fully allocated spectrum chart it seems like there is a spectrum scarcity. Recent studies have suggested that spectrum congestion is mainly due to the inefficient use of the spectrum rather than its scarcity. Two terminologies, namely Dynamic Spectrum Access (DSA) and Cognitive Radio (CR), are used within the research community in the context of improving spectrum efficiency by harnessing unused spectrum holes by primary users.

Cognitive radio is a paradigm for wireless communication in which either a network or a wireless node changes its transmission or reception parameters to communicate efficiently avoiding interference with licensed or unlicensed users. This alteration of parameters is based on the active monitoring of several factors in the external and internal radio environment, such as radio frequency spectrum, user behavior and network state.

The concept of cognitive radio was first presented officially in an article by Joseph Mitola III and Gerald Q. Maguire, Jr in 1999 [1]. It was a novel approach in wireless communications that Mitola later described as:

The point in which wireless personal digital assistants (PDAs) and the related networks are sufficiently computationally intelligent about radio resources and related computer-to-computer communications to detect user communications needs as a function of use context, and to provide radio resources and wireless services most appropriate to those needs.

It has been found by regulatory bodies in various countries (including the Federal Communications Commission in the United States, and Ofcom in the United Kingdom) that most of the radio frequency spectrum was inefficiently utilized[2]. For example, cellular network bands are overloaded in most parts of the world, but amateur radio and paging frequencies are not. Independent studies performed in some countries also confirmed this observation[3][4], and concluded that spectrum utilization depends strongly on time and place. Moreover, fixed spectrum allocation prevents rarely used frequencies (those assigned to specific services) from being used by unlicensed users, even when their transmissions would not interfere with the assigned service at all. This was the reason for allowing unlicensed users to utilize licensed bands whenever it would not cause any interference (by avoiding them whenever legitimate user presence is sensed). This paradigm for wireless communication is known as cognitive radio. Cognitive radio has been considered as an ideal goal towards which a software-defined radio platform should evolve: a fully reconfigurable wireless black-box that automatically changes its communication variables in response to network and user demands.

In Cognitive Radio, the interferences from the primary users and from other secondary users are well modeled by the lognormal distribution, since the microwave signal from those primary and secondary users experiences lognormal shadowing of the wireless communication environment. With the existence of numerous secondary users (and primary users) in the same frequency band, the accumulated interference will follow a lognormal sum distribution (the sum of multiple lognormal random variables). Hence, to accurately perform coexistence analysis and predict the performance of cognitive radio in a realistic environment, it is highly desired to generate lognormal sum random variables with high accuracy and easy implementation. However, the lognormal sum random variables are known to have no close-form Cumulative distribution function (CDF), and approximation methods have been proposed to approximate them.

1.2 Motivation

The approximation of lognormal sum random variables have been an long standing research topic in the wireless communication community before the emerging of cognitive radio technology [5] - [10]. Two other examples besides cognitive radio are the analysis of co-channel interference in cellular mobile systems and the computation of outage probabilities. It is difficult to estimate the entire sum of lognormal distributions because no close-form expression can be found. The well-known approach for obtaining the probability distribution of a sum of independent random variables uses the characteristic function (CF). This approach is totally general because the probability density function (PDF) of a sum of independent RV's has a CF equal to the product of the CFs of the summands [18]. However, the CF of a lognormal RV is not known. Moreover, numerical integration is difficult due to the oscillatory integrand and the slow decay rate of the tail of the lognormal density function [7], [19].

Several approximation solutions to the probability distribution of a sum of independent lognormal RV's have been reported, including Wilkinson's [5], Schwartz-Yeh's [5], and Farley's [5] methods, Minimax Approximation[11] [12] , LS approximation by quadratic functions (LSQ) [13] [14], Type IV Pearson Approximation [15] [16], Log Shifted Gamma Approximation (LSG) [17]. Different approaches have their own strengths and weaknesses to describe the sum of lognormal distributions. Depending on the applications, the approach of computation must be chosen wisely to get accurate estimation of certain range of the distribution, especially for independent lognormal distributions. It would be nice if we can have a method that works accurately for the entire distribution and under various conditions.

1.3 Thesis Outline

The rest of the thesis is organized as follows.

Chapter 2 is a description of sum of lognormal random variables and the model of the lognormal sum distribution, and an introduction to an important representation of the lognormal sum distribution, namely the "Lognormal Probability Paper".

Chapter 3 reviews existing approximation methods. We provide description, detail computations, and performance comparison of these methods.

Chapter 4 explains the novel proposed approximation method, namely LSN. We provide the close-form of the LSN approximation, derivation of the parameters calculation algorithm, and performance comparisons with other approximations. Numerical results confirm the high accuracy and performance of this novel approximation method.

Chapter 5 summaries the thesis and provides a few future research topics.

2

Sum of Lognormal Random Variables

This chapter describes the sum of lognormal random variable's and the model of the lognormal sum distribution. We also provide an introduction to an important representation of lognormal and lognormal sum distribution, namely the "Lognormal Probability Paper". "Lognormal Probability Paper" is a kind of scale paper like "Log-Scale", which is used to show the CDF of one Lognormal RV to be a straight line.

2.1 Lognormal Random Variables

In probability and statistics, the log-normal distribution is the single-tailed probability distribution of any random variable whose logarithm is normally distributed. If X is a random variable with a normal distribution, then $Y = exp(X)$ has a log-normal distribution; likewise, if Y is log-normally distributed, then $ln(Y)$ is normally distributed.

A random variable might be modeled as log-normal if it can be considered as the multiplicative product of many small independent factors. For example, the long-term return rate on a stock investment can be considered to be the product of the daily return rates. In wireless communication, the attenuation caused by shadowing or slow fading from random objects is often assumed to be log-normally distributed.

Given X, a Gaussian random variable with mean μ_X and variance σ_X^2 , $L = e^X$ is a lognormal random variable (RV) with probability density function (PDF):

$$f_L(l;\mu_X,\sigma_X) = \begin{cases} \frac{1}{\sigma_X\sqrt{2\pi}l}\exp\{\frac{-1}{2\sigma_X^2}[\ln(l)-\mu_X]^2\} & l>0 \\ 0 & l\leq 0 \end{cases} \tag{2.1}$$

In communications, when X usually represents power variation measured in dB, X_{dB}, $L = e^{\xi X} = 10^{X_{dB}/10}$ with PDF

$$f_L(l;\mu_L,\sigma_L) = \begin{cases} \frac{1}{\xi\sigma_L\sqrt{2\pi}l}\exp\{\frac{-1}{2\sigma_L^2}[10\log(l)-\mu_L]^2\} & l>0 \\ 0 & l\leq 0 \end{cases} \tag{2.2}$$

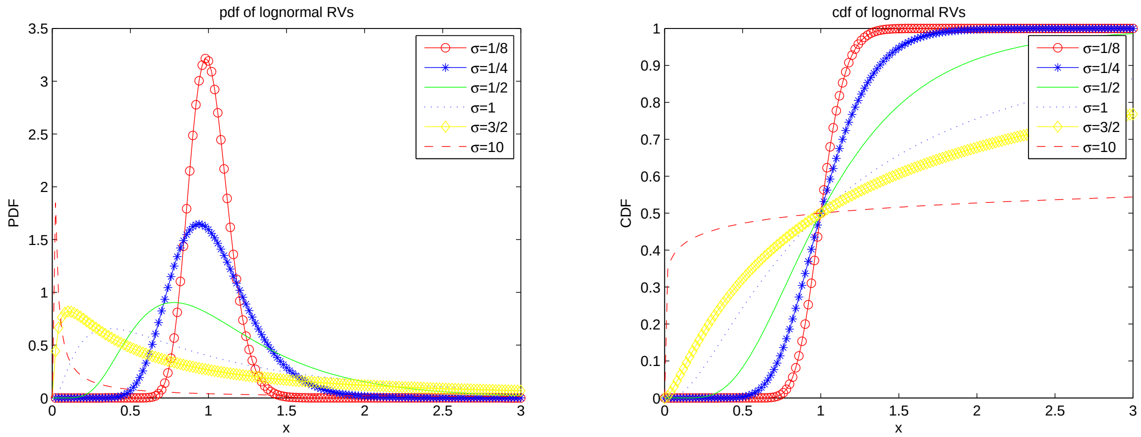

Figure 2.1: (a)PDF of Lognormal RV; (b) CDF of Lognormal RV

Fig. 2.1(a) and (b) show the PDF and CDF of one log-normal RV, respectively.

Assigning μ, σ^2, μ_{dB}, and σ_{dB}^2 to be mean and variance of $X$and X_{dB}, respectively. In wireless communications, σ is often called the decibel spread which typically ranges between 6 and 12 dB. The relationship between them are

$$\zeta = \ln(10)/10, \mu = \mu_{dB} * \zeta, \sigma^2 = \sigma_{dB}^2 * \zeta^2 \tag{2.3}$$

Sum of Lognormal RV's corresponds to the sum of some independent lognormal random variables. Various types of approximations have been suggested to approximate the sum of lognormal distributions. In [17], author mentioned that based on their variances, three types of lognormal distributions are identified: narrow ($\sigma^2 << 1$), moderately broad ($\sigma^2 < 1$), and very broad ($\sigma^2 >> 1$). It is shown that the sum of lognormal distributions can be approximated by Gaussian distribution for narrow case and lognormal distribution for moderately broad case. For very broad case, due to the asymptotic character of lognormal distribution, neither Gaussian nor lognormal approximation is appropriate.

2.2 Lognormal Sum Model

Consider the following model representing the RV as the sum of M lognormal RV's, $L_k s$,

$$\Lambda = \sum_{k=1}^{M} L_k = \sum_{k=1}^{M} e^{X_k} \tag{2.4}$$

where $X_k's$ are Gaussian RVs,Λ is the sum of the M lognormal RV's.

2.3 Lognormal Probability Paper

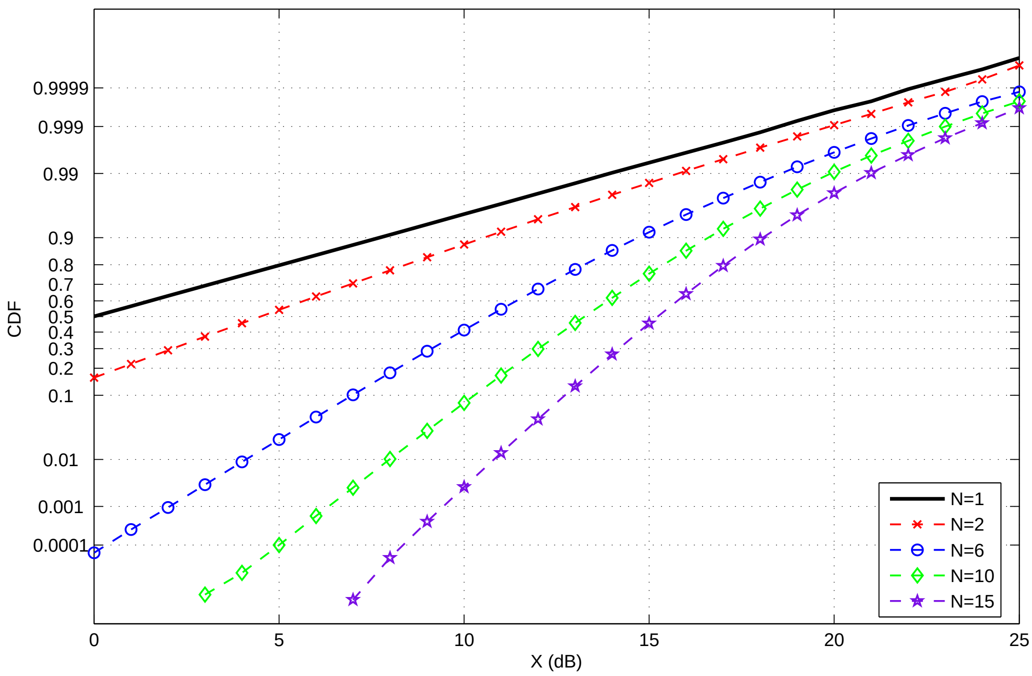

Figure 2.2: CDF of Lognormal RV

Fig. 2.2 shows the CDFs of for Λ values of $M = 1, 2, 6, 10, 15$ with $\sigma_{x_i} = \sigma = 6dB$, (indicated by the symbols) plotted on "lognormal probability paper". The plot for $N = 1$ in Fig. 2.2 is a lognormal CDF and is a straight line. The curves for other values of were computed using a numerical approach.

As a device to readily see how much an approximation deviates from a lognormal distribution, we use "lognormal probability scales" for our graphs. Recall that a Gaussian cdf plots as a straight line on probability paper [20]. Using "probability paper" but transforming the abscissa into $log(abscissa)$

yields graph scales on which a lognormal CDF plots as a straight line. Deviations from "lognormality" then are easily seen and appreciated. An explanation of how to design lognormal probability paper is given next.

On "lognormal probability paper", the CDF $F(x)$ of a distribution is transformed according to

$$g(x) = F_N^{-1}[F(x)] \tag{2.5}$$

where $F_N^{-1}(x)$ is the inverse function of the standard normal CDF $F_N(x)$ having zero mean and unit variance.

$$F_N(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx = \frac{1}{2}[1 + erf(\frac{x}{\sqrt{2}})] \tag{2.6}$$

We could see that if $F(x)$ is the zero mean, unit variance normal CDF, then $g(x) = x$. Also, if F(x) is a normal CDF with mean m and variance σ^2, then $F(x) = F_N((x-m)/\sigma)$, and in this case, $g(x) = (x-m)/\sigma$. Similarly, if $F(x)$ is a lognormal CDF with parameters m and σ, then $F(x) = F_N((\ln(x) - m)/\sigma)$, and in this case

$$g(x) = F_N^{-1}[F(x)] = \frac{\ln(x)}{\sigma} - \frac{m}{\sigma} = \frac{10 * \log_{10}(x)}{\frac{10}{\ln(10)} * \sigma} - \frac{m}{\sigma} = \frac{10 * \log_{10}(x)}{\sigma_{dB}} - \frac{m_{dB}}{\sigma_{dB}} \tag{2.7}$$

in which $g(x)$ is a linear function of $\ln(x)$. We plot the data pairs $(10 * \log_{10}(x), g(x))$ on a two-dimensional coordinate system and label the corresponding probability values on the vertical axis using the one-to-one transformation $F(x) = F_N[g(x)]$. So for the CDF curve on the "Lognormal Probability Paper", the x-axis is the x in dB scale, and the y-axis is the $g(x) = F_N^{-1}[F(x)]$ scale but using the one-to-one transformation $F(x) = F_N[g(x)]$. The advantage of plotting the CDF on the lognormal probability paper is that the CDF of a lognormal distribution turns out to be a straight line. As a direct result, it is easy to observe how far away a lognormal approximation assumption deviates from the true distribution.

3

Literature Review

This chapter reviews existing approximation methods in literature. Several approximate methods to the probability distribution of a sum of independent lognormal RV's have been reported, including Wilkinson's moment matching approach[5], Schwartz-Yeh's recursive approach[5], and Farley's strict lower bound on the complementary method [5] , Minimax Approximation [11] [12] to the CDF when plotted on lognormal probability paper (mentioned in 2.3) , LS approximation by quadratic functions (LSQ) [13] [14], Type IV Pearson Approximation [15] [16], and Log Shifted Gamma Approximation (LSG) [17]. Different approaches have their own strengths and weaknesses to describe the sum of lognormal distributions.

3.1 Wilkinson 's and Schwartz-Yeh 's Method

In this section, we consider two approaches to estimate lognormal sum probabilities: Wilkinson [5], Schwartz and Yeh [5]. These methods are based on the assumption that a sum of independent lognormal RV's can also be modeled as a lognormal RV.

As mentioned before, we have:

$$\Lambda = \sum_{k=1}^{M} L_k = \sum_{k=1}^{M} e^{X_k} = e^Z \tag{3.1}$$

where Z is a Gaussian RV, and these methods use the lognormal RV e^Z to approximate the Lognormal sum Λ.

3.1.1 Wilkinson 's Method

In Wilkinson's approach, the mean m_z, and the standard deviation, σ_z, of Z in (3.1) are obtained by matching the first two moments of L with the first two moments of $\sum_{k=1}^{M} L_k$.

The CDF of Λ can be written as

$$P_r(\Lambda \geq \gamma) = P_r(e^Z \geq \gamma) = F_N(\frac{\ln(\gamma) - m_z}{\sigma_z}) \tag{3.2}$$

where $F_N(x)$ is the CDF of a zero mean, unit variance Gaussian RV. In [5], it is stated that Wilkinson's approach is consistent with an accumulated body of evidence indicating that, for the values of M that are of interest, the distribution of the sum of a finite number of log-normal random variables is well-approximated, at least to first-order, by another lognormal distribution. But this approach is valid only for a limited range of small values of the dB spread σ_x. In particular, it is reported that the Wilkinson approach breaks down for $\sigma_x > 4dB$ which includes the range of most practical interest.

3.1.2 Schwartz-Yeh 's Method

In [5], Schwartz and Yeh examined the problem of computing the distribution of a sum of independent lognormal RV's and their approach is also based on assuming the sum is a lognormal RV.

The first and second moments of the RV Z, however, are not obtained by invoking the same assumption. Rather, exact expressions for the first two moments of the logarithm of the sum of two lognormal RV's were derived. By again assuming that the sum of two lognormal RV's is a lognormal RV, a recursive technique was developed for computing the first two moments of a sum $M > 2$ independent lognormal RVs.

In [6], it is proved that Wilkinson's approach is preferred over Schwartz-Yeh 's approach. Wilkinsons approach is easier to use than Schwartz and Yehs approach, especially for $M > 2$ cases, since to use the latter method one has to recursively use relatively complex expressions to compute the first two moments of $\sum_{k=1}^{M} L_k$, and for values of the CDF less than 10^{-1}, Wilkinsons approach gives more accurate results.

3.2 Farley 's Method

This approximation was mentioned in [5] and attributed to Farley. However, neither a physical intuitive basis nor a derivation of this result was given there. Reference [6] provides a basis for the derivation of this method.

Let $X_1, ..., X_M$ be M independent and identically distributed Gaussian RVs each with mean m_x and standard deviation σ_x. Consider the same model mentioned in Eq.(3.1),the Farley's Approximation is that:

$$Pr(\Lambda > r) = 1 - [1 - Q(\frac{\ln(r) - m_x}{\sigma_x})]^M \tag{3.3}$$

According to [5], this approach is valid for large variances, and this approximation is in fact a strict lower bound on the CDF for any variance [6].

3.3 Minimax Approximation

3.3.1 Method Description

As mentioned in 2.3 , we plot the data pairs $(10 * \log_{10}(x), g(x))$ on a two-dimensional coordinate system and label the corresponding probability values on the vertical axis using the one-to-one transformation $F(x) = F_N[g(x)]$. In “Lognormal Probability Paper”, the CDF of one lognormal RV is a straight line. Therefore, when considering a lognormal approximation to a lognormal sum distribution on the “Lognormal Probability Paper”, we need to find a linear function (i.e., a straight line) to best fit the CDF curve of the sum. The Minimax Approximation is the best in the sense that it minimizes the maximum absolute distance between the approximate function and the true function over a specified interval.

Consider Eq. (2.7), let $t = 10 * \log_{10}(x)$, and $G(t) = g(10^{(t/10)})$,we have

$$G(t) = g(10^{(t/10)}) = \frac{t}{\sigma_{dB}} - \frac{m_{dB}}{\sigma_{dB}} \tag{3.4}$$

A linear function on the lognormal probability paper $p(t) = c_0 + c_1 t$ is to be found with constraints c_0 and c_1 , determined according to :

$$\min_{c_0, c_1} \max_{t \in [a,b]} |G(t) - p(t)| \tag{3.5}$$

where $F(e^a) = 10^{-6}, F(e^b) = 1 - 10^{-6}$.

3.3.2 Computing Parameters

From the CDF curves in Fig. 2.3, in [12], it is implied that the CDF of a lognormal sum distribution is a concave function with $G''(t) < 0$ on lognormal probability paper. Therefore, $G'(t)$is a monotonically decreasing function. So , $E'(t) = G'(t) - p'(t) = G'(t) - c_1$ is also a monotonically decreasing function, which indicates that is a concave function. Considering that there exist at least three distinct points $a \leq t_1 < t_0 < t_2 \leq b$ that have the maximum error with alternating signs, the maximum error magnitude E_{max}must occur at the end points of the interval $[a, b]$. Hence, it is evident that $t_1 = a$ and $t_2 = b$. The third point of the maximum error magnitude t_0 is determined by setting $E'(t) = 0$, giving

$$G'(t_0) = c_1 \tag{3.6}$$

From Eq. (3.6), it shows that a line tangent to the lognormal CDF at the point is parallel to the minimax approximation. Further, this line is also parallel to the line through the end points $(a, 10^{-6})$, and $(b, 1 - 10^{-16})$. This property follows from the convexity of G(t) and the fact that $E(a) = E(b) = -E_{max}$. Finally, it shows

$$c_1 = \frac{G(b) - G(a)}{b - a} \tag{3.7}$$

And the constant c_0 can be determined using the fact that the point $(t_0, [G(t_0) + G(a) + c_1(t_0 - a)/2])$ is on the minimax line.

$$c_0 = \frac{1}{2}[G(a) + G(t_0)] - c_1\frac{a + t_0}{2} \tag{3.8}$$

wheret_0 is given by Eq. (3.6).

Since we have obtained the parameters c_0 and c_1, it is easy to find the parameters of the lognormal RV approximating the lognormal sum. From Eq. (2.7), it is shown that the CDF of the lognormal RV on "Lognormal Probability Paper" is the linear related to $10 * \log_{10}(x)$; and this curve is also described as $p(t) = c_0 + c_1 t$. Comparing two equations above, the slope and the intercept show that:

$$m_{dB} = -\frac{c_0}{c_1} \tag{3.9}$$

$$\sigma_{dB} = \frac{1}{c_1} \tag{3.10}$$

3.3.3 Results

Fig. 3.1 - Fig. 3.3 show approximations to the sum of lognormal RV's in different conditions, and compare with the simulation results which are from the Monte Carlo Simulation. They show that when $M = 6$, the straight line approximates the curve well, however when $M = 30$ or when the lognormal RVs have difference dB spreads σ, the straight line does not approximate the curve well any more.

Minimax Approximation uses one lognormal RV to approximate the sum of lognormal RV's, and this is using a straight line to approximate a curve on "Lognormal Probability Paper". From Fig. 2.2, it is evident that when M increases, the CDF of lognormal sum becomes more curving, which

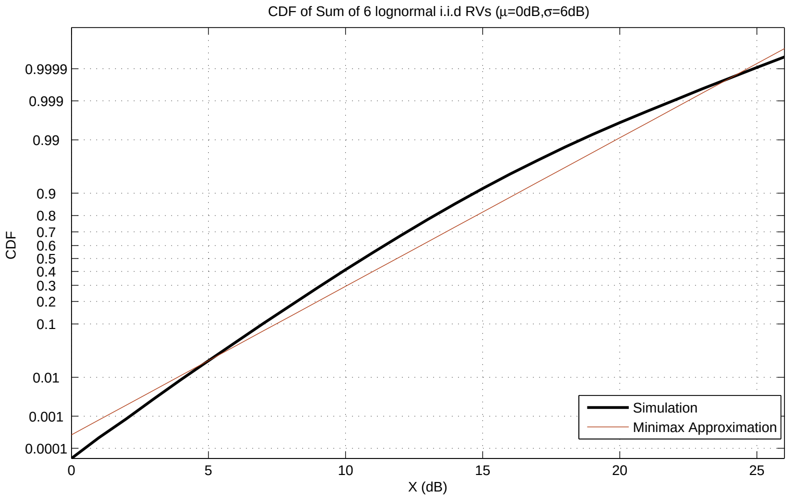

Figure 3.1: CDF of Sum of 6 lognormal i.i.d RV's ($\mu = 0dB, \sigma = 6dB$) for Minimax Approximation

means it is not accurate any more using a straight line to approximate a curve when M is large. There are also some other facts making the approximation worse. In [11] and [12], it was found that this assumption is good for sums of $N = 2$ i.i.d. summands, but is poor when the number of summands increases or the difference among the dB spreads of the summands increases.

3.4 LS Approximation by Quadratic functions (LSQ)

The development of the minimax algorithm is based on the assumption from the observations that the actual CDF of the sum of lognormal RV's is a strictly concave function on the lognormal probability paper. This assumption is hard, if not impossible, to prove because of the lack of more information on the characteristics of the sum distribution, which is what we are trying to find. Moreover, it is well known in approximation theory that the best approximation problem under the maximum norm for continuous functions is not easy to deal with because of the lack of differentiability of the resulting objective function. Therefore, in [13], authors derived a least squares (LS) approximation on the lognormal probability paper.

3.4.1 LS Approximation

The LS approximation based on the L^2 -norm

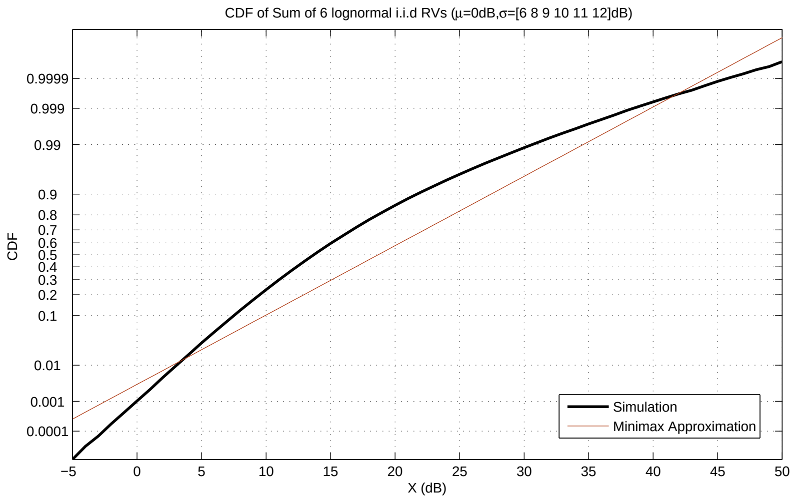

Figure 3.2: CDF of Sum of 6 lognormal i.i.d RV's ($\mu = 0dB, \sigma = [6, 8, 9, 10, 11, 12]dB$) for Minimax Approximation

$$\min_{p \in P_n} ||G - p||_2 = \min_{p \in P_n} \left(\int_a^b [G(r) - p(r)]^2 dr \right)^{\frac{1}{2}} \tag{3.11}$$

where P_n is the space of all polynomials of degree at most n, which is a smooth optimization problem. More specifically, if we define a function r of $n+1$ variables by

$$\begin{aligned} r(c_0, c_1, ...c_n) &= \int_a^b [G(r) - p(r)]^2 dr \\ &= \int_a^b [G(r) - (c_0 + c_1 r + c_2 r^2 + ... + c_n r^n)]^2 dr \end{aligned} \tag{3.12}$$

then r is a differentiable function of its variables, and the unique optimal solution $p(r)$ to Eq. (3.11) can be obtained by solving a system of $(n+1)$ equations in $(n+1)$ unknowns:

$$\frac{\partial}{\partial c_i} r(c_0, c_1, ..., c_n) = 0, \qquad i = 0, 1, ..., n \tag{3.13}$$

Here, comparing to the assumption in Minimax Approximation, no assumption about the concavity of G is assumed for obtaining the LS solution.

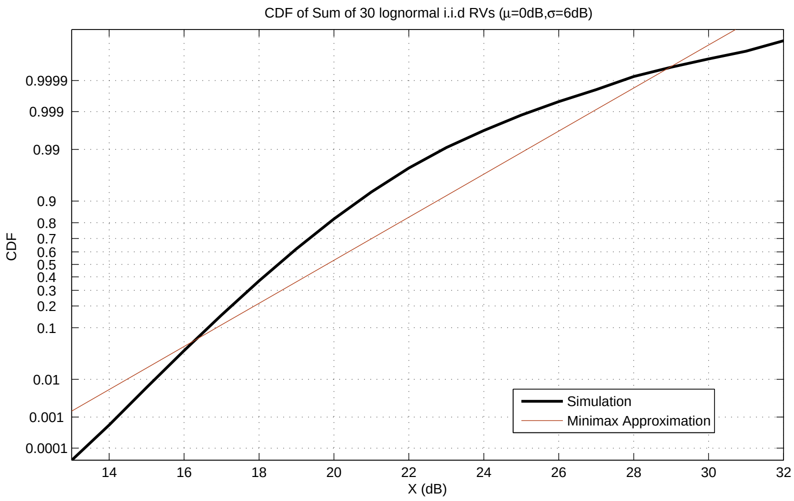

Figure 3.3: CDF of Sum of 30 lognormal i.i.d RV's ($\mu = 0dB, \sigma = 6dB$) for Minimax Approximation

3.4.2 LSQ Approximation

The Linear approximation scheme mentioned in 3.3.1 may not behave well when the number of summands increases or the difference among the dB spreads of the summands increases. Therefore, an LS approximation of $G(t)$ by a linear function may not achieve the desired accuracy on some required interval $[a, b]$ of interest. Higher order approximations are thus in need in applications. The LS Approximation by Quadratic functions is presented in [13] using the higher order to approximate the CDF of lognormal sum.

Similar to Minimax Approximation 3.3.1, consider the LS problem

$$\min_{c_0,c_1,c_2} \quad r(c_0, c_1, c_2) = \min_{c_0,c_1,c_2} \int_a^b [G(r) - c_0 - c_1 r - c_2 r^2]^2 dr \tag{3.14}$$

By Eq. (3.13), there are following three equations to solve c_0, c_1 and c_2:

$$\begin{cases} (b-a)c_0 + \frac{b^2-a^2}{2}c_1 + \frac{b^3-a^3}{3}c_2 = \int_a^b G(r)dr \\ \frac{b^2-a^2}{2}c_0 + \frac{b^3-a^3}{3}c_1 + \frac{b^4-a^4}{4}c_2 = \int_a^b G(r)rdr \\ \frac{b^3-a^3}{3}c_0 + \frac{b^4-a^4}{4}c_1 + \frac{b^5-a^5}{5}c_2 = \int_a^b G(r)r^2dr \end{cases} \tag{3.15}$$

Solving these equations, c_0, c_1 and c_2 are:

$$\begin{aligned} c_0 = \tfrac{3}{(b-a)^5}\{3[(a+b)^4 + 4a^2b^2]\int_a^b G(r)dr \\ -12(a+b)(a^2+3ab+b^2)\int_a^b G(r)rdr \\ +10(a^2+4ab+b^2)\int_a^b G(r)r^2dr\} \end{aligned} \tag{3.16}$$

$$\begin{aligned} c_1 = \tfrac{12}{(b-a)^5}\{-3(a+b)(a^2+3ab+b^2)\int_a^b G(r)dr \\ +4[4(a+b)^2 - ab]\int_a^b G(r)rdr \\ -15(a+b)\int_a^b G(r)r^2dr\} \end{aligned} \tag{3.17}$$

$$\begin{aligned} c_2 = \tfrac{30}{(b-a)^5}\{(a^2+4ab+b^2)\int_a^b G(r)dr \\ -6(a+b)\int_a^b G(r)rdr \\ +6\int_a^b G(r)r^2dr\} \end{aligned} \tag{3.18}$$

3.4.3 CDF and PDF

After we get the three coefficients for the curve in "Lognormal Probability Paper", we define the curve to be the CDF of a RV named Γ, and then we need also get the PDF and CDF expressions of this RV. As mentioned in Eq. (3.4) and (2.7) and the definition of "Lognormal Probability Paper" in 2.3, it is easy to find the CDF and the PDF of the Γ.

The relationship is:

$$\begin{aligned} & G(t) = p(t) = c_0 + c_1 t + c_2 t^2 \\ & t = 10 * \log_{10}(x) = \tfrac{\ln(x)}{\ln(10)/10} = \tfrac{\ln(x)}{\zeta} \\ & G(t) = F_N^{-1}(F(t)) \end{aligned} \tag{3.19}$$

And then the CDF $F(x)$ could be expressed as:

$$F(x) = F_N(G(\frac{\ln(x)}{\zeta})) = F_N(c_0 + c_1\frac{\ln(x)}{\zeta} + c_2(\frac{\ln(x)}{\zeta})^2) \tag{3.20}$$

Taking the first derivative to both sides above and manipulating yield the PDF of Γ as

$$\begin{aligned} f(x) = \tfrac{dF(x)}{dx} = (c_1 + 2c_2\tfrac{\ln(x)}{\zeta})\tfrac{1}{\zeta x} f_N(c_0 + c_1\tfrac{\ln(x)}{\zeta} + c_2(\tfrac{\ln(x)}{\zeta})^2) \\ = (c_1 + 2c_2\tfrac{\ln(x)}{\zeta})\tfrac{1}{\sqrt{2\pi}\zeta x}\exp\{-\tfrac{[c_0 + c_1\frac{\ln(x)}{\zeta} + c_2(\frac{\ln(x)}{\zeta})^2]^2}{2}\} \end{aligned} \tag{3.21}$$

where $f_N(x)$ is the PDF of Gaussian RV having zero mean and unit variance. The coefficient c_i can be obtained either from numerical results in Eq. (3.16).

3.4.4 Results

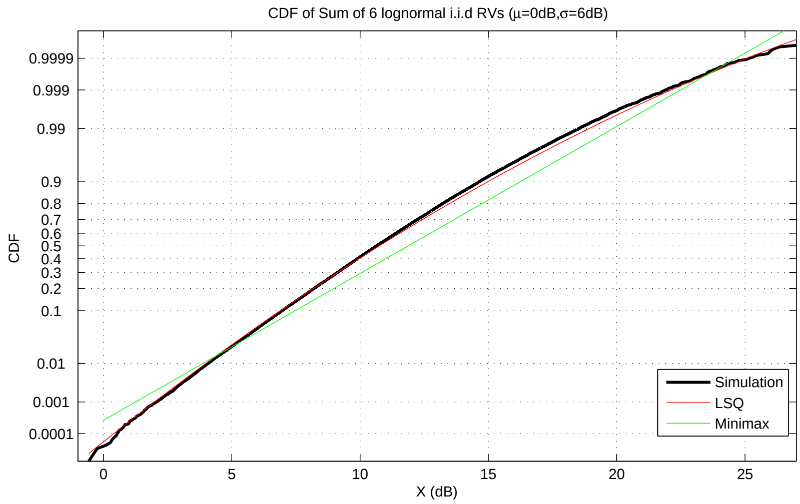

Figure 3.4: CDF of Sum of 6 lognormal i.i.d RV's ($\mu = 0dB, \sigma = 6dB$) for LSQ Approximation

In Fig. 3.4 - Fig. 3.5, they show approximations to the sum of lognormal RV's in different conditions, and compare with the simulation results which are from the Monte Carlo Simulation. Comparing with the Minimax Approximation, LSQ Approximation has better accuracy than the Minimax Approximation, especially when the CDF is curving.

Minimax Approximation and LSQ Approximation are all based on the simulation CDF $G(t)$, and then they use another curve to approximate the CDF. The LSQ method is better than Minimax method, however, their parameters cannot be determined until the numerical or empirical PDF and CDF of the lognormal sum distribution have been obtained, and hence the complexity to use this approximation method is high, while we don't want to do the Monte Carlo Simulation to get the whole curve first then get an approximation to it in most time.

3.5 Type IV Pearson Approximation

As mentioned before, it is desirable to avoid the Monte Carlo Simulation to get the whole curve first before obtaining the approximation. Hence, we need to find a distribution directly to approximate the CDF of the lognormal sum. Because general close-form expressions for the PDF or CDF of Λ are not available, Λ is usually approximated by a new lognormal RV [5],[11],[12], denoted as L. However, as a lognormal RV, L only has two independent parameters, μ_L and σ_L, so generally the lognormal approximation methods can only ensure that the mean and variance of L are close to

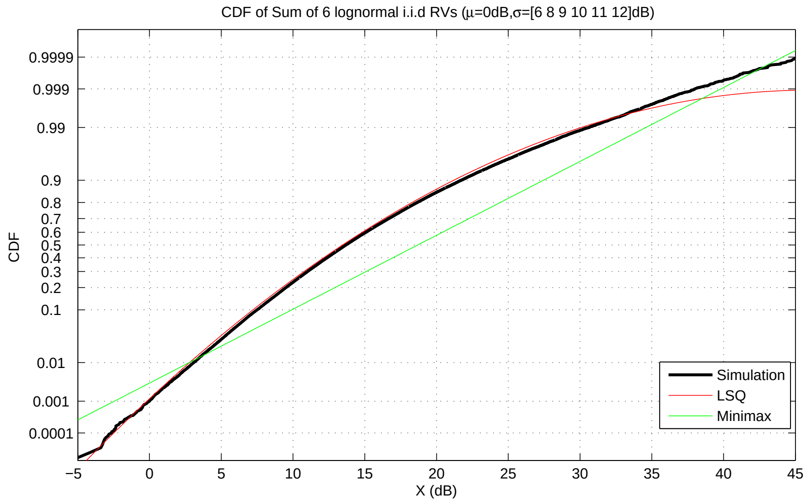

Figure 3.5: CDF of Sum of 6 lognormal i.i.d RV's ($\mu = 0dB, \sigma = [6, 8, 9, 10, 11, 12]dB$) for LSQ Approximation

those of Λ . However, serious discrepancy exists between the skewness and kurtosis of Λ and those of L [15]. Hence, it needs to consider the Skewness and Kurtosis when approximating. The Pearsons family consists of seven types of fundamental distributions which are tabulated in Table (3.1) for subsequent use, and it is nice if we could find one type to approximate the lognormal sum.

Table 3.1: Seven Types of Pearson Distributions

Model Type	pdf	Normal Name
I	$f(x) = \frac{1}{B(p,q)} x^{p-1}(1 - x^{q-1})\ x \in [0, 1]$	Beta Distribution
II	$f(x) = \frac{1}{aB(0.5,m+1)}(1 - \frac{x^2}{a^2})^m,\ x \in [-a, a]$	N/A
III	$f(x) = k(1 + \frac{x}{a})^p e^{-px/a},\ x \in [-a, \infty)$	Gamma Distribution
IV	$f(x) = \nu(1 + \frac{(x-\mu_4)^2}{\mu_3^2})^{-\mu_1} exp[-\mu_2 tan^{-1}(\frac{x-\mu_4}{\mu_3})],$ $x \in (-\infty, \infty)$	N/A
V	$f(x) = \frac{\gamma^{p-1}}{\Gamma(p-1)} x^{-p} e^{-\gamma/x}$	N/A
VI	$f(x) = \frac{1}{B(b,q)} \frac{x^{p-1}}{(1+x)^{p+q}},\ x \in [0, \infty)$	Beta of The Second Kind
VII	$f(x) = \frac{1}{aB(0.5,m-0.5)}(1 + \frac{x^2}{a^2})^{-m},\ x \in [-a, a]$	Student's t

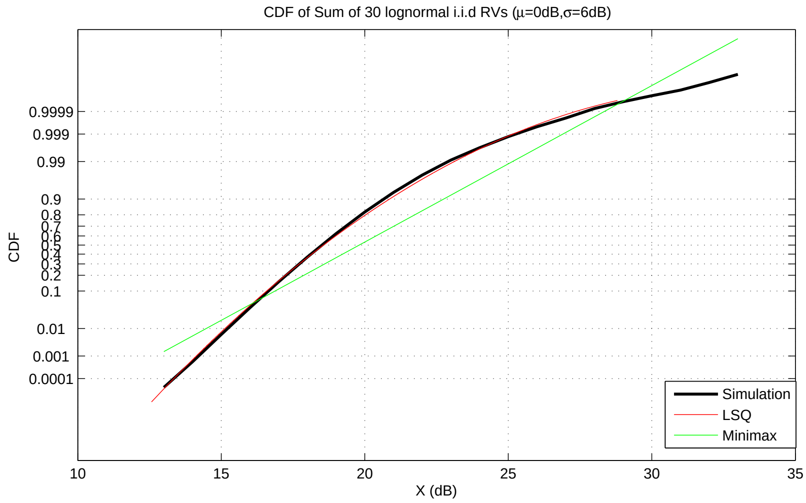

Figure 3.6: CDF of Sum of 30 lognormal i.i.d RV's ($\mu = 0dB, \sigma = 6dB$) for LSQ Approximation

3.5.1 Pearson Type Selection

The type I can be further divided into those of unimode (U), multimode (M), and J-shape (J). The pdf of type IV has a somewhat long expression:

$$f(x) = \nu(1 + \frac{(x-\mu_4)^2}{\mu_3^2})^{-\mu_1} exp[-\mu_2 tan^{-1}(\frac{x-\mu_4}{\mu_3})], \; x \in (-\infty, \infty) \tag{3.22}$$

Considering Eq. (2.4), let μ_1 denote the mean of Λ, and let μ_2, μ_3 and μ_4 denote its second, third and fourth moments about the mean, respectively. The selection of a particular model type is based on the moments ratios

$$\begin{aligned} \beta_1 &= \frac{\mu_3^2}{\mu_2^3} \\ \beta_2 &= \frac{\mu_4}{\mu_2^2} \\ k &= \frac{\beta_1(\beta_2+3)^2}{4(2\beta_2 - 3\beta_1 - 6)(4\beta_2 - 3\beta_1)} \end{aligned} \tag{3.23}$$

The decision criteria for different Pearsons distribution types are tabulated in Table (3.2). These criteria have been widely used to select the distribution models for empirical data and for approximating the algebraic functions of random variables. The model-type selection starts from the examination of k. From [15] and [16], Type IV Pearson Distribution is chosen to approximate the Lognormal Sum. Once the model type is identified, the remaining issue is to determine the model parameters.

Table 3.2: Selection Criteria For The Model Type

Model Type	Selection Criteria
I	$k < 0$
II	$k = 0, \beta_2 < 3$
III	$2\beta_2 - 3\beta_1 - 6 = 0$
IV	$0 < k < 1$
V	$k = 1$
VI	$k > 1$
VII	$k = 0,\ \beta_2 > 3$
Normal	$k = 0,\ \beta_2 = 3$

3.5.2 Computing Parameters

Considering Eq. (2.4) , define $a_i = exp(\zeta\mu_i)$ and $b_i = exp(\zeta^2\sigma_i^2)$, where μ_i and σ_i are mean and standard deviation in dB, then the mean of Λ is given by:

$$M_\Lambda = E(\Lambda) = \sum_{i=1}^{M} E(L_i) = \sum_{i=1}^{M} a_i\sqrt{b_i} \tag{3.24}$$

the variance of Λ is given by:

$$V_\Lambda = Var(\Lambda) = \sum_{i=1}^{M} E[(L_i - E(L_i))^2] = \sum_{i=1}^{M} a_i^2 b_i(b_i - 1) \tag{3.25}$$

the 3^{rd}-order central moment of Λ is given by:

$$S_\Lambda = \sum_{i=1}^{M} E[(L_i - E(L_i))^3] = \sum_{i=1}^{M} a_i^3 b_i^{3/2}(b_i - 1)^2(b_i + 2) \tag{3.26}$$

and the 4^{th}-order central moment of Λ is given by:

$$\begin{aligned} T_\Lambda &= \sum_{i=1}^{M} E[(L_i - E(L_i))^4] \\ &+ 6\sum_{i=1}^{M-1}\sum_{j=i+1}^{M} E[(L_i - E(L_i))^2]E[(L_j - E(L_j))^2] \\ &= \sum_{i=1}^{M} a_i^4 b_i^2(b_i - 1)^2(b_i^4 + 2b_i^3 + 3b_i^2 - 3) \\ &+ 6\sum_{i=1}^{M-1}\sum_{j=i+1}^{M} a_i^2 b_i(b_i - 1)a_j^2 b_j(b_j - 1) \end{aligned} \tag{3.27}$$

Correspondingly, the skewness and kurtosis of Λ can be obtained respectively as follows:

$$SK_\Lambda = \frac{S_\Lambda}{V_\Lambda^{3/2}} \tag{3.28}$$

$$KU_\Lambda = \frac{T_\Lambda}{V_\Lambda^2} \tag{3.29}$$

Since the mean, variance, skewness and kurtosis of Λ can be precisely evaluated from Eq. (3.24) - (3.29), a probability distribution that can properly exploit all those statistical characteristics should be able to approximate Λ with a much higher accuracy in a wide probability range.

Using Type IV Pearson RV to approximate the Lognormal Sum, we have

$$\Lambda \approx Pe \tag{3.30}$$

where Pe is a Type IV Pearson RV. Define M_P, V_P, SK_P and KU_P are respectively the mean, variance, skewness and kurtosis of Pe. By forcing that $M_P = M_\Lambda$,$V_P = V_\Lambda$,$SK_P = SK_\Lambda$ and $KU_P = KU_\Lambda$, Λ can be approximated with the Type IV Pearson RV Pe which has the same statistical characteristics as those of Λ.

From Eq. (3.22), there are 4 parameters needed to be decided. With the statistical characteristics matching, it is easy to represent the 4 parameters by M_P,V_P,SK_P and KU_P. The PDF of the Type IV Pearson RV Pe is given by Eq. (3.22), and the 4 parameters are:

$$\mu_1 = (r+2)/2 \tag{3.31}$$

$$\mu_2 = \frac{-r(r-2)SK_P}{\sqrt{16(r-1) - SK_P^2(r-2)^2}} \tag{3.32}$$

$$\mu_3 = \sqrt{V_P[r-1-SK_P^2(r-2)^2/16]} \tag{3.33}$$

$$\mu_4 = M_P - (r-2)SK_P\sqrt{V_P}/4 \tag{3.34}$$

$$\nu = [\mu_3 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^r(x) exp(-\mu_2 x) dx]^{-1} \tag{3.35}$$

where r is determined by:

$$r = \frac{6(KU_P - SK_P^2 - 1)}{2KU_P - 3SK_P^2 - 6} \tag{3.36}$$

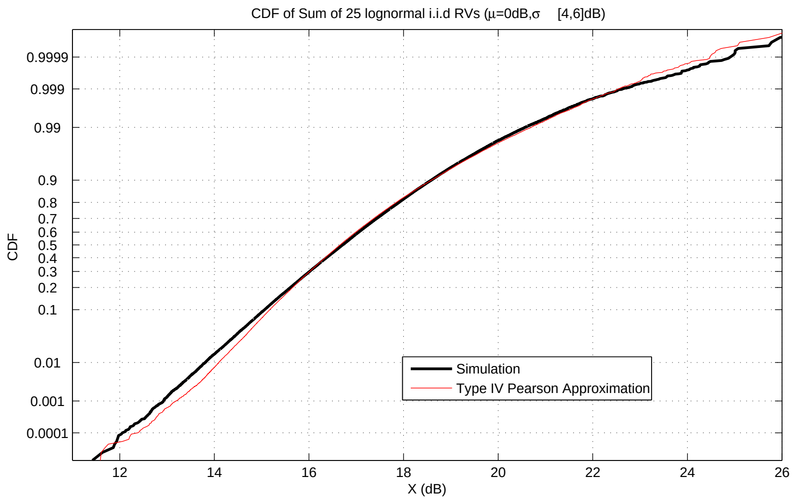

Figure 3.7: CDF of Sum of 6 lognormal i.i.d RV's ($\mu = 0dB, \sigma \in [4, 6]dB$) for Pearson Approximation

3.5.3 Results

In Fig. 3.7 - Fig. 3.9, they show approximations to the sum of lognormal RV's in different conditions, and compare with the simulation results which are from the Monte Carlo Simulation. The approximations are accurate in a wide region. This approximation method has three major advantages over the existing methods: first, it offers a close-form PDF solution to approximate the lognormal sum distribution; second, the parameter evaluations of the PDF solution are simple and straightforward; finally, a high accurate approximation can be achieved in a wide probability range.

However, this approximation has limitation $k < 1$ in Table (3.2) to use Type IV Pearson Distribution. In [15], the analysis shows that for the lognormal sum distribution, the value of k is always positive, and it increases with the value of σ_i, and decreases with the value of M. For example, if M is 50, $k < 1$ when $\sigma_i < 6.5dB$; if M increases to 100, $k < 1$ when $\sigma_i < 6.9dB$. Therefore, if the value of σ_i is large and the value of M is small, it could not use Type IV Pearson distribution to approximate lognormal sum distributions.

3.6 Log Shifted Gamma Approximation (LSG)

Pearson Approximation Method uses one Type IV Pearson RV to approximate the Lognormal Sum, however, this approximation has limitation so that it could not be used in some conditions. Hence, it is needed to find another approximation in wider conditions. Log Shifted Gamma Approximation uses LSG to approximate the lognormal sum and could be used in any conditions and also has good

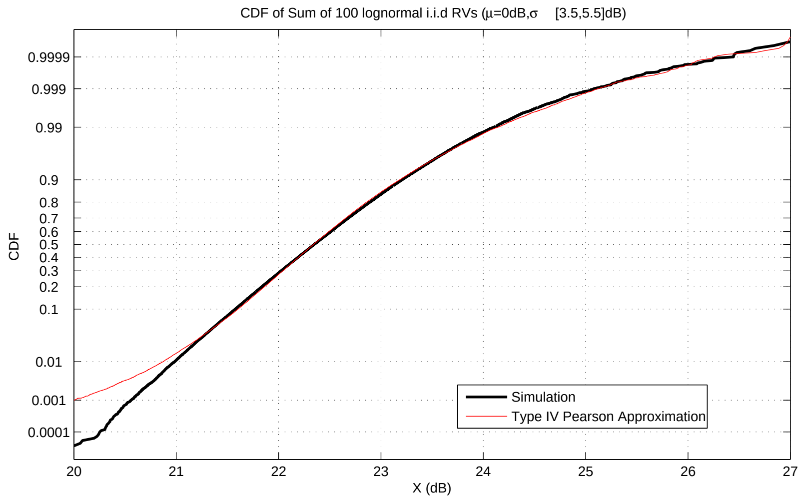

Figure 3.8: CDF of Sum of 30 lognormal i.i.d RV's ($\mu = 0dB, \sigma \in [3.5, 5.5]dB$) for Pearson Approximation

accuracy in wide ranges.

3.6.1 Log Shifted Gamma Distribution

Similar to the relation between normal and lognormal distributions, [17] considers a distribution called log shifted Gamma (LSG) to describe the *shifted Gamma* distribution in logarithmic domain, a LSG RV G defined as

$$G = e^{\zeta Z} = 10^{Z/10} \tag{3.37}$$

where Z is *shifted Gamma* distribution in dB with PDF

$$f_z(z;\alpha,\beta,\delta) = \begin{cases} \frac{1}{\beta^\alpha \Gamma(\alpha)}(z-\delta)^{\alpha-1}\exp[-\frac{z-\delta}{\beta}] & z > \delta \\ 0 & z \le \delta \end{cases} \tag{3.38}$$

and the CDF

$$F_z(z;\alpha,\beta,\delta) = \begin{cases} \frac{1}{\Gamma(\alpha)}\gamma(\frac{z-\delta}{\beta},\alpha) & z > \delta \\ 0 & z \le \delta \end{cases} \tag{3.39}$$

This is a little different from the *Gamma* Distribution, and the *"shifted"* means to introduce the offset δ in the above equations. Then the PDF of the LSG RV G can be given by:

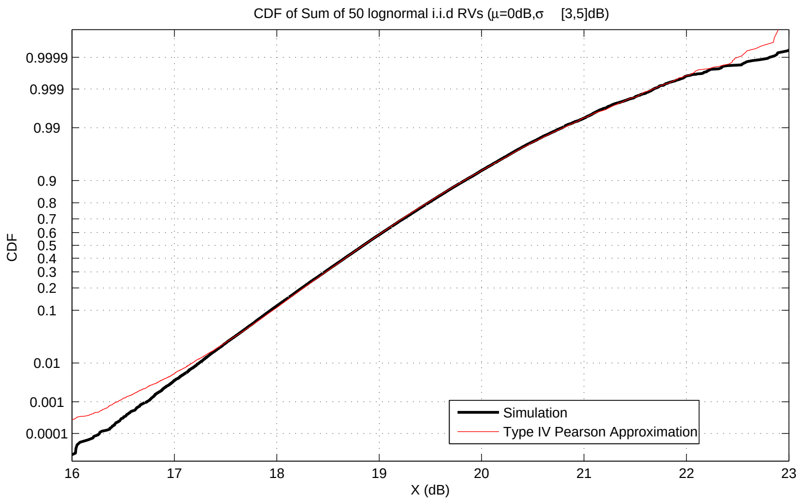

Figure 3.9: CDF of Sum of 6 lognormal i.i.d RV's ($\mu = 0dB, \sigma \in [3, 5]dB$) Pearson Approximation

$$f_G(g; \alpha, \beta, \delta) \begin{cases} \frac{(10 \log_{10} g - \delta)^{\alpha - 1}}{\Gamma(\alpha)} \exp[-\frac{10 \log_{10} g - \delta}{\beta}] & g > 10^{\delta/10} \\ 0 & g \leq 10^{\delta/10} \end{cases} \tag{3.40}$$

There are three parameters α, β and δ in dB, and they could control the CDF to approximate the CDF of the lognormal sum. The first parameter α controls the shape of the PDF. When α increases, the PDF becomes more Gaussian-like, while a more Gamma-like PDF is obtained with smaller $\alpha > 1$. When $\alpha = 1$, the RV corresponds to an exponential distribution. For $\alpha < 1$ and approaching to 0, the PDF approaches infinity, which is similar to chisquare distribution. The second parameter β is for scaling z or g of the distribution. And the last parameter δ is the offset allowing the flexibility to shift z or g for the best fitting of the resulting distribution.

LSG Approximation will use the introduced LSG distribution to approximate the sum of a number of lognormal RV's, and to derive the LSG parameters from the parameters of the individual lognormal RV's by matching the moments in both linear and logarithmic domains.

3.6.2 Computing Parameters

Considering the model in Eq. (2.4) again.

$$\Lambda = \sum_{k=1}^{M} L_k \approx e^z \tag{3.41}$$

3.6.2.1 Sum of Two Lognormal RV's

If M=2, LSG method takes the approach of Schwartz and Yeh method to compute the exact moments as follow. The Eq. (3.41) will be expressed as followed:

$$\Lambda_2 = L_1 + L_2 = e^{x_1} + e^{x_2} \approx e^{z_2} \tag{3.42}$$

To find the three parameters α, β and δ, three equations are needed to match the moments:

$$\begin{aligned} E[\Lambda_2] &\approx E[e^{z_2}] \\ E[\ln(\Lambda_2)] &\approx E[z_2] \\ Var[\ln(\Lambda_2)] &\approx Var[z_2] \end{aligned} \tag{3.43}$$

denoted as μ_{Λ_2}, μ_{z_2} and $\sigma^2_{z_2}$, respectively. From Eq. (3.38)-Eq. (3.40), we could find the expressions with the three parameters:

$$\begin{aligned} \mu_{\Lambda_2} &= E[\Lambda_2] = e^{\delta}/(1-\beta)^{\alpha} \\ \mu_{z_2} &= E[z_2] = \delta + \alpha\beta \\ \sigma^2_{z_2} &= Var[z_2] = \alpha\beta^2 \end{aligned} \tag{3.44}$$

The next step is to compute the μ_{Λ_2}, μ_{z_2} and $\sigma^2_{z_2}$ in terms of the mean μ_{x_i} and variance $\sigma^2_{x_i}$ which are known.

$$\mu_{\Lambda_2} = E[\Lambda_2] = E[e^{x_1}] + E[e^{x_2}] = \exp(\mu_{x_1} + \frac{\sigma^2_{x_1}}{2}) + \exp(\mu_{x_2} + \frac{\sigma^2_{x_2}}{2}) \tag{3.45}$$

Using the procedure by Schwartz and Yeh to calculate μ_{z_2} and $\sigma^2_{z_2}$, we define a new RV $Y \triangleq (x_2 - x_1)$:

$$\begin{aligned} \mu_{z_2} &= E[z_2] = \mu_{x_1} + G_1(\mu_Y, \sigma_Y) \\ \sigma^2_{z_2} &= Var[z_2] = \sigma^2_{x_1} - G_1^2(\mu_Y, \sigma_Y) - 2\sigma^2_{x_1} G_3(\mu_Y, \sigma_Y)/\sigma^2_Y + G_2(\mu_Y, \sigma_Y) \end{aligned} \tag{3.46}$$

where G_1, G_2 and G_3 are derived in [21] as:

$$
\begin{aligned}
G_1(\mu,\sigma) &= \mu\Phi(\tfrac{\mu}{\sigma}) + \tfrac{\sigma}{\sqrt{2\pi}}e^{-\frac{\mu^2}{2\sigma^2}} + \sum_{k=1}^{\infty} C_k[F(\sigma,\mu,k)+F(\sigma,-\mu,k)] \\
G_2(\mu,\sigma) &= (\mu^2+\sigma^2)\Phi(\tfrac{\mu}{\sigma}) + (\mu+\ln(4))\tfrac{\sigma}{\sqrt{2\pi}}e^{-\frac{\mu^2}{2\sigma^2}} \\
&\quad +2\sum_{k=1}^{\infty} C_k(\mu-k\sigma^2)F(\sigma,\mu,k) \\
&\quad +\sum_{k=2}^{\infty} B_{k-1}[F(\sigma,\mu,k)+F(\sigma,-\mu,k)] \\
G_3(\mu,\sigma) &= \sigma^2\sum_{k=1}^{\infty}(-1)^k[F(\sigma,\mu,k)+F(\sigma,-\mu,k+1)]
\end{aligned} \tag{3.47}
$$

where

$$
\begin{aligned}
F(\mu,\sigma,k) &= e^{-k\mu+\frac{k^2\sigma^2}{2}}\Phi(\tfrac{\mu-k\sigma^2}{\sigma}) \\
\Phi(x) &= \tfrac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{t^2}{2}}dt \\
C_k &= \tfrac{(-1)^{k+1}}{k} \\
B_k &= \tfrac{2(-1)^{k+1}}{k+1}\sum_{j=1}^{k}\tfrac{1}{j}
\end{aligned} \tag{3.48}
$$

Using Eq. (3.44),(3.45) and (3.46), we could solve the 3 equations to get the 3 parameters α, β and δ. However, they are not polynomial equations. After obtaining both μ_{z_2} and $\sigma_{z_2}^2$, we use Eq. (3.44) to express β and δ in terms of α.

$$
\begin{aligned}
\beta(\alpha) &= \alpha^{-1}\sqrt{(\alpha\sigma_{z_2}^2)} \\
\delta(\alpha) &= -\sqrt{(\alpha\sigma_{z_2}^2)} + \mu_{z_2}
\end{aligned} \tag{3.49}
$$

Then Eq. (3.44) could give one equation of α:

$$
f(\alpha) = \delta(\alpha) - \alpha\ln(1-\beta(\alpha)) - \ln(\mu_{\Lambda_2}) \tag{3.50}
$$

In [17], it gives a bisection method to solve them numerically The solution for α can be numerically found by using bisection method, narrowing down the interval of by halves, iteration by iteration. A predefined error tolerance e is set as a stopping condition. For $e = 10^{-3}$, and an initial interval of [1, 10000], α can be obtained after around 20 iterations.

3.6.2.2 Sum of More Than Two Lognormal RV's

For the condition of $M > 2$, we need proceed the $(j-1)^{th}$ iteration, $j = 3, 4, ..., M$. We do the same procedure:

$$
\Lambda_j = \Lambda_{j-1} + L_j \approx e^{z_{j-1}} + e^{x_j} \approx e^{z_j} \tag{3.51}
$$

Like Eq. (3.43), there are three equations needed to compute three parameters α_{z_j}, β_{z_j} and δ_{z_j} of z_j:

$$\begin{aligned}\mu_{\Lambda_j} &= E[\Lambda_j] = E[e^{z_j}] = e^{\delta_{z_j}}/(1-\beta_{z_j})^{\alpha}_{z_j} \\ \mu_{z_j} &= E[\ln(\Lambda_j)] = E[z_j] = \delta_{z_j} + \alpha_{z_j}\beta_{z_j} \\ \sigma^2_{z_j} &= Var[\ln(\Lambda_j)] = Var[z_j] = \alpha_{z_j}\beta^2_{z_j}\end{aligned} \tag{3.52}$$

where μ_{Λ_j} can be computed easily as:

$$\mu_{\Lambda_j} = E[\Lambda_j] = E[\Lambda_{j-1}] + \exp(\mu_{x_j} + \frac{\sigma^2_{x_j}}{2}) \tag{3.53}$$

However, z_{j-1} is not a Gaussian RV, so we could not find the PDF of the defined a new RV

$$W \triangleq (z_{j-1} - x_j) \tag{3.54}$$

like we define Y to directly compute μ_{z_j} and $\sigma^2_{z_j}$. We can approximate W by another shifted-Gamma RV by matching moments, which means W is a shifted-Gamma RV. From Eq. (3.54) and (3.38), the pdf of W is given by:

$$f_w(w) = \int_{-\infty}^{\infty} f_{x_j}(z-w) f_{z_{j-1}}(z) \approx \frac{(w-\delta_w)^{\alpha_w - 1}}{\Gamma(\alpha_w)\beta_w^{\alpha_w}} \exp(-\frac{(w-\delta_w)}{\beta_w}) \tag{3.55}$$

where α_w, β_w and δ_w are the three parameters for the shifted-Gamma RV W. The next step is to use moments matching to compute α_w, β_w and δ_w like Eq. (3.43):

$$\begin{aligned}&E[W] \approx E[z_{j-1} - x_j] = E[z_{j-1}] - E[x_j] \\ &Var[W] \approx Var[z_{j-1} - x_j] = Var[z_{j-1}] - E[x_j] \\ &E[(W - E[W])^3] \approx E[((z_{j-1} - x_j) - E[z_{j-1} - x_j])^3] = E[(z_{j-1} - E[z_{j-1}])^3] - E[(x_j - E[x_j])^3]\end{aligned} \tag{3.56}$$

where

$$\begin{cases} E[W] = \mu_w = \delta_w + \alpha_w\beta_w \\ E[z_{j-1}] - E[x_j] = \delta_{z_{j-1}} + \alpha_{z_{j-1}}\beta_{z_{j-1}} - \mu_{x_j} \\ Var[W] = \alpha_w\beta_w^2 \\ Var[z_{j-1}] - E[x_j] = \sigma^2_{x_j} + \alpha_{z_{j-1}}\beta^2_{z_{j-1}} \\ E[(W - E[W])^3] = 2\alpha_w\beta_w^3 \\ E[(z_{j-1} - E[z_{j-1}])^3] - E[(x_j - E[x_j])^3] = 2\alpha_{z_{j-1}}\beta^3_{z_{j-1}} \end{cases} \tag{3.57}$$

From above equations, α_w, β_w and δ_w can be derived as the term of the parameters of z_{j-1} and x_j:

$$\begin{aligned}\alpha_w &= \frac{(\sigma_{x_j}^2+\alpha_{z_{j-1}}\beta_{z_{j-1}})^3}{(\alpha_{z_{j-1}}\beta_{z_{j-1}}^3)^2}\\ \beta_w &= \frac{\alpha_{z_{j-1}}\beta_{z_{j-1}}^3}{\sigma_{x_j}^2+\alpha_{z_{j-1}}\beta_{z_{j-1}}^2}\\ \delta_w &= \delta_{z_{j-1}}+\alpha_{z_{j-1}}\beta_{z_{j-1}}-\mu_{x_j}-\frac{(\sigma_{x_j}^2+\alpha_{z_{j-1}}\beta_{z_{j-1}}^2)^2}{\alpha_{z_{j-1}}\beta_{z_{j-1}}^3}\end{aligned} \tag{3.58}$$

where $\alpha_{z_{j-1}}$, $\beta_{z_{j-1}}$ and $\delta_{z_{j-1}}$ are known from the previous iteration. Since W is known, the mean μ_{z_j} and variance σ_{z_j} can be obtained:

$$\mu_{z_j} = (E[z_j]) = E[\ln(e^{z_{j-1}}+e^{x_j})] = \delta_{z_{j-1}}+\alpha_{z_{j-1}}\beta_{z_{j-1}}+T_1(\alpha_w,\beta_w,\delta_w) \tag{3.59}$$

$$\begin{aligned}\sigma_{z_j}^2 &= E[z_j^2]-(E[z_j])^2\\ &= E[(z_{j-1}+\ln(1+e^{-W}))^2]-(E[z_j])^2\\ &= E[(z_{j-1})^2]+2E[z_{j-1}*\ln(1+e^{-W})]+E[(\ln(1+e^{-W}))^2]-(E[z_j])^2\\ &= \alpha_{z_{j-1}}\beta_{z_{j-1}}^2+(\delta_{z_{j-1}}+\alpha_{z_{j-1}}\beta_{z_{j-1}})^2+2T_3(\alpha_w,\beta_w,\delta_w)\\ &+T_2(\alpha_w,\beta_w,\delta_w)-(E[z_j])^2\end{aligned} \tag{3.60}$$

where the T_1, T_2 and T_3 are the intermediate term in the first moment in logarithmic domain, and they are given by:

$$\begin{aligned}&T_1(\alpha_w,\beta_w,\delta_w)\\ &= E[\ln(1+e^{-W})]\\ &= \sum_{k=1}^{\infty} C_k[H(\alpha_w,\beta_w,\delta_w,k)+H(\alpha_w,\beta_w,\delta_w,-k)]\\ &-\tfrac{\alpha_w\beta_w\gamma(-\delta_w/\beta_w,\alpha_w+1)}{\Gamma(\alpha_w+1)}-\tfrac{\delta_w\gamma(-\delta_w/\beta_w,\alpha_w)}{\Gamma(\alpha_w)}\end{aligned} \tag{3.61}$$

$$\begin{aligned}&T_2(\alpha_w,\beta_w,\delta_w)\\ &= E[\ln^2(1+e^{-W})]\\ &= \sum_{k=1}^{\infty} B_k[H(\alpha_w,\beta_w,\delta_w,k+1)+H(\alpha_w,\beta_w,\delta_w,-(k+1))]\\ &-2\sum_{k=1}^{\infty} C_k[\alpha_w\beta_w H(\alpha_w+1,\beta_w,\delta_w,k)+\delta_w H(\alpha_w,\beta_w,\delta_w,k)]\\ &+\alpha_w(\alpha_w+1)\beta_w^2\tfrac{\gamma(-\delta_w/\beta_w,\alpha_w+1)}{\Gamma(\alpha_w+1)}+\delta_w^2\tfrac{\gamma(-\delta_w/\beta_w,\alpha_w)}{\Gamma(\alpha_w)}\end{aligned} \tag{3.62}$$

$$T_3(\alpha_w,\beta_w,\delta_w) = \alpha_{z_{j-1}}\beta_{z_{j-1}}T_1(\alpha_{w2},\beta_{w2},\delta_{w2})+\delta_{z_{j-1}}T_1(\alpha_w,\beta_w,\delta_w) \tag{3.63}$$

where α_{w2} ,β_{w2} and δ_{w2} are obtained through matching moments as in Eq. (3.58) by replacing $\alpha_{z_{j-1}}$ by $\alpha_{z_{j-1}}+1$. $\gamma(z,\alpha)$ and $\Gamma(z,\alpha)$ are the "lower" and "upper" incomplete Gamma functions. In [17], $H(\alpha,\beta,\delta,k)$ is defined as:

$$\begin{aligned} H(\alpha,\beta,\delta,k) &= e^{k\delta}\frac{\gamma(-\frac{\delta}{\beta}+k\delta,\alpha)}{\Gamma(\alpha)(1-k\beta)^{\alpha}} \\ H(\alpha,\beta,\delta,-k) &= e^{-k\delta}\frac{\Gamma(-\frac{\delta}{\beta}-k\delta,\alpha)}{\Gamma(\alpha)(1+k\beta)^{\alpha}} \end{aligned} \tag{3.64}$$

From Eq. (3.53), (3.59) and (3.60) , μ_{Λ_j}, μ_{z_j}, and $\sigma^2_{z_j}$ have been obtained. We could use Eq. (3.56) to compute three parameters for the z_j: α_{z_j}, β_{z_j} and δ_{z_j}.

The PDF of the sum of M lognormal RV's can be derived by repeating the above procedure for (M-1) iterations.

3.6.3 Results

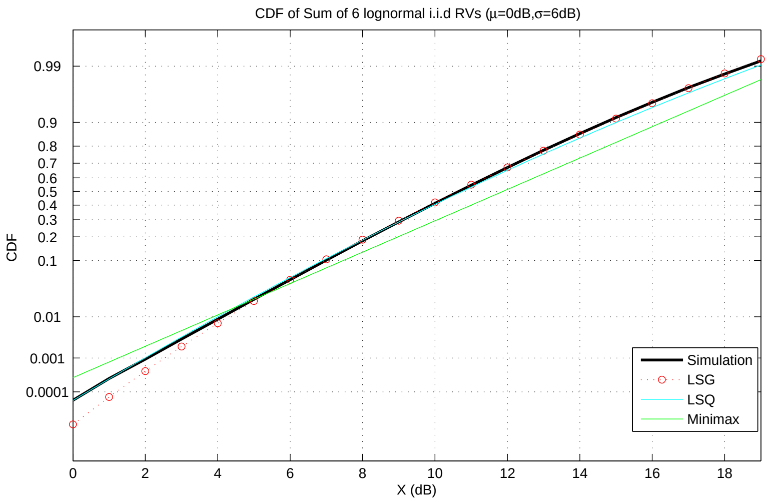

Figure 3.10: CDF of Sum of 6 lognormal i.i.d RV's ($\mu = 0dB, \sigma = 6dB$) for LSG Approximation

In Fig. 3.10 - Fig. 3.11, they show approximations to the sum of lognormal RV's in different conditions, and compare with the simulation results which are from the Monte Carlo Simulation. Comparing with other approximations, LSG is much better than Minimax Approximation, but not as good as LSQ approximation. However, Minimax and LSQ both need know the CDF curve from the Monte Carlo Simulation then use a curve to approximation while we don't want to do the Monte Carlo Simulation to get the whole curve first then get an approximation to it in most time. And LSG could be used to approximate in all the conditions with no limitation, while Type IV Pearson Approximation has one limitation.

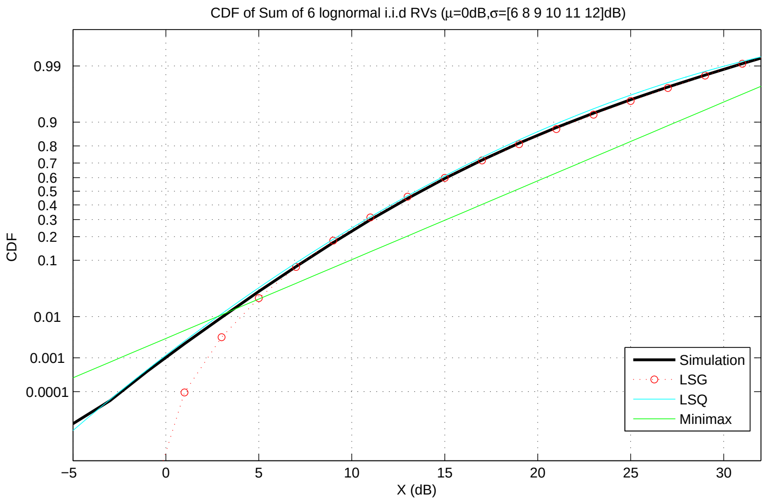

Figure 3.11: CDF of Sum of 6 lognormal i.i.d RV's ($\mu = 0dB, \sigma = [6, 8, 9, 10, 11, 12]dB$) for LSG Approximation

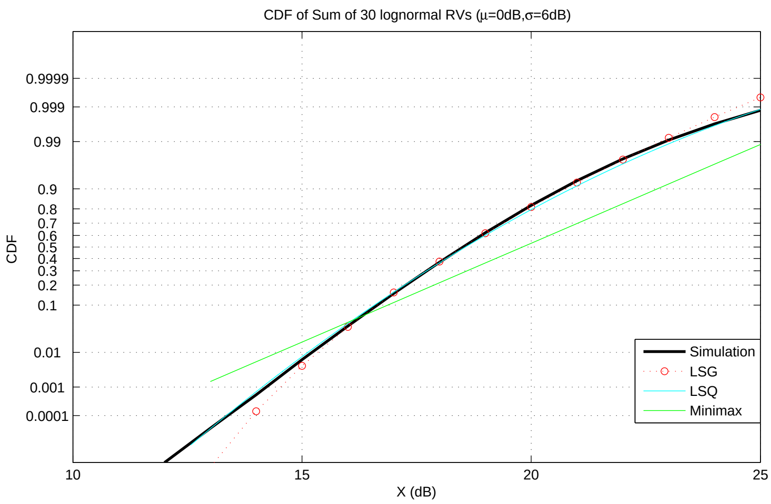

Figure 3.12: CDF of Sum of 30 lognormal i.i.d RV's ($\mu = 0dB, \sigma = 6dB$) for LSG Approximation

4

Log Skew Normal Approximation to Lognormal Sum Distributions

As mentioned in section 3, several approximations to the distribution have been proposed and employed in literature. Some widely used approximations (e.g., LSG) are not very accurate at the lower region while we don't have strong power in most time [17]. Some other approximations require the entire CDF curve from the Monte Carlo Simulation, then use the curve to obtain approximation [11]-[14]. Some other approximations have limitation preventing them from being used in all conditions. In this chapter, we propose a novel accurate approximation method, namely the Log Skew Normal (LSN) approximation, to model the sum of M lognormal distributed random variables. The proposed LSN approximation has good accuracy in most of the entire region of CDF, especially in the lower region. Furthermore, this approximation does not require the CDF curve, making it more desirable to engineers and researchers.

4.1 Log Skew Normal Distribution

4.1.1 Skew Normal Distribution

Let $f_N(x)$ denote the standard normal distribution function:

$$f_N(x) = N(0,1) = \frac{1}{\sqrt{2\pi}} \exp(-\frac{x^2}{2}) \tag{4.1}$$

with the CDF given by $F_N(x)$ in Eq. (2.6). Then the equivalent Skew-Normal distribution is:

$$f(x) = 2 f_N(x) F_N(\alpha x) \tag{4.2}$$

To add location and scale parameters to this (corresponding to mean and standard deviation for the normal distribution), one makes the usual transform $x \Rightarrow \frac{x-\epsilon}{\omega}$. This transform yields the general Skew-Normal Distribution PDF:

$$\begin{aligned} f_S(x) &= 2f_N(\frac{x-\epsilon}{\omega})F_N(\alpha\frac{x-\epsilon}{\omega}) \\ &= \frac{2}{\omega\sqrt{2\pi}}\exp(-\frac{(x-\epsilon)^2}{2\omega^2})\int\limits_{-\infty}^{\alpha(\frac{x-\epsilon}{\omega})}\frac{1}{\sqrt{2\pi}}\exp(-\frac{t^2}{2})dt \end{aligned} \tag{4.3}$$

4.1.2 Moments of Skew Normal Distribution

From the PDF of Skew Normal S, three moments of S could be obtain as follow.

Mean:

$$\begin{aligned} E[S] &= \int\limits_{-\infty}^{\infty} f_S(x)x dx \\ &= \epsilon + \omega\sqrt{2/\pi}\delta \end{aligned} \tag{4.4}$$

Variance:

$$\begin{aligned} Var[S] &= E[S^2] - (E[S])^2 \\ &= \int\limits_{-\infty}^{\infty} f_S(x)x^2 dx - (E[S])^2 \\ &= \omega^2(1-\frac{2\delta^2}{\pi}); \end{aligned} \tag{4.5}$$

Skewness:

$$SK[S] = \frac{\frac{4-\pi}{2}\frac{2}{\pi}\sqrt{\frac{2}{\pi}}\delta^3}{(1-2\frac{\delta^2}{\pi})^{\frac{3}{2}}}; \tag{4.6}$$

where

$$\delta = \frac{\alpha}{\sqrt{1+\alpha^2}} \tag{4.7}$$

For a sample of n values the sample, the skewness is

$$SK[X] = \frac{\frac{1}{n}\sum\limits_{i=1}^{n}(x_i-\bar{x})^3}{(\frac{1}{n}\sum\limits_{i=1}^{n}(x_i-\bar{x})^2)^{3/2}} \tag{4.8}$$

4.1.3 Log Skew Normal Distribution

Similarly to the relation between normal and lognormal distributions, we consider a distribution called log skew normal (LSN) to describe the skew normal distribution in logarithmic domain, i.e., a LSN RV LS defined as $LS = e^{\zeta S} = 10^{S/10}$, where S is Skew Normal distributed. The CDF of LSN RV LS could be computed as follow:

$$\begin{aligned} F_{LSN}(m) &= F(LS < m) \\ &= F(e^{\zeta S} < m) \\ &= F(S < \ln(m)/\zeta) \\ &= \int_{-\infty}^{\ln(m)/\zeta} f(x)dx \\ &= \int_{-\infty}^{\ln(m)/\zeta} 2f_N(\tfrac{x-\epsilon}{\omega})F_N(\alpha\tfrac{x-\epsilon}{\omega})dx \end{aligned} \tag{4.9}$$

Taking the first derivative to both sides above and manipulating yield the pdf of LSN as

$$f_{LSN}(m) = \frac{1}{\zeta m}2f_N(\frac{\ln(m)/\zeta - \epsilon}{\omega})F_N(\alpha\frac{\ln(m)/\zeta - \epsilon}{\omega}) \tag{4.10}$$

LSN distribution has three parameters α, ω and ϵ , since we get the parameters, we could know the PDF and CDF of the Log Skew Normal RV. We aim to use the introduced LSN distribution to approximately describe the sum of a number of lognormal RVs and to derive the LSN parameters (α,ω and ϵ) from the parameters (μ,σ) of the individual lognormal RVs by matching the moments in logarithmic domains.

4.2 Log Skew Normal Approximation

Consider the model in Eq. (2.4) again,

$$\Lambda = \sum_{k=1}^{M} L_k = \sum_{k=1}^{M} e^{X_k} \approx LS \tag{4.11}$$

where $X_k s$ are Gaussian RVs, LS is a Log Skew Normal RV to approximate the Lognormal Sum. LSN approximation using moments matching to compute the three parameters of the LS.

4.2.1 Moments Matching

Pearson Approximation [15] mentioned, Skewness is another factor which will affect the CDF of the lognormal sum, besides the Mean and Variance. There are three parameters to present the LSN RV , hence we will use Moments Matching (Mean, Variance and Skewness) to compute the three parameters α,ω and ϵ:

$$\begin{aligned} E[\ln(\Lambda)] &\approx E[\ln(LS)] \\ Var[\ln(\Lambda)] &\approx Var[\ln(LS)] \\ SK[\ln(\Lambda)] &\approx SK[\ln(LS)] \end{aligned} \tag{4.12}$$

where LS is LSN RV, and from the definition of LSN it is obvious that $\ln(LS)$ is a SN RV.

4.2.2 Computing Parameters

From Eq. (4.4) - (4.6), the right side parts of the equations (4.12) are be given as:

$$\begin{aligned} E[\ln(LS)] &= \varepsilon + \omega\sqrt{2/\pi}\delta; \\ Var[\ln(LS)] &= \omega^2(1 - \tfrac{2\delta^2}{\pi}); \\ SK[\ln(LS)] &= \frac{\frac{4-\pi}{2}\frac{2}{\pi}\sqrt{\frac{2}{\pi}}\delta^3}{(1-2\frac{\delta^2}{\pi})^{\frac{3}{2}}}; \end{aligned} \tag{4.13}$$

The left side parts could easily be obtained from the Monte Carlo Simulation denoted as:

$$\begin{cases} E[\ln(\Lambda)] = a; \\ Var[\ln(\Lambda)] = b; \\ SK[\ln(\Lambda)] = c; \end{cases} \tag{4.14}$$

Now we obtain three equations:

$$\begin{aligned} \varepsilon + \omega\sqrt{2/\pi}\delta &= a; \\ \omega^2(1 - \tfrac{2\delta^2}{\pi}) &= b; \\ \frac{\frac{4-\pi}{2}\frac{2}{\pi}\sqrt{\frac{2}{\pi}}\delta^3}{(1-2\frac{\delta^2}{\pi})^{\frac{3}{2}}} &= c; \end{aligned} \tag{4.15}$$

Solving these three equations, we have:

$$\begin{aligned} \delta &= \sqrt{\frac{c^3}{(\frac{4-\pi}{2}\frac{2}{\pi}\sqrt{\frac{2}{\pi}})^{\frac{2}{3}} + \frac{2}{\pi}c^{\frac{2}{3}}}} \\ \omega &= \sqrt{\frac{b}{1-\frac{2}{\pi}\delta^2}} \\ \varepsilon &= a - \omega\sqrt{\frac{2}{\pi}}\delta \end{aligned} \tag{4.16}$$

4.2.3 CDF and PDF of LSN

Since the three parameters α,ω and ϵ, the CDF and PDF are the same as defined in Eq. (4.9) and (4.10). In the logarithm scale, the PDF is the same as defined in Eq. (4.3).

4.3 Results

We examine the results of the proposed LSN and other approximation techniques in some representative cases. The cumulative distribution function (CDF) of the sum of M lognormal RV's generated by Monte Carlo simulation is used as reference.

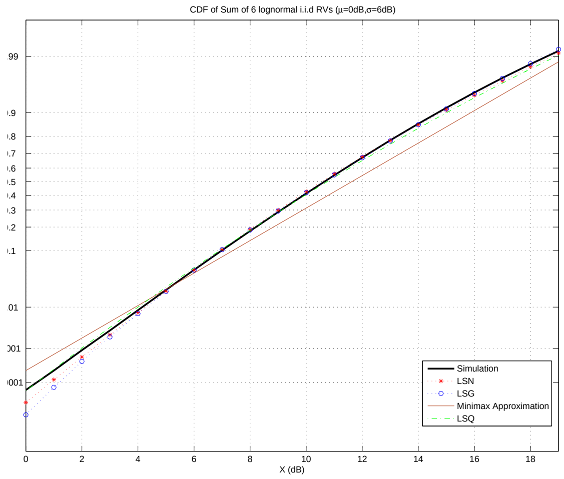

Figure 4.1: CDF of a sum of 6 i.i.d. lognormal RV's with $\mu = 0$ dB and $\sigma = 6$dB.

Fig. 4.1 and Fig. 4.2 show the results for the cases of 6 lognormal distributions with mean of 0dB and standard deviation of 6dB and 12dB, respectively. Fig. 4.3 is the figure for sum of 6 lognormal RV's having the same standard deviation of 12dB, but different means. The CDFs plotted in "lognormal Probability scale" [4] in these figures have less curvature.

In Fig. 4.4, we consider the sum of 6 lognormal RV's having the same mean of 0dB, but with different standard deviations. The curvature of the reference (simulation) CDF is increased in this case of lognormal RVs with different standard deviation values as compared to that in Fig. 4.1 , Fig. 4.3 and Fig. 4.2. The more severe the curvature, the worse the lognormal approximation is due to its straight line limitation. We can see that the lognormal approximation methods can only fit a part of the entire sum of distribution, but we could find that the LSN method is better than the LSG in the region of CDF is less than 0.1.

In Fig. 4.5, this is the figure for sum of 30 lognormal RV's having the same mean of 0dB and same standard deviation of 4, 6, 8, 10, 12dB. We could see that the LSN curve matches the simulation curve very well.

TABLE 4.1 lists the calculated LSN parameters for different cases of sum of lognormal RVs for reference.

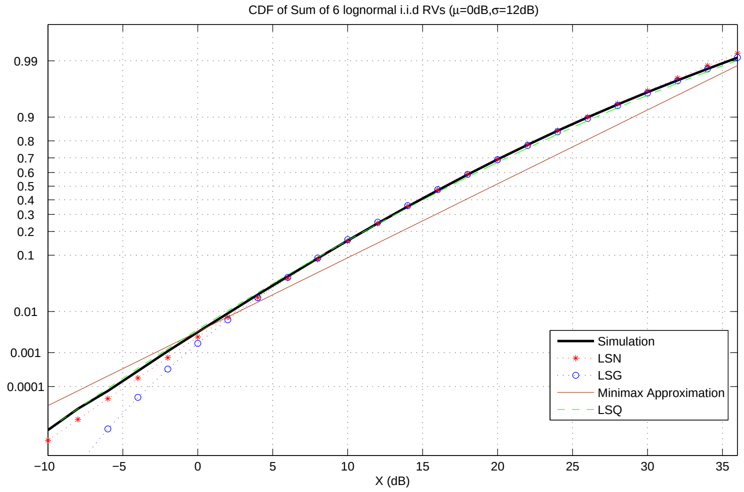

Figure 4.2: CDF of a sum of 6 i.i.d. lognormal RV's with $\mu = 0$dB and $\sigma = 12$dB.

4.4 Comparison

From the observation of these approximation methods, each method has its advantage and disadvantage:

- Minimax Approximation: This approximation method is very easy to obtain. It uses a straight line to approximate a curve on "Lognormal Probability Paper". However,when M increases, the CDF of lognormal sum becomes more curving, which means it is not accurate any more using a straight line to approximate a curve when M is large or when the difference among the dB spreads of the summands increases.

- LSQ Approximation: This approximation method make up the linearity limitation of Minimax Approximation, using high order approximation to get better approximating performance. The LSQ method is better than Minimax method, however, their parameters cannot be determined until the numerical or empirical PDF and CDF of the lognormal sum distribution have been obtained, and hence the complexity to use this approximation method is high, while we don't want to do the Monte Carlo Simulation to get the whole curve first then get an approximation to it in most time.

- Type IV Pearson Approximation: This approximation method uses Type IV Pearson RV to approximate the sum of lognormal RV's. Using this method, it is easy to find the parameters for the new Type IV Pearson RV then approximate the lognormal sum. However, this approximation has limitation $k < 1$ in Table (3.2) to use Type IV Pearson Distribution, thus, this approximation

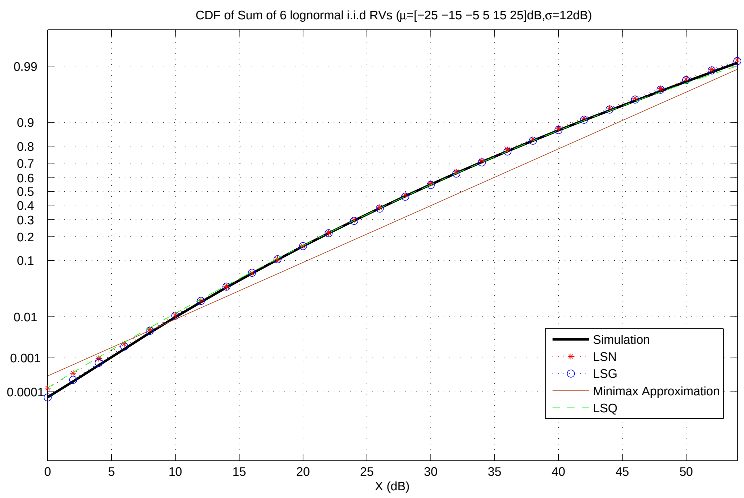

Figure 4.3: CDF of a sum of 6 i.i.d. lognormal RV's with μ =[-25 -15 -5 5 15 25]dB and σ = 12dB.

could not use to approximate under any conditions.

- LSG Approximation: This approximation method also uses another RV Log Shifted Gamma to approximate the sum of lognormal RV's, and could be used to approximate under any conditions. However, at the lower region of x (when power is lower), this approximation do not fit the curve very well and has more error with the approximation.
- LSN Approximation Method uses Log Skew Normal Approximation which has three parameters to control the shape and position of the CDF curve, and could fit the curve very well especially for the lower region of x.

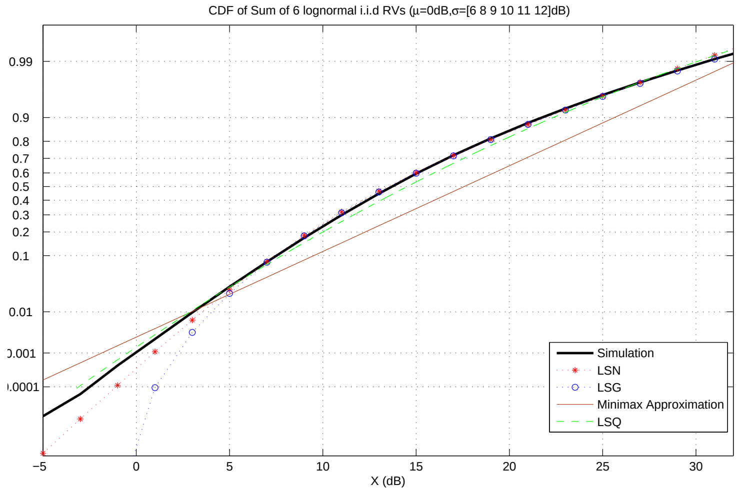

Figure 4.4: CDF of a sum of 6 i.i.d. lognormal RV's with μ = 0dB and σ = [6 8 9 10 11 12]dB.

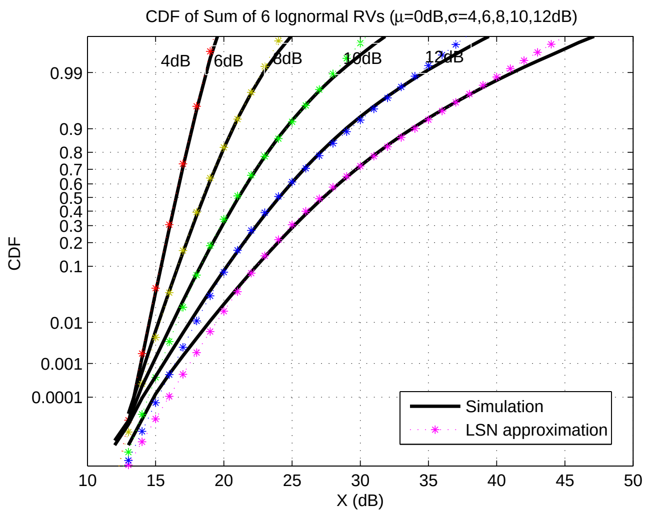

Figure 4.5: CDF of a sum of 30 i.i.d. lognormal RV's with μ =0dB and σ = 4,6,8,10,12dB.

Table 4.1: List of LSN Parameters of Different Cases of Sum of Lognormal RV's

Cases of Sum of Lognormal RVs	α	ω	ϵ
6 RVs $\mu = 0$dB, $\sigma = 6$dB (Fig. 4.1)	1.6480	0.9745	1.8191
6 RVs $\mu = 0$dB, $\sigma = 12$dB (Fig. 4.2)	1.6796	2.1754	2.4114
6 RVs $\mu =$ [-25,-15,-5,5,15,25]dB, $\sigma = 12$dB (Fig. 4.3)	1.3483	1.4895	5.0946
6 RVs $\mu = 0$dB, $\sigma =$ [6 8 9 10 11 12]dB (Fig. 4.4)	2.8453	2.0076	1.7630
30 RVs $\mu = 0$dB, $\sigma = 6$dB (Fig. 4.5)	1.9458	0.5193	3.9120

5

Conclusion

This thesis proposes a novel accurate approximation method to approximate the lognormal sum distribution random variables. A thorough survey of existing approximation methods to approximate the Sum of Lognormal RV's in literature is given and a performance comparison via numerical simulation is also provided. From the performance comparison, the advantages and disadvantages of these methods are analyzed. Next, the new approximation method, namely Log Skew Normal Approximation (LSN) is described. This new approximation method extracts the advantages of other methods and avoids their disadvantages, then builds an optimal approximation with high accuracy in most region especially in the low region which is the most important region for wireless communication applications. The close-form of the proposed LSN approximation method is given, with parameter calculation algorithms via moment matching method. Numerical results over a wide range of conditions confirm the benefits of the proposed low-complexity, high-accuracy approximation method.

With this novel approximation method, more accurate prediction of outage probabilities in cognitive radio and dynamic spectrum access network can be obtained. Moreover, by using this approximation method, more accurate and meaningful coexistence analysis of primary users and secondary users in cognitive radio is feasible. This method can also be applied to traditional co-channel cell interference analysis.

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