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Computational Analysis of Vortex Structures in Flapping Flight

Zongxian Liang

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Computational Analysis of Vortex Structures in Flapping Flight

A dissertation submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy

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Abstract


Vortex structures and vortical formation in flapping flight are directly related to the force production. To analyze the connection between vortex structures and aerodynamic performance of flapping flight, we have developed highly efficient algorithms for large-scale flow simulations with moving and deforming bodies. To further understand the underlying mechanisms of force generation caused by the coherent structures of the vortex formation, a new analysis method has been developed to measure the influence of Proper Orthogonal Decomposition (POD) modes on aerodynamic forces.

It is challenging to finish three-dimensional Direct Numerical Simulations (DNS) of insect flight in a limited amount of time. In the current work, the Modified Strongly Implicit Procedure (MSIP) has been implemented into an existing Computational Fluid Dynamics (CFD) solver, as a smoother for the multigrid method to solve the pressure equation and an iterative method to solve the momentum equation. The new solver is capable of performing a 17-million-mesh simulation within 10 days on a single core of an Intel i5-3570 chip at 3.4GHz, nearly 10 times faster than the traditional LSOR solver.

Based on this numerical tool, the free flight of a dragonfly for eight-and-a-half wing beats is studied in detail. The results show that the dragonfly has experienced two flight stages during the flight. In a maneuver stage, wing-wake interaction generated by the fore- and hindwings attenuates the total force by 8% (peak value). In contrast, in an escape stage, the fore- and hindwings collaborate to generate force which is 8% larger than when they flap
separately. Especially, the peak force on the forewing is significantly increased by 42% in a downstroke and this enhancement is known to associate with a distorted trailing edge vortex, as demonstrated by a theoretical model based on wake survey methods. The movement of the trailing edge vortex is a response to the motion of the hindwing. When the fore- and hindwings flap closely with only a short distance existing between them, the hindwing exerts a wall effect to the trailing edge vortex. Vortex formation of flapping flight and force generation are considered to be closely linked; however, it is difficult to accurately determine the influence of an individual vortex on the overall aerodynamic performance. Here, as an alternative, we examine the influence of coherent structures, which are thought as special types of vortices in terms of kinetic energy. First, wake structures are decomposed by the POD method and the most energetic vortices are extracted. Then, a pressure corrected POD Reduced-Order Models (ROM) method is used to verify that the POD modes can capture the dynamics of the flows. Finally, the force of POD modes is quantified by a new method, termed the POD mode Force Survey Method (POD-FSM).

The process is applied to investigate the flow field generated by a two- or three-dimensional plate undergoing a pitching-plunging motion. Superposition of force of the POD modes shows a good agreement with the DNS result. In addition, it is found that some POD modes have zero lift, and some have zero thrust. These force behaviors are related to symmetry of POD mode. According to the symmetry or antisymmetry about the streamwise line (or the crossflow plane in three-dimension), the POD modes can be qualitatively grouped into two sets. Combining POD modes in the same set can help to decompose the flow into thrust- and lift-producing flows. It is found that the force acting on the plate is a linear
combination of the force of the thrust- and lift-producing flows and their interactions. Because two flows have different frequency spectrum, it is possible to perform flow control with respect to frequency to achieve the desired aerodynamic performance.
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<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
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<tbody>
<tr>
<td>AR</td>
<td>aspect ratio</td>
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<tr>
<td>b</td>
<td>span length</td>
</tr>
<tr>
<td>c</td>
<td>chord length</td>
</tr>
<tr>
<td>(\bar{c})</td>
<td>mean chord length</td>
</tr>
<tr>
<td>(C_D)</td>
<td>drag coefficient</td>
</tr>
<tr>
<td>(C_L)</td>
<td>lift coefficient</td>
</tr>
<tr>
<td>DNS</td>
<td>direct numerical simulation</td>
</tr>
<tr>
<td>(\hat{e}_i)</td>
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</tr>
<tr>
<td>MSIP</td>
<td>modified strongly implicit procedure</td>
</tr>
<tr>
<td>(n)</td>
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<tr>
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<td>strongly implicit procedure</td>
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<td>(t)</td>
<td>time</td>
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<tr>
<td>(T)</td>
<td>viscous stress tensor</td>
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<tr>
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<td>trailing edge vortex</td>
</tr>
<tr>
<td>(U)</td>
<td>flow velocity</td>
</tr>
<tr>
<td></td>
<td>velocity on cell surface</td>
</tr>
<tr>
<td></td>
<td>upper triangular matrix</td>
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</table>
\( \bar{U} \) mean flow velocity
\( \bar{u} \) velocity
\( \bar{u}_m \) mean flow velocity
\( \bar{u}_S \) velocity at surface
\( u^* \) intermediate velocity
\( U_\infty \) incoming flow velocity
\( u \) x-direction velocity component
\( v \) y-direction velocity component
\( w \) z-direction velocity component
\( V_\infty \) infinite fluid domain
\( \alpha \) (AoA) angle of attack
\( \alpha_i \) temporal coefficient
\( \Gamma \) circulation
\( \Omega \) least squares plane
\( \bar{\omega} \) vorticity
\( \rho \) fluid density
\( \mu \) dynamic viscosity of the fluid
\( \bar{\tau} \) first moment of the vorticity
\( \Phi \) velocity POD modes
\( \lambda \) roll angle
\( \Theta \) pitch angle
\( \Psi \) yaw angle
\( \Psi_i \) voriticity POD modes
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Publications

1. Liang Z. and Dong H., “On Forces of Proper Orthogonal Decomposition Modes of Flapping Plates” (under preparation)
Chapter 1 Introduction

1.1 Motivation

Flapping flight is a mode of transportation widely adopted by natural fliers such as birds and insects. It has captured the interests of biologists and engineers because of several characteristics observed in natural fliers. From an energetic perspective, propulsive efficiency of flapping motion can be higher than 85% (Anderson, Streitlien et al. 1998). To meet the need for high efficiency in energy consumption, flapping flight can be chosen as an alternative solution for aircraft propulsion (Shyy et al. 1999). From the perspective of maneuverability and controllability, birds and insects have shown remarkable flying capabilities in very limited space with multiple obstacles. In contrast, conventional aircrafts cannot offer effective and reliable operations in such circumstance. It is natural to think Micro Air Vehicles (MAVs) can borrow these abilities from natural flyers by mimicking flapping flight.

Many aspects of flapping flight, such as wing kinematics, structural response of wing, power consumption, etc., are worth exploring. From the fluid dynamic point of view, studying vortex structures generated by flapping flight is of great importance for two reasons.

First, vortex formation is related to aerodynamic force, either indirectly or directly. Force on fluids exerted by bodies may deform fluids into various formations, leaving footprints of the propulsion in the wake (Platzer, Jones et al. 2008). Therefore, even without force sensors and directly measuring pressure field, some elemental conclusions on force generation can be drawn based on observed vortex formation in the wake. For example, it has been shown by Jones (Jones 1996) that location and orientation of vortex streets behind an oscillating airfoil
can indicate if the force acting on the airfoil is drag-indicative or thrust-indicative (Figure 1-1). When sufficient velocity and vorticity information around a submerged body is given, some wake survey methods (Wu 1981; Noca, Shiels et al. 1999; Wu, Ma et al. 2006) can accurately predict force acting on the body. Because force exerted by fluids on an immersed body is equal and opposite to the force exerted by the body to the fluids, controlling the vortex shedding in the wake of an airfoil is found to be effective in changing the aerodynamic performance of the airfoil (Ho, Nassef et al. 2003).

Figure 1-1. Vortex street behind an airfoil. (a) Drag-indicative; (b) thrust-indicative (Jones 1996).

Secondly, flapping flight usually operates in a Reynolds number (Re) range in which vortices are dominant flow features (10 < Re < 10^5 (Wegener 1992; Dudley 2000)) and many flow phenomena that are observed in flapping flight can be readily explained by using vortex dynamics. For instance, according to the well-known Biot-Savart law, vortex structures in far-field can induce velocity change near a flapping wing without direct interaction. Tijdeman and Seebass (Tijdeman and Seebass 1980) present an example of how vorticity in downstream induced velocity around an airfoil and caused the oscillation of lift lag behind the motion of the airfoil. A few near-field mechanisms for lift enhancement, i.e. leading edge
vortex (Ellington, vandenBerg et al. 1996), rotational forces and wake capture (Dickinson, Lehmann et al. 1999), have been accounted for the presence of vortex structures and their interactions with flapping wings.

Three approaches, theory, experiment, and numerical simulation, can be adopted for the study of vortex formation. A number of conventional steady and quasi-steady aerodynamic theories, e.g. blade element theory and momentum theory (disk actuator theory), have been used to explain a few phenomenons of flapping flight (Ellington 1984; Sane 2003) and depict the generated force. However, these theories were developed to solve steady flow problems. Some assumptions they take, such as irrotational and inviscid flow, do not satisfy the conditions of flapping flight, in which vortex motion is the theme and the generation and dissipation of vorticity is persistent. Ellington presented a comprehensive discussion on whether the steady and quasi-steady models can sufficiently explain the lift producing of flapping flight (Ellington 1984; Ellington 1984). It is found that these theories cannot accurately predict lift of flapping flight due to a lack of consideration of vortex structures and vortex interactions, e.g. leading edge vortex and wake capture. Zbikowski (Zbikowski 2002) proposed a mathematic model that linearly superposes the effects of a quasi-steady wing, added mass of the vortex structures near the wing, and a wake-induced contribution. However, an extension of this model by Pedersen and Zbikowski shows that the results from their theoretical model still deviate from experimental data by about 10% and 153% in mean lift and drag prediction, respectively (Ansari, Zbikowski et al. 2006). In general, the theoretical approach is incapable of accurately predicting aerodynamic performance of flapping flight.
The experimental approach includes observing animals flying in a laboratory environment (Brodsky 1991; Spedding, Rosén et al. 2003; Thomas, Taylor et al. 2004; Bomphrey, Lawson et al. 2006; Hedenström, Johansson et al. 2007; Altshuler, Princevac et al. 2009) and conducting robotic experiments with simplified kinematics (Von Ellenrieder, Parker et al. 2003; Buchholz and Smits 2006; Parker and Von Ellenrieder 2007; Buchholz and Smits 2008; Green, Rowley et al. 2011). Information on velocity field can be achieved by using Particle Image Velocimetry (PIV). However, there are still several difficulties in performing experiments. First, directly measuring full-field of pressure is not feasible with current experimental techniques. Second, flight of living animals is not easy to control and predict, especially insects. The experiments are therefore difficult to repeat with relative consistency. Third, robotic experiments are greatly constrained by flow conditions that experimental devices can create and kinematics that the robots can perform. Therefore, robotic experiments are limited to mimicking hovering or cruising flight.

As the development of numerical methodology and computational power, physics-based high-fidelity method, the Direct Numerical Simulation (DNS), has been widely employed in the recent years (Blondeaux, Fornarelli et al. 2005; Dong 2006; Aono, Liang et al. 2008). Compared to analytic modelling and experiments, DNS is able to provide full-field, spatially high-resolution and temporally continuous velocity and pressure information. However, this detailed information usually comes at a high cost of computational time and memory. Piomelli (Piomelli 1999) estimated that the number of grid points is required to be proportional to $Re^{9/4}$ in order to resolve all scales of motion. Mittal and Laccarino (Mittal and Laccarino 2005) concluded that the ratios to resolve a laminar boundary layer on two-dimensional and three-dimensional bodies are, respectively, $Re^1$ and $Re^{1.5}$ for Cartesian
grid-based immersed boundary methods. Furthermore, free flight of animals requires a corresponding increment of experimental zone size, which is nearly ten times body length in some experiments (Hedenström, Rosén et al. 2006; Rosén, Spedding et al. 2007; Johansson, Wolf et al. 2008), than tethered animals. The computational domain size has to increase accordingly when one attempts to perform comparative studies with the experiments. Consequently, this leads to a linear increment of the number of grid points with respect to desired grid density. Thus, more efficient computational methods are highly significant to study the free flight of animals.

As vortex analysis is the primary focus in this study, the ability to effectively perform the analysis is of great importance. Because of interaction of vortices, such as merging, stretching and separating, it is difficult to accurately track the development of individual vortex structures at high Reynolds numbers for a long-term history. In addition, to study the development of a whole vortex formation, it is required to track a large number of vortices to build the dynamical systems. This further complicates the problem. Currently, there is a lack of a method that can easily identify vortex structure and analyze vortex dynamics of flows. As an alternative, Proper Orthogonal Decomposition (POD) can extract coherent structures (POD modes) of flows based on kinetic energy measurement. In this context, the POD modes are general vortex formations in terms of average kinetic energy. The computation involves simple algebraic operations on the whole flow domain and requires no additional algorithms for vortex tracking. As a result, computational efficiency can be greatly improved. Because of these advantages, the POD method has been widely used to analyze flow fields with stationary boundaries such as driven cavity flows (Cazemier 1998), turbulent shear layers of a round jet nozzle (Citriniti and George 2000), flow past a cylinder (Konstantinidis, Balabani
et al. 2007; Perrin, Braza et al. 2007; Feng, Wang et al. 2011), a hemisphere (Manhart 1998), and an airfoil (Paik, Escauriaza et al. 2007; Kitsios, Cordier et al. 2011).

The study previously mentioned has indicated, however, that applications of the POD method on moving boundaries are infrequently reported, especially for multiple moving objects such as flapping wings. One of the largest concerns surrounding the POD method is that whether POD modes are able to capture the dynamics of flapping flight. To eliminate the concern, establishing a POD-based Reduced-Order Model (POD ROM) that can accurately predict the fluid dynamics of flapping flight is necessary. Furthermore, as an effective medium-fidelity method, POD ROMs can be used to study flapping flight in an acceptable accuracy and cost, compared to a high-fidelity method such as DNS.

It is notable that POD modes are only statistically representative of the average kinetic energy of flows (Blackmore, Krause et al. 2005). Analyses of POD modes are based on the sense of energy capture (Cazemier 1998). In contrast, a proportion of vortex analyses are focused on understanding the connection and interaction of vortex structures and force. A method that is capable of quantifying the influence of POD modes in terms of aerodynamic forces can significantly extend the POD analysis. In addition, because the POD method can decompose flows into a linear combination of spatial-temporal components and POD modes are spatial structures independent to time, forces of a POD mode are determined when the POD mode are computed. This property is of great importance to POD ROMs (Bergmann, Cordier et al. 2005) that monitor aerodynamic forces of flows because it can significantly simplify the calculation of forces.
1.2 Objective

As aforementioned, the research on flapping flight is in great need of powerful DNS tools and effective analysis approaches for vortex dynamics. In order to overcome these difficulties, we intend to develop: first, highly efficient numerical algorithms for an existing CFD solver to perform large-scale simulations; second, a pressure corrected POD ROM method to model the vortex formation of flapping flight; and third, a wake survey method for POD modes to quantify aerodynamic forces caused by POD modes. With these numerical and analysis tools, our goal to numerically analyze vortex structures of flapping flight could be accomplished.

1.3 Research Approach

The whole research consists of three major parts: multi-fidelity methodology, applications, and analysis, as shown in Figure 1-2. The first and second objectives correspond to high- and medium-fidelity methods, respectively. The third one is related to analysis technique.

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Figure 1-2. Chart of research approaches.
To build a computational platform for studying the connection between vortex structures and aerodynamic performance of flapping flight, highly efficient algorithms for solving Navier-Stokes equations with moving boundaries were developed. Literature review was conducted to examine the development and performance of iterative methods for regular flow domains. According to the literature, Modified Strongly Implicit Procedure (MSIP) could potentially be the fastest smoother and iterative method for immersed boundary. The review also explained the difficulties in implementing a smoother for multigrid method and pointed out that the boundary treatment is critical. The MSIP solver was accomplished with regard to these points. A number of validations and benchmarks were conducted to demonstrate the performance improvement of the solver.

As a practical application of the new high-fidelity tool, free flight of a dragonfly for eight-and-a-half wingbeats was studied in detail. Additional tools to analyze flight kinematics of real insects were developed. In order to gain insight into the influence of wing-wake interaction, simulations were conducted for wing combinations such as 4 wings, forewings only, and hindwings only. Based on a comparative study, development of near-field vortices and their influence to aerodynamics performance were examined by various analyses techniques. A model of near-field vortices was built to explain force change due to vortex motion.

Development of an accurate POD based ROM method for flapping flight was based on the discovery that the pressure term in traditional ROM methods was incorrectly ignored in previous studies. As a result of this, a ROM method with pressure term correction was developed. Comparative studies between the models with or without pressure term correction
were conducted for canonical flapping flights. Models for single or two wings flight were examined in terms of short- and long-term behaviors.

POD modes can be thought of as a special type of vortex structure in terms of kinetic energy. It has been proved that the superposition of all POD modes is a solution to the Navier-Stokes equations (Berkooz, Holmes et al. 1993; Holmes, Lumley et al. 1996). However, whether an individual POD mode is a solution of the Navier-Stokes equations is arguable. Theoretically solving this problem is out of the scope of the current research. Here, we hypothesize that each POD mode is associated with a virtual force that can be measured by wake survey methods. Based on this hypothesis, one can have a ‘force’ measurement for POD modes. Force superposition of all POD modes can reconstruct the force of the original flow. This method, termed the POD mode Force Survey Method (POD-FSM), was applied to flow past a low-aspect-ratio plate undergoing a pitching-plunging motion to analyze the force performance of POD modes. It was found that some POD modes always have zero thrust, while others always have zero lift. Further studies indicated that these force behaviors are related to symmetry of POD modes and vortex shedding frequency of the flow. A mathematic model was used to explain how symmetry of POD modes is caused by vortex shedding frequency.

1.4  Historical Perspective

1.4.1  Numerical Methods for Solving Elliptic Equations

An existing in-house CFD solver, which can solve unsteady, two- and three-dimensional fluid problems with moving and deforming boundaries, has been used in previous studies.
However, the solver is not efficient enough to support long-term, large-scale simulations of flapping flight. According to our performance profile, the time spent in solving a pressure-Poisson equation could be more than 90% of total computational time. Thus, the most effective approach to improve the overall efficiency is reducing the computational time in solving the pressure-Poisson equation.

1.4.1.1 Poisson equation solver for immersed boundary methods

The CFD solver uses an immersed boundary method (IBM), which is a class of methods that simulates viscous flows on Cartesian meshes with irregular immersed boundaries that do not conform to the meshes, to achieve a second order accuracy in space. The IBM has several advantages for flapping flight simulations: 1. The Navier-Stokes equations are solved in an Eulerian frame thus the conventional finite difference methods can still be used; 2. There is no need to re-mesh for the moving or deforming of bodies. Because of simplicity and flexibility, the IBM has received extensive attention and a large number of IBM-based solvers have been developed since it was first introduced by Peskin (Peskin 1972) to simulate flows of blood in a heart.

The IBM is widely implemented with a fractional step method, which usually includes one step of solving a pressure Poisson equation. A variety of methods have been used to solve the Poisson equation. Balaras (Balaras 2004) transferred the Poisson equation to a penta-diagonal matrix and then directly solved it. Ye (Ye, Mittal et al. 1999), Tseng (Tseng and Ferziger 2003), Gao (Gao, Tseng et al. 2007) and Taira (Taira and Colonius 2007) used a biconjugate gradient stabilized (Bi-CGSTAB) iterative method. Yang (Yang and Stern 2009) invoked a Krylov-based multigrid solver from the PETSc library. Marella (Marella, Krishnan et al. 2005) used an algebraic multigrid. Mittal (Mittal, Dong et al. 2008) used a geometric
multigrid with a Line-SOR smoother. From the literature, it can be seen that conventional numerical methods solving elliptic equations are still applicable to the immersed boundary methods. However, computational speed of different methods can be significantly different.

### 1.4.1.2 Iterative methods

An iteration is an asymptotic process to reach a solution of the linear algebraic equation $Ax = b$. It can be represented by some linear mapping function in the form of

$$x^{n+1} = g(x^n, b)$$  \hspace{1cm} (1-1)

Unlike direct methods, iterative methods can never solve the equation for an exact solution. However, an approximate solution satisfying certain error criterion is usually sufficient in numerical analysis.

Simple iterative methods are a group of iterative methods which has a linear function

$$g(x^n, b) = Mx^n + Nb$$  \hspace{1cm} (1-2)

It includes Jacobi method (1-3), Gauss-Seidel method (1-4), and successive over relaxation method (1-5):

$$x^{n+1} = D^{-1}(Bx^n + b)$$  \hspace{1cm} (1-3)


$$x^{n+1} = (D - L)^{-1}(Ux^n + b)$$  \hspace{1cm} (1-4)

$$x^{n+1} = (D - \omega L)^{-1}\left(\omega U + (1-\omega)D\right)x^n + \omega b$$  \hspace{1cm} (1-5)
where \([L]\) and \([U]\) are strictly lower and upper triangular matrices, respectively, \([D]-[L]-[U]=[A]\) and \(\omega\) is relaxation factor and always larger than 1. When \(\omega=1\), SOR degrades to the Gauss-Seidel method.

The rate of convergence of a simple iterative method can be calculated by inspecting the spectral radius of \([M]\):

\[
\rho(M) = \max \{ |\lambda| : \lambda \text{ is an eigenvalue of } M \}
\]  

(1-6)

An iterative method converges if and only if \(\rho(M) < 1\). A small \(\rho(M)\) is always desirable since the convergence rate determines how fast an iterative process can converge to a solution.

Incomplete factorization method is one class of iterative methods. Its strategy focuses on finding a decomposition which can factorize an approximation of the matrix \(A\) into the product of triangular matrices. Then, LU decomposition of a diagonal matrix can be executed quickly. The incomplete factorization method includes incomplete Cholesky factorization, Strongly Implicit Procedure (SIP) (Stone 1968), and Modified Strongly Implicit Procedure (MSIP) (Schneider and Zdan 1981). The latter one will be discussed in the next section.

The Krylov space method is one class of methods viewing the problem of linear equations \(Ax = b\) as an equivalent optimization problem of finding the minimum value of

\[
F = \frac{1}{2} \phi^T A \phi - \phi^T b
\]

(1-7)

with respect to all possible \(\phi\). The oldest Krylov space method is conjugate gradient method. It iterates \(x^{n+1}\) as
\[ x^{n+1} = x^n + \alpha_n d^n \]  

where \( d \) are the conjugate gradient directions that are orthogonal with respect to the matrix \([A]\), i.e., \( d^T \cdot Ad = 0 \), and \( \alpha \) is a parameter to construct new searching direction. The rate of convergence of the conjugate gradient method is determined by the condition number \( \kappa \) of the matrix \([A]\), where

\[ \kappa = \frac{\lambda_{\text{max}}}{\lambda_{\text{min}}} \]  

\( \lambda_{\text{max}} \) and \( \lambda_{\text{min}} \) are the maximum and minimum eigenvalues of \([A]\), respectively.

The conjugate gradient method is only applicable to symmetric matrix \([A]\). To solve an asymmetric matrix, one needs to construct a symmetric form for the asymmetric matrix, in the form of

\[
\begin{pmatrix}
0 & A \\
A^T & 0
\end{pmatrix}
\begin{pmatrix}
\varphi \\
\phi
\end{pmatrix}
= 
\begin{pmatrix}
b \\
0
\end{pmatrix}
\]  

This is so called bi-conjugate gradient method. Other Krylov space methods such as conjugate gradient squared (CGS) (Sonneveld 1989), bi-conjugate gradient stabilized (Bi-CGSTAB) (Vandervorst 1992), incomplete Cholesky conjugate gradient (ICCG) (Kershaw 1978), and general minimalized residual (GMRES) (Saad and Schultz 1986) are commonly seen in the literature, but not discussed here.

Multigrid method (Brandt 1977; Brandt 1982) is a special iterative technique that always works with other iterative methods to increase the rate of convergence. Many iterative methods mentioned above smooth the short wavelength components faster than the long wavelength of the error, thus, after a number of iteration steps, the short wavelength components have converged but the long wavelength components have not. Since the
wavelength is related to the mesh size of a grid, by using multiple grids, the multigrid method can smooth different bands of waves on different grids. Therefore, the all components of the error converge simultaneously. Details for the multigrid method will be presented in Section 2.1.2.

Table 1-1: Number of iterations required by various solvers to reduce the \( L_1 \) residual norm below \( 10^{-5} \) on a rectangular domain \( X \times Y = 10 \times 1 \) with uniform grid in X and Y (Ferziger and Peric 1996)

<table>
<thead>
<tr>
<th>Grid</th>
<th>GS</th>
<th>LGS-ADI</th>
<th>SIP</th>
<th>ICCG</th>
<th>MG-GS</th>
<th>MG-SIP</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 32^2 )</td>
<td>1164</td>
<td>18</td>
<td>13</td>
<td>11</td>
<td>242</td>
<td>5</td>
</tr>
<tr>
<td>( 64^2 )</td>
<td>4639</td>
<td>53</td>
<td>38</td>
<td>21</td>
<td>288</td>
<td>6</td>
</tr>
<tr>
<td>( 128^2 )</td>
<td>-</td>
<td>189</td>
<td>139</td>
<td>41</td>
<td>283</td>
<td>6</td>
</tr>
<tr>
<td>( 256^2 )</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>82</td>
<td>270</td>
<td>6</td>
</tr>
</tbody>
</table>

For solving elliptic equations on a regular computational domain, Ferziger et al. (Ferziger and Peric 1996) have reported a comparison on GS, SIP, ICCG, and multigrid methods with a GS or SIP smoother (see Table 1-1). It shows that the MG-SIP method can reduce the residual to a given criterion in the least number of iterations and the shortest time for grids with aspect ratio 10:1. Norris conducted a more comprehensive comparison that includes most well-known iterative methods, e.g. simple iterative methods, incomplete factorization methods, multigrid methods with different smoothers and Krylov space methods (Norris 2000) (see Figure 1-3). He also has reached the same conclusion that the MG-SIP method are among the fastest solvers.
1.4.1.3 Challenges to multigrid with MSIP smoother for IBM

Multigrid has been used with IBM to solve the fluid equations (Roma, Peskin et al. 1999; Zhu and Peskin 2002; Boyce, Richard et al. 2007). However, applications of multigrid with an MSIP smoother for IBM have not been reported, according to the author. This is understandable since coding a geometric multigrid for IBM is difficult work even for researchers who have had a strong background in multigrid coding (Tseng and Ferziger 2003). To incorporate an MSIP smoother that involves complicated boundary treatments into multigrid would be challenging work.

As mentioned by Guy (Guy and Philip 2012), there are two considerable difficulties in the implementation of multigrid for IBM. One is the coarsening and refinement operations between two levels of grids with irregular boundaries which are defined on Lagrangian grids. This difficulty is solvable by using a combination of semi-coarsening technique (Wesseling 1992) and stair-step approximation. In contrast to full coarsening which simultaneously coarsens finer meshes in all directions, semi-coarsening only coarsens in one or two
particular directions. Coarsening in the other direction is scheduled by algorithms (Wesseling 1992; Oosterlee 1995). Stair-step approximation is done by projecting the boundaries defined on Lagrangian grids to the boundaries of Eulerian grids. For cell-centered grids, the boundaries are at the interface of two neighboring grid points. Thus, the projected boundaries may take an appearance of stair steps. The semi-coarsening may generate highly stretched grids on the coarsest grid and increase the iteration number to reduce the errors to a desired magnitude (Ferziger and Peric 1996). Thus, a good smoother which is suitable for smoothing the errors on highly stretched grids is necessary. Compared to the instability problem caused by the full-coarsening method (McAdams, Sifakis et al. 2010), such a drawback is acceptable.

The second difficulty is how to smooth the errors in immersed boundary equations. This problem can be solved by imposing the immersed boundary equations only on the finest level, while solving the Navier-Stokes equations without immersed boundary equations on coarse levels. In the worst case scenario, the multigrid method may degenerate to one level method without smoothing the errors on the coarse levels. However, a convergent solution with the desired accuracy can be finally achieved as long as the immersed boundary equations are iteratively solved on the finest level. Moreover, no matter what order of interpolation is used, the coarsening operation usually causes additional errors in the vicinity of boundaries. This can be easily proved by comparing the boundaries on the finest and coarsest grids. Only in ideal cases where the boundaries on the finest and coarsest grids overlap there is no loss of accuracy. Generally, the loss of accuracy cannot be compensated by solving the immersed boundary equations on coarse levels because the immersed boundary method has usually achieve second order accuracy near boundaries (Mittal and Iaccarino 2005).
Whether MSIP can be a good smoother for multigrid method is also questionable. The sufficient condition for a smoother is

\[ \lambda(M) > -1 \]  

(1-11)

where \( M \) is given in (1-2). Note that the convergent condition (1-6) is different than the smoothing condition (1-11). Thus, not every iterative method can be used as a smoother, for instance, point Gauss-Seidel. However, by adding damping terms, every iterative method can be a smoother (Wesseling 1992). Wesseling also proved that the MSIP method, which belongs to a 7-point ILU family, can always be a smoother. However, it was pointed out that the boundary conditions may affect the smoothing efficiency.

1.4.2 POD Based Reduced-order Modeling Method for Moving Boundaries

Proper Orthogonal Decomposition (POD) is a technique providing optimal basis, which are called dominant features or coherent structures in the fluid field, for an ensemble of experimental data. It has been widely used to analyze flow fields with stationary boundaries such as driven cavity flows (Cazemier 1998), turbulent shear layer of a round jet nozzle (Citriniti and George 2000), flow past a cylinder (Konstantinidis, Balabani et al. 2007; Perrin, Braza et al. 2007; Feng, Wang et al. 2011), a hemisphere (Manhart 1998), and an airfoil (Paik, Escauriaza et al. 2007; Kitsios, Cordier et al. 2011).

Applications of POD to moving boundary problems are limited. A coordinate transformation technique is commonly used so that the POD methods, either the direction method or the snapshot method (Sirovich 1987; Smith, Moehlis et al. 2005), do not need to calculate the ‘fluidic energy’ of solid bodies. Galerkin projection is thus performed on a
modified momentum equation (Lewin and Haj-Hariri 2005; Liberge and Hamdouni 2010). Another approach is to use overlap grids. Laminar flow around a 3-D membrane wing is inspected (Utturkar, Zhang et al. 2005) by using a fixed uniform grid to compute POD modes around a membrane wing. The fluid velocity field is interpolated from the time-variant grid to the fixed uniform one, and the POD basis is computed on the fixed grid.

In numerous studies of the POD based Reduced-Order Models, i.e. (Graham, Peraire et al. 1999; Ma and Karniadakis 2002; Lewin and Haj-Hariri 2005), the outer boundaries of flow fields are set to either zero-velocity or zero-pressure in the normal direction of control surface. This approach can greatly simplify the ROM of flows since no pressure term is involved. However, zero-velocity condition at inflow is not physical. Noack et al. compared the ROM of shear flows that was imposed by correct velocity and pressure boundary conditions (Noack, Papas et al. 2005). They found that the effect of the pressure term is not neglectable. Utturkar et al. also showed that the aerodynamic performance of a moving membrane plate may be coarsened by moving dynamics, due to incomplete information of pressure (Utturkar, Zhang et al. 2005).

In the current study, general velocity and pressure boundary conditions are employed in DNS. Specifically, velocity of the moving object is not zero. As a result, the pressure term does not vanish. Our study found that the flows of flapping flight cannot be accurately modeled without including the pressure term. Thus the pressure term needs specific treatments.

There are a number of methods to compute the pressure term. One is to solve a pressure Poisson equation which is derived by taking the divergence of the Navier-Stoke equations
and then applying the divergence-free condition. Noack (Noack, Papas et al. 2005) proposed to solve a system of partial pressure equations, which expand the pressure term as a function of temporal coefficients of velocity field. Thus, the pressure term can be included in the Galerkin projection. Akhtar et al. (Akhtar, Nayfeh et al. 2009) proposed to conduct POD on pressure field. POD of velocity and pressure are substituted into the pressure-Poisson equation and then projected onto the pressure POD modes. This yields a set of ordinary differential equations that can predict the evolution of pressure.

1.4.3 Wake Survey Methods

The POD-FSM, which we proposed as one of the subject of study, is derived from Noca’s impulse equation (Noca, Shiels et al. 1997), a wake survey method. Wake survey methods are usually denoted by a class of methods by which aerodynamic forces can be computed without full knowledge of the instantaneous pressure field. In situations where using force sensors and directly measuring pressure field are not feasible, studying vortex formation in the wake is considered to be the most effective approach to discover the underlying mechanisms of force generation as the wake is a production of the propulsion (Platzer, Jones et al. 2008). Some wake survey methods (Wu 1981; Noca, Shiels et al. 1999; Wu, Ma et al. 2006) have been invented to connect vortex formation to force acting on submerged bodies.

Wu (Wu 1981) presented a method to calculate the aerodynamic force exerted by a fluid on solid bodies:

\[
\vec{F} = -\frac{1}{N-1} \rho \frac{d \vec{\tau}}{dt} + \rho \sum_{i=1}^{N} \frac{d}{dt} \int_{V} \vec{\nu} dV
\]  

(1-12)
where \( N \) is dimension of space (\( N=2 \) in 2-D and \( N=3 \) in 3-D), \( \rho \) is fluid density, \( N_b \) is the number of solid bodies in the fluid, \( V_i \) is volume of the \( i^{th} \) solid body, and \( \tau \) is the first moment of the vorticity field

\[
\vec{\tau} = \int_{V_c} \vec{r} \times \vec{\omega} dV
\]  

(1-13)

where \( \vec{r} \) is a position vector and \( V_c \) is infinite fluid domain.

Equation (1-13) is valid only in an infinite fluid domain and it requires the body and the flow to start from rest so that total vorticity in the flow field is conservative and equals zero. It states that the force on the body is determined by the time rate of change of the first moment of the vorticity field in the fluid and solid bodies plus an added mass term. Disregarding the constraints on (1-13), we can see that it is the time derivative, not the absolute value, of the first moment of vorticity affecting the aerodynamic force. In addition, the distance of vortex structures between each other is another key factor that may affect the force.

Based on (1-13), Wu et al. (Wu, Ma et al. 2006) derived three alternative force expressions for finite flow domains, written as

\[
\vec{F} = -\frac{\mu}{N-1} \int_{V(t)} \vec{r} \times \nabla^2 \vec{\omega} dV + F_S + F_{S_s}
\]  

(1-14)

\[
\vec{F} = -\frac{\rho}{N-1} \int_{V(t)} \vec{r} \times \frac{\partial \vec{\omega}}{\partial t} dV - \rho \int_{V(t)} \vec{l} dV
\]

\[
= -\frac{\rho}{N-1} \int_{S(t)+S_s(t)} \vec{r} \times \left( \vec{n} \times \vec{l} \right) dS + F_S + F_{S_s}
\]  

(1-15)

and
\[ \vec{F} = -\rho \oint_{S_b(t)} \vec{r} \times \left( \frac{1}{2} \sigma_p + \sigma_{vis} \right) dS \]  
(1-16)

where

\[ \vec{F}_{S_b} = \frac{\rho}{N-1} \int_{S_b(t)} \vec{r} \times \left( \frac{\partial^2 \vec{u}}{\partial t^2} \right) dS \]  
(1-17)

\[ \vec{F}_s = -\frac{\mu}{N-1} \int_{S} \vec{r} \times \left( \vec{n} \times \nabla \times \vec{\omega} \right) dS + \mu \oint_{S} \vec{\omega} \times \vec{n} dS \]  
(1-18)

\[ \sigma_p = \vec{n} \times \frac{\nabla p}{\rho} \]  
(1-19)

\[ \sigma_{vis} = \nu \left( \vec{n} \times \nabla \right) \times \vec{\omega} \]  
(1-20)

\[ \vec{l} = \vec{\omega} \times \vec{u} \] is the Lamb vector, \( S \) is the external surface of fluid domain \( V \) and \( S_b(t) \) is the body surface.

Noca et al. (Noca, Shiels et al. 1997) independently derived a wake survey method for finite domains, called impulse equation

\[ \vec{F} = -\frac{1}{N-1} \frac{d}{dt} \int_{V(t)} \vec{r} \times \vec{\omega} dV + \oint_{S(t)} \vec{n} \cdot \gamma_{imp} dS \] 
\[ + \frac{1}{N-1} \frac{d}{dt} \oint_{S_b(t)} \vec{r} \times \left( \vec{n} \times \vec{u} \right) dS - \oint_{S_b(t)} \vec{n} \cdot \left( \vec{u} - \vec{u}_s \right) \vec{u} dS \]  
(1-21)

\[ \gamma_{imp} = \frac{1}{2} \left[ u^2 I - \vec{u} \vec{u} - \frac{1}{N-1} \left( \vec{u} - \vec{u}_s \right) \left( \vec{r} \times \vec{\omega} \right) + \frac{1}{N-1} \vec{\omega} \left( \vec{r} \times \vec{u} \right) \right. \] 
\[ + \frac{1}{N-1} \left. \left[ \vec{r} \cdot \left( \nabla \cdot T \right) I - \vec{r} \left( \nabla \cdot T \right) \right] + T \]  
(1-22)
where $I$ is the unit tensor, $\vec{u}_s$ is velocity of the control surface and $T$ is the viscous stress tensor $T = \mu \left( \nabla \vec{u} + \nabla \vec{u}^T \right)$. Their method also has the vorticity moment term as Wu’s methods.

The surface integral term $\gamma_{imp}$ in (1-21) includes the information of external boundaries. The first two terms in $\gamma_{imp}$ represent the Kutta-Zhukovsky or vortex force.

Based on the impulse equation, Noca et al. (Noca, Shiels et al. 1999) developed another two equations, momentum equation

$$\vec{F} = -\frac{d}{dt} \int_{V(t)} \vec{u}dV + \oint_{S(t)} \vec{n} \cdot \gamma_{mom} dS$$

$$-\oint_{S_{b(t)}} \vec{n} \cdot (\vec{u} - \vec{u}_s) \vec{u} dS$$

(1-23)

$$\gamma_{mom} = \frac{1}{2} \vec{u}^2 I + \vec{u} \cdot \left[ \left( \frac{\partial}{\partial t} \vec{u} \right) \vec{u} \right] - \frac{1}{N-1} \vec{u} \left( \vec{r} \times \vec{\omega} \right) + \frac{1}{N-1} \vec{\omega} \left( \vec{r} \times \vec{u} \right)$$

$$- \frac{1}{N-1} \left[ \left( \vec{r} \cdot \frac{\partial}{\partial t} \vec{u} \right) I - \vec{r} \frac{\partial \vec{u}}{\partial t} \right] + \frac{1}{N-1} \left[ \vec{r} \cdot (\nabla \cdot T) I - \vec{r} \left( \nabla \cdot T \right) \right] + T$$

(1-24)

and flux equation

$$\vec{F} = \oint_{S(t)} \vec{n} \cdot \gamma_{flux} dS$$

$$-\oint_{S_{b(t)}} \vec{n} \cdot (\vec{u} - \vec{u}_s) \vec{u} dS - \frac{d}{dt} \oint_{S_{b(t)}} \vec{n} \cdot (\vec{u} \vec{r}) dS$$

(1-25)
\[
\gamma_{\text{flux}} = \frac{1}{2} u^2 I - \bar{u} \bar{u} - \frac{1}{N-1} \bar{u} (\bar{r} \times \bar{\omega}) + \frac{1}{N-1} \bar{\omega} (\bar{r} \times \bar{u}) \\
- \frac{1}{N-1} \left[ \frac{\partial \bar{u}}{\partial t} \right] \left[ I - \bar{r} \frac{\partial \bar{u}}{\partial t} + (N-1) \frac{\partial \bar{u}}{\partial t} \bar{r} \right] \\
+ \frac{1}{N-1} \left[ \bar{r} \cdot (\nabla \cdot T) I - \bar{r} \left( \nabla \cdot T \right) \right] + T
\]

(1-26)

1.5 Chapter Summaries

The remainder of this paper is organized as follows:

Chapter 2 discusses details of MSIP for the immersed boundary method used in the current numerical simulations. A number of validations and verifications have been conducted to demonstrate the performance improvement of the method. In addition, the POD-Galekin method is introduced.

Chapter 3 describes a comprehensive study on free flights of a dragonfly. The results show the fore- and hindwings can collaborate to generate larger forces on the forewing when two wings flap closely with only a short distance existing between them. The force enhancement of the forewings is associated with a distorted trailing edge vortex, as explained by a two-vortex model.

Chapter 4 presents a pressure corrected ROM method. The applicability of the POD method on flapping flight is discussed through studying projection error of the POD method and prediction error of ROM. The pressure corrected ROMs for single and tandem-wing flights were examined in terms of short- and long-term behaviors.

Chapter 5 contains a POD-FSM developed in this dissertation and an application of the method on types of flows which are characterized by half-periods. POD modes of such flows
can be grouped with respect to their symmetry attributes. Combining each group of modes can decompose the flow into thrust- and lift-producing flows. The force acting on the plate can be expressed as a linear combination of the force of the thrust- and lift-producing flows and their interactions.

Chapter 6 presents a summary of the current study and future work.
Chapter 2  Numerical Methodology

This chapter consists of three parts: numerical method for solving the Navier-Stokes equations, proper orthogonal decomposition (POD) method and Galerkin projection method for flapping flight. In the first part, some background information such as fractional step method, ghost-cell based immersed boundary method (IBM), multigrid method and line successive over-relaxation for IBM are given first. Modified Strongly Implicit Procedure and its coupling with IBM are presented next. Six numerical examples are considered for the validations and benchmarks of the method. In the second part, the POD method (snapshot method) is introduced. In the last part, we discussed how to use Galerkin projection to obtain Reduced-Order Models (ROM) of flows.

2.1  Numerical Method for Solving the Navier-Stokes Equations

2.1.1  Fractional Step (Projection) Method

The equations governing the motion of fluids can be derived from the statements of the conservation of mass, momentum, and energy. In fluids where the Mach number is below 0.3, compressibility of fluids can be neglected. Thus, we have the non-dimensional, viscous incompressible Navier-Stokes equations in forms of

\[ \nabla \cdot \mathbf{u} = 0 \]

\[ \frac{\partial \mathbf{u}}{\partial t} + \nabla \cdot (\mathbf{u} \mathbf{u}) = -\nabla p + \frac{1}{Re} \nabla^2 \mathbf{u} \]  \hspace{1cm} (2-1)

The vorticity equation can be obtained by taking the curl of the Navier-Stokes equations:
This equation does not contain the pressure term. In two-dimension, one can apply the vorticity-stream function approach to replace the velocity components with the stream function \( \psi \). This separates the mixed elliptic-parabolic 2-D incompressible Navier-Stokes equations into one parabolic equation and one elliptic equation, and both equations can be easily solved. Additionally, solving the vorticity equation is more appropriate in the context of vortex-dominant flows. However, stream functions do not exist for a three-dimensional flow. Therefore, we use the primitive variable approach, specifically the fractional step method (Chorin 1967; Chorin 1968), to solve the Navier-Stokes equations.

Expanding (2-1) in a three-dimensional Cartesian coordinate system yields:

\[
\begin{align*}
\frac{\partial u}{\partial t} + \frac{\partial uu}{\partial x} + \frac{\partial uv}{\partial y} + \frac{\partial uw}{\partial z} &= -\frac{\partial p}{\partial x} + \frac{1}{\text{Re}} \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) \\
\frac{\partial v}{\partial t} + \frac{\partial vu}{\partial x} + \frac{\partial vv}{\partial y} + \frac{\partial vw}{\partial z} &= -\frac{\partial p}{\partial y} + \frac{1}{\text{Re}} \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \right) \\
\frac{\partial w}{\partial t} + \frac{\partial uw}{\partial x} + \frac{\partial vw}{\partial y} + \frac{\partial ww}{\partial z} &= -\frac{\partial p}{\partial z} + \frac{1}{\text{Re}} \left( \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right)
\end{align*}
\] (2-3)

The fractional step method consists of three steps, providing a second-order accuracy in time. In the first step an intermediate velocity field \( u^* \) is calculated from the momentum equations without the contribution of the pressure gradient. A second-order central difference is used to discretize the first and second derivatives of velocity with respect to spatial
coordinates on a non-uniform Cartesian mesh, where primitive variables (velocity and pressure) are collocated at the cell centers. The convective terms are represented using an explicit Adams-Bashforth scheme and the diffusive terms is modeled with an implicit Crank-Nicolson scheme. Then the discretized momentum equations are written as:

\[
\frac{u_i^* - u_i^n}{\Delta t} = -\frac{1}{2} \left( 3N_i^n - N_i^{n-1} \right) + \frac{1}{2 \text{Re}} \left( L_i^* + L_i^n \right)
\]  

(2-4)

where the superscript \( n \) means time step \( n \), the star denotes the intermediate step between time step \( n \) and \( n+1 \), \( N \) and \( L \) denote, respectively, the advection and diffusion terms, both of which are discretized by a second-order central difference

\[
N_i = \frac{\delta (u_i u_j)}{\delta x_j}
\]  

(2-5)

\[
L_i = \frac{\delta}{\delta x_j} \left( \frac{\delta u_i}{\delta x_j} \right)
\]  

(2-6)

where \( \frac{\delta}{\delta x} \) corresponds to the second-order central difference.

For simplicity, our following discussions are based on two-dimensional cases. However, these discussions can be generally extended to three-dimensional space. An example of \( N_u \) shows discretization of the central difference for the first derivative term in the x-direction:

\[
N_u = \frac{u_e U_e - u_w U_w}{\Delta x} + \frac{u_u V_n - u_l V_s}{\Delta y}
\]  

(2-7)
where \( u_e \) and \( U_e \) can be interpolated by averaging two adjacent cells \( u_e = \frac{1}{2} (u_E + u_p) \) (see Figure 2-1 for name notation), or higher order interpolation such as QUICK scheme. The other terms are analogous.

Figure 2-1. Schematic describing the name convention used in the spatial discretization of the governing equation in two-dimension. In addition to the cell-centred velocities \((u_p, v_p)\), face-centred velocities that are normal to the cell face, such as \( U_e \) and \( V_n \), are computed, similar to a staggered arrangement.

The central difference for \( L_u \) on uniform grids can be written as

\[
L_u = \frac{u_E - u_p - u_p - u_w}{\Delta x} + \frac{u_N - u_p - u_p - u_s}{\Delta y}
\]  

(2-8)

The above formula can be simplified in forms of

\[
L_u = a_E u_E + a_w u_w + a_p u_p + a_N u_N + a_s u_s
\]

(2-9)

where
\[ a_E = a_W = \frac{1}{\Delta x^2} \]

\[ a_N = a_S = \frac{1}{\Delta y^2} \]  

(2-10)

\[ a_P = -(a_E + a_W + a_N + a_S) = -(\frac{2}{\Delta x^2} + \frac{2}{\Delta y^2}) \]

Rearrange (2-4) by moving unknown variables to the left-hand side and known variables to the right-hand side:

\[ u_i^* - \frac{\Delta t}{2 \text{Re}} L_i^* = u_i^n - \frac{\Delta t}{2} (3N_i^n - N_i^{n-1}) + \frac{\Delta t}{2 \text{Re}} L_i^n \]  

(2-11)

The coefficients of the intermediate velocity \( u^* \) form a 5- and 7-diagonal matrices in two- and three-dimension, respectively. These matrices can be solved iteratively by iteration methods such as Point Jacobi, Gauss-Seidel, Line-SOR, and SIP methods. For instance, given a grid stencil in forms of \( a_E u_E + a_W u_W + a_P u_P + a_N u_N + a_S u_S = RHS \), the point Gauss-Seidel method gives

\[ u_p^{k+1,*} = \frac{1}{a_p} \left( RHS - a_E u_E^{k,*} - a_W u_W^{k,*} - a_N u_N^{k,*} - a_S u_S^{k,*} \right) \]  

(2-12)

where the superscript \( k \) denotes the \( k^{th} \) iteration step.

Velocity \( u^* \) is not necessary to satisfy the divergence-free condition. However, for each time step \( n \) and \( n+1 \), velocity \( u^n \) and \( u^{n+1} \) has to satisfy the divergence-free condition so that the mass conservation holds at every time step. Thus, a projection step is necessary:

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\[
\frac{u_i^{n+1} - u_i^*}{\Delta t} = -\frac{\delta p^{n+1}}{\delta x_i}
\]  
(2-13)

Applying the divergence operator and invoking the divergence-free condition, the discretized Poisson equation for the pressure field is derived:

\[
\frac{\delta}{\delta x_i} \left( \frac{\delta p^{n+1}}{\delta x_i} \right) = \frac{1}{\Delta t} \left( \frac{\delta u_i^*}{\delta x_i} \right)
\]  
(2-14)

Note that the intermediate velocity \(u^*\) on the right-hand side of (2-14) is normal velocity on cell faces. It is calculated based on cell center velocity \(u^*\). Thus, we have the central difference scheme for pressure \(p\) written as

\[
a_E p_E + a_w p_w + a_p p_p + a_N p_N + a_S p_S = \frac{1}{\Delta t} \left( \frac{U_n^* - U_{n+1}^*}{\Delta x} + \frac{V_n^* - V_{n+1}^*}{\Delta y} \right)
\]  
(2-15)

where coefficient for pressure \(p\) are the same as (2-10).

The second step of the fractional step method is to solve the pressure-Poisson equation (PPE) in a compact second-order central difference stencil. For objects immersed in fluids, the projection method sets the boundary condition of \(u^*\) on the surface of the objects to be equal to that of \(u_n^{n+1}\), thus, the boundary condition for the pressure-Poisson equation can be derived from (2-13),

\[
\frac{\delta p^{n+1}}{\delta x_i} = 0
\]  
(2-16)

which is a homogeneous boundary condition of Neumann type. It is notable that \(p^{n+1}\) is an approximation to the theoretical pressure value \(p\) at time step \(n+1\) because boundary condition (2-16) is imposed by the projection method on the surface of objects.
Finally, the divergence-free velocity field \( u^{n+1} \) is obtained at the third step of the fractional step method by correcting the intermediate velocity field with the computed pressure gradient:

\[
u_i^{n+1} = u_i^* - \Delta t \frac{\delta p^{n+1}}{\delta x_i}
\]  

(2-17)

2.1.2 Multigrid Method

Solving the pressure-Poisson equation (2-14) with boundary conditions (2-16) is one of the most time-consuming steps in the projection method. For a simulation with \( N \) grids, direct methods such as Gauss-Jordan elimination cost tremendous time \( (O(N^3)) \) operations. Fourier and cyclic reduction methods can reduce the workload to \( O(N \log N) \) operations. But the most efficient one is multigrid method with \( O(N) \) operations. A geometric multigrid method is used in our solver.

The multigrid concept is based on an observation: long-wave (low-frequency) Fourier modes of solution decay slowly in iterative methods such as Jacobi, Gauss-Seidel, and successive over-relaxation (SOR), while short-wave (high-frequency) Fourier modes decay rapidly. If increasing the grid size, long waves on the old (finer) grid will become short waves on the new (coarser) grid. Note that ‘long’ or ‘short’ are relative terms with respect to grid number. Thus, the multigrid principle is to leave the long wavelength (smooth) part of the error approximated on coarser grids. The short wavelength (non-smooth) part is reduced by a small amount of smooth operations on the fine grid. Thus, the rate of convergence for different wavelengths is similar at all level of grids (Press, Flannery et al. 1992; Wesseling 1992). Generally, an iterative method can be a good smoother, but not vice versa.
We use a two-grid system to elucidate the multigrid methodology. Suppose we are trying to solve the linear elliptic equation on a uniform grid system with mesh size of \( h \)

\[
L_h u_h = f_h(x, y)
\]  

where \( L_h \) is some linear elliptic operator and \( f_h \) is the source term.

Let \( \phi_h \), which usually is set to zero, denote an initial guess solution to the elliptic equation (2-18). We make some steps of iteration on this level and get a better approximate solution \( \phi_h \) of the elliptic equation. The error between the approximate solution and exact solution \( u_h \) is

\[
\varepsilon_h = L_h^{-1}(f_h) - \phi_h = u_h - \phi_h
\]

The residual is

\[
r_h = f_h - L_h \phi_h
\]

Since the elliptic operator \( L_h \) is linear,

\[
L_h \varepsilon_h = L_h(u_h - \phi_h) = L_h u_h - L_h \phi_h = f_h - L_h \phi_h
\]

The error satisfies

\[
L_h \varepsilon_h = r_h
\]

We form an appropriate approximation \( L_H \) of \( L_h \) on a coarser grid with mesh size \( H \). The residual equation becomes

\[
L_H \varepsilon_H = r_H
\]
where $r_H$ and $\varepsilon_H$ are appropriate approximations of $r_h$ and $\varepsilon_h$ on a coarser grid $H$, respectively. Long-wave errors will be carried by the error $\varepsilon_H$ and be reduced after solving (2-23). On the coarse grid these errors damp faster than on the fine grid. Moreover, Solving $\varepsilon_H$ on the coarse grid is always faster than solving $\varepsilon_h$ on the fine grid since it involves fewer operations. We project $\varepsilon_H$ to $\varepsilon_h$ then the new approximation $\phi_h = \phi_h + \varepsilon_h$ is computed.

Not only on the two-grid system, can the multigrid method be applied to multiply grids. The above operations can be recursively performed until $\phi_h$ on the finest grid converges to $u_h$. A pseudo-code is given below for the recursive process of the multigrid method. Note that $L_h^{-1}(u_h, f_h)$ means to get an approximate solution by performing iterative operations with an initial value $u_h$, whereas $L_h^{-1}(f_h)$ represents solving $L_h$ directly to get an exact solution. $I_h^H$ denotes an interpolation from a fine grid to a coarse grid and $I_h^H$ denotes an opposite interpolation.
2.1.3 Modified Strongly Implicit Procedure (MSIP)

Both the momentum equation (2-11) and the pressure-Poisson equation (2-14) in 2-D can be discretized using the central difference scheme on a five-point stencil. This discretized equation can be conveniently written in a matrix form of

$$ Au = f $$

(2-24)

where \([A]\) is a five-diagonal coefficient matrix. Directly inverting \([A]\) may solve \(u\), however, it is not recommended because the fastest inversion of general matrix involves \(O(N^{\log_2 7})\) multiplication (Press, Teukolsky et al. 1992).
The LU decomposition can factorize a matrix as the product of a lower triangular matrix and an upper triangular matrix. Then the solution for \( u \) only involves forward and backward substitutions, as shown in (2-25).

\[
LUu = f
\]

\[
LV = f \quad (2-25)
\]

\[
Uu = V
\]

where \( V \) is an intermediate variable.

However, the general LU decomposition requires nearly the same operation as matrix inversion. It does not take advantage of the coefficient matrix \([A]\) to be a positive, five-diagonal matrix. Thus, a number of work has been done to construct an approximate matrix of \([A]\) which can be easily LU decomposed. MSIP (Schneider and Zedan 1981) is one of those incomplete LU decomposition algorithms.

![Figure 2-3](image)

Figure 2-3. (a) A five-point stencil (solid circles) and two auxiliary points (dashed circles) approximated in the MSIP method. (b) Schematic presentation of matrices \([L] [U] = [M]\). The diagonals of \([M]\) not found in \([A]\) are shown by dashed lines, which also represent the dashed circles in Figure 2-3a.
Suppose \([M]\), which corresponds to a seven-point stencil as shown in Figure 2-3a, is a

good approximation to the \([A]\) matrix and can be efficiently \(LU\) decomposed (see Figure

2-3b). Both the \([L]\) and \([U]\) matrices have only 4 nonzero diagonals (Note that the principal
diagonal of \([U]\) is the unity diagonal). The matrix \([M]\) has 7 nonzero diagonals, which is two

more than \([A]\). Thus, a complementary matrix \([N]\) is necessary to be constructed.

\[
M = LU = A + N
\]  

(2-26)

The original problem is converted to

\[
(A + N)u = f + Nu
\]

\[
\Rightarrow
\]

\[
(A + N)u = f + (M - A)u
\]  

(2-27)

\[
\Rightarrow
\]

\[
Mu^{n+1} = Mu^n - \omega(Au^n - f)
\]

where the superscript \(n\) is the \(n\)th iteration. MSIP presents a method to calculate the matrix

\([M]\). Then the \(LU\) decomposition is highly efficient since both the \([L]\) and \([U]\) matrices are

sparse, as shown in (2-28).

\[
LU \delta^{n+1} = R^n = \omega(f - Au^n)
\]

\[
LV = R^n
\]  

(2-28)

\[
U \delta^{n+1} = V
\]

Note that (2-28) is to solve a residual vector \(\delta^{n+1}\), which is defined by \(\delta^{n+1} = u^{n+1} - u^n\).
Therefore, the MSIP method can be considered as a two-stage method, as shown in Figure 2-4. In the first stage we construct the $[L]$ and $[U]$ matrices. And in the second stage we run simple forward and backward sweepings. The formulas to calculate the matrix $[M]$ can be seen in the reference (Schneider and Zedan 1981).

Using the big O notation (Cormen, Leiserson et al. 2009), we can have an estimation of orders of growth for the speed of MSIP as $O(n)$, the same as the other iteration methods. However, the actual operation number is affected by the convergence rate which is highly case-dependent. Slow convergence rates may cause more iterations, consequently more operations. However, the memory consumption of MSIP is strictly proportional to the grid numbers, written as $9n$ and $15n$ for 2-D and 3-D, respectively. Two arrays are allocated for the unknown variable $u$ and the known RHS $f$. Seven or thirteen auxiliary arrays are allocated to store the information of nonzero diagonals of the iterative matrix $[M]$. In contrast, Simple iterative methods such as Gauss-Seidel and successive over relaxation do not required any auxiliary arrays. Thus, their growth rate of memory is $2n$ for both 2-D and 3-D.

### 2.1.4 MSIP for IBM

As mentioned in chapter 1, one of our objectives is to improve the computational efficiency of the existing solver by implementing MSIP as an iterative method for the
momentum equation (2-11), and as a smoother for the multigrid method to solve the pressure Poisson equation (2-14). This work requires coupling MSIP with immersed boundaries. Our strategy for accomplishing this task is to transform an immersed boundary problem into an equivalent ‘non-immersed boundary’ problem and hide boundary information from MSIP. As a result, the original MSIP algorithm can be applied to the ‘non-immersed boundary’.

Details of the implementation are presented as follows.

On a 2-D cell-centered grid, the discretization of Eqs. (2-11) and (2-14) can be written in a form of

$$\frac{\delta}{\delta x_j} \left( \frac{\delta u}{\delta x_j} \right) = \frac{\delta u}{\delta x_j} \left( \left( \frac{\delta u}{\delta x_j} \right)_e - \left( \frac{\delta u}{\delta x_j} \right)_w \right) + \frac{\delta u}{\delta y} \left( \left( \frac{\delta u}{\delta y} \right)_n - \left( \frac{\delta u}{\delta y} \right)_s \right) = f(x, y)$$ (2-29)

where the lower case subscripts, namely $e$, $w$, $n$ and $s$, are the edges in the east, west, north and south directions, respectively, as shown in Figure 2-5.

Figure 2-5. Schematic describing the naming convention used in the spatial discretization of Eqs. (2-11) and (2-14).

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Discretizing (2-29) using the central difference scheme on a uniform mesh, where the grid spacing in the x and y directions are the same, yields a five-point formula that has a second order accuracy in space:

\[
u_{i+1,j} - 2u_{i,j} + u_{i-1,j} \frac{h^2}{h^2} + u_{i,j+1} - 2u_{i,j} + u_{i,j-1} \frac{h^2}{h^2} = f(x, y) \quad (2-30)
\]

\[
\frac{1}{h^2}(u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j}) = f(x, y) \quad (2-31)
\]

where \( h = \Delta x = \Delta y \).

Suppose that we have a 2-D \((4 \times 4)\) computational mesh shown in Figure 2-6a. The corresponding linear algebraic equations in matrix form are written as:

![Figure 2-6](image)

**Figure 2-6.** A 4 by 4 computational grid (a) without solid bodies and (b) with a submerged solid body.
where the blank entries are zero. Note that we have not applied any boundary condition to the left-hand side of matrix \([A]\), thus every entry in the principle diagonal is -4. The elements of two off-diagonals that represent the connection between different rows are one. Whereas in the superdiagonal and the subdiagonal that represent the connection between different columns, the values are either one or zero.

In cases when immersed bodies are in the fluid field, it is not required to solve the value of solid cells. For instance, a body occupies the grid points (3, 2) and (3, 3), as shown in Figure 2-6b. \(u_{3,2}\) and \(u_{3,3}\) do not have a fluidic meaning. However, simply removing \(u_{3,2}\) and \(u_{3,3}\) from the matrix \([A]\) may destroy its five-diagonal feature. Some portions of the off-diagonals shift closer to the main diagonal and cause the matrix \([A]\) seven-diagonal. This is just a simple case ignoring two grid points. In practical problems, the diagonal shifting caused by multi-bodies invalids the MSIP method that takes advantage of the matrix diagonals.
Our strategy to solve this problem for MSIP is to construct a boundary equation which contains the boundary information

$$Bu = f_B$$  \hspace{1cm} (2-33)

where $[B]$ is a five-diagonal matrix and $f_B$ is a vector. The ‘non-immersed boundary’ problem can be obtained by subtracting (2-24) by (2-33)

$$(A - B)u = f - f_B,$$  \hspace{1cm} (2-34)

or

$$A'u = f',$$  \hspace{1cm} (2-35)

where $[A']$ is a five-diagonal matrix that can be processed by the MSIP method.

This method can be called as implicit boundary treatment in contrast to explicit treatment which imposes boundary conditions at the beginning or the end of an iterative step by explicitly ‘correcting’ the value of grid points that are adjoining to the boundary. The explicit treatment works for a number of iterative methods such as Jacobi, point Gauss-Seidel, and successive over-relaxation (SOR). However, the explicit treatment is not compatible with the MSIP method, even for cases with exterior boundaries only. The iteration number of the MSIP method with the explicit treatment increases to be comparable to that of the SOR method. This contradicts to the performance reported in the literature.

There are several advantages using this method. First, since the new method does not change the algorithm of MSIP, the convergence speed of MSIP is maintained. Second, the iteration matrix $[M]$ need to be computed only once for each time step. This reduced a large amount of the computational time. Third, the new algorithm hardly modifies the geometrical multigrid method, which means its application is not limited to the current solver.
The construction of the boundary equation (2-33) is performed by assigning proper values to matrix $[B]$ and vector $f_B$ for fluid cells which are neighbors of solid cells and solid cells. Coefficients of fluid cells need different treatments with respect to different boundary conditions.

2.1.4.1 Coefficients for fluid cells

- Dirichlet boundary condition

![Diagram](image)

Figure 2-7. The boundary near the left wall.

An example of Dirichlet boundary condition (B.C.) is shown in Figure 2-7. An average representation of Dirichlet B. C. for cell-centered variables is
\[ \frac{u_{i-1,j} + u_{i,j}}{2} = g_w \]  

(2-36)

where \( g_w \) is a given Dirichlet B.C., \( u_{i,j} \) and \( u_{i-1,j} \) are both unknown variables. We are interested in the value of the fluid node \( u_{i,j} \) only, thus, \( u_{i-1,j} \) can be expressed as a function of \( g_w \) and \( u_{i,j} \) in forms of

\[ u_{i-1,j} = 2g_w - u_{i,j} \]  

(2-37)

Substituting (2-37) into the discretized equation (2-30) yields

\[ \frac{u_{i+1,j} - 2u_{i,j} + 2g_w - u_{i,j} + u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{h^2} = f(x, y) \]  

(2-38)

Moving all known variables to the right hand side of (2-38) gives the implicit treatment for Dirichlet B.C.:  

\[ \frac{u_{i+1,j} - 3u_{i,j} + u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{h^2} = f(x, y) - \frac{2g_w}{h^2} \]  

(2-39)

\[ \frac{1}{h^2} \left( u_{i+1,j} + u_{i,j+1} + u_{i,j-1} - 5u_{i,j} \right) = f(x, y) - \frac{2g_w}{h^2} \]  

(2-40)

If the Dirichlet B.C. is on the east wall, similarly, we have

\[ \frac{1}{h^2} \left( u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 5u_{i,j} \right) = f(x, y) - \frac{2g_e}{h^2} \]  

(2-41)

Comparing Eq. (2-41) to Eq. (2-31), we can conclude a method to construct the boundary equation (2-33) for Dirichlet B.C. Let (2-36) be in a form of
\[ \frac{1}{h^2} (u_{i-1,j} + u_{i,j}) = 2 \frac{g_w}{h^2} \]  

(2-42)

The coefficient of \([B]\) corresponding to \(u_{i,j}\) and \(u_{i,j-1}\) is specified to be equal to \(\frac{1}{h^2}\) and the RHS \(f_B\) is equal to \(2 \frac{g_w}{h^2}\).

- **Neumann B.C.**

Illustrated in Figure 2-7, suppose we have Neumann B.C. on the west wall:

\[ \left. \frac{\partial u}{\partial x} \right|_{x=w} = g_w \]  

(2-43)

Substituting (2-43) into (2-29) yields

\[ \frac{u_{i+1,j} - u_{i,j}}{h} - f_w + \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{h^2} = f(x,y) \]  

(2-44)

Rearranging the above equation yields

\[ \frac{u_{i+1,j} - u_{i,j}}{h^2} + \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{h^2} = f(x,y) + \frac{g_w}{h} \]  

(2-45)

\[ \frac{1}{h^2} (u_{i+1,j} + u_{i,j+1} + u_{i,j-1} - 3u_{i,j}) = f(x,y) + \frac{g_w}{h} \]  

(2-46)

Therefore, for Neumann B.C. cases we can write (2-43) in a form of

\[ \frac{1}{h^2} (u_{i,j} - u_{i,j-1}) = \frac{g_w}{h} \]  

(2-47)

The coefficient of \([B]\) corresponding to \(u_{i,j}\) and \(u_{i,j-1}\) is \(\frac{1}{h^2}\) and \(\frac{1}{h^2}\), respectively. And the RHS \(f_B\) equals to \(-\frac{g_w}{h}\).
Note that in multigrid method on the coarsest grid where only one fluid grid point exists, (2-46) may degenerate to

\[ 0 = f(x, y) + \frac{1}{h}(g_w + g_e + g_s + g_n) \]  

(2-48)

No variable can be solved. Thus the coarsest grid must have at least two grid points.

- **Dirichlet B.C. and Neumann B.C.**

![Diagram of grid boundaries](image)

Figure 2-8. The left (inlet) and bottom (wall) boundaries.

It can be frequent to encounter a mixed boundary in simulations, especially for internal flows such as channel flows, where Dirichlet B.C. is given at the inlet and Neumann B.C. is given at walls or immersed bodies. Assume Dirichlet B.C. and Neumann B.C. are imposed on the west and south boundaries, respectively, as shown in Figure 2-8.
\[
\frac{u_{i-1,j} + u_{i,j}}{2} = g_w \tag{2-49}
\]
\[
\frac{\delta u}{\delta y} = g_s \tag{2-50}
\]

Substituting these two equations into (2-29) yields
\[
\frac{u_{i+1,j} - 2u_{i,j} + 2g_w u_{i,j}}{h^2} + \frac{\left( \frac{\delta u}{\delta y} \right)}{h} = \frac{g_s}{h} = f(x, y) \tag{2-51}
\]

Leaving all unknown variables to the left hand side and moving all known variables to the right hand side yield
\[
\frac{u_{i+1,j} - 3u_{i,j} + u_{i,j+1} - u_{i-1,j}}{h^2} = f(x, y) - \frac{2g_w}{h^2} + \frac{g_s}{h} \tag{2-52}
\]
\[
\text{or} \quad \frac{1}{h^2} \left( u_{i+1,j} + u_{i,j+1} - 4u_{i,j} \right) = f(x, y) - \frac{2g_w}{h^2} + \frac{g_s}{h} \tag{2-53}
\]

It can be seen that the coefficients of matrix \([B]\) and vector \(f_B\) for Dirichlet B.C. and Neumann B.C. can be linear superposition.

2.1.4.2 Coefficients for solid cells

It seems that the MSIP algorithm should be able to handle any five-diagonal matrix. However, this is not true. The construction of the \([L]\) and \([U]\) matrices is a chaining process. Computation of the coefficients in the \([L]\) and \([U]\) matrices for a grid point \((i, j)\) invokes the values of the grid points \((i, j-1)\), \((i+1, j-1)\), \((i-1, j)\), despite that the cell is of fluid or solid. For instance, in the 2-D \((4 \times 4)\) case shown in Figure 2-6b, grid points \((4, 2)\), \((3, 3)\), \((2, 3)\) are
affected by grid point (3, 2). Grid points (4, 3), (3, 4), (2, 4) are affected by grid point (3, 3).

There are five fluids nodes invoking the values of two solid nodes in this case.

Values of solid nodes affect the convergence rate of MSIP. Because the shape of solid bodies can be arbitrary in a simulation, there is not a universal method to determine the optimal values for solid nodes. Our choice is setting the entries on the main-diagonal of the grid points (3, 2) and (3, 3) in the matrix \([A']\) to one, and the other entries on the off-diagonal to be zero. Also, set the RHS \(f'_{3,2}\) and \(f'_{3,3}\) to zero. This approach is slightly different from the SIP case that has an adiabatic region, in which zero clearing is employed for the entire row of solid nodes. This difference is to avoid divided by zero during constructing the \([L]\) and \([U]\) matrices. Assume the exterior boundaries are all Neumann type. The coefficient matrix \([A']\) will be modified to the form as (2-54). This approach guarantees the convergence of the MSIP method with immersed boundary; however, it still takes time to calculate the values in solid bodies.
We have mentioned that the MSIP method is a two-stage method. The first stage is the construction of the $[L]$ and $[U]$ matrices and the second stage is the iteration. Time spent in the first stage is much less than that in the second stage. To save the computational time to the greatest extent, we use a tag for solid nodes. In the first stages, whenever the tag is met, we set the rows for solid nodes in the matrices $[A']$, $[L]$ and $[U]$ equal to zero. The MSIP method is only used to build the $[L]$ and $[U]$ matrices for fluid nodes. In the second stage, which indeed is to calculate residual (2-30), we set the residual of solid nodes equal to zero because from the physical point of view the values on the solid nodes do not change, their residuals should be zero. The MSIP method is still applied to the fluid nodes. This specific treatment for solid nodes is essential for the iteration since fluids nodes may invoke the residual of solid nodes during the forward and backward sweepings.

2.1.4.3 Error accumulation for IBM
A practical problem we have met in simulations is that multigrid-MSIP may diverge in certain circumstance. Suppose that we have a moving boundary case where a solid body continuously moves leftwards (see Figure 2-9). The grid points where a solid body occupied in the last time step but left in the current time step is called fresh cells in the context of immersed boundary method. We usually take the pressure field from last time step as an initial guess for the current time step since it is thought that they should have similar values and it will not take many iterations to converge from the pressure in last time step to the current time step. Assume the initial guess for fresh cells $F_1$ and $F_2$ is zero and the values in neighboring fluid nodes are extremely low pressure, e.g. -10000. It is possible since all the boundary condition is Neumann boundary condition (2-16). If the pressure gradient satisfies the Poisson equation, it does not matter what the actual pressure is. When the body keeps generating new fresh cells and the values of neighboring points are getting lower and lower, the multigrid with MSIP method may suddenly diverge. We compared the solution of MG-
MSIP with that of MG-LSOR. At a time step $n$, two solutions are approximately the same. But from time step $n+1$, they begin to show certain difference. The divergence occurs after tens of time steps. After investigations, we found that the difference between fresh cells and their neighboring grid points leads to the divergence because MSIP is a point iterative method that relies on the values of neighboring points. For example, the fresh cell $F_2$ invokes values from a solid node, the fresh cell $F_1$ and a fluid node. The multigrid with LSOR method that solves a row or a column of grids simultaneously had never diverged in such a situation. The solution to this problem is interpolating the values of fresh cells $F_1$ and $F_2$ from neighboring fluid nodes, which are also none fresh cells, so that the initial values are not too distinguished to their neighbors.

2.1.5 Validation, Verification and Performance Benchmark

In this section we have chosen six problems to demonstrate the validity and efficiency of the MSIP and multigrid-MSIP methods. The first two problems are model problems testing the convergence of Poisson equation solvers. The third and fourth are simulations of flow past a membrane plate in two-dimension and three-dimension, respectively, focusing on the overall efficiency of the MSIP and multigrid-MSIP methods. The last two are flows past two-dimensional circular cylinders and three-dimensional spheres, respectively, examining convergent speed of the multigrid-MSIP method on different numbers of submerged bodies.

2.1.5.12-D model problem

The first problem is a 2-D model problem on a rectangular domain
\[ \nabla p = -2 \cos x \cos y \text{ in } \Omega = (0, 2\pi) \times (0, 2\pi) \]  \hspace{1cm} (2-55)

with exterior Neumann boundary conditions

\[ \frac{\partial p}{\partial x} = 0 \text{ on } x = 0 \text{ and } x = 2\pi \]  \hspace{1cm} (2-56)

\[ \frac{\partial p}{\partial y} = 0 \text{ on } y = 0 \text{ and } y = 2\pi \]  \hspace{1cm} (2-57)

It has an analytic solution

\[ p = \cos x \cos y + p_a \]  \hspace{1cm} (2-58)

where \( p_a \) is ambient pressure, an arbitrary constant. Thus, the analytic solution is not unique.

The convergence criterion is defined by \( L_\infty \)-Norm of residual:

\[ \left\| \nabla p^n - \nabla p \right\|_\infty \leq 10^{-m} \]  \hspace{1cm} (2-59)

where \( p^n \) is discrete solution at the \( n^{th} \) iteration.

We numerically solved (2-55) with boundary conditions on various uniform grids. All runs were conducted on an Intel(R) Core(TM) i7 CPU 950 at 3.07GHz, initialized from a field of \( p = 0 \). A comparison of discrete solution of the multigrid-MSIP method, when the \( L_\infty \)-norm of residual is reduced to \( 10^{-6} \), with the exact solution on a 256 \( \times \) 256 grid is shown in Figure 2-10. The discrete solution is slightly different from the theoretic solution because the boundary conditions are Neumann boundary at four edges and the ambient pressure cannot be solved out. The minimum and maximum values of the discrete solution are -0.99 and 1.01, respectively. The difference is 2.00.
Figure 2-10. Contours of (a) theoretic and (b) discrete solutions on a 256 × 256 grid.

Convergence criterion is $10^{-6}$.

We compared the performance of four iterative methods: the line-SOR (LSOR), the MSIP, the multigrid with Line-SOR (MG-LSOR) and the multigrid with MSIP (MG-MSIP). Table 2-1 shows the number of iterations to reduce the $L_\infty$ residual norm below $10^{-4}$ with respect to different grid numbers. The MG-LSOR and MG-MSIP methods employ the same number of iterations. Both are far less than the LSOR and MSIP methods because multigrid method employs $O(N)$ operations for $N$ grids. The iteration number of the MSIP method is nearly one third of the LSOR method on different grid number. The computational time for the LSOR method on a 256 × 256 grid is too long thus we did not finish this case.

Figure 2-11a shows the variation of the $L_\infty$-norm of residual as a function of the iteration number for various solvers on a 64 × 64 grid. It is obvious that residuals of the multigrid method with the LSOR or MSIP smoothers decrease linearly against the iteration number. The behavior of the LSOR and MSIP methods are different. Their residuals reduce very slowly in the first a few iterations but accelerate thereafter. One of explanations for this
behavior is that the LSOR and MSIP methods solve the short-wave components first, and then the long-wave components. The current case has only long-wave components (2-58), thus, the first a few iterations are not efficient.

Table 2-1: Number of iterations required by various solvers to reduce the $L_\infty$ residual norm below $10^{-4}$ for the 2-D Poisson equation

<table>
<thead>
<tr>
<th>Grid</th>
<th>LSOR</th>
<th>MSIP</th>
<th>MG-LSOR</th>
<th>MG-MSIP</th>
</tr>
</thead>
<tbody>
<tr>
<td>$32^2$</td>
<td>141</td>
<td>42</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>$64^2$</td>
<td>360</td>
<td>126</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>$128^2$</td>
<td>1184</td>
<td>373</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>$256^2$</td>
<td>-</td>
<td>1115</td>
<td>3</td>
<td>3</td>
</tr>
</tbody>
</table>
Figure 2-11. (a) Variation of the $L_{\infty}$-norm of residual as a function of the iteration number for various solvers on a $64 \times 64$ grid and (b) the $L_{\infty}$-norm of the error versus the computational grid size for the MG-MSIP method. The dash-dot-dot line denotes a second order slope.

We also perform a grid independent study on the order of convergence for the MG-MSIP method, as shown in Figure 2-11b. The error norm is defined by $L_{\infty}$-Norm of error between the discrete solution $p^n_k$ and the analytic solution $p$:

$$ E = \| p^n_k - p \|_{\infty} \quad \text{when} \quad \| \nabla p^n_k - \nabla p \|_{\infty} \leq 10^{-m} $$

(2-60)

The order of convergence can be obtained from the slope of the curve of $\log(E)$ versus $\log(h)$, where $h$ is grid size. It can be seen that the MG-MSIP method is of second order accuracy. It is because the spatial discretization used in the current solver is the central difference scheme, which is a second order scheme. The Neumann boundary conditions are discretized by a linear interpolation; however, the influence is small in this range of grid size.
2.1.5.23-D model problem

Problem 2 is a three-dimensional version of Problem 1. Since it has been shown that the computational efficiency of the LSOR and MSIP methods are not comparable to the MG-LSOR and MG-MSIP methods in the two-dimensional case, only the MG-LSOR and MG-MSIP methods are studied in this problem.

\[

\nabla p = -3\cos x \cos y \cos z \text{ in } \Omega = (0,2\pi) \times (0,2\pi)

\]

(2-61)

\[

\frac{\partial p}{\partial x} = 0 \text{ on } x = 0 \text{ and } x = 2\pi

\]

(2-62)

\[

\frac{\partial p}{\partial y} = 0 \text{ on } y = 0 \text{ and } y = 2\pi

\]

(2-63)

\[

\frac{\partial p}{\partial z} = 0 \text{ on } z = 0 \text{ and } z = 2\pi

\]

(2-64)

The computation is also conducted on a personal computer with Intel(R) Core(TM) i7 CPU 950 chip. Figure 2-12a shows the time curves for the MG-LSOR and MG-MSIP solvers to reduce the \( L_\infty \) norm of residual below 10^{-6} on various grids, 32 \times 32 \times 32 (32768 meshes), 64 \times 64 \times 64 (nearly 26 thousand meshes), 128 \times 128 \times 128 (2 million meshes) and 256 \times 256 \times 256 (12.8 million meshes). The MG-MSIP method is obviously superior to the MG-LSOR method on every grid in terms of residue reducing speed. Moreover, the time consumption ratio of the MG-LSOR method to the MG-MSIP method increases as the grid number increases, as shown in Table 2-2. This implies that the MG-MSIP method may have more advantages than the MG-LSOR method for large scalar problems.
Figure 2-12. (a) Computational time for the MG-LSOR and MG-MSIP solvers to reduce the $L_{\infty}$-norm of residual below $10^{-6}$ on various grids and (b) computational time versus grid numbers for the MG-LSOR and MG-MSIP solvers. The dash-dot-dot line represents a linear increment.

Table 2-2: Cumulative time ratio of the MG-LSOR to the MG-MSIP when the $L_{\infty}$-norm of residual reaches $10^{-6}$ on various grids

<table>
<thead>
<tr>
<th>Grid number</th>
<th>32$^3$</th>
<th>64$^3$</th>
<th>128$^3$</th>
<th>256$^3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ratio</td>
<td>2.2</td>
<td>2.7</td>
<td>4.1</td>
<td>7.6</td>
</tr>
</tbody>
</table>

Computational time scales monotonously with grid numbers. Assuming that the operation number is linearly proportional to the computational time, this ratio also reflects the relation between the operation number and the grid number. We draw the curves of computational time as a function of grid numbers for the MG-LSOR and MG-MSIP solvers with respect to the $L_{\infty}$ residual norm of $10^{-3}$ and $10^{-6}$, as shown in Figure 2-12b. Compared to the dash-dot-
dot line, which represents a linear increment in the log-log plot, it shows that the operation number for our multigrid method is not exact $O(N^1)$ for $N$ grids. We perform a linear regression on those curves and calculate the actual power factor. The results are listed in Table 2-3. Thus the operation numbers are $O(N^{1.47})$ and $O(N^{1.35})$ for the MG-LSOR and MG-MSIP methods, respectively. Although the MG-MSIP method does not reach the ideal number $O(N^1)$, its power factor is still smaller than that of MG-LSOR. This also demonstrates the computational efficiency of the MG-MSIP methods.

Table 2-3: Power increment of operation number for various solvers

<table>
<thead>
<tr>
<th>Convergent criterion</th>
<th>$10^{-3}$</th>
<th>$10^{-6}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Method</td>
<td>MG-LSOR</td>
<td>MG-MSIP</td>
</tr>
<tr>
<td>Power factor</td>
<td>1.47</td>
<td>1.30</td>
</tr>
</tbody>
</table>

We record the time cost in every iteration step, as shown in Figure 2-13a. It is found that the first iteration of the MG-MSIP method takes more time than the others, especially on a large grid, e.g. $256 \times 256 \times 256$. The time spent in the other iterations is nearly the same, thus we can see a sharp descent followed by a flat curve. Compared with the MG-LSOR method, the time cost of the MG-MSIP method is nearly one order of magnitude smaller on all testing grids.
2.1.5.3A 2-D stationary flat plate with 30° angle of attack

The third problem is a flow past a two-dimensional stationary membrane plate with angle of attack of 30 degree. The chord length of the plate and the incoming flow speed are all one unit. The Reynolds number is 300. The mesh number is $353(X) \times 258(Y)$ and the domain size
is $18 \times 15$ (see Figure 2-14). The smallest grid spacing is $1/32$ around the foil and in the wake zone. This choice of domain size and grid spacing is based on previous works done in (Taira, Dickson et al. 2007), where a smaller grid size of $400 \times 200$ is used on a larger domain of $30 \times 15$ for Re=300. Since both solvers use immersed boundary method, the grid density in the current simulation should be sufficient for Re=300.

Dirichlet velocity boundary condition is applied except at the downstream, where it is of Neumann boundary condition; the pressure boundary condition is homogeneous Neumann condition on all the boundaries.

In this case we have two sets of configuration of the solvers:

1. The LSOR and MG-LSOR methods for the momentum equations and the Poisson equation, respectively;
2. The MSIP and MG-MSIP methods for the momentum equations for the Poisson equation, respectively.

The convergence criterions are $10^{-6}$ and $10^{-3}$ for the momentum equations and the Poisson equation, respectively. Because these two equations are independent and convergence testing is also an independent module, switching algorithms for each equation does not affect the time cost of each algorithm.
Figure 2-14. (a) The grid used for the 2-D simulation of flow past a fixed plate and (b) an instantaneous vorticity field.

Figure 2-15. (a) Computational time of single time step for different configurations. Convergence criterion for intermediate velocity \( u^* \) and pressure \( p \) are the \( L_\infty \) residual norm of \( 10^{-6} \) and \( 10^{-3} \), respectively. (b) Cumulative time for different configurations.
The computation runs on an Intel(R) Core(TM) i5-2500K CPU at 3.30GHz. Figure 2-15a shows the computation time of each time step (the $x$-axis is the time step products $\Delta t$). The flow field is initialized with a uniform streamwise velocity equal to one. The flow gradually arrives at an unsteady state where vortices shed periodically. This process takes thousands of time steps. Thus, we can see the descending trend of the computational time because the iteration number in each time step decreases, especially for the MG-LSOR method solving the Poisson equation.

It can also be observed in Figure 2-15a that the time cost of the MG-MSIP method for the Poisson equation is even less than that of the LSOR method for the momentum equation. The difference between two sets of configurations is also observed in the cumulative time, as shown in Figure 2-15b. The total computational time for the combination of the MSIP and MG-MSIP methods is even less than solving the momentum equation using LSOR. The time ratio of the configuration 2 is 4.2 times faster than configuration 1.

Transient history of drag and lift coefficients of two configurations are shown in Figure 2-16. Because the function checking error criterion is the same and the convergence criterions are set to small values, the curves of two configurations show no difference. For the purpose of validating our simulations, we also compare lift and drag coefficients with the literature (Taira, Dickson et al. 2007). The maximum
lift coefficient in the plot is 2.24, almost the same as Taira’s 2.25. The average lift and drag coefficients are 1.45 and 0.896, respectively. These two values are higher than Taira’s, which are 1.21 and 0.86, respectively. It is because Taira’s simulations last for a non-dimensional time length of 70 and ours are only 10. It can see that both curves of lift and drag coefficients have a trend of reducing oscillating peaks. We believe that simulating for a longer time will reduce the average lift and drag coefficients in this case.

2.1.5.4 A Low-aspect-ratio, stationary flat plate with 30° angle of attack

Problem 4 is a flow past a finite-aspect-ratio, fixed plate with angle of attack of 30 degree (see Figure 2-17). The plate is a rectangular membrane of zero thickness and its aspect ratio equals 2. The Reynolds number is 300. The mesh number is 272 ($X$) \times 104($Y$) \times 144($Z$) and the domain size is 10.1 \times 10 \times 10. Compared with our setup, Taira (Taira, Dickson et al. 2007) used the same domain size but with a smaller grid size of 125 \times 55 \times 80. Other than the aforementioned differences we have the same simulations setup, the convergence criterions and the solver configurations, as the 2-D case.
The iteration numbers of the first ten time steps for two configurations are listed in Table 2-4. The iteration number of the LSOR method is always less or equal to that of the MSIP method. This is different from what we have seen in the 2-D model problem (Section 2.1.5.1), where the iteration number of the MSIP method is far less than that of the LSOR method, especially for large grid numbers. But it is notable that the LSOR method can converge in only several iterations, which is far less than the 2-D model problem. This implies the advantage of the MSIP method in convergent speed may not be utilized in practical simulations. On the contrary, the iteration number of the MG-MSIP method is approximately 50% larger than that of the MG-LSOR method. It is probably because the shifting of immersed boundaries in multigrid has more negative influences on the performance of MSIP method.
Despite the larger numbers of iteration of the MSIP-class methods, their computational time of each time step is still less than that of the LSOR-class methods (see Figure 2-18a) due to comparatively less time in a single iteration, as illustrated in Figure 2-13. The time cost of the MG-MSIP method for the Poisson equation is comparable to that of the LSOR method for the momentum equation. Thus the total computation time for the MSIP-class methods is less than the LSOR-class methods with a time ratio of 1:5 (see Figure 2-18b).
Figure 2-18. Curves of (a) computational time of single time step and (b) cumulative time with respect to two configurations. Convergence criterion for intermediate velocity \( u^* \) and pressure \( p \): the \( L_\infty \) residual norm of \( 10^{-6}, 10^{-3} \), respectively.

Figure 2-19 shows the lift history of the MSIP-class methods. The curve rises to a maximum 1.3 at an early time \( t=1.7 \) due to an impulsive start-up, then it decays. Nearly after \( t=20.0 \), when the initial transient ends, it begins a platform region, where only small oscillations occurs. The global maximum and time average of the lift coefficient are 1.31 and 0.71, respectively. The curve shows a qualitatively agreement with Taira’s simulation result (Taira, Dickson et al. 2007), however, quantitatively larger than his maximum and average (1.25 and 0.56, respectively). Given that Taira’s simulation results are also smaller than his experimental results, especially at angle of attack of 30 degree, and his simulations employ a membrane with finite thickness, the difference between ours and his results are acceptable.
2.1.5.5 Flow past cylinders

We have designed two groups of numerical simulations about the flow past 2-D circular cylinders. The first group has one cylinder located at the center of the flow field (8, 5) (see Figure 2-20a). We add two more cylinders, which are located respectively at (7, 4.25) and (7, 5.75) (see Figure 2-20b), in the second group. Both groups of simulation are performed on five uniform grids with grid numbers of $300 \times 200$, $600 \times 400$, $960 \times 620$, $1200 \times 800$ and $1440 \times 960$. The purposes of experiments are: 1. to estimate the ratio between the grid number and computational time; 2. to understand the influence of immersed bodies to the MSIP and MG-MSIP algorithms. The diameter of the cylinders is one. The incoming flow speed is one. The Reynolds number is 200. Simulations last for 1000 time step with $\Delta t$ equal to 0.001. The flow boundary conditions are the same as Problem 3. The momentum and Poisson equation solvers are the MSIP and MG-MSIP methods, respectively.
Figure 2-20. Typical pressure contours of (a) 1 cylinder and (b) 3 cylinders on a 1440 × 960 grid.

The log-log plots of cumulative time vs. the computational grid numbers are shown in Figure 2-21. Linear regression is performed on the data to calculate the scaling of computing time with respect to grid number. The time for the MSIP method solving the momentum equation increases linearly against the grid number ($O(N^{1.00})$) when only one cylinder is presented in the flow field. Adding two more cylinders slightly increases the operation number to $O(N^{1.09})$. This shows a good computational efficiency for the MSIP method in solving the momentum equation.

The operation numbers for the MG-MSIP method solving the Poison equation are $O(N^{1.18})$ and $O(N^{1.14})$ for one and three cylinders, respectively. In contrast to the MSIP method solving the momentum equation, adding objects can slightly achieve better power scaling for MG-MSIP. However, it is notable that the computational time of MG-MSIP for the one-cylinder case is much less than that of MSIP when the grid number is less than 1 million. It is also less than the time of MG-MSIP for the three-cylinder case at the same grids.
Therefore, we may conclude that more objects can increase the computational time of MG-MSIP, while this increment decreases as the grid number increases.

![Graph](image)

Figure 2-21. Cumulative time versus the computational grid numbers for the case of (a) 1 cylinder and (b) 3 cylinders. Convergence criterions for intermediate velocity $u^*$ and pressure $p$ are the $L_\infty$ residual norm of $10^{-6}, 10^{-3}$, respectively. The dash-dot-dot line represents a linear increment.

**2.1.5.6 Flow past spheres**

In addition to the 2-D cylinder simulation, we design another two groups of flow past 3-D spheres and measure the influence of MSIP and MG-MSIP with respect to the number of 3-D bodies. The flow parameters such as the diameter of the spheres, the incoming flow speed, the Reynolds number, and $\Delta t$ are the same as the 2-D simulations. Time step of simulation reduces to 100. Similar to the 2-D simulation, the first group has one sphere located at the center of the flow field (8, 5, 5). Four more spheres are added to the upstream of the first
sphere. The arrangement is shown in Figure 2-22. Five uniform grids with grid numbers of $100 \times 100 \times 100$, $120 \times 120 \times 120$, $140 \times 140 \times 140$, $160 \times 160 \times 160$ and $200 \times 200 \times 200$ are used.

![Figure 2-22. Arrangement of spheres in the second group viewed from (a) xy-view and (b) yz-view.](image)

The log-log plots of cumulative time vs. the computational grid numbers are shown in Figure 2-23. The time of MSIP increases at the order of $O(N^{1.38})$ for both groups. Adding spheres seldom affects the power factor. However, comparing with 2-D cases, the power factor is much larger. This is probably because MSIP algorithms are different for 2-D and 3-D. In 2-D, MSIP factorizes a seven-diagonal matrix, where two diagonals are complementary. While in 3-D, it factorizes a thirteen-diagonal matrix with six complementary diagonals.

The time of MG-MSIP increases at the order of $O(N^{1.14})$ and $O(N^{1.24})$ for groups one and two, respectively. The time consumption of MG-MSIP for the five-sphere case is more than the one-sphere case on the same grids. In addition, the time consumption abruptly rises
on the grid $140 \times 140 \times 140$. This indicates that mismatch of objects and grids position may cause the performance loss of the multigrid method.

![Figure 2-23](image.png)

Figure 2-23. Cumulative time versus the computational grid numbers for the (a) one-sphere and (b) five-sphere configurations. Convergence criterions for intermediate velocity $u^*$ and pressure $p$ are the $L_\infty$ residual norm of $10^{-6}$, $10^{-3}$, respectively. The dash-dot-dot line represents a linear scaling of time with respect to grid number.

### 2.2 Proper Orthogonal Decomposition

The proper orthogonal decomposition (POD) method extracts a set of modes from large quantities of high-dimensionally computational or experimental data through an eigenvalue analysis. It is optimal in the sense of capturing the largest possible amount of certain quantity based on researchers’ interests: kinetic energy from velocity, enstrophy from vorticity (Kostas, Soria et al. 2005) and enthalpy (Rowley, Colonius et al. 2004). In the current study,
we adopt traditional kinetic energy concept in the POD analysis of flow fields. Thus, the velocity vector is written as an ensemble of spatial and temporal components in terms of:

\[ U(x,t) = \bar{U}(x) + \sum_{i=1}^{N_m} \alpha_i(t) \Phi_i(x) \]  

(2-65)

where \( \bar{U} \) is the mean (time averaging) velocity, \( \Phi_i \) denotes individual, \( \alpha_i \) is temporal coefficient of the associated mode, and \( N_m \) is the total number of modes ensembled. Thus, each spatio-temporal component can be interpreted as a spatial fluctuating structure \( \Phi_i \) whose intensity depends on a temporal coefficient \( \alpha_i \). Because every velocity field in the ensemble is a linear combination of these spatio-temporal components, POD is also considered as a method decoupling spatial coherent structure \( \Phi_i \) from the temporal variations.

We define \( (f,g) \) as the inner product for the linear Hilbert space \( L^2(\Omega_x) \) of square-integrable functions

\[ (f,g) = \int_{\Omega_x} f \cdot g^* \, dx \]  

(2-66)

where the dot denotes a standard vector dot product, and \( * \) denotes complex conjugation.

To form a set of the modes \( \Phi_i \in L^2(\Omega_x) \) in the context of proper orthogonal decomposition method, it is required to maximize the quantity:

\[ \frac{\langle (u, \Phi)^2 \rangle}{\| \Phi \|^2} \]  

(2-67)
subjected to a constraint $\|\Phi\|=1$ for each mode $\Phi_i$ in the set, where $\|\Phi\|=(\Phi,\Phi)^{1/2}$ and $\langle f \rangle$ denotes a time average operation on the data ensemble. Note that $u=U-\bar{U}$ represents the turbulent fluctuating velocity of instantaneous flow fields from the time average. The POD method in this scenario produces the POD modes $\Phi_i$ that maximize the resolution of the fluctuating kinetic energy.

Apply the calculus of variations to recast this problem as the solution of the following integral equation (see (Holmes, Lumley et al. 1996) for further justification and details)

$$
\int \langle u(x) \otimes u^*(x') \rangle \Phi(x') dx' = \lambda \Phi(x)
$$

where $\otimes$ is the outer or tensor product. In practice, (2-68) can be solved by transformed into a matrix problem through suitable discretization,

$$
AV = \lambda V
$$

Two kinds of POD methods, the direct method and the method of snapshots (Sirovich 1987), are available to calculate the POD modes. The size of matrix $A$ is respectively proportional to the number of grid points for the direct method, or to the number of snapshots for the method of snapshots. The present study has used the method of snapshots, which is suitable to process the datasets from CFD simulations, especially in three-dimensional scenario, where the number of snapshots $N_s$ is far less than the number of grid points.

By defining

$$
b_i = (u, \Phi) = \int_{\Omega} u_{(i)}^* (x') \Phi(x') dx' \quad i = 1, ..., N_s
$$

$$
a_{ij} = (u_i, u_j) = \int_{\Omega} u_{(i)}^* (x) u_{(j)} (x) dx \quad i, j = 1, ..., N_s
$$
where $u^{*}_i$ is the complex conjugation of velocity field of the $i^{th}$ snapshot. Then (2-69) is written as

$$
\begin{bmatrix}
    a_{11} & \cdots & a_{1N_s} \\
    \vdots & \ddots & \vdots \\
    a_{N_s1} & \cdots & a_{N_sN_s}
\end{bmatrix}
\begin{bmatrix}
    b_1 \\
    \vdots \\
    b_{N_s}
\end{bmatrix}
= \lambda
\begin{bmatrix}
    b_1 \\
    \vdots \\
    b_{N_s}
\end{bmatrix}
$$

(2-72)

where the size of the correlation matrix $A$ is $N_s \times N_s$. Solving the eigenvalue problem provides eigenvalues and eigenvectors for constructing the POD modes $\Phi_i$:

$$
\Phi_i = \frac{1}{\sqrt{\lambda_i}} \sum_{j=1}^{N_s} b_j^i u_j
$$

(2-73)

where $\lambda_i$ are the $i^{th}$ eigenvalues in a descendent order, $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{N_s} \geq 0$ since the coefficient matrix $A$ in (2-72) is non-negative definite, and $b_j^i$ are the $j^{th}$ elements of the $i^{th}$ eigenvector $V_j$. From the physical point of view, $\lambda_i$ measures the amount of kinetic energy captured by the corresponding mode $\Phi_i$.

The POD modes are orthogonal to each other:

$$
(\Phi_i, \Phi_j) = \delta_{ij}
$$

(2-74)

Given the POD modes, the $n^{th}$ snapshot $U^n$ can be reconstructed from a linear combination of the POD modes:
\[ U^n = \bar{U} + \sum_{i=1}^{N}\alpha_i^n \Phi_i \]  

(2-75)

where temporal coefficients \( \alpha_i^n = (\Phi_i, u_n) \). Because the POD modes \( \Phi_i \) are normalized by square root of \( \lambda_i \), to satisfy the constraint \( \|\Phi\| = 1 \), the temporal coefficients \( \alpha_i^n \) takes a unit of velocity. Also, we have the following property for \( \alpha_i^n \)

\[ \langle \alpha_i^n \alpha_j^* \rangle = \delta_{ij} \lambda_i \]  

(2-76)

The construction method of the POD modes in (2-73) shows that each POD mode \( \Phi_i \) can be expressed as a linear combination of snapshots. It implies that any property shared by the ensemble members is inherited by the POD modes if the property is linear and homogeneous. For instance, each POD mode generated from snapshots of an incompressible flow is still incompressible; mass conservation for the flow domain is preserved globally; homogeneous and linear boundary conditions are kept. Rigorous proofs can be found in Holmes, 1996.

### 2.3 Galerkin Projection

The Galerkin projection is based on projecting the governing equation (the Navier-Stokes equations in fluid dynamics) onto subspace consisting of basis eigenfunctions which optimally capture the average energy content. It yields a simple set of ordinary differential equations from which the characteristics of the expected solution can be achieved. We write the instantaneous velocity field as the sum of the mean flow \( u_m = (u_m, v_m, w_m) \) and the velocity fluctuations. The mean flow \( u_m \) is the time average of the ensemble data. The fluctuation part is processed using POD. Then the original velocity field is expressed as
\[
\ddot{u}(x,t) \approx \ddot{u}_m + \sum_{i=1}^{N_M} \alpha_i(t) \Phi_i(x) \tag{2-77}
\]

where \( \ddot{u} = (u,v,w) \), \( \Phi_i = (\Phi_{x,i}, \Phi_{y,i}, \Phi_{z,i}) \) and \( N_M \) is the mode number adopted. The Galerkin approximation to the momentum equation is

\[
\left( \Phi_i, \frac{\partial \ddot{u}}{\partial t} + (\ddot{u} \cdot \nabla) \ddot{u} \right) = -\nabla p + \frac{1}{\text{Re}} \nabla^2 \ddot{u} \tag{2-78}
\]

where \( \Phi \) is modal vector and the subscript \( i \) is the \( i \)th mode.

The inner product maps vector to scalars, including the boundary data. A system of coupled ordinary differential equations (ODEs) of function \( \alpha_i(t) \) is generated. The following shows the details of decomposition of (2-78).

The inner product is a linear operator, thus we have

\[
\left( \Phi_i, \frac{\partial \ddot{u}}{\partial t} \right) + \left( \Phi_i, (\ddot{u} \cdot \nabla) \ddot{u} \right) = \left( \Phi_i, \nabla p \right) + \frac{1}{\text{Re}} \left( \Phi_i, \nabla^2 \ddot{u} \right) \tag{2-79}
\]

**Term 1: Time derivative term**

\[
\left( \Phi_i, \frac{\partial \ddot{u}}{\partial t} \right) = \left( \Phi_i, \frac{\partial}{\partial t} \left( \ddot{u}_m + \sum_{j=1}^{N_M} \alpha_j(t) \Phi_j(x) \right) \right) \tag{2-80}
\]

Since \( \ddot{u}_m \) and \( \Phi \) are only function of spatial variables, their derivatives with respect to time are zero.
\[
\left( \Phi_i, \frac{\partial u}{\partial t} \right) = \left( \Phi_i, \sum_{j=1}^{N_u} \frac{d\alpha_j}{dt} \Phi_j (x) \right) = \sum_{j=1}^{N_u} \frac{d\alpha_j}{dt} (\Phi_i, \Phi_j)
\] (2-81)

Because POD modes are orthogonal to each other, (2-74) is used to further simplify (2-82). Finally for the time derivative term we have

\[
\left( \Phi_i, \frac{\partial \bar{u}}{\partial t} \right) = \frac{d\alpha_i}{dt}
\] (2-82)

**Term 2: Convection term**

\[
(\Phi_i, (\bar{u} \cdot \nabla) \bar{u}) = (\Phi_i, \left( \bar{u}_m + \sum_{j=1}^{N_u} \alpha_j \Phi_j \right) \cdot \nabla \left( \bar{u}_m + \sum_{k=1}^{N_u} \alpha_k \Phi_k \right))
\]

\[
= (\Phi_i, (\bar{u}_m \cdot \nabla) \bar{u}_m) + \left( \Phi_i, (\bar{u}_m \cdot \nabla) \left( \sum_{k=1}^{N_u} \alpha_k \Phi_k \right) \right)
\]

\[
+ \left( \Phi_i, \left( \sum_{j=1}^{N_u} \alpha_j \Phi_j \right) \cdot \nabla \bar{u}_m \right) + \left( \Phi_i, \left( \sum_{j=1}^{N_u} \alpha_j \Phi_j \right) \cdot \nabla \left( \sum_{k=1}^{N_u} \alpha_k \Phi_k \right) \right)
\]

\[
= (\Phi_i, (\bar{u}_m \cdot \nabla) \bar{u}_m) + \sum_{k=1}^{N_u} \alpha_k (\Phi_i, (\bar{u}_m \cdot \nabla) \Phi_k) + \sum_{j=1}^{N_u} \alpha_j (\Phi_i, (\Phi_j \cdot \nabla) \bar{u}_m) + \sum_{j=1}^{N_u} \sum_{k=1}^{N_u} \alpha_j \alpha_k (\Phi_i, (\Phi_j \cdot \nabla) \Phi_k)
\] (2-83)

**Term 3: Pressure term**

The general formula for integration by parts in high dimensions is
\[
\int_{\Omega} \nabla u \cdot \tilde{v} d\Omega = \int_{\Gamma} u \tilde{v} \cdot \bar{n} d\Gamma - \int_{\Omega} u \nabla \cdot \tilde{v} d\Omega \tag{2-84}
\]

where \( \bar{n} \) is the outward unit surface normal to the boundary \( \Gamma = \partial \Omega \). Applying (2-84) to the pressure term yields:

\[
(\Phi_i, \nabla p) = \int_{\Omega} \nabla p \cdot \Phi_i d\Omega = \int_{\Gamma} p \Phi_i \cdot \bar{n} d\Gamma - \int_{\Omega} p \nabla \cdot \Phi_i d\Omega \tag{2-85}
\]

From (2-73) and the divergence-free condition we can have

\[\nabla \cdot \Phi_i = 0 \tag{2-86}\]

for \( i = 1, 2, ..., N_M \). Substituting (2-86) into (2-85) yields

\[
(\Phi_i, \nabla p) = \int_{\Gamma} p \Phi_i \cdot \bar{n} d\Gamma \tag{2-87}
\]

We denote \([ \ ]\) for boundary integral. Now (2-87) can be written in the form of boundary integral

\[
(\Phi_i, \nabla p) = \left[ p \Phi_i \right] \tag{2-88}
\]

**Term 4: Diffusion term**

From (2-84) we can derive first Green's identity by letting \( \tilde{v} = \nabla w \)

\[
\int_{\Omega} \nabla u \cdot \nabla w d\Omega = \int_{\Gamma} u \nabla w \cdot \bar{n} d\Gamma - \int_{\Omega} u (\nabla \cdot \nabla w) d\Omega \tag{2-89}
\]

Applying integration by parts to the above equation yields

\[
\int_{\Omega} u \nabla^2 w d\Omega = \int_{\Gamma} u \nabla w \cdot \bar{n} d\Gamma - \int_{\Omega} \nabla u \cdot \nabla w d\Omega \tag{2-90}
\]

The definition of inner product gives

\[
(\Phi_i, \nabla^2 u) = (\Phi_{x,i}, \nabla^2 u) + (\Phi_{y,i}, \nabla^2 v) + (\Phi_{z,i}, \nabla^2 w) \tag{2-91}
\]
Applying (2-90) to (2-91) yields

\[ \left( \Phi_i, \nabla^2 \bar{u} \right) = \int_{\Gamma} \Phi_{x,i} \nabla u \cdot \vec{n} d\Gamma - \int_{\Omega} \nabla \Phi_{x,i} \cdot \nabla ud\Omega \\
+ \int_{\Gamma} \Phi_{y,i} \nabla v \cdot \vec{n} d\Gamma - \int_{\Omega} \nabla \Phi_{y,i} \cdot \nabla vd\Omega \\
+ \int_{\Gamma} \Phi_{z,i} \nabla w \cdot \vec{n} d\Gamma - \int_{\Omega} \nabla \Phi_{z,i} \cdot \nabla wd\Omega \]  

(2-92)

Define

\[ [\Phi_i \nabla \bar{u}] = [\Phi_{x,i} \nabla u] + [\Phi_{y,i} \nabla v] + [\Phi_{z,i} \nabla w] \]  

(2-93)

and

\[ (\nabla \Phi_i, \nabla \bar{u}) = (\nabla \Phi_{x,i}, \nabla u) + (\nabla \Phi_{y,i}, \nabla v) + (\nabla \Phi_{z,i}, \nabla v) \]  

(2-94)

Thus, (2-92) can be simplified as

\[ \left( \Phi_i, \nabla^2 \bar{u} \right) = [\Phi_i \nabla \bar{u}] - (\nabla \Phi_i, \nabla \bar{u}) \]  

(2-95)

Substituting (2-77) into the above equation yields

\[ \left( \Phi_i, \nabla^2 \bar{u} \right) = \left[ \Phi_i \nabla \left( \bar{u}_m + \sum_{j=1}^{N_u} \alpha_j \Phi_j \right) \right] \\
- \left[ \nabla \Phi_i, \nabla \left( \bar{u}_m + \sum_{j=1}^{N_u} \alpha_j \Phi_j \right) \right] \\
= [\Phi_i \nabla \bar{u}_m] - (\nabla \Phi_i, \nabla \bar{u}_m) \\
+ \sum_{j=1}^{N_u} \alpha_j [\Phi_i \nabla \Phi_j] - \sum_{j=1}^{N_u} \alpha_j (\nabla \Phi_i, \nabla \Phi_j) \]  

(2-96)

Ordinary differential equations of full-system dynamics

Substituting each term back to equation (2-79) yields:
\[
\frac{d\alpha_i(t)}{dt} = a_i + d_i(t) + \sum_{j=1}^{N_M} b_{ij} \alpha_j(t) + \sum_{j=1}^{N_M} \sum_{k=1}^{N_M} c_{ijk} \alpha_j(t) \alpha_k(t)
\] (2-97)

where \( i = 1, 2, \ldots, N_M \)

\[
a_i = -\left( \Phi_i, (\bar{\mathbf{u}}_m \cdot \nabla) \bar{u}_m \right) + \frac{1}{\text{Re}} \left\{ [\Phi_i \nabla \bar{u}_m] - (\nabla \Phi_i, \nabla \bar{u}_m) \right\}
\] (2-98)

\[
b_{ij} = -\left( \Phi_i, (\bar{\mathbf{u}}_m \cdot \nabla) \Phi_j \right) - \left( \Phi_i, (\Phi_j \cdot \nabla) \bar{u}_m \right) + \frac{1}{\text{Re}} \left\{ [\Phi_i \nabla \Phi_j] - (\nabla \Phi_i, \nabla \Phi_j) \right\}
\] (2-99)

\[
c_{ijk} = -\left( \Phi_i, (\Phi_j \cdot \nabla) \Phi_k \right)
\] (2-100)

\[d_i = -\left[ \rho \Phi_i \right]
\] (2-101)

It is a system of the first order nonlinear ordinary differential equations. Given the initial conditions, these equations can be solved to generate the time histories of basis functions by using many implicit and explicit iterative methods. In our computation, a fourth order Runge-Kutta method is used.
Chapter 3  Dragonfly in Free Flight

Four-winged dragonflies are one of the most well-known insects involving ipsilateral wing-wake interaction. The influence of wing-wake interaction in slow and hovering flight has been investigated in previous studies. Using Local Momentum Theory (LMT), Azuma et al. (1985) predicted that the complex wake structures produced by the forewing would affect the lift production of the hindwing in slow flight. This influence was confirmed by experiments (Thomas, Taylor et al. 2004) and numerical simulations (Sun and Lan 2004; Wang 2007; Liang and Dong 2009; Dong and Liang 2010). In other studies using mechanical models, it is found that the performance of the forewing remains nearly constant, while lift of the hindwing may vary by a factor of two during hovering flight (Maybury and Lehmann 2004; Lehmann 2008). However, hovering and slow flight are only two typical flight modes used by dragonflies. With each wing controlled independently, dragonflies can fly toward almost any direction (Thomas, Taylor et al. 2004). Knowledge of wing-wake interaction obtained from the previous studies may not be applicable to other flight modes. How wing-wake interaction affects aerodynamic performance of dragonflies is still unknown for take-off and maneuver. With particular interest in this problem, we studied a freely flying dragonfly, which experienced two flight stages: an escape (take-off) stage and a maneuver stage. We selected different wing combinations such as four wings, two forewings, and two hindwings to perform CFD simulations. Comparison between three simulations helps to disclose the effect of ipsilateral wing-wake interaction on vortex formation and associated aerodynamic performance. With supports from the newly developed solver, these long-term, large-scale simulations were accomplished.
This chapter consists of four sections. Flight kinematics related techniques are discussed first, followed by wing deformation metrics. The third section introduces simulation related topics. At the end, the simulation results are presented and discussed.

### 3.1 Body Movement and Orientation

This section describes the approaches to quantify flight kinematics using some parameters that can represent the characteristics of flapping motion.

#### 3.1.1 Coordinate Systems

Two sets of orthonormal coordinate systems are used to describe body movement and orientation. One is a fixed inertial coordinate system, or earth-fixed system, with orthonormal basis \( \hat{e}_i \), for \( i=1,2,3 \). It is the one built in surface reconstruction. The other one is fixed in the body, known as the body-fixed system or non-inertial coordinate system, with orthonormal basis \( \hat{e'}_i \). The origin of the body-fixed system is located on the mass center of the dragonfly body with the \( X \) axis through the body axis from the tail to the head and the \( Z \) axis from the dorsal to the abdomen (see Figure 3-1 for the coordinate notation of the body-fixed system). The body-fixed system is used to define relative movement and orientation of the body to the earth-fixed system. It also helps to identify some kinematic and dynamic parameters such as moment, local flow velocity, and local forces.
Figure 3-1. Body orientation in the body-fixed system, viewing from (a) top, (b) right side of the body, (c) head to tail and (d) perspective view.

A vector \( \vec{r}_e = (r_{e1}, r_{e2}, r_{e3}) \) in the earth-fixed system can be expressed by

\[
\vec{r}_e = \sum_i r_{ei} \hat{e}_i \tag{3-1}
\]

In the body-fixed system, the same vector \( \vec{r}_e \) can be written as

\[
\vec{r}_e = \sum_i r'_{ei} \hat{e}'_i + \sum_i \vec{R}_i \hat{e}_i \tag{3-2}
\]
where \( r_b' = (r_{b1}, r_{b2}, r_{b3}) \) is the coordinates in the body-fixed system and \( \tilde{R} = (R_x, R_y, R_z) \) is the vector from the origin of the earth-fixed system to that of the body-fixed system. \( \tilde{R} \) is the movement of mass center of the dragonfly in our experiment.

\( r_b' \) and \( \tilde{R} \) are with respect to different basis. We can rearrange the equation by moving terms under the same basis:

\[
\sum_i r_b' \hat{e}_i = \sum_j (r_j - \tilde{R}_j) \hat{e}_j
\]

Because both two coordinate systems consist of orthonormal basis, one set of basis can be expressed as a linear combination of the other set. Here we represent the body-fixed basis in terms of the earth-fixed basis, written as

\[
\hat{e}_i' = \sum_j A_{ij} \hat{e}_j
\]

where \( A_{ij} \) is rotation matrix.

(3-4) can be written in matrix notation

\[
A^T r_b' = r_e - \tilde{R}
\]

The rotation matrix has properties such that

\[
AA^{-1} = AA^T = I
\]

It can help to transform Eq. (3-5) into a form of

\[
r_b' = A(r_e - \tilde{R})
\]
3.1.2 Euler Angles

According to Euler’s rotation theorem, any rotation can be made by three elemental rotations about three Euler angles. There are twelve possible Euler angle sequences with respect to rotation axes. Six of them involve a rotation about an axis twice, e.g. the sequence ‘z-x-z’ which means rotating about the Z axis by angle $\Psi$ first, then about the X axis by angle $\Theta$, and finally about the Z axis by angle $\Phi$. The other six sequences rotate against three different axes, e.g. the sequence ‘z-y-x’ (pitch-roll-yaw) which is widely used in the scenario of airplane flight dynamics (Roskam 1995). As dragonflies are natural flyers, it is convenient to adopt the pitch-roll-yaw sequence in the current study to analyze the flight dynamics.

The elemental rotations of the pitch-roll-yaw sequence can be written in matrix forms as

\[
\begin{align*}
B &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \Phi & \sin \Phi \\ 0 & -\sin \Phi & \cos \Phi \end{bmatrix} \\
C &= \begin{bmatrix} \cos \Theta & 0 & -\sin \Theta \\ 0 & 1 & 0 \\ \sin \Theta & 0 & \cos \Theta \end{bmatrix} \\
D &= \begin{bmatrix} \cos \Psi & \sin \Psi & 0 \\ -\sin \Psi & \cos \Psi & 0 \\ 0 & 0 & 1 \end{bmatrix}
\end{align*}
\]

The result is a simply matrix product, written as

\[
A = B \cdot C \cdot D
\]

Expanding Eq. (3-11) yields
\[
A = \begin{bmatrix}
\cos \Theta \cos \Psi & \cos \Theta \sin \Psi & -\sin \Theta \\
\cos \Psi \sin \Theta \sin \Phi - \cos \Phi \sin \Psi & \cos \Phi \cos \Psi + \sin \Theta \sin \Phi \sin \Psi & \cos \Theta \sin \Phi \\
\cos \Phi \cos \Psi \sin \Theta + \sin \Phi \sin \Psi & -\cos \Psi \sin \Phi + \cos \Phi \sin \Theta \sin \Psi & \cos \Theta \cos \Phi
\end{bmatrix}
\] (3-12)

Its transpose and inverse are the same, written as

\[
A^{-1} = A^T =
\begin{bmatrix}
\cos \Theta \cos \Psi & \cos \Psi \sin \Theta \sin \Phi - \cos \Phi \sin \Psi & \cos \Phi \cos \Psi \sin \Theta + \sin \Phi \sin \Psi \\
\cos \Theta \sin \Psi & \cos \Phi \cos \Psi + \sin \Theta \sin \Phi \sin \Psi & -\cos \Psi \sin \Phi + \cos \Phi \sin \Theta \sin \Psi \\
-\sin \Theta & \cos \Theta \sin \Phi & \cos \Theta \cos \Phi
\end{bmatrix}
\] (3-13)

The angular rotational rate can be calculated by the following equation:

\[
\Omega' = -A(t) \cdot \frac{d}{dt} A^{-1}(t) = \frac{d}{dt} A(t) \cdot A^{-1}(t)
\] (3-14)

Expanding Eq. (3-14) yields

\[
\Omega = \begin{bmatrix}
0 & \Psi \cos \Theta \cos \Phi - \Theta \sin \Theta & -\Theta \cos \Phi - \Psi \cos \Theta \sin \Phi \\
-\Psi \cos \Theta \cos \Phi + \Theta \sin \Theta & 0 & \Phi - \Psi \sin \Theta \\
\Theta \cos \Phi + \Psi \cos \Theta \sin \Phi & -\Phi - \Psi \sin \Theta & 0
\end{bmatrix}
\] (3-15)

The rotational tensor \( \Omega' \) is antisymmetric with three degrees of freedom. It is equivalent to a rotational vector \( \omega = (\omega_1, \omega_2, \omega_3) \), where

\[
\omega_1 = \Omega'_{23} = \dot{\Phi} - \Psi \sin \Theta
\] (3-16)

\[
\omega_2 = \Omega'_{31} = \dot{\Theta} \cos \Phi + \Psi \cos \Theta \sin \Phi
\] (3-17)

\[
\omega_3 = \Omega'_{42} = \dot{\Psi} \cos \Theta \cos \Phi - \dot{\Theta} \sin \Phi
\] (3-18)

The rotational vector \( (\omega_1, \omega_2, \omega_3) \) is defined in the body-fixed system and may be called as body axis angular rates \((P,Q,R)\) in some engineering references.

### 3.1.3 Calculation of Euler Angles
The expansion of ration matrix $A$ has a complex form in which three Euler angles are non-linearly coupled. This causes the major difficulty to solve for the Euler angles. However, it can be seen that some entry of $A$ only involves one Euler angle and some involve two angles. This yields possibilities to solve Euler angles by carefully selecting points whose rotation only invokes those simple entries.

The points we select are vectors $(x,0,0)$ and $(0,y,0)$ in the body-fixed system. They have corresponding coordinates $(x_1,y_1,z_1)$ and $(x_2,y_2,z_2)$ in the earth-fixed system.

Substituting $(x,0,0)$ and $(x_1,y_1,z_1)$ into Eq. (3-5) yields

$$A^{-1} \begin{pmatrix} x \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} x \cos \Theta \cos \Psi \\ x \cos \Theta \sin \Psi \\ -x \sin \Theta \end{pmatrix} = \begin{pmatrix} x_1 - R_x \\ y_1 - R_y \\ z_1 - R_z \end{pmatrix}$$

(3-19)

From the last row of the above equation, we can solve for the pitch angle $\Theta$ as

$$\Theta = \sin^{-1} \left( \frac{z_1 - R_z}{-x} \right)$$

(3-20)

After knowing the pitch angle $\Theta$, we can solve for the yaw angle $\Psi$ using equations in any other two rows. Here the first row equation is used:

$$\Psi = \cos^{-1} \left( \frac{x_1 - R_x}{x \cos \Theta} \right)$$

(3-21)

The roll angle $\Phi$ is solved by substituting $(0,y,0)$ and $(x_2,y_2,z_2)$ into Eq. (3-5),

$$A^{-1} \begin{pmatrix} 0 \\ y \\ 0 \end{pmatrix} = \begin{pmatrix} y (\cos \Psi \sin \Theta \sin \Phi - \cos \Phi \sin \Psi) \\ y (\cos \Phi \cos \Psi + \sin \Theta \sin \Phi \sin \Psi) \\ y \cos \Theta \sin \Phi \end{pmatrix} = \begin{pmatrix} x_2 - R_x \\ y_2 - R_y \\ z_2 - R_z \end{pmatrix}$$

(3-22)
The third row equation yields

\[
\Phi = \sin^{-1}\left( \frac{z_2 - R_z}{y \cos \Theta} \right)
\] (3-23)

At this step, there is still some problem about the solution of Euler angles. Because the pitch angle \( \Theta \) calculated from the arcsine function is in the range \( -\frac{\pi}{2} \) to \( \frac{\pi}{2} \), \( \cos \Theta \) will always be positive. To find the real solution of Euler angles, angles outside of the range have to be considered.

So far, our discussion is based on the rotation of the axes. However, there is another convention of rotation, so called rotation of objects relative to fixed axes. For instance, rotating about the Z axis by yaw angle 45° using Eq. (3-10) can be interpreted as rotating the basis \( \hat{e}'_1 \) and \( \hat{e}'_2 \) of the body-fixed system to \( \left( \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right) \) and \( \left( -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right) \) of the earth-fixed system, respectively. When the same operation is applied to vectors \( (1,0) \) and \( (0,1) \), they are rotated by -45° to \( \left( -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right) \) and \( \left( \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right) \), respectively. The same rotational parameter causes moving in the opposite direction. To move vector \( (1,0) \) to \( \left( \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right) \), we can still substitute \( \Psi = -45° \) into the rotational matrix \( A \). In many mathematic textbooks, rotation of the axes is usually the way preferred to explain rotation. However, when dealing with visualization and engineering work, rotation of the body is more commonly used.
3.2 Wing Deformation Metrics

In order to accurately make repeatable measurements of the deformations, such as twist and camber, undergone by a reconstructed flapping insect wing, a universal reference plane must be established. Some works have attempted to do this by defining wing deformation with respect to a plane that is fixed to several points on the wing or with respect to the straight line between the wing root and tip. However, these methods do not account for span-wise deformations and will vary based on wing size and shape. We address this by defining a least-squares plane as a universal reference plane for each wing that can aid in describing the physical position of the wing at any point in the stroke from a statistical perspective.

To study the deformation and topography of an individual wing surface \( \Pi \), a least squares reference plane \( \Omega \) is generated from the points on the reconstructed wing. The wing surface \( \Pi \) is first discretized as an element \( \{(X_l, A_l) : l = 1,2,\ldots,m\} \) with \( X_l = (x_l, y_l, z_l) \) and \( A_l \) standing for the centroid and area of the \( l^{th} \) element respectively. The centroid of the entire wing is defined as follows:

\[
X_0 = \frac{\sum_{l=1}^{m} X_l A_l}{\sum_{l=1}^{m} A_l} = (x_0, y_0, z_0)
\]  
(3-24)

The least squares plane corresponding to the wing surface is then defined as the plane passing through centroid \( X_0 \) with unit normal vector \( n \), such that the following quantity \( e_0^2 \) is minimized:

\[
e_0^2 = \sum_{l=1}^{m} A_l^2 \left( X_l - X_0 \right) n^T n \]  
(3-25)

\[
e_0^2 = n^T \sum_{l=1}^{m} A_l^2 \left( X_l - X_0 \right) \left( X_l - X_0 \right)^T n = n^T Mn
\]
The symmetric matrix $M$ in (3-25) is the central moment tensor. The components of normal vector $n$ can be found as the principal axes of inertia of $M$ after singular value decomposition is applied. More details on this least squares plane-fitting method can be found in (Ahn, 2004).

![Diagram](image)

Figure 3-2. Conceptual illustration of wing deformation measurements taken in the chord-wise direction with respect to the least squares plane $\Omega$. The local angle of chord-wise twist $\alpha^c_k$ is illustrated at the cross-section plane 80% from the wing root and the camber-to-chord ratio parameters are illustrated in the cross-section 20% from the root. The gray dotted silhouette of the wing is the projection of the entire deformed wing onto the least squares plane (Koehler, Liang et al. 2012).

The obtained least squares plane $\Omega$ and reconstructed wing surface $\Pi$ shown in Figure 3-2, are used together to define several wing deformation metrics such as the local angle of twist, the span-wise camber, and the chord-wise camber. Points A and B on $\Omega$ denote the projections of the user selected wing root and tip onto $\Omega$. The cross-section plane $\chi_k$ can
then be defined normal to line segment AB, where the subscript \( k \) denotes the percentage of the distance between points A and B. Lower values of \( k \) indicate cuts taken close to the root of the wing, and higher values indicate cuts taken near the wing tip.

The camber line \( S_k \) is defined as the intersection of the plane \( \chi_k \) and the wing surface \( \Pi \). The chord \( C_k \) is then the line segment connecting the trailing and leading points of \( S_k \). The line \( l_k \) is then defined as the intersection of \( \Omega \) and \( \chi_k \). The local angle of chord-wise twist \( \alpha_i^c \) at the cross-section plane \( \chi_k \) can then be defined as the angle between the chord \( C_k \) and line \( l_k \). These definitions are explained visually in Figure 3-2.

A similar process is used to define deformation metrics in the span-wise direction. The superscript \( c \) denotes a cross-sectional cut in the chord-wise direction and the superscript \( s \) denotes one in the span-wise direction. Instead of using points A and B to define a line along which cross-section planes are placed, the projection of the endpoints of \( C_{50} \) onto \( \Omega \) is used. Thus, the span \( P_i^c \) and local angle of span-wise twist \( \alpha_i^c \) can be measured relative to \( \Omega \) in cuts approximately parallel to the leading edge of the wing.

### 3.3 Simulations

Experimental data were recorded during a dragonfly conducted escape using our high-speed photogrammetry system. The length of the whole video was 0.66 second. A fragment of 0.231 seconds was adopted, starting from the instant that the dragonfly’s feet left the ground and ending at a point that the dragonfly flew out of the focus of cameras. Based on the video, three dimensional kinematics data were then reconstructed in Autodesk Maya with
95% confidence in spatial accuracy (Koehler, Liang et al. 2012). The data processing process yielded highly accurate kinematical motion of dragonfly escape and maneuver.

3.3.1 Flow Parameters

In numerical simulations, researchers usually match the non-dimensional parameters, Reynolds number, with experiments to reproduce the same physical features. The mid-chord length of the left forewing is used as the primary length scale for the current flow and is denoted by $L$. Based on the time averaged velocity $U$ at the mid-chord of the left forewing, we define the Reynolds number as $Re=UL/\nu$, where $\nu$ is the kinematic viscosity of air. $U$, $L$ and $\nu$ are 1.47 m/s, 7.39E-3 m and 1.57E-6 m$^2$/s, respectively. Thus the experimental Reynolds number is estimated as 693. The Reynolds number can vary in a wide range with respect to different selections of reference velocity and length. If we choose the reference length and velocity at the $\frac{3}{4}$ chord, the Reynolds number will increase to approximately 1100. Based on a previous study on fish swimming which employs three Reynolds number, 540, 1440 and 6300 (Bozkurttas, Mittal et al. 2009), the main difference between different Reynolds numbers is the force amplitude. Pressure thrust may increase about 20% when the Reynolds number increases from 540 to 1440, and increase additionally about 20% as the Reynolds number increases from 1440 to 6300. However, the thrust history of three Reynolds numbers shows a similar trend.

As mentioned in Chapter one, the grid number needs to be proportional to $Re^{\frac{3}{4}}$ to fully capture flow features. It can conclude that the grid number must be increased by at least 2.8 times if the Reynolds number increases from 693 to 1100. Regarding the current
computational power, we select two values of the Reynolds number, 693 and 69.3, which span one order-of-magnitude, to simulate and study. Simulations for a larger Reynolds number such as 6930 will be performed in the future.

3.3.2 Computational Domain, Grid and Boundary Conditions

Figure 3-3 presents the computational domain, a $30 \times 30 \times 30$ cuboid, used in simulations, showing the location of the first and last frames of the dragonfly. A stretching Cartesian grid with grid numbers equal to $231 \times 231 \times 326$ was used in the simulations. To provide the highest resolution for the flow around the wings, a uniform grid with grid size equal to 0.03125 was selected as the dense region where the grid number was $198 \times 198 \times 293$. Homogenous Neumann boundary was specified at six faces of the cuboid for both velocity and pressure boundary condition. No-slip velocity boundary and homogeneous Neumann boundary were specified on the surface of dragonfly body and wings for both velocity and pressure, respectively. Comprehensive studies had been carried out to assess the effect of the grid resolution and domain size of the salient features of the flow and also to demonstrate the accuracy of the nominal grid (Dong, Mittal et al. 2006; Bozkurttas, Mittal et al. 2009)
3.3.3 Case Summary

Five cases were studied with respect to the combination of the Re number and wings. In order to assess the scaling effects on the flow, two Reynolds numbers (69.3 and 693) were chosen. The effect of wing-wake interaction was studied by changing wing combination as 4 wings, forewings only and hindwings only. Table 3-1 summarizes simulation cases in this work.
Table 3-1: Simulations parameters to investigate the influence of wing-wake interaction

<table>
<thead>
<tr>
<th>Re</th>
<th>Case name</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>69.3 4-wing</td>
<td>one body, two forewings and two hindwings</td>
</tr>
<tr>
<td>2</td>
<td>69.3 Forewing-only</td>
<td>one body and two forewings</td>
</tr>
<tr>
<td>3</td>
<td>69.3 Hindwing-only</td>
<td>one body and two hindwings</td>
</tr>
<tr>
<td>4</td>
<td>693 4-wing</td>
<td>one body, two forewings and two hindwings</td>
</tr>
<tr>
<td>5</td>
<td>693 Forewing-only</td>
<td>one body and two forewings</td>
</tr>
</tbody>
</table>

3.4 Results and Discussions

3.4.1 Flight Kinematics Analysis

3.4.1.1 Motion of the dragonfly body

The position of mass center and Euler angles of the dragonfly body as a function of time are shown in Figure 3-4. From the video, we can think that the complete flight consists of two successive stages: an escape (take-off) stage and a maneuver stage. In the escape stage, the dragonfly left the mounted rod it had been stayed because the dragonfly sensed that the mounted rod was stroked horizontally by another rod. Therefore, its movement was mainly in the Z direction to avoid the striking. The climbing movement in the Y direction was to maintain dragonfly’s altitude so that it did not touch the ground. After several wingbeats at approximate 60ms, the dragonfly found that it was safe from the rod striking. Thus, it began to turn around for better observing the environment. During this maneuver stage, the pitch angle changes relatively smaller than the yaw and roll angels so that the dragonfly can maintain certain incline angle and look around.
Indeed, the escape and maneuver motions were performed simultaneously. We identified 60ms as the transition time from the escape to maneuver stages because the pitch, roll and yaw angles experienced turning at that moment. In addition, the X position of the body mass center had a local minimum at 60ms. This means that the dragonfly was moving back and forth at that time.

![Graphs showing position and Euler angles](image)

Figure 3-4. (a) Position of mass center and (b) Euler angles of the dragonfly body.

### 3.4.1.2 Wing stroke plane

Because of the deformation of wings, the process of determining a suitable wing stroke plane is much complicated than those with rigid wings. Ellington (Ellington 1984) defined a stroke plane by the linear regression line of wing tips in addition to a wing root with an implicit assumption that the tip did not deviate from the stroke plane. Tobalske et al. (Tobalske, Warrick et al. 2007) defined a stroke plane by three points: wing root, wingtip at the start of a downstroke and wingtip at the end of the downstroke. These methods were suitable and accurate for rigid wings. However, stroke amplitudes are overestimated for
flexible wings because wing tips always bend to an extent larger than any other portion of the wings at the reversal time. In our study, we defined a stroke plane by the wing root and the projections of the wing tip on the least-squares plane $\Omega$ (points B in Figure 3-2) at the reversal time of a half-stroke. Using the projection points can reduce the overestimation of stroke amplitudes.

The process consisted of two rounds of trial and error. A body-fixed coordinate system on the dragonfly body (see Figure 3-1) was constructed to detect the reversal time. In the first round, the reversal time was determined by the maximum or minimum position of the projected wing tip in the X direction of the local system. The initial stroke plane was defined by the wing root and the other two projections of the wing tip. Resulting from possible non-zero deviation angles, the initial stroke plane might not always span over the largest stroke amplitude for the half-stroke. Thus, a second round was to ensure that the largest stroke amplitude was achieved. This was done by selecting time frames near the initial reversal time as candidates to find the local extreme of stroke amplitude.

Stroke angle $\phi$, deviation angle $\theta$ and geometrical angle of attack $\alpha$ can be defined with respect to the stroke plane $X'Y'$, as shown in Figure 3-5a. The deviation angle is the angle between the rotational axis of the wing and the stroke plane. The geometric angle of attack is the angle between the least-squares plane and the stroke plane. For the hindwings, the $Y'Z'$ plane is approximately parallel to the body center plane. In contrast, the angle between the $Y'Z'$ planes of the forewings and the body center plane ranges from $10^\circ$ to $30^\circ$. Figure 3-5b shows a 2-D sketch of slice cuts across wing span. The wings are projected on the least-squares plane, thus are shown as line segments with circles representing the leading edge.
The horizontal dash-dotted lines represent the X axis in the body-fixed system. The angle between the body axis and the stroke plane is denoted by the incline angle $\gamma$.

(a) (b) Figure 3-5. (a) Coordinate system for wing motions, where the X’Y’-plane is stroke plane.

(b) Illustration for the geometric angle of attack $\alpha$.

3.4.1.3 Timing of wing stroke

The timing of wing reversals was determined by the aforementioned method. Evidences in wingbeats of the dragonfly confirmed that the dragonfly adopted different flight modes in escape and maneuver. We observed that both forewings were synchronized within one millisecond to reverse at the same time despite that their starting stroke angles were different. Table 3-2 and Table 3-3 respectively show the stroke timing of left and right forewing strokes identified from the reconstructed kinematics in the body-fixed system. Before time node 1, it was a downstroke motion. Because of this synchronization, we consider the forewings as one system independent of the hindwings system. Wings rotation may include pronation and supination stages. However, it is difficult to accurately determine them in free flight.
Table 3-2: Stroke timing of left forewing

<table>
<thead>
<tr>
<th>Time Node</th>
<th>Beginning of Time (ms)</th>
<th>Time (ms)</th>
<th>Interval of U to U (ms)</th>
<th>Frequency $f$ (Hz)</th>
<th>Frequency variation $f - f_{RMS}$</th>
<th>Frequency variation rate $\frac{f - f_{RMS}}{f_{RMS}}$</th>
</tr>
</thead>
<tbody>
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<td>21.4</td>
<td>46.7</td>
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<td>D</td>
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<td>11.3</td>
<td></td>
<td></td>
<td></td>
</tr>
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Table 3-3: Stroke timing of right forewing

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<th>Interval (ms)</th>
<th>Interval of U to U (ms)</th>
<th>Frequency f (Hz)</th>
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<td></td>
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<td>-1.0%</td>
</tr>
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<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>U</td>
<td>73.1</td>
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<td>25.0</td>
<td>40.1</td>
<td>0.1</td>
<td>0.4%</td>
</tr>
<tr>
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<td>D</td>
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<td>11.9</td>
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<td></td>
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<td></td>
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<td>33.7</td>
<td>-6.2</td>
<td>-15.5%</td>
</tr>
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<td>D</td>
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<td>11</td>
<td>U</td>
<td>127.7</td>
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<td>26.3</td>
<td>38.0</td>
<td>-1.9</td>
<td>-4.8%</td>
</tr>
<tr>
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<td>D</td>
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<td>U</td>
<td>183.3</td>
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<td>38.2</td>
<td>-1.7</td>
<td>-4.2%</td>
</tr>
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<tr>
<td>17</td>
<td>U</td>
<td>209.5</td>
<td>13.0</td>
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<td></td>
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<td></td>
</tr>
<tr>
<td>18</td>
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<td>222.5</td>
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</tr>
</tbody>
</table>

Minimum: 33.7 -6.2
Maximum: 47.2 7.3
RMS: 39.9 4.6

We define a complete stroke as an upstroke followed by a downstroke. Flapping frequency of dragonfly’s forewings can be statistically divided into three modes based on behaviors of frequency variation rate $\frac{f - f_{RMS}}{f_{RMS}}$: 1) sign of the frequency variation rate is
positive or negative and 2) value of the frequency variation rate in each stage has relatively small variations. Stroke 1 and Stroke 2 use an escape mode. Stroke 3 and Stroke 4 use a transition mode. The end time of third stroke is at 60ms, which is identified as the transition time from the escape phase to the maneuver phase. From Stroke 5 to Stroke 8, a maneuver mode is used.

The escape mode employs the largest flapping frequency 46.4 and 47.1Hz for the left and right forewings, respectively, while the maneuver mode includes the lowest frequency 33.6Hz (both forewings in Stroke 5). The frequency of the transition mode is approximately the RMS values 39.7 and 39.7Hz, which is between that number of the escape and maneuver modes. Both the escape and transition modes have small variations in frequency (less than 4%). As a comparison, frequency fluctuation of the maneuver mode is almost up to 10%.

Synchronization of hindwing reversals within one millisecond is also observed. The stroke timing of the left and right hindwing strokes is tabulated in Table 3-4 and Table 3-5, respectively. Before time node 1, it is an upstroke motion whose starting time cannot be known.

Based on the same criterions of identifying flapping modes, we classify the hindwing stroking into two modes. The first mode is an escape mode used from Stroke 1 to Stroke 3, and the second is a maneuver mode used from Stroke 4 to Stroke 8. Similar to the escape mode of the forewings, the flapping frequency of the hindwings in the escape mode is also relatively higher than those in the maneuver mode. It can be found that the RMS frequency of the hindwings is approximate the same as that of the forewings. However, the frequency variation of the forewings is larger than the hindwings.
There is no obvious evidence indicating a transition mode in the hindwing strokes as in the forewings. During the escape mode, the forewings try to be in the same phase with the hindwings to generate larger forces for escape. The starting time of Stroke 3 (near 48ms) differs for 3 ms, which is less than ¼ of an average stroke time. However, stroking at the same phase of the fore- and hindwings has never been achieved.
Table 3-4: Stroke timing of left hindwing

<table>
<thead>
<tr>
<th>Time Node</th>
<th>Beginning of Time (ms)</th>
<th>Time (ms)</th>
<th>Interval (ms)</th>
<th>Interval of U to U (ms)</th>
<th>Frequency $f$ (Hz)</th>
<th>Frequency variation $f - f_{RMS}$</th>
<th>Frequency variation rate $\frac{f - f_{RMS}}{f_{RMS}}$</th>
</tr>
</thead>
<tbody>
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<td>11.6</td>
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<td></td>
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<td></td>
</tr>
<tr>
<td>2</td>
<td>22.1</td>
<td></td>
<td>11.6</td>
<td>23.7</td>
<td>42.3</td>
<td>3.1</td>
<td>7.9%</td>
</tr>
<tr>
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<td>12.1</td>
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<td></td>
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</tr>
<tr>
<td>4</td>
<td>45.8</td>
<td></td>
<td>11.5</td>
<td>23.2</td>
<td>43.2</td>
<td>4.0</td>
<td>10.2%</td>
</tr>
<tr>
<td>5</td>
<td>57.2</td>
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<td>11.7</td>
<td></td>
<td></td>
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<td></td>
</tr>
<tr>
<td>6</td>
<td>68.9</td>
<td></td>
<td>12.2</td>
<td>24.3</td>
<td>41.2</td>
<td>2.0</td>
<td>5.1%</td>
</tr>
<tr>
<td>7</td>
<td>81.2</td>
<td></td>
<td>12.1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>U</td>
<td>93.2</td>
<td>14.0</td>
<td>27.3</td>
<td>36.6</td>
<td>-2.6</td>
<td>-6.6%</td>
</tr>
<tr>
<td>9</td>
<td>D</td>
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<td></td>
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<td>27.1</td>
<td>36.9</td>
<td>-2.3</td>
<td>-5.8%</td>
</tr>
<tr>
<td>11</td>
<td>D</td>
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<td>14.5</td>
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<td></td>
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<td></td>
</tr>
<tr>
<td>12</td>
<td>U</td>
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<td>-6.3%</td>
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<td></td>
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<tr>
<td>14</td>
<td>U</td>
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</tr>
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<td></td>
</tr>
<tr>
<td>16</td>
<td>U</td>
<td>201.4</td>
<td>14.0</td>
<td>26.3</td>
<td>38.1</td>
<td>-1.1</td>
<td>-2.8%</td>
</tr>
<tr>
<td>17</td>
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<td>12.3</td>
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<td></td>
</tr>
<tr>
<td>18</td>
<td>U</td>
<td>227.6</td>
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<td></td>
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<td></td>
<td></td>
<td></td>
<td>43.2</td>
<td>4.0</td>
<td></td>
</tr>
<tr>
<td>RMS</td>
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<td></td>
<td>39.2</td>
<td>2.5</td>
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### Table 3-5: Stroke timing of right hindwing

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<th>Time (ms)</th>
<th>Interval (ms)</th>
<th>Interval of U to U (ms)</th>
<th>Frequency $f$ (Hz)</th>
<th>Frequency variation $f - f_{RMS}$</th>
<th>Frequency variation rate $\frac{f - f_{RMS}}{f_{RMS}}$</th>
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<td></td>
</tr>
<tr>
<td>2</td>
<td>U</td>
<td>22.9</td>
<td>10.8</td>
<td>22.6</td>
<td>44.2</td>
<td>4.9</td>
<td>12.5%</td>
</tr>
<tr>
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<td>11.8</td>
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<td></td>
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<td></td>
</tr>
<tr>
<td>4</td>
<td>U</td>
<td>45.5</td>
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<td>22.7</td>
<td>44.0</td>
<td>4.6</td>
<td>11.8%</td>
</tr>
<tr>
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<td>57.5</td>
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<tr>
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<td>68.3</td>
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<td>40.8</td>
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<td>3.6%</td>
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<td>120.4</td>
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<td>26.7</td>
<td>37.4</td>
<td>-1.9</td>
<td>-4.9%</td>
</tr>
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<td>U</td>
<td>147.2</td>
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<td>26.8</td>
<td>37.4</td>
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<td>-5.0%</td>
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<td>U</td>
<td>173.9</td>
<td>14.1</td>
<td>27.6</td>
<td>36.3</td>
<td>-3.1</td>
<td>-7.8%</td>
</tr>
<tr>
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<td>188.0</td>
<td>13.5</td>
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<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>16</td>
<td>U</td>
<td>201.5</td>
<td>13.8</td>
<td>26.7</td>
<td>37.5</td>
<td>-1.8</td>
<td>-4.7%</td>
</tr>
<tr>
<td>17</td>
<td>D</td>
<td>215.3</td>
<td>12.9</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>18</td>
<td>U</td>
<td>228.2</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

| Minimum   |              | 36.2      |               |                         |                  | -3.2                              |                                   |
| Maximum   |              |           |               |                         | 44.2             | 4.9                               |                                   |
| RMS       |              |           |               |                         |                  | 39.3                              | 3                                 |

#### 3.4.1.4 Stroke amplitude

Stroke amplitudes of 17 half-strokes are shown in Figure 3-6. In general, the stroke amplitude ranges from $40^\circ$ to $85^\circ$ for all four wings during the complete flight. In the escape
stage, stroke amplitudes of the left-side wings are larger than that of the right-side wings, especially at half-strokes 3 and 4 of the forewings and half-strokes 2 and 3 of the hingwings. This implies that force generation from one side is different from the other side. Given that the stroke motion of the wings is synchronized, the smaller stroke amplitudes, the smaller maximum wing tip velocity. If the angle of attack is same for the wings, the left-side wings would generate larger forces than the right-side wings and the dragonfly would roll to the right-side. As a result, we can see that the roll angle $\Phi$ greatly increases from 30ms to 65ms, which span the half-strokes with significant difference in stroke amplitudes. In contrast, there is no obvious trend in the maneuver stage.

![Figure 3-6. Stroke amplitudes of (a) the forewings and (b) the hindwings.](image)

### 3.4.1.5 Phase between fore- and hindwings

In literature, the phase difference between fore- and hindwings of dragonflies has been classified into three categories: counterstroking ($180^\circ$), parallel stroking ($0^\circ$) and phase-shifted stroking ($54$-$100^\circ$) (Wang, Zeng et al. 2003; Maybury and Lehmann 2004). The
phase-shifted stroking is used both in maneuver flight and a special escape motion observed on tethered dragonflies. For robotic wing systems, the phase difference is usually a constant. In the current study, the flapping frequency changes non-uniformly. This results in a continuously phase shift between the fore- and hindwings. In order to quantify the phase difference and make comparisons with literature, we normalized the stroke angle $\phi$ by the stroke amplitude in such a way that the start of the upstroke and downstroke was equal to -1 and +1, respectively. Therefore, the normalized stroke angle can be written as

$$
\begin{align*}
\phi_H &= -\cos(t) \\
\phi_F &= -\cos(t - \phi_p)
\end{align*}
$$

where $\phi_p$ is the phase difference between the forewing and hindwing. Taking an inverse of cosine can directly compute the phase difference.

Figure 3-7 shows the phase between the left-side wings and phase between the right-side wings at reversals of wings. A phase-catch-up process is indicated by the turning point at $t=48$ms. The stroke starting time difference between the fore- and hindwings is less than 3ms, a quarter of one half-stroke time. At an earlier or later reversal time, the time difference is larger than 3ms.

The range of phase difference in the escape and maneuver stages is from $36^\circ$ to $62^\circ$ and from $52^\circ$ to $122^\circ$. Thus, the total range is from $36^\circ$ to $122^\circ$, which is larger than the range defined for the phase-shifted stroking ($54^\circ$-$100^\circ$) (Wang, Zeng et al. 2003). The difference is probably because the rigid wing assumption was used and flapping frequency of the fore- and hindwings is unified to a constant value (Wang, Zeng et al. 2003) or the phase angle of the hindwing is normalized with respect to the stroke amplitude of the forewing (Wakeling and Ellington 1997).
3.4.2 Aerodynamic Force

3.4.2.1 Total force magnitude of wings

In order to study the influence of wing-wake interaction, we compare the forces acting on four wings of case 4-wing with the forces acting on two wings of case Forewing and two wings of case Hindwing. It can be seen that these curves have a few characteristics corresponding to wing flapping modes in the escape (t=0 to 60ms) and maneuver stages (t=60 to 231ms), as shown in Figure 3-8. The time-average and the largest peak forces of two stages are given in Table 3-6.
Table 3-6: Comparison of the time-average and the largest peak forces of the escape and maneuver stages between 4-wing and Forewing plus Hindwing at Re=69.3

<table>
<thead>
<tr>
<th>Force (N x 10^3)</th>
<th>Escape</th>
<th>Maneuver</th>
<th>Escape</th>
<th>Maneuver</th>
</tr>
</thead>
<tbody>
<tr>
<td>Average</td>
<td>12.53</td>
<td>7.29</td>
<td>12.00</td>
<td>7.35</td>
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<tr>
<td>Peak</td>
<td>30.87</td>
<td>18.19</td>
<td>28.57</td>
<td>19.72</td>
</tr>
</tbody>
</table>

Figure 3-8. History of total force magnitude of 4-wing in comparison with Forewing plus Hindwing at Re=69.3.

Several common features can be seen in the curves of both the 4-wing and Forewing plus Hindwing cases. First of all, the largest force peaks at t=17.8 and 40.4ms appear in the downstrokes of the fore- and hindwings during the escape stage. The peak values are up to approximately 30mN, which is about 12 times of body weight. In contrast, the peak value of the upstrokes in the escape stage is only 11mN, nearly 2 times smaller than those in the downstrokes. In addition, there is only one dominant peak in a downstroke.
Secondly, the peak value of the third upstroke (right before t=60ms) increases to 17mN and the peak value of the third downstroke conspicuously decreases to 19mN. This verifies our prior judgment that the transition time from the escape to the maneuver was at t=60ms. It also manifests the effect of the transition mode of the forewings. The time interval of the transition mode was from t=48 to 98ms, which overlapped with some portions of the escape and maneuver modes of the hindwings. The cooperation between the forewings and hindwings caused the changes in the forces.

Third and the last, there is no obvious pattern to identify upstrokes and downstrokes based on force peaks in the maneuver stage. The force peaks range from 7 to 19mN. The force valleys are relatively larger than those in the escape stage. However, the average and peak forces are both smaller.

Based on Table 3-6, one may conclude that the influence of wing-wake interaction is not significant because the difference in the time-average and peak forces are less than 10% during the escape and maneuver stages. However, significant differences can be seen at a number of moments, as shown in Figure 3-8. The difference at the peak of second downstroke (t=40ms) is 3.62mN, which is more than 10% of the peak force. Wing-wake interaction acted oppositely for different stages. It generally enhanced the force generation in the escape stage, while attenuated the force in the maneuver stage.

3.4.2.2 Force of the left-side wings

We take the left fore- and hindwings as examples to further examine the effect of wing-wake interaction. Observations and conclusions can be applied to the right-side wings. Figure
3-9a shows the total force acting on the left forewing of case 4-wing and the force acting on the left forewing of case Forewing at Re=69.3. The time-average force of two stages is given in Table 3-7. The time-average force of Re=693 case is approximately 12% larger in the escape stage than the one at Re=69.3. However, both cases follow the same trend with or without wing-wake interaction, as shown in Figure 3-9b.

Table 3-7: Comparison of the time-average force (N \times 10^3) of the escape and maneuver stages between the left forewing of the cases 4-wing and Forewing at Re=69.3 and Re=693

<table>
<thead>
<tr>
<th>Re</th>
<th>Escape</th>
<th>Maneuver</th>
<th>Total</th>
<th>Escape</th>
<th>Maneuver</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>69.3</td>
<td>3.60</td>
<td>2.46</td>
<td>2.76</td>
<td>3.43</td>
<td>2.56</td>
<td>2.79</td>
</tr>
<tr>
<td>693</td>
<td>4.04</td>
<td>2.44</td>
<td>2.86</td>
<td>3.89</td>
<td>2.66</td>
<td>2.98</td>
</tr>
</tbody>
</table>

Difference between the force curves is significant in the escape stage. The largest peak force difference is in the second wingbeat (26% and 42% in the up- and downstroke, respectively). Generally, the force of Forewing is larger than that of 4-wing in the upstrokes. However, it is smaller in the downstrokes. It means the wing-wake interaction enhanced the force generation of the forewing in the downstrokes. On the contrary, it attenuated the force in the upstrokes.

In several wingbeats of the maneuver stage, e.g. at t=81 and 191ms, the forewing was seldom affected by the hindwing. Other than that, the influence of wing-wake interaction was
non-trivial and visible. In addition, the interaction usually attenuated the force in both the upstrokes and downstrokes. This behavior was different from the escape stage.

The influence of wing-wake interaction on the hindwing was not as significant as on the forewing in the escape stage, but successively significant in the maneuver stage. From Figure 3-9c, it can be seen that the interaction slightly attenuated the force in the downstrokes of the maneuver stage. The influence in the upstrokes was relatively larger and complicated. The interaction could change the force trend, increasing the number of force peak from one to two, or shifting the position of peaks.
Figure 3-9. Comparison of the total force on (a) the left forewing of 4-wing and Forewing at Re=69.3, (b) the left forewing of 4-wing and Forewing at Re=693 and (c) the left hindwing of 4-wing and Hindwing at Re=69.3.
Figure 3-10. Comparison of the force in the Z direction. (a) The left forewing of 4-wing and Forewing. (b) The left hindwing of 4-wing and Hindwing. Re=69.3.

Table 3-8: Comparison of the time-average Z-force ($N \times 10^3$) of the escape and maneuver stages between the left forewing of the cases 4-wing and Forewing at Re=69.3 and Re=693

<table>
<thead>
<tr>
<th></th>
<th>4-wing</th>
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<th>Forewing</th>
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<tr>
<td></td>
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<td>Escape</td>
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<td>Re</td>
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<tr>
<td>69.3</td>
<td>1.72</td>
<td>-0.30</td>
<td>0.22</td>
<td>1.40</td>
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<tr>
<td>693</td>
<td>2.06</td>
<td>-0.30</td>
<td>0.31</td>
<td>1.68</td>
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</table>
Figure 3-10 shows the force in the Z direction between the left forewing of the 4-wing case and that of the two-wing cases at Re=69.3. The time-average forces generated by the forewing during the escape and maneuver stages are 1.72 and -0.30mN (see Table 3-8), respectively. As we can see in Figure 3-4a, the largest displacement of the body mass center during the escape stage is in the Z direction, thus the Z force exhibits large bumps in the first four downstrokes for the fore- and hindwings. In the other wingbeats of the maneuver stage, the negative force generated in the upstrokes dominants the positive force generated in the downstrokes. Thus, from approximately t=90ms, the upward trend of the Z position of the body mass center continued, however, not as strong as before. The influence of wing-wake interaction is mainly at the downstrokes of the forewing during the escape stage and the upstrokes of the hindwing during the maneuver stage. It implies that different interaction mechanisms were used in two stages.
Animals fly against the gravity, thus, a positive lift is necessary to support the flight. Most of birds get all the lift from downstrokes (Norberg 1990). Hummingbirds generate 75% of lift during their downstrokes and 25% during upstrokes (Warrick, Tobalske et al. 2005; Liang, Dong et al. 2010). Fruitfly can generate 50% and 50% during up- and downstrokes, respectively (Dickinson, Lehmann et al. 1999; Sun and Tang 2002). It can be seen in Figure 3-11 that the lift of the fore- and hindwings is positive during most of the up- and downstrokes. As the dragonfly continuously changed its body orientation direction during the whole flight, the positive lift implies the dragonfly was aware of its body orientation and intentionally adjusted its wing motion so that enough lift was generated. In addition, the lift
generated in the upstrokes is even larger than in the downstrokes, especially on the forewing in the maneuver stage. It suggests that dragonflies produce lift by using wing kinematics similar to the other insects.

![Graph](image)

**Figure 3-12.** Comparison of the force in the X direction. (a) The left forewing of 4-wing and Forewing. (b) The left hindwing of 4-wing and Hindwing. Re=69.3.

The time-average X-force generated by the forewing during the escape and maneuver stages is -0.27 and 0.60mN, respectively. The force generated by the hindwing is -0.41 and 0.06mN for the two stages, respectively. The influence of wing-wake interaction in the X-force has the similar characteristic as the Z-force. It is significant in the downstrokes of the forewing during the escape stage and the upstrokes of the hindwing during the maneuver
stage, as shown in Figure 3-12. Further inspection reveals that the interaction increases the magnitude of the time-average force of the forewing to -0.41 and 0.81mN.

### 3.4.3 Aerodynamic Power

In incompressible flow, instantaneous aerodynamic power consumption of a body can be written as the surface integral of the inner product between the stress tensor and velocity on body surface:

\[
P = \int_S \left( -p\delta_{ij} + 2\mu e_{ij} \right) \cdot udS
\]  

(3-27)

where \( p \) is interpreted as the mean pressure on the element \( dS \), \( e_{ij} \) is strain rate tensor and \( S \) is surface area. The aerodynamic power represents the power requirement for a body to overcome pressure and stress forces from the air. In the current study, positive power means the body consumes energy to do work. Negative power means that the body receives work from the air.

#### 3.4.3.1 Total wing power

We compare the three cases in terms of the power consumption of wings, as shown in Figure 3-13a, where the solid line represents the power summation of four wings from 4-wing and the dashed line represents the power summation of two forewings from Forewing and two hindwings from Hindwing. In general, the curves of the total power have some common features as the curves of the total force. From \( t=0 \) to 45ms, the curves experience a small bump followed by a large bump for two strokes, in which peak-to-peak values of the
large bumps change slightly. From Figure 3-13b it can be seen that the small and large bumps are caused by the up- and downstrokes of the wings, respectively. The third period is from t=45 to 70ms, during which the first peak rises up and the second peak descends greatly, comparing to the preceding stroke. From t=70ms, the curves show no obvious periodical pattern as the first three strokes. Both two curves generally follow the same trend in the whole flight.

Influence of wing-wake interaction between ipsilateral wings of the dragonfly is indicated by the difference of two curves, especially at peaks and valleys. For the first two periods, the wing-wake interaction causes the dragonfly consumes more energy. While in the other time, the interaction generally causes energy saving. The whole history of power consumption has positive value. This suggests that the dragonfly is continuously consuming energy to sustain the flight. However, from the detailed plot Figure 3-13b, we can see negative power values at some valleys. This means that both fore- and hindwings may gain a small amount of energy from the air during their reversals, probably due to the flexible structure of the wings. However, reversal of fore- and hingwings are not at the same time. The magnitude of the gained energy is so small that it can only trivially compensate the energy expense.
Figure 3-13. Comparison of power consumption of wings between 4-wing and Forewing plus Hindwing at Re=69.3. (a) The complete history and (b) the time interval from $t=0$ to 80ms also showing the curves of Forewing and Hindwing.

### 3.4.3.2 Aerodynamic power of left-side wings

Figure 3-14 shows the power consumption of the left fore- and hindwings, which are taken as examples of the fore- and hindwings, respectively, for the first three wingbeats. Right-side wings have the same behavior as their left parts in terms of power. In the first two strokes, the power is mainly consumed during the downstrokes. While in the third stroke, consumed power in the upstroke is comparable to that in the downstroke.

It has been shown that total energy consumption of 4-wing is always larger than the summation of Forewing and Hindwing during the first two strokes (see Figure 3-13b). A straightforward explanation is that the wings without wing-wake interaction consume less energy than with interaction. However, this explanation is not supported by the observed power consumption of an individual wing. For the forewing, the power consumption without
wing-wake interaction is a little bit larger in upstrokes but significantly smaller in downstrokes than with the interaction. The difference for the hindwing is not as significant as the forewing, except in the second upstroke. It can be seen that the power consumption without the interaction is slightly smaller in the upstrokes and larger in the downstrokes. This is opposite to the forewing trend.

3.4.4 Wake structure analysis

In the current study, a special feature in force and power is observed in the first two wingbeats of the forewings. The force is significantly enhanced in the downstrokes when the forewings work together with the hindwings. This enhancement, specifically in the Z-force, is thought to be favored by the dragonfly because it was trying to escape from striking. A larger force could help it moving away faster. As this feature appears in both the first and the second wingbeats, it is not caused by the starting motion of the wings but by some other
factors. In this section, we will present some results on the near field of the wings and analyze the development of local vortices. The investigation consists of four steps. The first step is to compare three-dimensional vortex structures with or without wing-wake interaction. It can give us some basic information about the position at which the wing-wake interaction significantly affects the vortices. The second is performing slice cut on the most affected position to view the development of vorticity on the cross-section plane. The third is to identify the major vortices near the forewing on the slice cuts and track their circulation development. And the last step is to examine the surface pressure on the forewing to understand how the wing-wake interaction affects the pressure distribution on the whole wing.

We choose the first downstroke to study the flow fields because it is less affected by the vortices formed in previous strokes. The orientation of the dragonfly at t=20ms is shown in Figure 3-15. The dragonfly was inclining with the stroke plane of the fore- and hindwings approximately parallel to the Z-axis and a positive Z-force is greatly generated.

The global view of three-dimensional vortex structures in the flow field is shown by iso-surfaces of $Q$-criterion (Hunt, Wray et al. 1988), which defines vortex as regions with $Q>0$, where
\[
Q = \frac{1}{2} \left[ |\Omega|^2 - |S|^2 \right] = \frac{1}{4} \left( \omega^2 - 2 |S|^2 \right)
\]

(3-28)

\(\Omega\) is the vorticity tensor and \(S\) is the tensor of rate of strain. In incompressible flows, the \(Q\)-criterion is related to pressure through the pressure-Poisson equation:

\[
Q = \frac{1}{2\rho} \nabla^2 p = \frac{\partial u_i}{\partial x_j} \frac{\partial u_j}{\partial x_i}
\]

(3-29)

Instantaneous iso-surfaces of \(Q\)-criterion and streamlines at \(t=14\) and \(20\)ms are shown in Figure 3-16. The streamlines are generated by integrating over a velocity field in the plane. We define LEV and TEV as leading edge vortex and trailing edge vortex, respectively. Subscription \(F\) means forewing and \(H\) means hindwing. There are six major vortices in this short period. Four vortices are related to the forewing. Leading edge vortex \(LEV_{F1}\) and trailing edge vortex \(TEV_{F1}\) are the vortex pair generated during the preceding upstroke of the forewing, while leading edge vortex \(LEV_{F2}\) and trailing edge vortex \(TEV_{F2}\) are generated during the ongoing downstroke. Two vortices, leading edge vortex \(LEV_{H}\) and trailing edge vortex \(TEV_{H}\), are related to the hindwing.

At \(t=14\)ms, the forewing is at pronation stage and the hindwing is in the downstroke. We can see clearly the leading edge vortex \(LEV_{F1}\) and \(LEV_{F2}\) and the trailing edge vortex \(TEV_{F2}\) attaching to the forewing, although their size is small. In contrast, the size of the leading edge vortex \(LEV_{H}\) and the trailing edge vortex \(TEV_{H}\) is much larger. Additionally, \(LEV_{H}\) and \(TEV_{H}\) are connected through wing tip, forming a whole vortex structure in space. The forewing vortices are hardly affected by the hindwing at this moment.
Figure 3-16. Instantaneous streamlines and iso-surfaces of $Q$ criterion ($Q=14$) for 4-wing at $t=14$ and 20ms. Left column: 4-wing case with wing-wake interaction. Right column: Forewing case without interaction. Re=69.3.

At $t=20$ms, the leading edge vortex LEV$_{F2}$ and the trailing edge vortex TEV$_{F2}$ have connected through wing tip. It can be observed that the size of LEV$_{F2}$ is relatively larger in case 4-wing. TEV$_{F2}$ has detached the trailing edge of the forewing and contacts with a vortex filament which bifurcates from the leading edge vortex LEV$_{H}$. The shape of TEV$_{F2}$ in case
Forewing is a straight cone. However, this straight cone distorts in case 4-wing due to wing-wake interaction, specifically the wall effect caused by the hindwing. The movement of TEV$_{F2}$ is induced by the motion of the hindwing and the other vortices generated by the hindwing. This inducing effect is not uniform along the trailing edge and it is maximal at approximately mid-span. Therefore, the largest distortion happens at mid-span.

Next, we perform slice cut at mid-span to track the vortex development near the forewing. Four successive time instants are highlighted, showing some critical moments at which the vorticity near the forewing is affected by the hindwing. The phase difference between the fore- and hindwing is about 60°, which corresponds to a phase-shift value of 16% in (Maybury and Lehmann 2004; Lehmann 2009) (0% indicates an in-phase motion; 25% indicates the hindwing leads the forewing by 90°).
Figure 3-17. Time sequence of the wake showing instantaneous streamlines and vorticity contours. Left column: 4-wing case with wing-wake interaction. Right column: Forewing case without interaction. From top to bottom row: t=14, 15, 17 and 20ms. The contour levels, as shown in Figure 3-17a, are the same in all plots. Vorticity unit is $s^{-1}$. Re=69.3.

Figure 3-17 shows instantaneous streamlines and vorticity contours on a plane that intersects the mid-span of the forewing at four successive time $t=14$, 15, 17 and 20ms. The vorticity contours show the component of vorticity vectors normal to the plane, indicating the strength and range of vortices. The arrows are pointing to the center of vortices, which are indicated as node or spiral points by streamlines. Note that not all visualized vorticity contours converge to nodes. And some nodes may not align to points with the highest or the lowest strength locally. These mismatches are normal fluid phenomena that reflect the difference of velocity and vorticity.

At $t=14$ms, the forewing is performing pronation and at the beginning of downstroke. We can observe the leading edge vortex $\text{LEV}_{F2}$, which can be confirmed by converged streamlines, is being generated at the forewing. The hindwing has already finished pronation and is accelerating in its downstroke. Compared with the newborn $\text{LEV}_{F2}$, the leading edge vortex of the hindwing $\text{LEV}_{H}$ is well-developed and its strength is much larger. Note that at this moment, the vortex $\text{TEV}_{H}$ has shed from the trailing edge of the hindwing. Compared with the Forewing case, the streamlines on the upper surface of the forewing change directions when they are away from the forewing. As the distance between the fore- and hindwing is still large, the vorticity around the forewing is slightly changed due to the wing-wake interaction.
At $t=15$ms, the leading edge of the hindwing is approaching the trailing edge of the forewing and the two wings are at the nearest distance to each other. The strength of the leading edge vortex $\text{LEV}_{F2}$ continuously increases, in contrast to the decrease of $\text{LEV}_{F1}$, which partially concentrates at the trailing edge of the forewing to form a trailing edge vortex. The leading edge vortex $\text{LEV}_{H}$ partially contacts with $\text{TEV}_{F1}$. Compared with the Forewing case, the shape of $\text{LEV}_{F1}$ at the upper surface of the forewing is slightly different.

At $t=17$ms, the leading edge vortex $\text{LEV}_{F2}$ is gradually developing on the upper surface of the forewing. While a part of the leading edge vortex $\text{LEV}_{F1}$ converts to an independent vortex structure $\text{TEV}_{F2}$ that can be indicated by a streamline node. The leading edge vortex of the hindwing $\text{LEV}_{H}$ is fully interacting with the trailing edge vortex of the forewing $\text{TEV}_{F1}$. Because these two vortices have the same direction, two vortices merge together. Compared with the Forewing case, the area of $\text{LEV}_{F2}$ and $\text{TEV}_{F2}$ is much larger.

At $t=20$ms, as both the fore- and hindwings move at about their maximal translational velocity, both trailing edge vortices $\text{TEV}_{F2}$ and $\text{TEV}_{H}$ have shed away from the trailing edges for more than one chord length with their center marked by streamlines. On the contrary, both the leading edge vortices $\text{LEV}_{F2}$ and $\text{LEV}_{H}$ are still attached to the corresponding leading edges. Compared with the Forewing case, the direction of streamlines near the forewing is not deflected by the hindwing. The shedding distance of $\text{TEV}_{F2}$ is larger and the area of $\text{LEV}_{F2}$ is larger too.

We choose a closed contour line with a threshold as the boundary of $\text{LEV}_{F2}$ and $\text{TEV}_{F2}$. The threshold is 1% of the largest vorticity of the vortices. If vorticity at $(x, y)$ is positive (negative) and is larger (smaller) than the threshold, the point $(x, y)$ is considered as part of
TEV\(F_2\) (LEV\(F_2\)). Algorithms were developed to distinguish LEV\(F_2\) and TEV\(F_2\) from other vortices. After the boundary is determined, circulation of LEV\(F_2\) and TEV\(F_2\) can be calculated by taking area integral of vorticity. Figure 3-18 shows the circulation history of LEV\(F_2\) and TEV\(F_2\) with or without wing-wake interaction. It can be seen that LEV\(F_2\) and TEV\(F_2\) experienced a process that its strength increased from \(t=14\) to 23ms and decreased thereafter. This trend is a result of a competition between new vorticity generated at the surface of the forewing and viscous diffusion of vorticity \(\nu \nabla^2 \omega\) in the flow field. In the 4-wing case, the presence of the hindwing increased the strength of both LEV\(F_2\) and TEV\(F_2\). The curve of TEV\(F_2\) stopped at \(t=21.5\)ms is because TEV\(F_2\) merged with TEV\(_H\) and could not be extracted separately.

Figure 3-18. Circulation history of the leading edge vortex LEV\(F_2\) (blue lines with open circles) and the trailing edge vortex TEV\(F_2\) (black lines without symbols) in a comparison of 4-wing (solid line) and Forewing (dashed line).
Distribution of pressure difference and pressure on the lower and upper surfaces are respectively shown in Figure 3-19 for cases 4-wing and Forewing at t=20ms when the largest Z-force is achieved. The pressure difference is defined as the pressure on the upper surface minus the pressure on the lower surface. There are several interesting observations on the forewing.

First, a global extreme of the pressure difference is located at approximately 75% from the root of the forewings. It is caused by the upper surface pressure according to the location. A local pressure extreme, which is highly related to the lower surface pressure, is located at approximately 90% for the forewing. The extremes are on the leading edge. Therefore, the leading edge would experience larger stress than other portions of the wing. Correspondingly, the structure of the leading edge is strengthened by wing veins to build corrugated wing architecture, as shown in Figure 3-20.

Second, pressure difference changes non-uniformly along the wing span from root to tip. Contour lines indicate pressure gradient, which is mainly caused by the upper surface pressure due to the contour pattern and magnitude, inclined to the chord-wise at about 50% wing span. Accordingly, wing veins are inclined approximately at the same angle to reinforce the structure.

Third, pressure on the upper surface of case 4-wing is generally lower than that of Forewing. It is generally higher on the lower surface. As a result, the pressure difference is lower and the total force acting on the forewing of 4-wing is larger than that of Forewing. In addition, the contour patterns of the pressure difference, the upper and lower surface pressure are very similar on the forewing of 4-wing and Forewing cases.
Figure 3-19. Pressure distributions on wing surfaces at $t=20$ms. Top row: pressure difference of the upper and lower surface of the wings. Middle row: pressure on the lower surface of the wings. Bottom row: pressure on the upper surface of the wings. Negative pressure means pressure on the upper surface is smaller than on the lower surface. Positive pressure has the opposite meaning. Pressure unit is Pa ($\text{N/m}^2$).
LEVF₂ is on the upper surface. TEVF₂ is on the lower surface. Solid and dashed lines denote the shape of vortices for cases 4-wing and Forewing, respectively. Arrows denote the change of center of the first moment of vorticity.

So far, as we have known the range, area and circulation of the leading edge vortex LEVF₂ and the trailing edge vortex TEVF₂, and had a general idea about the vortex formation affected by wing-wake interaction or not, we made a simple model explaining why the total force is enhanced instead of attenuated or unaffected during the downstroke if TEVF₂ sheds further away from the trailing edge. Figure 3-21 shows a sketch of the forewing and its attaching vortices LEVF₂ and TEVF₂. According to the study done by (Li and Lu 2012), the force and power are mainly contributed by local vortex structures in the vicinity of the wing, while the local vortex structures also contain the full induced effect of the other structures in
the flow. Thus, other vortices such as TEV_{F1} has detached from the wing thus is not modeled. Solid and dash lines denote the shape of LEV_{F2} and TEV_{F2} for cases 4-wing and Forewing, respectively. According to Figure 3-17g and h, area of LEV_{F2} of 4-wing is relatively larger than that of Forewing. TEV_{F2} of 4-wing has shed further than that of Forewing. The first moment of vorticity is defined in (1-13), reproducing here

\[ \bar{r} = \int_V \vec{r} \times \vec{\omega} \ dV \]  

(3-30)

where \( \vec{r} \) is a position vector and \( V = V_{TEV} + V_{LEV} \). In inviscid flows, the total vorticity of a system starting from rest is always zero (Batchelor 1967). This statement is generally applicable to viscous flows if a domain retains all vorticity (Wu 1981). Because we do not model the starting vortex, the volume integral of vorticity is not zero. In the context of two-dimensional vector fields, we can write \( \Gamma = \int_V \vec{\omega} \ dV \). If \( \Gamma \) is non-zero, a non-trivial vector \( \vec{R} \) satisfying

\[ \vec{R} \times \int_V \vec{\omega} \ dV = \dot{\vec{R}} \times \Gamma = \int_V \vec{r} \times \vec{\omega} \ dV \]  

(3-31)

exists, where \( \vec{R} \) is called center of the first moment of vorticity. Therefore, (3-30) can be transformed into a form of

\[ \bar{r} = \vec{R} \times \Gamma \]  

(3-32)

where \( \vec{R} = (R \cos \theta, R \sin \theta) \) in a polar coordinate system with the origin fixed at the leading edge of the forewing. Based on (1-12), the magnitude of the force acting on the forewing exerted by LEV_{F2} and TEV_{F2} is
\[
\sqrt{\textbf{F}}^2 = \sqrt{\left(\frac{d\tau}{dt}\right)^2} = \left[ \frac{dR^2}{dt} \cdot \Gamma^2 + R^2 \cdot \frac{d\Gamma^2}{dt} \right]^\frac{1}{2}
\]  

(3-33)

Note that this formula is used to quantify the effect of specific vortices in terms of force. The force calculated by (3-33) is not the same as the force exerted by all vortices in the flow field.

Let \( \Gamma \) be the same for cases 4-wing and Forewing at an initial time, e.g. \( t=14\text{ms} \), respectively. Assuming the non-derivative terms in (3-33) are invariant, an approximation of the force difference between 4-wing and Forewing can be obtained by

\[
\Delta \sqrt{\textbf{F}}^2 \approx \left[ \delta \left( 2R \frac{dR}{dt} \right) \cdot \Gamma^2 + \delta \left( 2\Gamma \frac{d\Gamma}{dt} \right) \cdot R^2 \right]^\frac{1}{2}
\]  

(3-34)

where \( \delta \) denotes the change between 4-wing and Forewing at a later time. In the current study, the amount of negative vorticity generated during a limited time is approximately equal to the amount of positive vorticity. Thus, the rate of vorticity generation is nearly the same for \( \text{LEV}_{F2} \) and \( \text{TEV}_{F2} \). Assuming the diffusion rate of \( \text{LEV}_{F2} \) and \( \text{TEV}_{F2} \) is also the same, we obtain \( \frac{d\Gamma_{TEV}}{dt} + \frac{d\Gamma_{LEV}}{dt} \approx 0 \). Therefore, \( \frac{d\Gamma}{dt} \) is equal to zero. The second term in (3-34) can be ignored.

It can be seen from Figure 3-21 that travelling distance of \( \text{TEV}_{F2} \) is larger in 4-wing than in Forewing during the same time interval. Thus \( \delta \frac{dR}{dt} \) is larger than zero. As a result, the total force of 4-wing is accordingly larger.
3.4.5 Wall Effect

A few studies have been done to investigate the influence of wing-wake interaction on the forewings. Maybury and Lehmann found that the interaction between ipsilateral wings could affect the aerodynamic performance of the forewings (Maybury and Lehmann 2004). However, the influence on the forewings was less pronounced than on the hindwings. As a result, their study focused on the hindwings and only force history of the hindwings was shown.

Theoretically, the hindwings can be easily affected by the forewings because the downwash generated by the forewing may change the effective angle of attack of the hindwing. In contrast, the way the hindwings can affect the forewings is by interrupting the downwash of the forewings. This may change the stagnation point location on the forewing. However, it is thought to be considerably less effective since the interference is to affect the upstream of the downwash.

We have shown in the previous section, the force on the forewing was significantly enhanced in the first two downstrokes when the forewing did not separately move. Correspondingly, the motion and strength of the leading edge vortex $LEV_{F2}$ and the trailing edge vortex $TEV_{F2}$ were affected. The only reason that caused these changes is the presence of the hindwing. However, the hindwing was flapping for the whole eight-and-half strokes. The force of the other strokes in the maneuver stage did not behave like the first two. Maybury and Lehmann suggested that at certain phase difference the hindwing can act as a wall, distorting flow direction of the forewing downwash (Maybury and Lehmann 2004). Thus, we examined the phase difference between the left fore- and hindwings (see Figure 3-7). It is found that the phase difference cannot be the critical factor because the phase
difference of the left-side wings is 56-62° and 52-67° in the first and third downstrokes, respectively. The third downstroke has a similar range of phase difference as the first downstroke, however, the force is not accordingly enhanced. In addition, the phase difference of the right-side wings is 65° at the start of the second downstroke. It is relatively higher than at the start of the third stroke (55°). Therefore, it is not convincible that the phase difference can totally determine the wall effect.

This concept of wall effect is from the ground effect, which is thought to be used in the take-off and landing of birds to enhance lift at low speed. The distance between the wing and the ground is the only factor determining whether the ground effect comes into effect. Implied by this, we measured the distance between two wings, which is computed as the distance between the midpoint of the trailing edge of the forewing and the midpoint of the leading edge of the hindwing (see Figure 3-22).

![Figure 3-22. Sketch for the distance between the two wings.](image-url)
Figure 3-23. Distance between the trailing edge of the forewings and the leading edge of the hindwings at mid-span. The shadowed regions are the downstrokes of the left forewing.

The measurement is shown in Figure 3-23. It can be seen that the distance in the first two downstrokes has a different pattern in the other downstrokes. The minimum distance between the right-side wings appears in the middle of the first two downstrokes. It means the fore- and hingwings were getting closer (less than 1 chord length) during the first half of downstrokes. Hence the wall effect was continuously effective. The left-side wing has a small peak in the first two downstrokes. However, these peaks are much smaller than in the other downstrokes. From the figures showing the development of the leading- and trailing edge vortices (see Figure 3-17e-h), we can see the wall effect played an important role when new vortices were being generated. The continuous approaching of the hindwings to the trailing edge vortices changed local flow conditions at the reversal, thus, changed the strength and movement of the trailing edge vortex. If the distance was as large as in the other downstrokes, the vortices were less affected by the wall effect.

Still, a question remains: how the wall effect changes the motion and strength of LEV\textsubscript{F2} and TEV\textsubscript{F2}. We examined the pressure and in-plane velocity vector fields of 4-wing at the moments at t=17 and 20ms, in comparison with Forewing, as shown in Figure 3-24. Generally, on the upper surface of the hindwing, LEV\textsubscript{H} caused a low pressure region, which
consequently increased the flow velocity of the forewing downwash. As the fore- and hindwing moved together in a short distance, the downwash blew TEV$_{F2}$ further away from the trailing edge of the forewing in the direction indicated by a large open arrow, as shown in Figure 3-24a and c. It can be seen that both the velocity on the upper and lower surfaces of the forewing was induced to increase. It is notable that this wall effect is different from the well-known ground effect because: first, the flow direction of the downwash is like a ‘Z’ shape, as the downwash is induced by the hindwing; second, a low pressure region at LEV$_{H}$ exerts efforts on accelerating the downwash of the forewing. As a comparison, there is no such positive effect in the ground effect.
Figure 3-24. Flow field around the forewing showing velocity vectors and pressure contours.

Left column: 4-wing with wing-wake interaction. Right column: Forewing without interaction. The contour levels, as shown in Figure 3-24a, are the same in all plots. Pressure unit is Pa (N/m$^2$). Re=69.3.
Chapter 4  POD and Reduced-Order Modeling of Flapping Flight

In the previous chapter, a variety of qualitative analysis techniques, such as the two-dimensional streamlines, vorticity contours, and three-dimensional iso-surfaces of $Q$ criterion, have been used to examine the flow topology of a freely flying dragonfly. The quantitative analysis on the circulation of near-field vortices has also been conducted to understand the connection between aerodynamic forces and vortex structures. However, it is notable that such analyses are limited, despite of their effectiveness. For instance, vortex merging of leading- and trailing edge vortices nullifies the circulation calculation of these two vortices. At high Reynolds numbers, interaction of vortex structures can be more frequent. In such situations, tracking circulation of individual vortices is significantly affected.

As an alternative analysis technique, Proper Orthogonal Decomposition (POD) can extract coherent structures from the flows based on statistical measurement of energy. Thus, POD analysis have been used in the analysis of turbulent shear layers (Citriniti and George 2000), driven cavity flows (Cazemier 1998), flow past a cylinder (Konstantinidis, Balabani et al. 2007; Perrin, Braza et al. 2007; Feng, Wang et al. 2011). The current study intends to use the traditional POD method (the snapshot method) in an inertial coordinate system to analyze the flow of flapping flight. Such an attempt would be invalid if POD modes cannot capture the dynamics of flapping flight. To eliminate this concern, establishing a POD-based Reduced-Order Model (ROM) that can accurately model the fluid dynamics of flapping flight is necessary.
In this chapter, we present a pressure corrected POD ROM method for the flow with moving boundaries at low Reynolds numbers. The basic techniques of ROM have been introduced in section 2.3. Here, the pressure correction to the ROM method is discussed. The proposed method is examined by flows past single or multiple moving bodies.

4.1 Effect of Pressure Term in Galerkin Projection

In numerous studies, i.e. (Graham, Peraire et al. 1999; Ma and Karniadakis 2002; Lewin and Haj-Hariri 2005), the outer boundaries of flow fields are set to either zero-velocity or zero-pressure in the normal direction of control surface. And the velocity at the surface of stationary obstacle is zero for no-slip boundary conditions. Because the POD modes $\Phi_i$ preserve the homogeneous velocity boundary condition, $\Phi_i$ at the surface of stationary obstacle is zero too. As a result,

$$\left(\Phi_i, \nabla p\right) = \left[p\Phi_i\right] \approx 0$$  \hspace{1cm} (4-1)

This can greatly simplify the ROM of flows since no pressure term is involved. However, zero-velocity condition at inflow is substantially not physical. Noack et al. compared the ROM of shear flows that was imposed by correct velocity and pressure boundary conditions(Noack, Papas et al. 2005). They found that the effect of the pressure term is not negligible. In current study, general velocity and pressure boundary conditions are employed in DNS. Specifically, velocity of the moving object is not zero. Thus the pressure term $\left(\Phi_i, \nabla p\right)$ does not vanish and needs specific treatments.
There are a number of methods to compute pressure fields. One is to solve a pressure Poisson equation

$$\nabla^2 p = -\frac{\partial u_i}{\partial x_j} \frac{\partial u_j}{\partial x_i}$$

(4-2)

which is derived by taking the divergence of Navier-Stoke equations then applying the divergence-free condition.

To evaluate the effect of the pressure term in the Galerkin projection for shear flows, Noack (Noack, Papas et al. 2005) solved a system of partial pressure equations in forms of

$$\nabla^2 p_{kl} = -\frac{\partial u_i^k}{\partial x_j} \frac{\partial u_j^l}{\partial x_i}$$

(4-3)

where the superscripts $k$ and $l$ represent the $k^{th}$ and the $l^{th}$ snapshots, respectively. Then the pressure $p$ can be expanded as a function of $\alpha_k(t)$ and $\alpha_l(t)$ and the pressure term can be included in the Galerkin projection. However, this approach employs computational work even more than direct numerical simulation since the partial pressure equation needs to be solved for $N_m^2$ times.

Akhtar et al. (Akhtar, Nayfeh et al. 2009) proposed to conduct POD on pressure field:

$$p(x,t) \approx p_m + \sum_{i=1}^{N_{\beta}} \beta_i(t) \Psi_i(x)$$

(4-4)

They substitute POD of velocity (2-77) and pressure (4-4) into the pressure-Poisson equation (4-2) and obtain

$$\frac{\partial^2 p_m}{\partial x_i^2} + \sum_{k=1}^{N_{\beta}} \beta_k \frac{\partial^2 \Psi_k}{\partial x_k^2} = - \left[ \frac{\partial u_{m,d}}{\partial x_j} + \sum_{k=1}^{N_{\beta}} \alpha_k \frac{\partial \Phi_{k,d}}{\partial x_j} \right] \left[ \frac{\partial u_{m,j}}{\partial x_j} + \sum_{k=1}^{N_{\beta}} \alpha_k \frac{\partial \Phi_{k,j}}{\partial x_j} \right]$$

(4-5)
Using an approach similar to the velocity projection on the Navier-Stokes equations equation, they project (4-5) onto the pressure POD modes, which finally yields

$$\sum_{j=1}^{N_f} e_{ij} \beta_j(t) = f_i + \sum_{j=1}^{N_m} g_{ij} \alpha_j(t) + \sum_{j=1}^{N_m} \sum_{k=1}^{N_m} h_{ijk} \alpha_j(t) \alpha_k(t)$$  \hspace{1cm} (4-6)

where

$$e_{ij} = (\Psi_i, \nabla^2 \Phi_j)$$  \hspace{1cm} (4-7)

$$f_i = - (\Psi_i, \nabla^2 p_m) - (\Psi_i, \overline{u}_m : \overline{u}_m)$$  \hspace{1cm} (4-8)

$$g_{ij} = - (\Phi_i, \nabla u_m : \nabla \Psi_j) - (\Phi_i, \nabla \Psi_j : \nabla u_m)$$  \hspace{1cm} (4-9)

$$h_{ijk} = - (\Phi_i, \nabla \Psi_j : \nabla \Psi_k)$$  \hspace{1cm} (4-10)

This approach has a problem that the temporal coefficients $\alpha_i$ and spatial functions $\Phi_i$ of velocity fields have to be known a priori. Moreover, the spatial functions $\Psi_i$ of pressure fields have to be known a priori, too. If the basis functions $\Psi_i$ are known, their temporal coefficients $\beta_i$ are known simultaneously when the POD method is performed on pressure fields. Therefore, (4-6) only shows the mathematic connection between the temporal coefficients $\beta_i$ of pressure fields and the other POD coefficients of velocity and pressure fields. It cannot independently solve any problem as a system of evolution equations.

In our DNS simulations, the fractional method is used to solve the Navier-Stokes equations. The solved pressure field is an approximation to the exact solution of pressure basically because of the difference at boundaries. We reproduce here the boundary condition of numerical method in differential form:
\[ \frac{\partial p^{n+1}}{\partial x_i} = 0 \]  

(4-11)

This boundary condition is to satisfy the velocity boundary at time step n+1 and intermediate step. In DNS, we set the velocity at intermediate step equal to the one at time step n+1. Therefore, the RHS of (4-11) is equal to zero.

In contrast, the boundary condition derived from the Navier-Stokes equations is, for instance at a stationary wall with normal vector \( n \),

\[ \frac{\partial p}{\partial n} = \frac{1}{Re} n \cdot \nabla^2 u \]  

(4-12)

(4-11) and (4-12) are exactly consistent only in inviscid flows. The difference between the theoretical pressure and the numerical pressure could be trivial because in many situations the magnitude of the viscous term in the normal direction is much smaller than that of the pressure term. For instance, in our simulations for flow past a pitching-plunging plate, the averaged magnitude of the viscous term in the normal direction is less than 2% of the pressure term at Re=200.

Regarding the small difference between the pressure of DNS and possible exact solutions, we propose a general pressure correction ROM method based on pressure fields generated from DNS. This method is examined by the study of flow past a pitching-plunging plate with pitching amplitude of 30°, followed by a comparison with pitching amplitude 10° and 20°. The ROM method in this work is also applied to two-dimensional flows with multiple moving bodies.
For the purpose of error estimation, we use the definitions of projection error and prediction error from (Graham, Peraire et al. 1999). The prediction error represents how accurate the POD method reconstructs the original flows. It is given by

\[
E = \frac{(u_e - u_p, u_e - u_p)}{(u_e - u_m, u_e - u_m)}
\]  
(4-13)

where \(u_e\) and \(u_p\) are exact flows from DNS and reconstructed flows from the POD method, respectively. The error function is normalized to avoid the effect of averaged flow because the POD method is to resolve the energy of fluctuating velocity.

Similarly, prediction error is defined as

\[
\hat{E} = \frac{(u_e - \hat{u}_p, u_e - \hat{u}_p)}{(u_e - u_m, u_e - u_m)}
\]  
(4-14)

where \(\hat{u}_p\) is reconstructed flows from Galerkin projection, in which the temporal coefficients \(\alpha_i\) are obtained by solving (2-97).

### 4.2 Results and Discussions

#### 4.2.1 Two-dimensional Flow Past a Pitching-plunging Plate

##### 4.2.1.1 Pitching amplitude equal to 30°

A two-dimensional membrane (zero thickness) plate undergoing a pitching-plunging motion is placed in flow fields with incoming free stream of velocity \(U_{\infty} = 1\). For velocity, Dirichlet boundary is used at all boundaries except at right-hand boundary where it is outflow.
The pressure on all the four boundaries is of homogeneous Neumann boundary condition. The motion of membrane airfoil can be prescribed by

\begin{align}
h(t) &= H \sin(2\pi f t - \phi + \delta) \\
\alpha(t) &= A \sin(2\pi f t + \delta)
\end{align}

where $H$ is the plunging amplitude, $A$ is the amplitude of the sinusoidal pitch angle variation, $f$ is the frequency, $\phi$ is the phase difference between the pitching and plunging motion and $\delta$ is phase difference between two tandem wings. In the single-wing study, the plunging amplitude $H$ is set to 0.5. The pitching amplitude $A$ is $30^\circ$. The phase difference of pitching and plunging motion $\phi$ is equal to $\pi/2$ and the phase difference of wings $\delta$ is zero. Reynolds number if defined as $Re = U_\infty c/\nu = 200$ where $c$ is the chord length of the wing, which is one unit length. Strouhal number $St = 2H f / U_\infty = 0.6$. Five datasets are extracted from a DNS simulation in which the vortex shedding has already reached a quasi-steady state at 960 frames per period. From the global data we select one snapshot every 1, 10 and 30 frames. For better understanding the effect of the pressure term in ROM, we also choose snapshots every 1 and 10 time frames in two periods. The case index and configurations are listed in Table 4-1. In order to make a comparison with the interference of mode number, 32 modes are used for the POD ROM in all the cases.
Figure 4-1. Vorticity contours at $t/T=1$ for a membrane plate undergoing pitching-plunging motion with pitch amplitude $30^\circ$.

<table>
<thead>
<tr>
<th>Case Index</th>
<th>$A$</th>
<th>Dataset</th>
<th>Mean of Error</th>
<th>Standard deviation</th>
</tr>
</thead>
<tbody>
<tr>
<td>C1</td>
<td></td>
<td>1 Period, 960 Frames</td>
<td>9.46E-02</td>
<td>1.50E-01</td>
</tr>
<tr>
<td>C2</td>
<td></td>
<td>1 Period, 96 Frames</td>
<td>2.06E-01</td>
<td>3.66E-01</td>
</tr>
<tr>
<td>C3</td>
<td>$30^\circ$</td>
<td>1 Period, 32 Frames</td>
<td>7.16E-01</td>
<td>1.43E+00</td>
</tr>
<tr>
<td>C4</td>
<td></td>
<td>2 Periods, 192 Frames</td>
<td>3.50E-01</td>
<td>3.98E-01</td>
</tr>
<tr>
<td>C5</td>
<td></td>
<td>2 Periods, 64 Frames</td>
<td>8.20E-01</td>
<td>8.89E-01</td>
</tr>
<tr>
<td>C6</td>
<td>$20^\circ$</td>
<td>1 Period, 96 Frames</td>
<td>1.74E-01</td>
<td>2.39E-01</td>
</tr>
<tr>
<td>C7</td>
<td>$10^\circ$</td>
<td>1 Period, 96 Frames</td>
<td>4.89E-01</td>
<td>1.39E+00</td>
</tr>
</tbody>
</table>
Figure 4-1 shows a comparison of vorticity contour at t/T=1 between DNS result and ROM result of case C2 (see Table 4-1 for reference). It shows the vortex structures in wake are accurately captured by the current ROM method after shedding for one period. Although near the trailing edge of the flapping wing the shape of vortex at the upper surface is a little bit distorted and the vortex structure at lower surface is discontinues, the major feathers of the vortices are still clear and similar to the DNS result. The vorticity plot from case C1 is also checked, however, it does not show obvious difference to Figure 4-1b.

Eigenvalues matrix $A$ in equation (2-69) are normalized of in a form of $\frac{\lambda_k}{\sum_{k=1}^{N_e} \lambda_k}$. It represents the contribution of individual eigenvalue to the total turbulent kinetic energy. Energy resolved by the first $i$ modes can be represented by $\frac{\sum_{k=1}^{i} \lambda_k}{\sum_{k=1}^{N_e} \lambda_k}$. The eigenvalue spectrum is shown in Figure 4-2. First two modes contain nearly 90% of total fluctuating kinetic energy. First four modes contain about 95% of the total energy. The accumulation of first six modes increases to nearly 98%.

The velocity contours for the mean flow and the first four modes are shown in Figure 4-3. In the mean flow there is a strong jet in x-direction and no obvious flow in y-direction can be
found in far field. Note the pairing of similar patterns in the contours of modes, shifted spatially, as a result of the convective nature of the flow. Modes 1 and 2 come in pairs and have the similar pattern, in both x and y directions, except a phase shift of $\pi$ approximately. Modes 3 and 4 also form a pair and have similar pattern with an approximate phase shift of $\pi/2$, in both x and y direction.
Figure 4-3. Mean velocity contours and POD modes for pitching amplitude of 30°.

As majority of kinetic energy is captured by the first four modes, it is sufficient to check the amplitudes of the first four basis functions, as shown in Figure 4-4. The curves of first
four modes show a good match between projected and predicted results. The eigenvalue spectrums of the other cases for pitching amplitude $30^\circ$ are almost identical to that of case C2 except high order modes. This behavior shows a kind of convergence in the calculation of eigenvalue. The amplitudes of basis function of other cases are similar to C2.

![Graphs showing time history of amplitude of basis functions of case C2.](image)

Figure 4-4. Time history of amplitude of basis functions of case C2.

We examine the long-term behavior of the POD modes by study the phase portraits of the coefficients of higher modes $a_n$ against that of the first mode $a_1$ for 10 periods, as shown in Figure 4-5. Both the phase portraits of $a_1$ vs. $a_2$ and $a_1$ vs. $a_3$ indicate limited cycles approximating to the projection results. The curve of $a_1$ vs. $a_2$ diverges slightly from the projection result in the first two periods. Then it converges to a stable elliptic orbit which has a smaller radius. This trend can be verified by the prediction error, as shown in Figure 4-6. The prediction error with the pressure term correction becomes periodic from the third period. As a comparison, the prediction error without the correction does not converge at the 10th period. Therefore, it concludes that the pressure term acts like an error damper in the dynamic system and help to stabilize the ROM of the flows. In addition, it can be seen that
the pressure term correction reduces the mean prediction error for at least one order of magnitude.

Figure 4-5. Phase portrait of basis functions of case C2 for 10 periods.

Figure 4-6. Time history of prediction error $\hat{E}$ of case C2 for 10 periods.
Figure 4-7. Prediction results with only first four POD modes for 10 periods. (a) and (b) without pressure correction. (c) and (d) with pressure correction.

We also examine the effect of pressure correction when only 4 POD modes (captured 95% of total kinetic energy) are used in the ROM, as shown in Figure 4-7. Without the pressure term correction, the basis functions are significantly damped and the limited cycle in the phase portrait is attracted to the orbit center. On the contrary, the ROM with the pressure term correction still correctly predicts the trend. Its limited cycle shows a good agreement to
the projection result. This result indicates that the pressure term is more important than high order POD modes for the dynamic system of flapping flight.

The effects of the number of snapshots and periods on the POD ROM method are also examined. Figure 4-8 shows the time history of the projection error $E$. When constructing the entry of matrix $A$ in equation (2-69), the snapshots should be uncorrelated and linearly independent, however, the method we extract the snapshots determines that the uncorrelation between two snapshots which are temporally successive is deteriorated when the sampling frequency increases. Meanwhile, keeping the sampling frequency but increasing the sampling period generates similar sets of successive snapshots. It also deteriorates the uncorrelation of snapshots. So it is not surprising to see that the case of 2 periods, 192 frames has the global maximum projection error while the case of 1 period, 32 frames has the global minimum.

As one of few successful attempts in modelling flapping wing using the POD method, Lewin and Hariri (Lewin and Haj-Hariri 2005) applied coordinate transformations to an airfoil undergoing heaving, lagging and pitching motions and performed POD in a body-fixed (non-inertial) coordinate system. Their averaged projection error is approximately at the order of $10^{-2}$, similar to that of the case C2. This indicates that the accuracy of the POD method in rebuilding the original flow fields is not greatly affected by coordinate transformations.
Figure 4-8. History of projection error $E$ of 30° cases.

Figure 4-9. History of prediction error $\hat{E}$ of 30° cases.

Figure 4-9 shows the history of the prediction error. Generally, the curves of the prediction error show a trend of increasing; especially the highest error appears near the end of time, similar to the result in (Graham, Peraire et al. 1999). The mean and standard deviation of the prediction error are tabled in Table 4-1. It can be found that the improvement in mean and standard deviation of the errors gained by increasing the number of snapshots is obvious; however, this cannot be achieved by increasing the number of period. The effect of periodicity on the prediction error is complex. For C5, the standard deviation is smaller than that of C3. But it is not true for C4 and C2. Further investigation on it is still needed.

4.2.1.2 Pitching amplitude equal to 10° and 20°

A parametric study on the pitching angle $A$ of 10° and 20° is conducted. The flow boundary conditions and the controlling equations for wing kinematics are same for the DNS simulations. The datasets of 96 snapshots are extracted uniformly from 960 frames of one period. It shows that the eigenvalues can be group as pairs, especially for first six eigenvalues
(Figure 4-10). The energy decay of case 10° is faster than the other two cases after six eigenvalues. An integrated comparison of the prediction and projection error is shown in Figure 4-11 and Table 4-1. It can be seen the prediction errors generally increase as time goes, however, the prediction errors of case 10° and 20° has the same order of magnitude of case 30°. This shows that the pressure correction method can be generally used for flapping plates undergoing pitching-plunging motions at low Reynolds numbers.

![Figure 4-10. Eigenvalue spectrum of 10°, 20° and 30° cases.](image1)

![Figure 4-11. Projection and prediction error of 10°, 20° and 30° case.](image2)

**4.2.2 Flow Past Two Tandem Wings**

The investigation on the tandem wings is motivated by the design of quad-wing MAVs that mimic dragonfly or damselfly flight due to their superior performance in maneuver and endurance (Wakeling 1993; Ho, Nassef et al. 2003; Wang, Zeng et al. 2003; Maybury and Lehmann 2004). The effect of complex wake structures produced by the tandem wings has been studied experimentally (Lehmann 2009) and numerically (Sun and Lan 2004; Wang 2007; Liang and Dong 2009; Dong and Liang 2010). These researches provide great insight
into the flow physics such as wing-wake interaction, vortex formation and vortex shedding, and assist the development of more comprehensive aerodynamic models.

The same kinematics is used for the tandem wings as for the single wing in previous section. The phase angle $\delta$ is 0° for the forewing, and -48° for the hindwing. Both wings pitch about their mid-chord. The streamwise distance between the mid-chords of two wings is twice the wing chord length. The Reynolds number is 600, based on the free stream velocity and the chord length of the hindwing. The forewing leads the hind wing by 48° of phase difference. The parameters of motion and flow field are similar to those described in (Akhtar, Mittal et al. 2007). More control parameters of the motion are listed in Table 4-2. The vortex sheet in the wake is shown in Figure 4-12. The interaction of forewing shed vortices and the hindwing is obvious. The wake structure shows great difference to the single wing.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Forewing</th>
<th>Hindwing</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\theta_m$</td>
<td>6.4°</td>
<td>9.5°</td>
</tr>
<tr>
<td>A/c</td>
<td>0.32</td>
<td>0.56</td>
</tr>
<tr>
<td>$\delta$</td>
<td>0°</td>
<td>-48°</td>
</tr>
<tr>
<td>Chord ratio ($c_f/c_h$)</td>
<td>1.0</td>
<td>—</td>
</tr>
<tr>
<td>$Re = U_{\infty}c/\nu$</td>
<td>600</td>
<td>—</td>
</tr>
<tr>
<td>$St = 2fA/U_{\infty}$</td>
<td>—</td>
<td>0.28</td>
</tr>
</tbody>
</table>

Table 4-2: Control parameters of two tandem wings
Figure 4-12. Vorticity of tandem wings undergoing pitching and plunging.

Similar to the single wing cases, the datasets of 96 snapshots are extracted uniformly from 960 frames of one period and 32 modes are used for the POD ROM. The eigenvalue spectrum of the tandem wing case is shown in Figure 4-13a. As the single wing cases, the modes are paired by energy level. The distinction in tandem wings is that the energy levels of modes 1 and 2 only accounts for 60% of kinetic energy, which is lower compared to that in single wing case (see Figure 4-10). The energy levels of mode 3 to 6 are higher. The cumulative energy of the first 6 modes in the tandem wings is similar in the single wings, which is approximately 95%. We examined the amplitudes of the first four basis functions, as shown in in Figure 4-13b. The prediction curves of the first two modes show a good agreement to the projection curves. This indicates that that the pressure-correctttion based ROM can capture the dynamics of multiple flappling wings at low Reynolds numbers.
Figure 4-13. (a) Eigenvalue spectrum in tandem wings. (b) Time history of amplitude of basis functions of the first four modes.

The velocity modes of tandem wings in pitching and plunging are shown in Figure 4-14. It is interesting to see in higher order modes (3 to 6) that some high energy components exist in the far field. This is different from the single-wing flow. It implies the increasingly importance of high order mode in the far field.
Figure 4-14. Velocity mode of tandem wings in pitching and plunging.
Chapter 5  POD Mode Force Survey Method and Effect of Symmetry of POD Modes

The pressure corrected POD ROM method discussed in the previous chapter is well suited to model flapping flight. This allows the further analysis on the POD modes. In this chapter, we discuss an analysis technique, termed the POD mode Force Survey Method (POD-FSM), by which the influence of POD modes to aerodynamic force can be defined and quantified. The POD-FSM can be used in two- and three-dimensional flow problems with stationary or moving boundaries. A three-dimensional flow generated by a pitching-plunging plate is presented as a demonstration.

A brief outline of the chapter is given below. First, we present the theorem of POD-FSM, followed by a validation for the impulse equation (Noca, Shiels et al. 1997). Next, POD-FSM is applied to a three-dimensional flow past a plate undergoing a pitching-plunging motion. A brief introduction to numerical simulation set-up, which provides flow data required for the subsequent POD process, is given. The following sections discuss vortex formation of the flow and POD modes, followed by flow reconstructions by different POD modes. The reduced-order model method discussed in the previous chapter is applied to this three-dimensional flow. Then, we validate the impulse equation and POD-FSM. Some special behaviors seen in the force of the POD modes lead to a discussion on the spatial structures of the POD modes, specifically symmetry. The discussion is followed by force of the POD modes in the same symmetry set. At the end, an explanation for the connection between symmetry of the POD modes and the flow is presented.
5.1 POD Mode Force Survey Method (POD-FSM)

5.1.1 The Theorem

POD modes can be thought as a special type of vortex structure in terms of kinetic energy. Assuming that each POD mode is associated to a virtual force that can be measured by wake survey methods, one can have a ‘force’ measurement for POD modes. Force superposition of all POD modes can reconstruct the force of the original flow.

The derivation of POD-FSM is straightforward. The POD method can decompose velocity fields into a linear combination of POD modes

\[
\bar{u}(\bar{r},t) = \bar{u}(\bar{r}) + \sum_{i=1}^{N_m} \alpha_i(t) \Phi_i(\bar{r})
\]

(5-1)

where \( \bar{u}(\bar{r}) \) is time average of velocity flow fields, \( N_m \) is number of POD modes, \( \alpha_i(t) \) are temporal coefficients and \( \Phi_i(\bar{r}) \) are POD modes in a vector form of \( (\Phi_{x,i}, \Phi_{y,i}, \Phi_{z,i}) \).

Taking the curl of (5-1) yields the vorticity fields:

\[
\bar{\omega}(\bar{r},t) = \bar{\omega}(\bar{r}) + \sum_{i=1}^{N_m} \alpha_i(t) \Psi_i(\bar{r})
\]

(5-2)

where \( \bar{\omega}(\bar{r}) = \nabla \times \bar{u}(\bar{r}) \) and \( \Psi_i(\bar{r}) = \nabla \times \Phi_i(\bar{r}) \).

We substitute (5-2) into Wu’s wake survey method (1-12). Because we model the flows generated by infinite thin plates, the term which involves the bodies with finite volume can be omitted. It yields
\[ F = -\frac{1}{N-1} \rho \frac{d}{dt} \int_{V_0} \bar{r} \times \left( \vec{\omega} + \sum_{i=1}^{N_n} \alpha_i(t) \Psi_i(\bar{r}) \right) dV \]
\[ = -\frac{\rho}{N-1} \sum_{i=1}^{N_n} \frac{d\alpha_i(t)}{dt} \int_{V_0} \bar{r} \times \Psi_i(\bar{r}) dV \quad (5-3) \]

Denote \( F_i \) the force of the \( i^{th} \) POD mode acting on the immersed body. (5-3) can be written as

\[ F = \sum_{i=1}^{N_n} F_i \quad (5-4) \]

where

\[ F_i = -\frac{\rho}{N-1} \frac{d\alpha_i(t)}{dt} \int_{V_0} \bar{r} \times \Psi_i dV \quad (5-5) \]

(5-4) means the superposition of virtual forces of POD modes. The equation is only applicable for an infinite large domain.

Analogously, we can substitute the POD expressions of velocity and vorticity into the impulse equation (1-21) (Noca, Shiels et al. 1999). Because the body surface is impermeable, we can omit the last term \( \oint_{S_0(\bar{r})} \bar{n} \cdot \left( \bar{\vec{u}} - \bar{\vec{u}}_z \right) \bar{\vec{u}} dS \). We obtain the force expression

\[ F = -\frac{1}{N-1} \sum_{i=1}^{N_n} \frac{d\alpha_i(t)}{dt} \int_{V_0} \bar{r} \times \Psi_i(\bar{r}) dV \\
+ \oint_{S(\bar{r})} \bar{n} \cdot \gamma_{\text{imp}} dS \\
+ \frac{1}{N-1} \frac{d}{dt} \oint_{S_0(\bar{r})} \bar{r} \times (\bar{n} \times \bar{\vec{u}}) dS \quad (5-6) \]

The first term takes the same form as (5-3), while the second term is extremely complicated because it involves nonlinear interactions between different modes. Some sub-
terms related to the viscous stress tensor $T$ in $\gamma_{imp}$ can still be decomposed into linear combinations of corresponding term of POD modes:

$$
\bar{r} \cdot (\nabla \cdot T) I = \sum_{i=0}^{N_m} \alpha_i (t) [\bar{r} \cdot (\nabla \cdot T_i)] I
$$

(5-7)

$$
\bar{r} (\nabla \cdot T) = \sum_{i=0}^{N_m} \alpha_i (t) [\bar{r} (\nabla \cdot T_i)]
$$

(5-8)

$$
T = \sum_{i=0}^{N_m} \alpha_i (t) T_i
$$

(5-9)

where $T_i = \mu \left[ \nabla \Phi_i (\bar{r}) + \nabla \Phi_i (\bar{r})^T \right]$. Note that the mean velocity is denoted as mode zero ($i=0$), whose temporal coefficient $\alpha_0$ is equal to one.

We organize all linear terms as the force of the $i^{th}$ POD mode $F_i$, written as

$$
F_i = -\frac{1}{N-1} \frac{d\alpha_i}{dt} \int_v \bar{r} \times \Psi_i dV + \alpha_i \int_{S(t)} \bar{n} \cdot \chi_i dS
$$

(5-10)

where

$$
\chi_i = \frac{1}{N-1} \left[ \bar{r} \cdot (\nabla \cdot T_i) I - \bar{r} (\nabla \cdot T_i) \right] + T_i
$$

(5-11)

The nonlinear terms $u^2 I$, $u\bar{u}$, $(\bar{u} - \bar{u}_S) (\bar{r} \times \bar{\omega})$ and $\bar{\omega} (\bar{r} \times \bar{u})$ have a complex expansion. For instance,
It can be seen that this expansion involves products of different POD modes, which represent the interaction between different POD modes and are not neglectable in the calculation of forces. Denote \( F_{ij} \) the force caused by the interaction between the \( i^{th} \) and \( j^{th} \) POD mode at the external surface of control volume. We obtain

\[
F_{ij} = \oint_{S(t)} \bar{n} \cdot \eta_{ij} dS \tag{5-13}
\]

where

\[
\eta_{ij} = \frac{1}{2} \langle \Phi_i, \Phi_j \rangle | I - \Phi_i \Phi_j |
- \frac{1}{N-1} (\Phi_i - \bar{u}_i)(\bar{r} \times \Psi_j) \\
+ \frac{1}{N-1} \Psi_i (\bar{r} \times \Phi_j) \tag{5-14}
\]

In addition, submerged bodies may move in fluids. This movement interacts with each POD mode and generates additional forces, written as

\[
F_{B,i} = \frac{1}{N-1} \frac{d}{dt} \oint_{S_h(t)} \bar{r} \times [\bar{n}(t) \times \alpha_i \Phi_i] dS \tag{5-15}
\]

Therefore, the total force is the summation of all aforementioned forces:
\[ F = \sum_{i=0}^{N_m} (F_i + F_{B,i}) + \sum_{i=0}^{N_m} \sum_{j=0}^{N_m} F_{ij} \]  \hspace{1cm} (5-16)

If the immersed body is a zero-thickness plate, \( F_{B,i} \) is equal to zero. This is because the flow velocity is the same at the upper and lower surfaces as the moving velocity of the plate. The normal vectors of the upper and lower surfaces are opposite. As a result, the surface integral on the body is zero. The total force can be written as

\[ F = \sum_{i=0}^{N_m} F_i + \sum_{i=0}^{N_m} \sum_{j=0}^{N_m} F_{ij} \]  \hspace{1cm} (5-17)

5.1.2 Validation for Impulse Equation

We have developed numerical tools for the impulse equation and POD-FSM. The tool for the impulse equation is first validated by a flow past a two-dimensional stationary membrane plate. Further validation of both methods for three-dimensional flows with moving boundary can be seen in section 5.2.6.

A two-dimensional plate (zero thickness) is placed in flow fields with angle of attack equal to 30°. Incoming free stream of velocity \( U_\infty = 1 \). The chord length of the stationary membrane plate is one. The Reynolds number of the flow is 200. The size of flow domain is 18 (X) × 15 (Y) with the grid number equal to 353 × 258. Figure 5-1 shows history of drag and lift coefficients up to 40 units of non-dimensional time. Table 5-1 lists time-average and RMS of drag and lift coefficients. The DNS result is computed by surface integral of the pressure and viscous terms. Computation of the impulse equation uses a domain size equal to the simulation domain. Smaller computational domain for the impulse equation may change
the force magnitude up to 10%, which is in a range similar to the results given by (Noca, Shiels et al. 1999). In general, despite that the impulse method slightly underestimates the lift force at valleys; it shows a good agreement with the DNS result in both lift and drag curves.

![Figure 5-1. Force coefficients history from t/T=0 to 40. The legend ‘Impulse’ means the impulse equation.](image)

Table 5-1: Comparison of average and RMS of drag and lift coefficients between DNS and the impulse equation

<table>
<thead>
<tr>
<th></th>
<th>DNS</th>
<th>Impulse</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Average</td>
<td>RMS</td>
</tr>
<tr>
<td>$C_D$</td>
<td>0.74</td>
<td>0.74</td>
</tr>
<tr>
<td>$C_L$</td>
<td>1.14</td>
<td>1.14</td>
</tr>
</tbody>
</table>

5.2 Results and Discussions

5.2.1 Simulation Set-up

Interests at the high propulsive efficiency have driven studies on a foil undergoing a pitching and plunging motion (Anderson, Streitlien et al. 1998). Three-dimensional vortex
formation behind a pitching-plunging foil has been studied experimentally (Von Ellenrieder, Parker et al. 2003; Buchholz and Smits 2006; Buchholz and Smits 2008) and numerically (Blondeaux, Fornarelli et al. 2005; Dong, Mittal et al. 2006). The current work applied POD-FSM to investigate the wake formation behind a low-aspect-ratio, pitching-plunging foil. POD-FSM can also be used in two-flow problems with stationary or moving boundaries.

We select a coordinate system with the X-axis along the downstream direction, the Z-axis along the spanwise direction and the Y-axis perpendicular to the other two axes. The elliptic foil undergoing a pitching-plunging motion is assigned in the middle of the flow field with its span axis along the Z direction. The spanwise axis of the elliptic plate serves as a pivot axis. Its Y position is defined by:

\[ y(t) = -H \cos(2\pi ft) \]  \hspace{1cm} (5-18)

where \( H \) is the plunging amplitude and \( f \) is the frequency.

The pitching angle of the plate, defined as the angle between the foil and the \( XZ-plane \) that crosses the pivot axis, is prescribed by

\[ \alpha(t) = A\sin(2\pi ft) \]  \hspace{1cm} (5-19)

where \( A \) is the amplitude of the sinusoidal pitch angle.

Reynolds number and Strouhal number are defined as \( Re=U_\infty c/\nu \) and \( St=2Hf/U_\infty \), respectively, where \( U_\infty \) is incoming flow velocity, \( c \) is chord length of the plate and \( \nu \) is the kinematic viscosity. In this study, \( U_\infty \) and \( c \) are both one unit. The plunging amplitude \( H \) is equal to 0.5. The pitching amplitude \( A \) is 30°. The flapping frequency \( f \) is 0.6. Thus, Reynolds number and Strouhal number are 200 and 0.6, respectively. The spanwise diameter

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of the plate is 0.5. This number corresponds to a low aspect ratio of 0.64, which has not been addressed in (Dong, Mittal et al. 2006). Pitch-bias angle is set to zero to generate symmetric flow. The plate is a zero-thickness membrane. Flow validations for such a plate have been done in Mittal et al. (2008) and it has been used to model the pectoral fin of bluegill sunfish (Dong, Mittal et al. 2010).

Grid independent study at Re=200 has been conducted in (Dong, Mittal et al. 2006). The size of flow domain used in the current study increases to 18 (X) × 15 (Y) × 15 (Z) for a larger visualization window in the wake. As a result, the grid number increases to 353 × 258 × 130 (or 11.8M).

5.2.2 Vortex Formation of the Flow

Flow visualization in this study is by rendering iso-surfaces of the \( Q \)-criterion (Hunt, Wray et al. 1988) with intensity of certain vorticity components. This approach is convenient to identify the direction of vortices in canonical problems. Furthermore, we apply the \( Q \)-criterion and iso-surface rendering to POD modes to visualize vortex structures.

Figure 5-2 shows typical vortex topologies of the simulation results at time \( t/T=0 \) and 1/2. The vortex structures are two series of vortex rings which convect in opposite vertical directions during travelling to downstream. The upward vortex rings are counterclockwise (CCW) and those downward are clockwise (CW) when viewed from the top. At time \( t/T=0 \), the plate is at the lowest position and starting to move upward. A downward vortex ring has shed out its trailing edge vortex near the plate. A complete vortex ring will form after the leading vortex is shed at \( t/T=1/4 \). Then another vortex structure is beginning to form at the
leading edge of the lower surface of the plate. Its movement is induced by the upward motion of the plate, thus finally it will form an upward travelling vortex ring. The profile of vortex structures has a small aspect ratio when it attaches to the plate. However, after it sheds out, it is quickly elongated in the Z-direction and gradually round off. One interesting feature is the so-called vortex ‘contrails’ (Dong, Mittal et al. 2006) that are vertical vortices developed from tip vortices. They gradually move from the trailing edge to nearly the leading edge of vortex rings and finally dissipate.

![Perspective and Side views of vortex formation.](image)

*Figure 5-2. The vortex formation viewed from different angles, rendered by X-direction vorticity $\omega_x$. $Q=0.25$. The plate is at time $t/T=0$ (first row) and 0.5 (second row), respectively.*

### 5.2.3 POD Modes

The POD method is applied to the full velocity field when vortices shed periodically. 48 snapshots of the instantaneous velocity fields are selected from a data ensemble of one period to generate 48 modes. A time average operation is performed on the data ensemble. Figure
5-3a shows the iso-surface of \textit{Q-criterion} of the mean flow. The major structures in the wake are four straight vortex tubes that are spatially antisymmetric about the \textit{XY-plane} and the \textit{XZ-plane} when colored by the X-direction vorticity $\bar{\omega}_x$. The vorticity components in Y- and Z-directions are much smaller on these tubes and may change the sign on any individual tube. Thus, these vortex tubes are considered to be dominant by the X-direction vorticity, and they mainly represent the flow of the X-direction vorticity. This is different from two-dimensional flows in the previous chapter, where only the Z-direction vorticity is nontrivial.

Figure 5-3. (a) Iso-surface of the \textit{Q-criterion} of mean vorticity rendered by the X-direction vorticity $\bar{\omega}_x$, \textit{Q}=0.25; (b) Normalized eigenvalues (circle) and captured kinetic energy by the first $i$ modes (square) versus mode number $i=1,2,\ldots,24$.

Contribution of individual eigenvalue to the total turbulent kinetic energy can be expressed as normalized eigenvalues in a form of $\frac{\lambda_i}{\sum \lambda_k}$. Captured kinetic energy by the first $i$ modes can be represented by $\frac{\sum \lambda_k}{\sum \lambda_k}$. The values of the first six modes are listed in Table 5-2. Curves of normalized eigenvalue and captured kinetic energy with respect to the number of POD modes are shown in Figure 5-3b. It can be seen that every two
eigenvalues form a pair with approximate values for the first several modes. The first two modes contain about 41% and 35% of total turbulent kinetic energy, respectively. The third and fourth modes account for lower energy at about 7%. And up to the 6th mode the cumulative energy from corresponding modes is over 96%, which means that the first six modes capture a majority of large flow features including the vortices in the vicinity of the flapping wing and the vortices in the far wake zone.

Table 5-2: Contribution of the first six eigenvalues to the turbulent kinetic energy

<table>
<thead>
<tr>
<th>Mode</th>
<th>Contribution(%)</th>
<th>Accumulation(%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>41.4</td>
<td>41.4</td>
</tr>
<tr>
<td>2</td>
<td>34.7</td>
<td>76.1</td>
</tr>
<tr>
<td>3</td>
<td>8.3</td>
<td>84.4</td>
</tr>
<tr>
<td>4</td>
<td>7.8</td>
<td>92.2</td>
</tr>
<tr>
<td>5</td>
<td>2.3</td>
<td>94.5</td>
</tr>
<tr>
<td>6</td>
<td>2.0</td>
<td>96.5</td>
</tr>
</tbody>
</table>

Figure 5-4 shows the POD modes from modes 1 to 6. The topology of the first two modes exhibits a pattern of vortex ring structures. However, the first mode is a CCW, doughnut-shaped vortex structure followed by two rows of chain knots of vortices, which is convected downstream with an inclination angle equal to 16° in the $XY$-plane, similar to that of vortex formation in the flow fields. The chain vortices are formed by connecting the leading and trailing edges of individual vortex ring, whose direction switches from CW to CCW in turn in the X- direction. The first two vortex rings in the chain look different to the others due to their individual inclination angles and connecting position.
The POD modes are essentially the superposition of spatial waves of different wave numbers in different directions. These waves form unique spatial structures depending on the flow features, e.g., if the vortex formation in the far wake of the flow is mainly vortex rings then the POD modes are analogous ring structures. However, the wave behavior can still be observed if the modes are colored by suitable vorticity components. For instance, in the side view of mode 1 a sinusoidal wave of the vorticity $\Psi_x$ with varying wavelength starts propagating from the leading edge of the doughnut-shaped vortex along two rows of chain vortices. Peaks and valleys of the sinusoidal wave are rendered by red and blue color, respectively. Another vorticity component propagates along the X-direction is the vorticity $\Psi_z$, which is the dominate component of leading and trailing edges of vortex rings. Its wave behavior is not as clear as the vorticity $\Psi_x$ because the chain vortices are mainly streamwise and the spanwise vortices are only a small portion of the vortex rings in mode 1.

The vortex formation of mode 2 is similar to mode 1. The position of each vortex rings has one quarter of wavelength shifting along the vortex chains. The phase difference between modes 1 and 2 can be illustrated by the vorticity $\Psi_z$ in the side view.

The major structures of modes 3 and 4 are still a bifurcated vortex formation, which are formed by series of vortex structures instead of round rings. It is difficult to determine the length of the vortices in the formation because the vortices deform and diffuse in the far-field, and the inclination angle of individual vortex is not exactly the same as that of the vortex chains. However, the number of spanwise vortex structures can be counted and it is approximately to be twice of that in mode 1. The widths of the chains are approximately the same in the Z-direction. One interesting feature is that the vertical vortices connecting two
chains in the near wake. They are suspected to reconstruct the vortex contrails in the flow field. The $\pi/2$ phase difference between modes 3 and 4 can be observed in the vorticity $\Psi_y$.

Modes 5 and 6 are structurally similar to modes 3 and 4. The vortex number in the chain is triple of that in mode 1. The spanwise width of the vortices changes slightly. Vertical vortices can also be seen in the near wake. This suggests that the vortex contrails are coupled with the vortex rings but they are not strong vortices with high energy. The phase shifting is viewed by the vorticity $\Psi_x$. It is notable that the turbulent kinetic energy gradually concentrates to the flapping plate region as the mode number increases.
Figure 5-4. Iso-surfaces of \textit{Q-criterion} of POD modes from 1 to 6. The iso-surfaces of modes 1, 2, 5 and 6 are rendered by the vorticity $\Psi_x$. Modes 3 and 4 are rendered by the vorticity $\Psi_y$. $Q=0.0625$. 
The history of temporal coefficients $\alpha_i$ is shown in Figure 5-5a. The pairs $(\alpha_1, \alpha_2)$, $(\alpha_3, \alpha_4)$ and $(\alpha_5, \alpha_6)$ have a frequency of $f$, $2f$ and $3f$, respectively, with a $\pi/2$ phase difference in each pair. The phase portraits of the coefficients of higher modes $\alpha_n$ against that of the first mode $\alpha_1$ show the trajectories of ellipse and unsymmetrical lobe (Figure 5-5b), which means the harmonics and the phase difference between the temporal coefficients.

![Figure 5-5. (a) Time history of the temporal coefficients $\alpha_i$, i=1,…,6. (b) Phase portraits, $\alpha_1$ vs. $\alpha_2$, $\alpha_1$ vs. $\alpha_3$, and $\alpha_1$ vs. $\alpha_5$.](image)

### 5.2.4 Flow Reconstructed from POD modes

Table 5-3: Reconstructed flows with different modes

<table>
<thead>
<tr>
<th>Flow</th>
<th>R1</th>
<th>R2</th>
<th>R3</th>
<th>R4</th>
<th>R5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Combination</td>
<td>Mean+1+2</td>
<td>Mean+1~4</td>
<td>Mean+1~6</td>
<td>Mean+3+4</td>
<td>Mean+5+6</td>
</tr>
</tbody>
</table>

Flow reconstructions are performed by adding the mean flow and individual POD modes. Respectively, Table 5-3, Figure 5-6 and Figure 5-7 show the combination and corresponding flow reconstruction. Reconstruction $R1$ depicts a rough sketch of the vortex formation, in which the $XY$-plane symmetry and the half-period symmetry have been established. However,
there are several major differences compared with the DNS result: the vortex at the flapping plate region is still of doughnut shape; the upward and downward vortex rings connect to the leading vortex; the vortex lengths in the chain are larger; the vertical contrails cannot be seen. Although the first two modes contain 76% of turbulent kinetic energy, they miss the vortex contrails whose energy is low. With the supplement of modes 3 and 4, reconstruction $R_2$ shows some improvements, globally. The upward and downward vortex rings disconnect from the leading vortex. The vortex lengths in the chain are smaller. The vertical contrails are still not clear until modes 5 and 6 are added ($R_3$), when the captured energy has increased to 96%. In general, adding higher modes gradually removes large vortices from the flapping plate region and the vortex rings, but sometimes small vortices are added. For example, small vortices are added to the leading edge of vortex rings in reconstruction $R_2$ and some vortices are added to the flapping plate region in reconstruction $R_3$. This indicates that information of these vortex structures is usually contained in several POD modes. A reconstruction with all modes resembles the DNS result, thus not shown here.
Some other reconstructions are performed to study the half-period symmetry with respect to the individual mode pair. A reconstruction with the mean flow and modes 3 and 4 is shown in Figure 5-7. Two vortex chains are reconstructed with a spatial antisymmetry about the XZ-plane. The chains vortices shed simultaneously in a frequency of $2f$, controlled by the temporal coefficient pair $(\alpha_3, \alpha_4)$. The numbers in the figure mark vortex rings shedding in different period. The spatial intervals between the vortex rings of periods 2 and 3 are visible; however the vortex rings in periods 1 and 2 connect to each other and the spanwise vortices at the leading edge of vortex rings dissipate. Thus, it takes an appearance of a ladder with vortex sheets joining two wingtip vortices in periods 1 and 2.

Vortices in ladder formation can also be seen in reconstruction with the mean flow and modes 5 and 6. In this reconstruction, ladders immediately form after vortex rings leave the flapping region. The transition from separate vortex rings to ladder formation happens naturally with shedding frequency increasing. Reconstructions with the mean flow and high
order modes are not shown in the section since they have the similar ladder shape as reconstructions $R4$ and $R5$.

One of interesting observations is that the vertical vortex structures in modes 3 and 4 do not form the vertical contrails in reconstruction $R4$; neither do modes 5 and 6 in reconstruction $R5$. The vertical contrails appear only when modes of different frequency join together, as shown in reconstruction $R3$. This indicates that the composition of the contrails spans several frequencies.

![Reconstructed flow fields $R4$ and $R5$ at $t/T=0$.](image)

**Figure 5-7.** Reconstructed flow fields $R4$ and $R5$ at $t/T=0$.

### 5.2.5 Reduced-order Model

We applied the pressure corrected POD ROM method, which has been successfully applied to two-dimensional cases in the previous chapter, to this three-dimensional flow. It can be seen that the prediction curves of the first two reconstruction coefficients agree well with the projection curves. The prediction curves of modes three and four gradually deviate
from the projection curves as time increases. These results partially proved the applicability of pressure corrected projection algorithm in three-dimensional flapping flight.

![Figure 5-8. Time history of amplitude of basis functions.](image)

5.2.6 Force of POD Modes

The time histories of the thrust and lift coefficients and their power spectrum, as shown in Figure 5-9, are investigated. The time-average thrust coefficient is 0.274. From Figure 5-9b, it is notable that the lift and thrust forces are contributed from odd and even harmonics, respectively. This is similar to the well-known observation that lift and drag acting on a translating cylinder consist of odd and even harmonics, respectively.
POD-FSM (5-17) is applied to the flow. Adding all forces of POD modes can exactly calculate the forces acting on the plate, as shown in Figure 5-10. Force reconstruction with 48 modes shows a good agreement with the results of the impulse equation. Influence of high order POD modes in terms of force is examined by using 24 modes and 48 modes. It can be seen that the forces computed from 24 and 48 modes can match the results of the impulse equation both in lift and thrust, despite that the thrust from 24 modes has only several large displacements to the impulse equation near $t/T = 0$. Generally speaking, because high order POD modes only can capture a small amount of fluctuating energy, their interaction with high energy modes can hardly affect the total force.
Figure 5-10. Thrust and lift coefficients from DNS, the impulse equation and POD modes.

Figure 5-11. Thrust and lift coefficients of the mean flow, POD modes 1 and 3.
Because a zero-thickness plate is used in the simulation, $\vec{F}_{B,i}$, which represents the force caused by direct interactions between POD mode and body surface, is equal to zero. Here, force related to the $k^{th}$ POD mode is written as

$$NF_k = F_k + \sum_{j=0}^{N} F_{kj}$$  \hspace{1cm} (5-20)

where $NF$ means non-linear forces because it includes both the linear and non-linear terms. Note the mean flow is the $0^{th}$ mode. The non-linear force history of the mean flow, POD modes 1 and 3 are shown in Figure 5-11. It can be seen that although the temporal coefficient of the mean flow is a constant, the lift and thrust coefficient of the mean flow is still non-zero, due to the interaction between the mean flow and other POD modes. It is interesting to see that the thrust of mode 3 is non-zero, while its lift is zero. On the contrary, only the lift of mode 1 is non-zero.

It is notable that Wu’s wake survey method (Wu 1981) is only applicable to infinite flow domain, in which $\int_{V_n} \omega dV = 0$. In our study, it is found that if a finite domain satisfying $\int_{V} \omega dV = 0$ is used, Wu’s method is still valid. When a plate starts moving from rest, we can always select a control volume in which the volume integral of vorticity is zero. If (5-5) is applied to the volume, the summation of individual force over all POD modes can converge to the force predicted by Wu’s method and the DNS result. However, if any vortex flows out of the control volume, it may cause $\int_{V} \omega dV$ not equal to zero. Wu’s method is thus invalid. The summation of individual force, however, still converges to the force predicted by Wu’s method.
5.2.7 Symmetry of POD Modes

The force of POD modes is determined by the vortex formation of POD modes. Thus we examined the topology of the POD modes. It reveals certain kinds of symmetry property, as listed in Table 5-4. The terms “S” and “A” denote symmetric and antisymmetric, respectively. Note that each pair of POD modes that have approximate eigenvalues has the same symmetric and antisymmetric attributions. Furthermore, the POD modes can be categorized into two sets according to symmetry of three vorticity components with respect to the XY-plane and the XZ-plane. The first symmetry set can be defined as \( G_1 = \{ x : \text{mean or modes } 3, 4, 7, 8, ..., 19, 20, 23, 24 \} \) and the second one is \( G_{-1} = \{ x : \text{modes } 1, 2, 5, 6, ..., 17, 18, 21, 22 \} \). This classification holds for all modes in the current study and can be observed in the cases in (Dong, Mittal et al. 2006) and (Liang, Wan et al. 2012), as well.

<table>
<thead>
<tr>
<th></th>
<th>Mean</th>
<th>Mode 1, 2</th>
<th>Mode 3, 4</th>
<th>Mode 5, 6</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Velocity</strong></td>
<td>( \bar{u} ) ( \bar{v} ) ( \bar{w} )</td>
<td>( \Phi_x ) ( \Phi_y ) ( \Phi_z )</td>
<td>( \Phi_x ) ( \Phi_y ) ( \Phi_z )</td>
<td>( \Phi_x ) ( \Phi_y ) ( \Phi_z )</td>
</tr>
<tr>
<td><strong>XY-plane</strong></td>
<td>S</td>
<td>S</td>
<td>S</td>
<td>S</td>
</tr>
<tr>
<td><strong>XZ-plane</strong></td>
<td>S</td>
<td>A</td>
<td>S</td>
<td>A</td>
</tr>
<tr>
<td><strong>Vorticity</strong></td>
<td>( \bar{\omega}_x ) ( \bar{\omega}_y ) ( \bar{\omega}_z )</td>
<td>( \Psi_x ) ( \Psi_y ) ( \Psi_z )</td>
<td>( \Psi_x ) ( \Psi_y ) ( \Psi_z )</td>
<td>( \Psi_x ) ( \Psi_y ) ( \Psi_z )</td>
</tr>
<tr>
<td><strong>XY-plane</strong></td>
<td>A</td>
<td>A</td>
<td>S</td>
<td>A</td>
</tr>
<tr>
<td><strong>XZ-plane</strong></td>
<td>A</td>
<td>S</td>
<td>A</td>
<td>A</td>
</tr>
</tbody>
</table>

Denote \( XY\)-plane and \( XZ\)-plane the central plane in the X- and Y-directions, respectively.

Spatial symmetry of POD modes about the \( XY\)-plane can be written as:

\[ ... \]
\[ \Phi_x(x, y, z) = \Phi_x(x, y, -z) \]
\[ \Phi_y(x, y, z) = \Phi_y(x, y, -z) \]
\[ \Phi_z(x, y, z) = -\Phi_z(x, y, -z) \] (5-21)

where \((x, y, z, t)\) are dependent variables of velocity component \(\Phi_x\), \(\Phi_y\) and \(\Phi_z\). The spatial symmetry about the \(XY\)-plane can be conveniently written in a combined form of group actions and vector field:

\[ r_{xy} \cdot (\Phi_x, \Phi_y, \Phi_z)(x, y, z) = (\Phi_x, \Phi_y, -\Phi_z)(x, y, -z) \] (5-22)

where \(r_{xy}\) is a matrix representing the spatial reflection about the \(XY\)-plane.

Similarly, \(XZ\)-plane symmetry of the symmetry sets \(G_1\) and \(G_{-1}\) can be, respectively, written as:

\[ r_{xz} \cdot (\Phi_x, \Phi_y, \Phi_z)(x, y, z) = (\Phi_x, -\Phi_y, \Phi_z)(x, -y, z) \] (5-23)

\[ -r_{xz} \cdot (\Phi_x, \Phi_y, \Phi_z)(x, y, z) = (-\Phi_x, \Phi_y, -\Phi_z)(x, -y, z) \] (5-24)

Vorticity symmetry can be calculated based on (5-22)-(5-24).

\[ r'_{xy} \cdot (\Psi_x, \Psi_y, \Psi_z)(x, y, z) = (-\Psi_x, -\Psi_y, \Psi_z)(x, y, -z) \] (5-25)

\[ r'_{xz} \cdot (\Psi_x, \Psi_y, \Psi_z)(x, y, z) = (-\Psi_x, -\Psi_y, -\Psi_z)(x, -y, z) \] (5-26)

\[ -r'_{xz} \cdot (\Psi_x, \Psi_y, \Psi_z)(x, y, z) = (\Psi_x, -\Psi_y, \Psi_z)(x, -y, z) \] (5-27)

(5-25) represents the \(XY\)-plane symmetry for all of the POD modes. (5-26) and (5-27) represent the \(XZ\)-plane symmetry of the symmetry sets \(G_1\) and \(G_{-1}\), respectively.

As mentioned in (Deane, Kevrekidis et al. 1991), it is not necessary for half-period flows to have the every-two-mode pattern as in this case. But asymmetric modes were not reported in their cases. Nevertheless, with regard to the symmetry of POD modes, we can always
attribute the modes satisfying (5-22) and (5-23) to \( G_1 \) and the modes satisfying (5-22) and (5-24) to \( G_{-1} \).

Temporal coefficients are also symmetric or antisymmetric. Define \( \tau \) the symmetry for temporal coefficient \( \alpha_i \) in \( G_1 \):

\[
\tau \cdot \alpha_i(t) = \alpha_i(t + \frac{T}{2}),
\]

(5-28)

Analogous to the spatial operator \(-r_{xz}\) in (5-24), the group action for \( G_{-1} \) can be written as

\[
(-\tau) \cdot \alpha_i(t) = -\alpha_i(t + \frac{T}{2}).
\]

(5-29)

The \( XY\)-plane symmetry of POD modes is invariant with respect to time, thus the corresponding temporal coefficient has a trivial group, which only has an identity element \( I \):

\[
I \cdot \alpha_i(t) = \alpha_i(t),
\]

(5-30)

5.2.8 Force with respect to Symmetry Set

The original flow can be decomposed into the flows which only include the members of \( G_1 \) and \( G_{-1} \). Therefore, the total force can be decomposed as

\[
F = F_1 + F_{-1} + F_{1,1} + F_{-1,1} + F_{-1,-1} + F_{-1,1}
\]

(5-31)

where \( F_a = \sum_{i \in G_a} F_i \) and \( F_{a,b} = \sum_{i \in G_a} \sum_{j \in G_b} F_{ij} \) (\( a \) and \( b = 1 \) or \(-1\)). It can be found that POD modes in the symmetry set \( G_1 \) are only related to the X-direction force, as shown in Figure 5-12a, and those in \( G_{-1} \) are only related to the Y-direction force (see Figure 5-12b), because of the symmetries of \( G_1 \) and \( G_{-1} \). In other words, POD modes in \( G_1 \) and \( G_{-1} \) can be called thrust and
lift modes, respectively. The flows $G_l$ and $G_{-l}$ are thus thrust- and lift-producing flows, respectively. It is worth to note that the sets $G_l$ and $G_{-l}$ only determine force directions.

![Diagram](image)

Figure 5-12. Thrust and lift coefficients of (a) $F_l$ and (b) $F_{-l}$.

Interactions between POD modes at the surface of the control volume also generate forces of flow interactions, $F_{1,1}$, $F_{1,-1}$, $F_{-1,1}$ and $F_{-1,-1}$. The force history is shown in Figure 5-13. $F_{1,1}$ has non-zero thrust and zero lift through the whole period. In contrast, $F_{1,-1}$ and $F_{-1,1}$ have non-zero lift and zero thrust. $F_{-1,-1}$ has zero thrust and lift. In addition, thrust of $F_{1,1}$ shows a periodic pattern with frequency equal to two, while lift of $F_{1,-1}$ and $F_{-1,1}$ take an appearance of sinusoid with frequency equal to one.
We perform reconstructions $R6$ and $R7$, which exclusively include the members of $G_1$ and $G_{-1}$, respectively. Both reconstructions are coherent structures of the flow in a form of chained vortex rings but with different vorticity directions, as shown in Figure 5-14. It can be observed that $R6$ and $R7$ are half-period symmetric.

Figure 5-13. Thrust and lift coefficients of (a) $F_{1,1}$, (b) $F_{1,-1}$, (c) $F_{-1,1}$ and (d) $F_{-1,-1}$.
5.2.9 Symmetry of Flow

To understand why $G_1$ and $G_{-1}$ have different symmetry, symmetry of the flow is examined and two types of symmetry are identified.

\[
R_{xy} \cdot (u, v, w)(x, y, z, t) = (u, v, -w)(x, y, -z, t) \quad (5-32)
\]

\[
R_{xz} \cdot (u, v, w)(x, y, z, t) = (u, -v, w)(x, -y, z, t + \frac{T}{2}) \quad (5-33)
\]

where $\frac{T}{2}$ is the half of a period. The $XZ$-plane symmetry is a half-period symmetry, which means reflecting the flow fields about the $XZ$-plane after half a period. Because both the spatial and spatio-temporal symmetries are reflective symmetry and two consecutive reflections have the same effect as no action, these two symmetries are identified as $Z_2$ symmetry. Respectively, the groups for the spatial symmetry and the spatio-temporal symmetry are \( \{ I, R_{xy} \} \) and \( \{ I, R_{xz} \} \), where $I$ is the identity element of groups. As a result, the symmetry group of the flow can be written as $Z_2 \times Z_2$. Multiplication of two groups
generates the Klein four-group, which has the fourth element other than $I$, $R_{xy}$, and $R_{xz}$. Because the $XY$-plane symmetry is preserved with respect to arbitrary $t$, the fourth element can be calculated by taking the $XY$-plane reflection after the half-period reflection, written as $R_x = R_{xy} \cdot R_{xz}$, where $R_x$ acts on the velocity field as follows:

$$R_x \cdot (u,v,w)(x,y,z,t) = (u,-v,-w)(x,-y,-z,t + \frac{T}{2}) .$$  \hfill (5-34)

This yields a rotational symmetry, which means rotating the flow fields about the X-axis by $180^\circ$ after half a period.

Corresponding vorticity symmetry can be calculated from (5-32) to (5-34):

$$R'_{xy} \cdot (\omega_x, \omega_y, \omega_z)(x,y,z,t) = (-\omega_x, -\omega_y, \omega_z)(x,y,-z,t) ,$$  \hfill (5-35)

$$R'_{xz} \cdot (\omega_x, \omega_y, \omega_z)(x,y,z,t) = (-\omega_x, \omega_y, -\omega_z)(x,-y,z,t + \frac{T}{2}) ,$$  \hfill (5-36)

$$R'_x \cdot (\omega_x, \omega_y, \omega_z)(x,y,z,t) = (\omega_x, -\omega_y, -\omega_z)(x,-y,-z,t + \frac{T}{2}) .$$  \hfill (5-37)

Thus, the complete Klein four-groups are $\{I, R_{xy}, R_{xz}, R_x\}$ and $\{I, R'_{xy}, R'_{xz}, R'_x\}$ for velocity and vorticity, respectively. It is interesting to see that the rotational symmetry has the same sign for velocity and vorticity components while the reflective symmetry has opposite signs. Calculating one vorticity component involves four variables. If two of them change the sign, the vorticity change the sign; if all changes the sign, the sign of vorticity components keeps the same as velocity. It is also notable that the velocity field is equivariant under the group $\{I, R_{xy}, R_{xz}, R_x\}$, while the vorticity field is not equivariant under the group $\{I, R'_{xy}, R'_{xz}, R'_x\}$.
5.2.10 Cause of Symmetry of POD Modes

The \textit{XY-plane} symmetry (5-32), which is invariant with respect to time, is inherited by all
the POD modes. While the half-period symmetry (5-33) is inherited by the symmetry sets $G_1$
and $G_{-1}$. Because reconstruction $R1$ maintains the symmetry of the flow, we expand the
velocity field to the first two POD modes by modelling temporal coefficients $\alpha_1$ and $\alpha_2$ by
sinusoidal functions $\sin[2\pi f (t + t_0)]$ and $\sin[2\pi f (t + t_0) + \frac{\pi}{2}]$ respectively, where $t_0$ means
initial phase,

\begin{equation}
\begin{align*}
  u(x, y, z, t) &= \bar{u}(x, y, z) \\
                   &+ \sin[2\pi f (t + t_0)] \Phi_{x,1}(x, y, z) \\
                   &+ \sin[2\pi f (t + t_0) + \frac{\pi}{2}] \Phi_{x,2}(x, y, z)
\end{align*}
\end{equation}

and

\begin{equation}
\begin{align*}
  u(x, -y, z, t + \frac{T}{2}) &= \bar{u}(x, -y, z) \\
                           &+ \sin[2\pi f (t + \frac{T}{2} + t_0)] \Phi_{x,1}(x, -y, z) \\
                           &+ \sin[2\pi f (t + \frac{T}{2} + t_0) + \frac{\pi}{2}] \Phi_{x,2}(x, -y, z)
\end{align*}
\end{equation}

According to equation (5-33), the right-hand side (RHS) of (5-38) and (5-39) should be
equal. By matching the terms one obtains that the mean velocity $\bar{u}$ is symmetric about the
\textit{XZ-plane}, and

\begin{equation}
\begin{align*}
  \sin[2\pi f (t + t_0)] \Phi_{x,1}(x, y, z) &= \sin[2\pi f (t + \frac{T}{2} + t_0)] \Phi_{x,1}(x, -y, z) \\
  \sin[2\pi f (t + t_0) + \frac{\pi}{2}] \Phi_{x,2}(x, y, z) &= \sin[2\pi f (t + \frac{T}{2} + t_0) + \frac{\pi}{2}] \Phi_{x,2}(x, -y, z)
\end{align*}
\end{equation}

which yield

\begin{equation}
\begin{align*}
  \Phi_{x,1}(x, y, z) &= -\Phi_{x,1}(x, -y, z) \\
  \Phi_{x,2}(x, y, z) &= -\Phi_{x,2}(x, -y, z)
\end{align*}
\end{equation}
Similarly, the \textit{XZ-plane} symmetry for the other velocity components can be verified.

Following the same strategy, one can expand the velocity field to higher order modes with modeled temporal coefficients, e.g. \( \sin[4\pi f (t + t_0)] \) and \( \sin[4\pi f (t + t_0) + \frac{\pi}{4}] \) for \( \alpha_3 \) and \( \alpha_4 \), respectively, and match the corresponding terms:

\[
\begin{align*}
\sin[4\pi f (t + t_0)] \Phi_{x,3}(x, y, z) &= \sin[4\pi f (t + \frac{T}{2} + t_0)] \Phi_{x,3}(x, -y, z) \\
\sin[4\pi f (t + t_0) + \frac{\pi}{4}] \Phi_{x,4}(x, y, z) &= \sin[4\pi f (t + \frac{T}{2} + t_0) + \frac{\pi}{4}] \Phi_{x,4}(x, -y, z)
\end{align*}
\tag{5-42}
\]

The product of the half period \( \frac{T}{2} \) and \( 4\pi f \) is \( 2\pi \), which does not change the phase of the sinusoidal functions on the RHS of (5-42). Thus, it yields

\[
\begin{align*}
\Phi_{x,3}(x, y, z) &= \Phi_{x,3}(x, -y, z) \\
\Phi_{x,4}(x, y, z) &= \Phi_{x,4}(x, -y, z)
\end{align*}
\tag{5-43}
\]

It concludes that the different sign of the two symmetry sets is caused by the frequencies of temporal coefficients of the sets, which are odd and even multiples of frequency \( f \), respectively. Thus, the information for the half-period symmetry is carried by both symmetry sets \( G_1 \) and \( G_-1 \). Indeed, every POD mode with its temporal coefficient is half-period symmetric regardless its symmetry set.
Chapter 6  Summary and Conclusion

6.1 Summary of Research

The goal of this research is to numerically analyze vortex structures of flapping flight. To satisfy the computational power requirement for large-scale simulations, a highly efficient CFD solver was developed. With this solver, free flights of a dragonfly were simulated, and the vortex formation was studied with particular interest in wing-wake interaction. A pressure corrected POD ROM method was developed to model the vortex formation of flapping flight accurately. To measure the aerodynamic forces caused by POD modes, a POD mode Force Survey Method (POD-FSM) was also developed. This method was examined in the flow past a plate undergoing a pitching-plunging motion, in which symmetric POD modes were found to generate zero lift or zero thrust.

6.2 Modified Strongly Implicit Procedure for the Immersed Boundary Method

Modified Strongly Implicit Procedure (MSIP) for the immersed boundary method (IBM) was developed for an existing Computational Fluid Dynamics (CFD) solver which is highly capable in solving three-dimensional fluid problems with moving and deforming boundaries. The MSIP method is used as a smoother for multigrid method to solve the pressure equation and as an iterative method to solve the momentum equation in the context of immersed boundaries. In the current research, it is found that the boundary treatment is critical for the MSIP method in the context of IBM. To avoid changing the diagonal structure of iteration
matrix, entries of the matrix are modified by assigning proper values based on types of immersed boundaries. This so-called implicit boundary treatment also avoids re-calculating the iteration matrix in each iteration. The solver was validated by a number of theoretic model problems and immersed boundary problems such as flows past a plate or multiple cylinders. The results show that 8 times speedup can be achieved for large-scale simulations, comparing to the LSOR method.

6.3 Free Flight of a Dragonfly

The free flight of a dragonfly for eight-and-a-half wingbeats was studied in detail. The results show that the flight can be divided into two successive stages: an escape (take-off) stage and a maneuver stage. The wing flapping frequency in the escape stage can be up to 47.2 Hz and the average force can be approximately 5 times the dragonfly weight. In contrast, the wing flapping frequency in the maneuver stage can be down to 33.7 Hz and the average force is approximately 3 times the weight.

Flights with different wing combinations, such as four wings, two forewings, and two hindwings, were numerically simulated to examine the influence of wing-wake interaction on the fore- and hindwings. It was found that because of the interaction, the average force on the 4 wings was slightly increased in the escape stage and slightly decreased in the maneuver stage. However, the force on the forewings can be significantly enhanced by 42% (peak value) during the second downstroke of the escape stage. This enhancement was related to a distorted trailing edge vortex of the forewing, as demonstrated by a theoretical model based on wake survey methods. When the fore- and hingwings flap closely with only a short
distance (less than 1 chord length) existing between them, the hindwing acts as a wall, changing the flow direction of the forewing downwash and accelerating the downwash to push the trailing edge vortex further away from the forewing.

6.4 Pressure Corrected POD ROM Method

A pressure corrected POD ROM method, which can accurately model the fluid dynamics of flapping flight, was presented and examined by two-dimensional flows past a pitching-plunging plate with the pitching amplitude equal to 10°, 20° and 30°. POD modes of the flows were computed in an inertial coordinate system. This approach produces projection errors at the same order of magnitude as the one computed in a non-inertial coordinate system, according to literature.

Prediction errors of the pressure corrected POD ROM were examined with respect to the number of snapshots and periods in terms of short- and long-term behaviors. It was found that the prediction errors were on the order of $10^{-1}$, which was at least one order of magnitude smaller than the one produced by a ROM method without pressure correction. The long-term behavior of a 4-mode ROM was examined and it was found that the pressure term stabilized the ROM of the flows.

The pressure corrected POD ROM method was also examined by a two-dimensional flow past two pitching-plunging plates in tandem. The prediction curves showed a good agreement to the projection curves. This indicates that that the pressure-correction based ROM can capture the dynamics of multiple flapping wings at low Reynolds numbers.
6.5 Force Survey of POD Modes

To analyze the force performance of POD modes, POD-FSM was developed. It was found that the aerodynamic force acting on an immersed body can be decomposed into a linear combination of the force of individual POD modes and their interactions. This method was demonstrated by a three-dimensional flow past a plate undergoing a pitching-plunging motion. Force superposition of all POD modes accurately predicted the aerodynamic force acting on the plate. It was also found that some POD modes always have zero thrust, while others always have zero lift. Categorizing POD modes with respect to their force behaviors can help to decompose the original flow into thrust- and lift-producing flows ($G_1$ and $G_{-1}$, respectively). As a result, the force acting on the plate can be expressed as a linear combination of the force of the thrust- and lift-producing flows and their interactions.

6.6 Symmetry of Half-period Flows and POD Modes

Based on POD-FSM, the force behavior of POD modes is determined by the structure of modes. Particularly, symmetric POD modes can always generate zero force in a certain direction. Further investigation on the aforementioned flow showed that the symmetry of POD modes is caused by the flow symmetry: a reflectional symmetry about the central plane in the spanwise direction and a half-period symmetry about the central plane in the crossflow direction. The half-period symmetry causes the POD modes to be symmetric or antisymmetric about the crossflow central plane (or the streamwise line in 2-D). POD modes thus can be categorized into two sets $G_1$ and $G_{-1}$ with respect to the symmetry in the crossflow direction. $G_1$ includes the mean (averaged) flow and POD modes with even
multiples of vortex shedding frequency \( f \). Members of \( G_l \) only generate thrust. The other POD modes that are odd multiples of frequency \( f \) consist of \( G_{-l} \). Members of \( G_{-l} \) only generate lift. Because \( G_l \) and \( G_{-l} \) have different frequency spectrum, it is possible to perform flow control with respect to frequency to achieve the desired aerodynamic performance.

### 6.7 Overview of Contributions

The main contributions of this dissertation are as follows:

- Development of an MSIP iterative method solving the momentum equations and an MSIP smoother for geometric multigrid methods solving the pressure-Poisson equation for immersed boundary method (IBM).
- Study of kinematics, aerodynamic performance, and vortex formation of a dragonfly in free flight, and investigation of forewing force enhancement caused by wing-wake interaction.
- Development of a pressure corrected POD ROM method for moving bodies.
- Development of a POD mode Force Survey Method (POD-FSM) to measure the contribution of POD modes to aerodynamic forces.
- Investigation of symmetry of half-period flows and POD modes

### 6.8 Future Work

One possible extension to this work would be the parallelization of MSIP for immersed boundary method. Actually, this work has been partially finished using Message Passing Interface (MPI) and domain decomposition techniques. However, the efficiency of the
parallel MSIP is about 60% to 75% on four computational nodes. The low efficiency is caused by two problems. One is that the multigrid methods employ multiple levels to reduce errors simultaneously. Communications between nodes are extremely frequent at coarser grids. Our profile shows that the communication time could be 10% of total computational time on four nodes. The other problem is that the MSIP method is a serial algorithm. Domain decomposition usually increases the iteration number by 30% to 50%. To solve these problems, particular work on developing parallel algorithms for the MSIP and multigrid methods is needed. With the parallel computing, simulations for the freely flying dragonfly with Re equal to 6930 can be conducted.

Another possible future work is to extend the pressure corrected POD ROM method to flexible wings. The current method can accurately predict the short-term behavior of the flow with a prediction error on the order of $10^{-1}$. However, the error increases immediately to the order of $10^2$ after 1 period. We suspect that high order modes can significantly affect the dynamics of the fluid because energy of high order modes concentrates near the wing. Further investigation of this problem is required.
Appendix A: CFD Solver Validation with Experiments (1)

We compare results of a rectangle plate between three-dimensional simulations and experimental measurements from an oil tow-tank experiment. The experiments were carried out by Dr. Xinyan Deng’s group at Bio-robotics Lab, Purdue University. The experimental step-up is shown in Figure A-1. The plate used in the experiments has a rectangle planform with chord length $c_e$ equal to 100mm and span $b_e$ equal to 200mm. Thus the aspect ratio of the plate ($AR = \frac{b_e^2}{b_e \cdot c_e}$) equals 2. The thickness of the plate is 2mm and the ratio of the thickness to the chord length of the plate is 2%. The plate was vertically placed in the tank with angle of attack $\alpha$ equal to 45°. The plate impulsively started from rest. After 0.5sec, it travelled steadily at a constant velocity $U_e$ along the oil tank for a travel distance equal to 3 chord lengths. The translational velocity $U_e$ of the plate was 99mm/s. The kinematic viscosity $\nu_e$ was measured to be 20.2cSt. The Reynolds number, defined as $Re = U_e c_e / \nu_e$, is thus 495.

Figure A-1. Experimental configuration for the moving plate.
Our simulations match the Reynolds number of the flow, angle of attack $\alpha$ and the aspect ratio $AR$ of the plate. A rectangle planform with non-dimensional chord length $c_s=1$ is used. As the plate in the experiments has a small thickness ratio (2%), we model the plate with an infinitely thin membrane in our simulations. The translational velocity $Us$ of the plate equals 1. The computational domain size is $18(X) \times 12(Y) \times 18(Z)$ (see Figure A-2 for reference). The grid number is $157 \times 103 \times 113$ (nearly 1.8 million). To provide the highest resolution for the flow around the plate, a dense grid ($3.8 \times 2 \times 2.5$) is used in the region where the plate travels. Velocity and pressure boundary conditions are specified as homogenous Neumann boundary condition on all the external boundaries.

During the experiments, the flow field is captured with stereo digital particle image velocimetry (DPIV) at mid-span of the plate. Typical vorticity fields ($\omega_z$) at $t=0.8$ and $2.0\text{sec}$ from the experiments are shown in Figure A-3, as well as our simulation results for a
comparison. At $t=0.8$sec (0.3sec after the impulsive start ended), the leading edge vortex formed and the trailing edge vortex (starting vortex) formed and detached. At $t=2.0$sec, the leading edge vortex also detached. It can be seen that the shape, strength, and position of leading- and trailing edge vortices show a good agreement between the PIV and DNS results at both moments.

![PIV and DNS Comparison](image)

**Figure A-3.** Spanwise vorticity contours ($\omega_z$) at the mid-span of the plate of $AR=2$ at $\alpha=45^\circ$.

The plate, illustrated by solid black line, was moving from right to left.
Figure A-4. (a) Drag and (b) lift coefficients as a function of time.

A six component force/torque sensor (ATI Nano 17) is mounted at the base of the plate to measure instantaneous dynamic forces. The measured forces are filtered at a cutoff frequency about 800 to 900 HZ at 100dB and 200dB. Drag and lift coefficients from our simulations are compared with the force from experiments, as shown in Figure A-4. The impulsive start produces a peak force at nearly $t=0.2$ sec. Shortly after that, both drag and lift decay rapidly to a valley at $t=0.5$ sec. The forces oscillate one more time with the second peak at approximately $t=0.8$ sec. After $t=1.0$ ms, the drag and lift acting on the plate enter a platform region. Our simulation results accurately capture the first and second peak force, and the platform.

Ren Yan and Yun Liu made major contributions to the simulation and experimental work, respectively.
Appendix B: CFD Solver Validation with Experiments (2)

In order to demonstrate the validity of the CFD solver used in the current study, simulations of flow around a robotic fruit fly wing were conducted. The robotic wing replicates *Drosophila Melanogaster* wings (Dickinson et al., 1999) with wing area 0.0167 m$^2$, span 0.25 m, and average chord $\bar{c} = 8.79$ cm. In the experiment of Dickinson et al. (1999), the wing sweeps in the horizontal plane and rotates at the end of each stroke. The stroke amplitude is 160°, the angle of attack at mid-stroke is 40° and flapping frequency is 0.145 Hz. The wing is either advanced rotated or delayed rotated by 8% of the stroke cycle with respect to the stroke reversal. The Reynolds number is 136, with the average translational velocity at the wing mid-chord 0.15 m/s. A non-uniform Cartesian grid of size 181×241×181 is used, and the computational domain is chosen to be $30\bar{c} \times 30\bar{c} \times 30\bar{c}$ to get domain independence results. The simulation is conducted for six flapping cycles. The lift coefficient during each stroke is virtually identical after the third cycle. Figure B-1 shows the lift coefficient from the fourth cycle for cases of advanced rotation and delayed rotation, together with the experimental results (Dickinson et al., 1999). There are two lift peaks produced around the beginning and the end of the half-stroke for the case of advanced rotation, and there is only one lift peak near the end of the half-stroke for the delayed rotation case. As can be seen, our simulations capture the experimental results well. This proves that the current CFD solver can accurately predict the instantaneous force for flow past a 3D flapping wing.
Figure B-1. Lift production from the current study and previous works on a robotic fruit fly wing. (a) Advanced rotation. (b) Delayed rotation.
Appendix C: Parameters of Dragonfly Specimen

The data used for the simulation is from an Eastern Pondhawk, *Erythimus Simplicicollis*, dragonfly, which was taken from a lake near the research lab during the period of August–September 2010. The mean body mass of the specimen used to demonstrate our reconstruction was 0.265±0.0005g. The body length was 43.95±0.005 mm, and the forewing and hindwing lengths were 35.03±0.005 mm and 33.21±0.005 mm respectively. The mid-span chord lengths for forewing and hindwing were 7.39±0.005 mm and 9.21±0.005 mm respectively.

Prior to photographing, the forewings and hindwings were each marked in a grid pattern with 15 black ink marker points of approximately 0.5 mm diameter on each forewing and 16 on each hindwing. Marker points were spaced 5.3 mm apart with a spacing accuracy of 0.6 mm in the span-wise direction and 3.6 mm apart with a spacing accuracy of 0.8 mm in the chord-wise direction. The marker points aid in reconstructing the wings in 3D because the subtle natural pattern of wing veins is indiscernible at most time steps. The marker points were applied with a felt-tipped ink pen, adding negligible mass to the insect wings.

The initial 3D wing template models were generated with Catmull-Clark subdivision surfaces by aligning surface points corresponding to the first level of the subdivision surface hierarchy with the marker points on the wings in a top down image. The dragonfly's body was manually created with Autodesk Maya based on several still images of the dragonfly taken with calibrated cameras, and was assumed rigid due to how slowly it moves relative to the wings. Figure C-1 shows an initial configuration of a dragonfly's wings and body based on a top down view prior to takeoff.
Figure C-1. Initial configuration of a dragonfly template mesh: (a) Top down image showing the marker points on the wings. (b) Wing and body template models with the surface points corresponding to the top level of the subdivision hierarchy marked in blue (Koehler, Liang et al. 2012).
References


