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On the Spectra of Momentum Operators

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ON THE SPECTRA OF MOMENTUM OPERATORS

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I HEREBY RECOMMEND THAT THE THESIS PREPARED UNDER MY SUPERVISION BY Cody E. Watson ENTITLED On The Spectra of Momentum Operators BE ACCEPTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF Master of Science.

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Abstract

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The selfadjoint momentum operators acting in two intervals are parametrized by the unitary two-by-two matrices. Fixing a parametrization of the unitary two-by-two matrices we study how the spectra of the selfadjoint momentum operators depends on these parameters as well as on the lengths of the two intervals.

Contents

1. Introduction

1.1. General Introduction

Classical physics deals with systems on a scale familiar to our usual experience. Towards the end of the 19th century physicist realized that that certain observations of system on a very small scale could not be explained by classical physics. This lead to the development of quantum mechanics.

In quantum mechanics, the analogue of Newton's law is Schrödinger's equation. It is a linear partial differential equation:

$$i\frac{h}{2\pi}\frac{\partial}{\partial t}f = Hf$$

where t is time, h is Plank's constant and H is the Hamiltonian. The Hamiltonian H is a selfadjoint operator characterizing the total energy of the system, its exact form depends on the system. The wave function f is the most complete of a physical system. For a single particle in an electric field Schrödinger's equation takes the form

$$i\frac{h}{2\pi}\frac{\partial}{\partial t}f(x,t) = \frac{-h^2}{8\pi^2 m}\nabla^2 f(x,t) + V(x,t)f(x,t)$$

where m is the mass of the particle, ∇ is the gradient, V is the potential energy, and the wave function f is a complex valued function.

In quantum mechanics all observalbles are represented by selfadjoint operators acting on the wavefunction. The eigenvalues (more generally, the spectrum) of the operator are the possible values of the observable, that is the possible values that can be measured in an experiment.

We will restrict attention to a particle living in a one-dimension space, more precisely, to a particle confined to the union of two disjoint intervals $[a,b]\cup[c,d]$. Quantum tunneling allows the particle to be in $[a,b]$ at some times and in $[c,d]$ at other times. Further, we will only consider a particle at a specific time t_0, hence we will ignore the time varible t in the wave function $f(x,t)$.

Chosing units such that $h=1$, the operator corresponding to the momentum is

$$(Pf)(x) = \frac{1}{i2\pi}f'(x)$$

acting in the space $L^2([a,b]\cup[c,d]])$ of complex valued square integrable function on $[a,b]\cup[c,d]$. There are two problems associated with this: (i) arrange that P is selfadjoint and (ii) calculate the eigenvalues of each selfadjoint realization of P. In particular, if different selfadjoint realizations of P have different sets of eigenvalues, the physics corresponding to these selfadjoint realizations must be different. Denote the eigenstates (also called eigenvectors or eigenfunctions) of P by e_k and the corresponding eigenvalues by λ_k, i.e., $(Pe_k)(x) = \lambda_k e_k(x)$. Write the wave function f in terms of the eigenstates e_k as

$$f(x) = \sum c_k e_k(x).$$

If the eigenstates are normalized in the sense that $\int |e_k(x)|^2\,dx = 1$ for all k. Then, the probability that a measurement of the momentum gives the value λ_n is

$$\frac{|c_n|^2}{\sum_k |c_k|^2}.$$

In particular, if the wave function $f(x)$ is not an eigenfunction for P, measuring the momentum may have more than one outcome. For example, if $f(x) = c_1 e_{\lambda_1}(x) + c_2 e_{\lambda_2}(x)$. Then a measurement of the momentum will give the outcome λ_1 with probability $|c_1|^2 / \left(|c_1|^2 + |c_2|^2\right)$ and the outcome λ_2 with probability $|c_2|^2 / \left(|c_1|^2 + |c_2|^2\right)$. For this and much more see [16].

1.2. **Notation**

We start by defining the two closed intervals $I_1 = [a,b]$ and $I_2 = [c,d]$ with $a < b \leq c < d$. Consider the Hilbert space of square integrable functions $L^2(I_1 \cup I_2)$, equpped with the inner product

$$\langle f, g \rangle := \int_{I_1 \cup I_2} f \bar{g}.$$

Let $\overset{o}{I}_k$ be the interior of I_k, $k = 1, 2$. The maximal momentum operator is

$$P_{max} := \frac{1}{i2\pi} \frac{d}{dx}$$

acting in the set $\mathscr{D}(P_{max})$ of functions on $\overset{o}{I}_1 \cup \overset{o}{I}_2$, that are absolutely continuous on $\overset{o}{I}_k$, for $k = 1, 2$. For f in $\mathscr{D}(P_{\max})$ we set $f(a) := \lim_{x \searrow a} f(x)$ and $f(b) := \lim_{x \nearrow b} f(x)$, and similarly, $f(c) := \lim_{x \searrow c} f(x)$ and $f(d) := \lim_{x \nearrow d} f(x)$. The limits exists since any absolutely continuous function is uniformly continuous. We may have $f(b) \neq f(c)$, even if $b = c$. We also have the boundary form associated with P,

$$B(f,g) := \langle Pf, g \rangle - \langle f, Pg \rangle\,,$$

on $\mathscr{D}(P)$. For $f \in \mathscr{D}(P)$, let $\rho_1(f) := (f(a), f(d))$ and $\rho_2(f) := (f(a), f(c))$. Then

$$B(f,g) = \langle \rho_1(f), \rho_1(g) \rangle - \langle \rho_2(f), \rho_2(g) \rangle\,.$$

The set of selfadjoint restrictions of P are parametrized by the set of unitary 2×2 matrices, see [2]. By setting B to be any 2×2 unitary matrix, the seladjoint restriction P_B of P is

$$\mathscr{D}(P_B) := \{f \in \mathscr{D}(P) \,|\, B\rho_1(f) = \rho_2(f)\}\,. \tag{1.1}$$

We consider the two copies of the two-demensional Hilbet space $\mathbb{C}^2$:

$$\rho_2(\mathscr{D}(P)) = \mathscr{B}_L = \left\{ \begin{pmatrix} f(a) \\ f(c) \end{pmatrix} \,|\, f \in \mathscr{D}(P) \right\},$$

$$\rho_1(\mathscr{D}(P)) = \mathscr{B}_R = \left\{ \begin{pmatrix} f(b) \\ f(d) \end{pmatrix} \,|\, f \in \mathscr{D}(P) \right\}.$$

where ρ_1 and ρ_2 are the respective boundary-restrictions.

For $f \in \mathscr{D}(P)$. Decompose f, as $f = \varphi_0 + f_+ + f_-$, where $\varphi_0 \in \mathscr{D}_0$, $f_\pm \in \mathscr{D}_\pm$. If $f \in \mathscr{D}(P)$, then

$$B(f,f) = \| f_R \|^2_{\mathscr{B}_R} - \| f_L \|^2_{\mathscr{B}_L} = \| f_+ \|^2 - \| f_- \|^2\,.$$

Set

$$\mathscr{C}_b = \{B : \mathscr{B} \to \mathscr{B}_R, \text{ isometric}\}\,.$$

Theorem 1.1. *In the two-interval case, there is a natural isomorphism $\mathscr{C}_b \cong \mathscr{C}(L)$, where $\mathscr{C}(L)$ is defined as $\mathscr{C}(L) = \left\{U : \mathscr{D}_+ \to \mathscr{D}_- \,|\, U^*U = P_{\mathscr{D}_+},\, UU^* = P_{\mathscr{D}_-}\right\}$. See* [6]

Corollary 1.2. *Let U, $b_\pm$, and B be the operators in Theorem 1, then $B : \mathscr{B}_L \to \mathscr{B}_R$ is $B = b_- U b_+^{-1}$. See* [6]

Theorem 1.3. *We can parametrize the collection of unitary 2×2 by*

$$B = \begin{pmatrix} we(\phi) & -\sqrt{1-w^2}e(\theta-\psi) \\ \sqrt{1-w^2}e(\psi) & we(\theta-\phi) \end{pmatrix}, \tag{1.2}$$

where $0 \le w \le 1$, ϕ, ψ, $\theta \in \mathbb{R}$, and

$$e(x) := e^{i2\pi x}.$$

Proof. Let $T = \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix}$ be a unitary matrix with coefficients from $\mathbb{C}$. Assume $\alpha = we(\phi)$, $\beta = \| \beta \| e(\psi)$, $\gamma = \| \gamma \| e(x)$, $\delta = \| \delta \| e(y)$, where $0 \le w \le 1$. Since T is unitary we have the following equations:

$$\alpha\bar{\alpha} + \beta\overline{\beta} = 1, \tag{1.3}$$

$$\gamma\overline{\gamma} + \delta\overline{\delta} = 1, \tag{1.4}$$

$$\alpha\overline{\alpha} + \gamma\overline{\gamma} = 1, \tag{1.5}$$

$$\beta\overline{\beta} + \delta\overline{\delta} = 1, \tag{1.6}$$

and

$$\alpha\delta - \beta\gamma = 1. \tag{1.7}$$

(1.3) tells us that $\| \beta \| = \sqrt{1-w^2}$, (1.5) and (1.6) tells us $\| \alpha \| = \| \delta \|$ and $\| \beta \| = \| \gamma \|$. (1.7) Gives us the following

$$w^2 e(y+\phi) - (1-w^2)e(x+\psi) = 1.$$

Thus

$$e(y+\phi) + e(x+\psi) = 0,$$

and

$$e(x+\psi) = -1.$$

Hence, $y = \theta - \phi$, $x = \theta - \psi$, and $\gamma = - \| \gamma \| e(x)$, where $\theta \in \mathbb{R}$. □

The following three results are from [6], that the spectrum is equal to the set of eigenvalues, the boundary condition, and an equation for the eigenvalues.

Lemma 1.4. *The spectrum of any selfadjoint restriction $\tilde{P}$ of P equals the set of eigenvalues of $\tilde{P}$ and each eigenvalue has multiplicity one or two.*

The proof of this provides a valuable corollary that we will use later. We are now assuming $I_1 = [0,1]$, and $I_2 = [\alpha, \beta]$, with $1 < \alpha < \beta$. Let $\ell = \beta - \alpha$.

Corollary 1.5. *λ is an eigenvalue for P_B, the selfadjoint restriction of P, iff there are complex numbers a,b such that $f = e_\lambda^{(a,b)}$ satisfies the boundary condition,*

$$\begin{pmatrix} we(\phi) & -\sqrt{1-w^2}e(\theta-\psi) \\ \sqrt{1-w^2}e(\psi) & we(\theta-\phi) \end{pmatrix} \begin{pmatrix} ae(\lambda) \\ be(\lambda\beta) \end{pmatrix} = \begin{pmatrix} a \\ be(\lambda\alpha) \end{pmatrix}. \tag{1.8}$$

Consequently $\lambda \in \mathbb{R}$.

Theorem 1.6. *Fix $1 < \alpha < \beta$. Let $I_1 = [0,1]$ and $I_2 = [\alpha, \beta]$, $\ell = \beta - \alpha$. Suppose P_B is the selfadjoint restriction associated with (1.2) via (1.1) and $0 < w < 1$. Then a point λ in $\mathbb{R}$ is an eigenvalue of P_B if and only if it is a solution to the equation*

$$e(-\theta - \frac{1}{2} + \phi - \ell\lambda) = \frac{e(\phi+\lambda) - w}{1 - we(\phi+\lambda)}. \tag{1.9}$$

When λ is a real solution to (1.9) then any corresponding eigenfunction is a multiple of

$$e_\lambda^{(a,1)}(x) = (a\chi_{I_1}(x) + \chi_{I_2}(x))e_\lambda(x),$$

where a is determined by $\frac{\sqrt{1-w^2}e(\theta-\psi+\beta\lambda)}{we(\phi+\lambda)-1} = \frac{1-we(\theta-\phi+\ell\lambda)}{\sqrt{1-w^2}e(\psi+\lambda-\alpha\lambda)}$. In particular, the spectrum has uniform multiplicity equal to one.

Proof. Corollary 1.5 tells us that

$$awe(\phi+\lambda) - b\sqrt{1-w^2}e(\theta-\psi+\beta\lambda) = a$$
$$a\sqrt{1-w^2}e(\psi+\lambda) + bwe(\theta-\phi+\beta\lambda) = be(\alpha\lambda).$$

The first equation shows that $a = 0 \iff b = 0$. So, we set $b = 1$, and solve for a.

$$\frac{\sqrt{1-w^2}e(\theta-\psi+\beta\lambda)}{we(\phi+\lambda)-1} = \frac{1-we(\theta-\phi+\ell\lambda)}{\sqrt{1-w^2}e(\psi+\lambda-\alpha\lambda)}.$$

Cross-multiplying gives

$$(1-w^2)e(\theta+\lambda+\ell\lambda) = (1-we(\theta-\phi+\ell\lambda))(we(\phi+\lambda)-1).$$

Hence by expanding the right hand side we have

$$(1-w^2)e(\theta+\lambda+\ell\lambda) = 1 - w^2e(\theta+\lambda+\ell\lambda) + w(e(\theta-\phi+\ell\lambda)+e(\phi-\lambda)).$$

Thus

$$e(\theta+\lambda+\ell\lambda) = -1 + w(e(\theta-\phi+\ell\lambda)+e(\phi+\lambda)).$$

Subtracting $we(\theta-\phi+\ell\lambda)$ we get

$$e(\theta+\lambda+\ell\lambda) - we(\theta-\phi+\ell\lambda) = -1 + we(\phi+\lambda).$$

Facting the left hand side we get

$$(e(\lambda) - we(-\phi))e(\theta+\ell\lambda) = -1 + we(\phi+\lambda),$$

or equivalently

$$(e(\phi+\lambda) - w)e(\theta-\phi+\ell\lambda) = -1 + we(\phi+\lambda).$$

Rearranging

$$e(-\theta+\phi-\ell\lambda) = \frac{e(\phi+\lambda)-w}{we(\phi+\lambda)-1}.$$

Multipying by $e(-\frac{1}{2}) = -1$,

$$e(-\theta-\frac{1}{2}+\phi-\ell\lambda) = \frac{e(\phi+\lambda)-w}{1-we(\phi+\lambda)}.$$

□

1.3. **Summary of results**

We are concerned with how the spectrum of a selfadjoint restriction P_B depends on the lengths of the two intervals and on the parameters w, ϕ, ψ, and θ in (1.2). In turn out that the spectrum of P_B is independent of ψ. We show in Section 2.1 that it is sufficient to consider that cases where $\phi = 0$ and $b-a = 1$. To simplify the notation we set $\xi := \frac{1}{2}+\theta$. Equation (2.6) leads to a natural enumeration $n \to \lambda_n$, $n \in \mathbb{Z}$ of the eigenvalues. For irrational ℓ, the distribution of the eigenvalues modulo one is calculated in Theorem 3.8. A consequence of Theorem 3.8 is that the eigenvalues modulo one are dense in $[0,1]$. This was established by a different method in [6].

The present paper contains several estimates of the eigenvalues λ_n, for example, if $w < 1$ we shown, in Proposition 3.2, that

$$\left\lfloor \frac{n-\xi}{L} \right\rfloor \leq \lambda_n < \left\lfloor \frac{n-\xi}{L} \right\rfloor ,$$

where $L := 1 + \ell$ is the total length of the two intervals. See also, Proposition 4.1 and Example **??**. Theorem **??** states that

$$\lambda_n = \frac{n-\xi}{1+\ell} + \sum_{k=1}^{\infty} a_k w^k$$

where $|a_k| \leq \frac{1}{\pi k}$ depends on n, w, ℓ, and ξ. We show (Corollary 4.3) that, if $n(a,k)$ is the number of eigenvalues in the open interval $]a, a+k[$ of lenth k, then

$$\frac{n(a,k)}{k} \to 1 + \ell$$

as $k \to \infty$. Hence, the asymptotic density of the eigenvalues equals the total lenght of the two intervals. This is knows for spectral sets [3, 4, 5, 7, 8, 11, 10, 12, 13, 15], but the union of two intervals need not be a spectral set, see e.g., [6].

For fixed ℓ and ξ, the range and monotonicity of $w \to \lambda_n$ is established in Theorem 4.6.

For the convenience of the reader and for the sake of completeness, there is some overlap between [6] and this paper. Where there are overlaps, typically, our results are either stronger and/or our proofs are simpler.

2. Preliminaries

2.1. **Unitary Equivalence**

The following lemmas can be used to reduce the discussion of the spectrum, from the general cases $I_1 = [a,b]$, $I_2 = [c,d]$, $a < b \leq c < d$ to the cases where $I_1 = [0,1]$ and $I_2 = [1,\ell]$ with $\ell \geq 1$. We split this into three lemmas. In the lemmas we keep track of the spectrum and of the boundary unitaries.

The first lemma shows how the spectrum changes by a dilation of $I_1 \cup I_2$ by a positive scalar.

Lemma 2.1. *Fix a unitary 2×2 matrix B, let P be the momentum operator in $L^2([a,b] \cup [c,d])$ determined by the boundary condition*

$$B \begin{pmatrix} f(b) \\ f(d) \end{pmatrix} = \begin{pmatrix} f(a) \\ f(c) \end{pmatrix} \tag{2.1}$$

and let P' be the momentum operator in $L^2([0,1] \cup [\alpha,\beta])$ determined by the boundary condition

$$B \begin{pmatrix} f(1) \\ f(\beta) \end{pmatrix} = \begin{pmatrix} f(0) \\ f(\alpha) \end{pmatrix}, \tag{2.2}$$

then P is unitary equivalent to $\frac{1}{b-a}P'$. In particular, λ is in the spectrum of P' if and only if $(b-a)\lambda$ is in the spectrum of P.

Proof. Let $I_1 := [a,b]$, $I_2 := [c,d]$, $m := \frac{1}{b-a}$ and $\mu(x) := m(x-a)$. Then μ maps $I_1 \cup I_2$ onto $[0,1] \cup [\alpha,\beta]$, where $\alpha := \mu(c) > 1$ and $\beta := \mu(d)$. The map $Uf := \sqrt{m} f \circ \mu$ is a unitary operator mapping $L^2([0,1] \cup [\alpha,\beta])$ onto $L^2(I_1 \cup I_2)$. It is easy to see that f satisfies (2.2) if and only if Uf satisfies (2.1). Since f is absolutely continuous if and only if Uf is absolutely continuous, it follows that $U\mathscr{D}(P') = \mathscr{D}(P)$. Let $f \in \mathscr{D}(P')$, then

$$\begin{aligned} PUf &= \frac{1}{i2\pi}(Uf)' = \frac{1}{i2\pi}\sqrt{m}\,(f \circ \mu) \\ &= \frac{1}{i2\pi} m^{3/2} f' \circ \mu = m^{3/2}(P'f) \circ \mu = mUP' \end{aligned}$$

Hence, P is unitary equivalent to mP'. □

The next lemma shows that a reflection of $I_1 \cup I_2$ leads to a reflection of the spectrum.

Lemma 2.2. *Fix a unitary 2×2 matrix B, let P be the momentum operator in $L^2([a,b] \cup [c,d])$ determined by the boundary condition (2.1) and let P' be the momentum operator in $L^2([-d,-c] \cup [-b,-a])$ determined by the boundary condition*

$$\widetilde{B}\begin{pmatrix} f(-c) \\ f(-a) \end{pmatrix} = \begin{pmatrix} f(-d) \\ f(-b) \end{pmatrix}, \tag{2.3}$$

where

$$\widetilde{B} := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} B^{-1} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

then P is unitary equivalent to $-P'$. In particular, λ is in the spectrum of P' if and only if $-\lambda$ is in the spectrum of P.

Proof. Similar to the proof of Lemma 2.1, but with $\mu(x) := -x$ and $Uf := f \circ \mu$. □

The following lemma shows that the location of the intervals does not influence the spectrum of the momentum operators. Hence, only the choice of a 2×2 unitary matrix and the lengths of the intervals can influence the spectrum.

Lemma 2.3. *Fix a unitary 2×2 matrix B and a real number $\gamma \geq b - c$. Let P be the momentum operator in $L^2([a,b] \cup [c,d])$ determined by the boundary condition (2.1) and let P' be the momentum operator in $L^2([a,b] \cup [c+\gamma, d+\gamma])$ determined by the boundary condition*

$$B\begin{pmatrix} f(b) \\ f(d+\gamma) \end{pmatrix} = \begin{pmatrix} f(a) \\ f(c+\gamma) \end{pmatrix}, \tag{2.4}$$

then P is unitary equivalent to P'.

Proof. Similar to the proof of Lemma 2.1, but with

$$\mu(x) := \begin{cases} x & \text{if } a \leq x \leq b \\ x + \gamma & \text{if } c \leq x \leq d \end{cases}$$

and $Uf := f \circ \mu$. □

2.2. **The Spectrum as Solutions to an Equation**

By Lemma 2.1 and Lemma 2.3, the general case of two intervals $I_1 = [a,b]$, $I_2 = [c,d]$, and $a < b \leq c < d$ can be reduced to the case where $I_1 = [0,1]$ and $I_2 = [1, 1+\ell]$, with $\ell > 0$. The boundary unitary 2×2 matrix $B = B_{w,\phi,\psi,\theta}$ is unchanged and the spectrum is rescaled, specifically

$$\text{spectrum}\,(P_{\text{general}}) = \frac{1+\ell}{b-a+d-c}\text{spectrum}\,(P_{\text{specific}})$$

For $0 \leq w < 1$, let

$$g(t,w) := \int_0^t \frac{1-w^2}{1-2w\cos(2\pi u)+w^2}\,du. \tag{2.5}$$

Recall, $P_w(u) = \frac{1-w^2}{1-2w\cos(2\pi u)+w^2}$ is the Poisson kernel. It is shown in [6, Corollary 3.15] that, if $\lambda_n = \lambda_n(\ell, w, \phi, \theta)$ is the solution, for t, to the equation

$$n - \tfrac{1}{2} - \theta + \phi - \ell t = g\left(t+\phi, w\right), \tag{2.6}$$

then $(\lambda_n)_{n\in\mathbb{Z}}$ is the collection of eigenvalues for P. In particular, the spectrum is independent of ψ. It is clear that by exponentiating (2.6) with $g(t+\phi, w)$ written as in (2.9), we arrive at (1.9). Hence the equations, (1.9) and (2.6), are equivalent.

Writing (2.6) as

$$n - \tfrac{1}{2} - (\theta - \phi - \ell\phi) - \ell\left(t+\phi\right) = g\left(t+\phi, w\right)$$

shows that

$$\lambda_n\left(\ell, w, \phi, \theta\right) = \lambda_n\left(\ell, w, 0, \theta - \phi - \ell\phi\right) - \phi.$$

Hence, we will restrict attention to the case where $\phi = 0$. To simplify the notation further we set $\xi := \frac{1}{2} + \theta$. Thus we study the solutions $\lambda_n = \lambda_n(w, \ell, \xi)$, for t, to

$$n - \xi - \ell t = g(t,w), \tag{2.7}$$

with $0 < \ell < \infty$, $0 \leq w < 1$, and $\xi \in \mathbb{R}$. Some instances of (2.7) are illustrated in Figure 2.1.

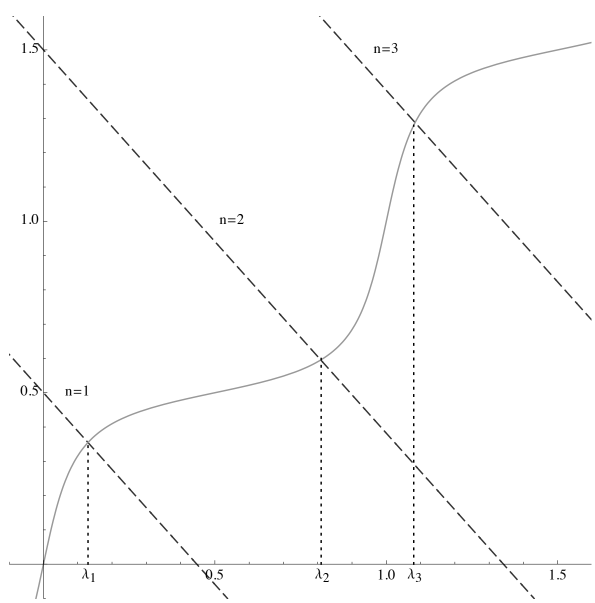


FIGURE 2.1. $g(t, 0.65)$ and the lines $n - \xi - \ell t$, $\xi = \frac{1}{2}$, $\ell = \sqrt{5}$, and $n = 1, 2, 3$.

Clearly, it is sufficient to consider ξ in some half-open interval of length one, e.g., $0 \leq \xi < 1$, but it is convenient not to impose a restriction of this nature on ξ.

Remark 2.4. When $w = 1$, the boundary conditions (2.2) reduce to $e(\phi) f(1) = f(0)$ and $e(\theta - \phi) f(\beta) = f(\alpha)$. Hence, see e.g., [14], the spectrum of P is the union of $-\phi + \mathbb{Z}$ and $\frac{\phi-\theta}{\ell} + \frac{1}{\ell}\mathbb{Z}$. The points in the intersection of $-\phi + \mathbb{Z}$ and $\frac{\phi-\theta}{\ell} + \frac{1}{\ell}\mathbb{Z}$ have multiplicity two, all other points have multiplicity one. □

2.3. Some properties of g from (2.5)

To obtain a more detailed understanding of $(\lambda_n)_{n\in\mathbb{Z}}$ we need a better understanding of the function g from (2.5).

Lemma 2.5. *For all $t \in \mathbb{R}$ and $0 \le w < 1$,*

$$g(t,w) = t + \sum_{k=1}^{\infty} w^k \frac{\sin(2\pi k t)}{\pi k} \tag{2.8}$$

$$= t + \frac{1}{2\pi i} \ln\left(\frac{1 - w\, e(-t)}{1 - w\, e(t)}\right) \tag{2.9}$$

$$= t + 2\sum_{s=0}^{\infty} (-1)^s \frac{(2\pi)^{2s}}{(2s+1)!} Li_{-2s}(w)\, t^{2s+1} \tag{2.10}$$

where

$$Li_\alpha(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^\alpha}$$

is the polylogarithm function.

Remark 2.6. For integer values of the polylogarithm order, explicit expressions are obtained by using $\mathrm{Li}_1(z) = -\ln(1-z)$ and $\mathrm{Li}_{n-1}(z) = z\frac{d}{dz}\mathrm{Li}_n(z)$ inductively. See, [1]. Consequently, $\mathrm{Li}_{-n}(z)$ is a rational function of z. The examples,

$$\mathrm{Li}_0(z) = \frac{z}{1-z}$$

$$\mathrm{Li}_{-2}(z) = \frac{z(1+z)}{(1-z)^3}$$

$$\mathrm{Li}_{-4}(z) = \frac{z(1+z)\left(1+10z+z^2\right)}{(1-z)^5}$$

etc., are of interest in (2.10). □

Proof of Lemma 2.5. We have

$$\begin{aligned}\int_0^t \frac{1-w^2}{1-2w\cos(2\pi x)+w^2}dx &= \int_0^t \left(\sum_{k\in\mathbb{Z}} w^{|k|} e(ks)\right) ds \\ &= \int_0^t \left(1 + 2\sum_{k=1}^{\infty} w^k \cos(2\pi k s)\right) ds \\ &= t + \sum_{k=1}^{\infty} w^k \frac{\sin(2\pi k t)}{\pi k}.\end{aligned}$$

Expanding $\sin x$ into power series, we get

$$\sum_{k=1}^{\infty} w^k \frac{\sin(2\pi k t)}{\pi k} = 2\sum_{s=0}^{\infty} (-1)^s \frac{(2\pi)^{2s}}{(2s+1)!} \left(\sum_{k=1}^{\infty} w^k k^{2s}\right) t^{2s+1}$$

$$= 2\sum_{s=0}^{\infty}(-1)^s \frac{(2\pi)^{2s}}{(2s+1)!} Li_{-2s}(w)\, t^{2s+1}.$$

Moreover,

$$\frac{1}{2\pi i}\ln\left(\frac{1-w\,e(-t)}{1-w\,e(t)}\right) = \sum_{k=1}^{\infty} w^k \frac{\sin(2\pi k t)}{\pi k}$$

as a Fourier series expansion. □

It follows directly from (2.5) that

$$g(0,w) = 0 \tag{2.11}$$

$$g(-t,w) = -g(t,w) \tag{2.12}$$

$$g(t,w) \text{is an increasing function of } t \in \mathbb{R} \tag{2.13}$$

It follows from simple properties of the sine function and (2.8) that,

$$g\left(\tfrac{1}{2},w\right) = \tfrac{1}{2}, \tag{2.14}$$

and

$$g(t+1,w) = g(t,w)+1, \tag{2.15}$$

for all $t \in \mathbb{R}$. In particular, $g(\cdot,w)$ is a continuous increasing function mapping the closed interval $\left[\frac{k}{2},\frac{k+1}{2}\right]$ onto itself, for any integer k.

Fix a real number s. For any real number t, the *s–fractional decomposition* is

$$t = \langle t\rangle_s + \lfloor t\rfloor_s$$

where $s \le \langle t\rangle_s < s+1$ and $\lfloor t\rfloor_s$ is an integer. We call $\langle t\rangle_s$ the *s–fractional part* of t and $\lfloor t\rfloor_s$ the *s–integer part* of t. The standard fractional decomposition of t is obtained by setting $s=0$. We will simplify the notation by omitting the subscript when $s=0$, i.e., by setting $\langle t\rangle := \langle t\rangle_0$ and $\lfloor t\rfloor := \lfloor t\rfloor_0$. In addition to $s=0$, we also find it convenient to use $s=-\frac{1}{2}$. Note,

$$\langle t\rangle_{-1/2} := \left\langle t+\tfrac{1}{2}\right\rangle - \tfrac{1}{2} \qquad \lfloor t\rfloor_{-1/2} := \left\lfloor t+\tfrac{1}{2}\right\rfloor.$$

For $0 \le w < 1$ and $-\frac{1}{2} < t < \frac{1}{2}$ let

$$\varphi(t,w) := \frac{1}{\pi}\arctan\left(\frac{(1+w)\tan(\pi t)}{1-w}\right). \tag{2.16}$$

Since

$$\varphi(t,w) \nearrow \tfrac{1}{2} \text{ as } t \nearrow \tfrac{1}{2} \text{ and } \phi(t,w) \searrow -\tfrac{1}{2} \text{ as } t \searrow -\tfrac{1}{2}.$$

Setting

$$\varphi\left(\tfrac{1}{2},w\right) := \tfrac{1}{2} \text{ and } \varphi\left(-\tfrac{1}{2},w\right) := -\tfrac{1}{2}, \tag{2.17}$$

extends $\phi(t,w)$ to a continuous function on $\left[-\frac{1}{2},\frac{1}{2}\right]\times[0,1[$.

Lemma 2.7. *If $0 \le w < 1$ and t is a real number, then*

$$g(t,w) = \lfloor t\rfloor_{-1/2} + \varphi(\langle t\rangle_{-1/2}, w), \tag{2.18}$$

where φ is determined by (2.16) and (2.17). In particular,

$$g_w(t,w) = \partial_w g(t,w) = \frac{1}{\pi}\cdot\frac{\sin(2\pi t)}{1-2w\cos(2\pi t)+w^2} \tag{2.19}$$

for all $0 \le w < 1$ and all real t.

Proof. If $0 \le w < 1$ and t is a real number, then

$$g(t,w) = g(\langle t\rangle_{-1/2}, w) + \lfloor t\rceil_{-1/2},$$

by (2.15). Since

$$\int \frac{1-w^2}{1-2w\cos(2\pi u)+w^2}du = \frac{1}{\pi}\arctan\left(\frac{(1+w)\tan(\pi u)}{1-w}\right) + C$$

and $g(0,w)=0$, by (2.11), the proof is complete. $\square$

Remark 2.8. Taking the limit as $w \nearrow 1$ in (2.18) we see that

$$g(t,1) := g(t,1-) = \lim_{w\nearrow 1} g(t,w) = \begin{cases} \lfloor t\rceil_{-1/2} - \frac{1}{2} & \text{if } \langle t\rangle_{-1/2} < 0 \\ \lfloor t\rceil_{-1/2} & \text{if } \langle t\rangle_{-1/2} = 0 \\ \lfloor t\rceil_{-1/2} + \frac{1}{2} & \text{if } \langle t\rangle_{-1/2} > 0 \end{cases}.$$

For fixed t, it follows from (2.19) that $g(t,w)$ is a decreasing function of w when $\langle t\rangle < 0$ and $g(t,w)$ is a increasing function of w when $\langle t\rangle > 0$. These properties of g are illustrated in Figure 2.2. In particular, g does not extend from a continuous function on $\mathbb{R}\times[0,1[$ to a continuous function on $\mathbb{R}\times[0,1]$. $\square$

Remark 2.9. We saw in Remark 2.8 that $g(t,1-) = \lim_{w\nearrow 1}\int_0^t \frac{1-w^2}{1-2w\cos(2\pi u)+w^2}du \neq 0$ when $t \neq 0$. Interchanging the limit and integral gives $\int_0^t \lim_{w\nearrow 1}\frac{1-w^2}{1-2w\cos(2\pi u)+w^2}du = 0$ for all t. $\square$

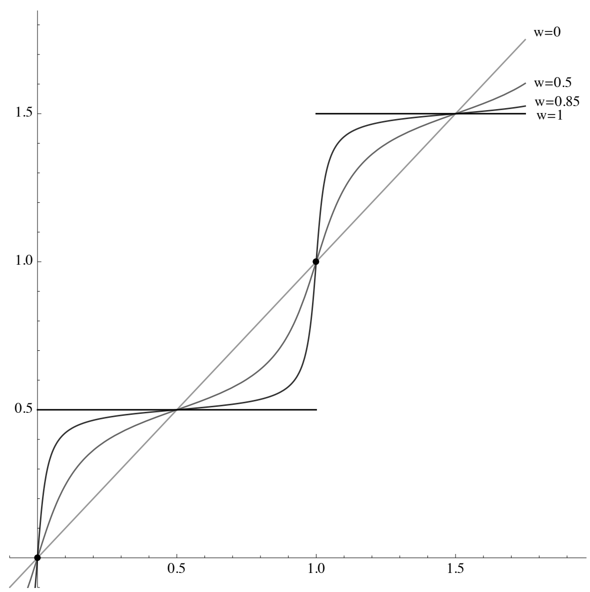


FIGURE 2.2. $g(t,w)$ as a function of t, for $w = 0, 0.5, 0.85, 1$.

Remark 2.10. Most of the results in this paper do not depend on the specific formula for g. Rather they are established under (various subsets of) (2.11)–(2.15) and the properties listed in Remark 2.8. Proposition 4.4 uses that $g_t(0,w)$ is the largest value of $g_t(t,w)$. However, we leave it to the interested reader to note which properties are required in each instance. □

3. Distribution of Eigenvalue Modulo One

In this section we investigate properties of the eigenvalues in terms of n. First, in Proposition 3.2 we calculate the integer part of $\lambda_n = \lambda_n(w,\ell,\xi)$. In Theorem 3.8, we find the distribution of the fractional parts of λ_n.

Lemma 3.1. *If $w = 0$, then $\lambda_n = \lambda_n(0,\ell,\xi) = \frac{n-\xi}{1+\ell}$.*

Proof. When $w = 0$, $g(t) = t$. Hence, (2.7) takes the form $n - \xi - \ell t = t$, solving for t gives the result. □

Proposition 3.2. *Fix $\xi \in \mathbb{R}$, $0 < \ell$, and $0 \le w < 1$. For any integer n, we have $a_n \le \lambda_n < b_n$, where*

$$a_n := \begin{cases} \left\lfloor \frac{n-\xi}{1+\ell} \right\rfloor & \text{if } 0 \le \left\langle \frac{n-\xi}{1+\ell} \right\rangle < \frac{1}{2} \\ \left\lfloor \frac{n-\xi}{1+\ell} \right\rfloor + \frac{1}{2} & \text{if } \frac{1}{2} \le \left\langle \frac{n-\xi}{1+\ell} \right\rangle < 1 \end{cases} \quad \text{and}$$

$$b_n := \begin{cases} \left\lfloor \frac{n-\xi}{1+\ell} \right\rfloor + \frac{1}{2} & \text{if } 0 \le \left\langle \frac{n-\xi}{1+\ell} \right\rangle < \frac{1}{2} \\ \left\lfloor \frac{n-\xi}{1+\ell} \right\rfloor + 1 & \text{if } \frac{1}{2} \le \left\langle \frac{n-\xi}{1+\ell} \right\rangle < 1 \end{cases}$$

Consequently, $\left\lfloor \frac{n-\xi}{1+\ell} \right\rfloor \le \lambda_n < \left\lfloor \frac{n-\xi}{1+\ell} \right\rfloor + 1$. In particular, the integer part of λ_n is independent of w.

Remark 3.3. Note $b_n - a_n = \frac{1}{2}$ for all n. □

Proof of Proposition 3.2. Let k be the integer satisfying

$$k \le \lambda_n < k+1,$$

i.e., $k := \lfloor \lambda_n \rfloor$. By (2.7), λ_n is the solution for t to

$$n - \xi - \ell t = g(t,w).$$

Write $\lambda_n = t = k + t_0$, where $0 \le t_0 < 1$. Then

$$n - \xi - \ell(k+t_0) = g(k+t_0,w) = k + g(t_0,w)$$

using (2.15). Solving for k yields

$$k = \frac{n-\xi}{1+\ell} - \frac{g(t_0,w) + \ell t_0}{1+\ell}.$$

Since $0 \le g(t_0,w) + \ell t_0 < 1 + \ell$, it follows that

$$\frac{n-\xi}{1+\ell} - 1 < k \le \frac{n-\xi}{1+\ell}.$$

Hence,

$$k = \left\lfloor \frac{n-\xi}{1+\ell} \right\rfloor$$

and consequently $\left\lfloor \frac{n-\xi}{1+\ell} \right\rfloor \le \lambda_n < \left\lfloor \frac{n-\xi}{1+\ell} \right\rfloor + 1$.

Letting k be the integer satisfying

$$k - \tfrac{1}{2} \le \lambda_n < k + \tfrac{1}{2},$$

and repeating the argument above shows that

$$\left\lfloor \frac{n-\xi}{1+\ell}+\frac{1}{2}\right\rfloor-\frac{1}{2}\le\lambda_n<\left\lfloor \frac{n-\xi}{1+\ell}+\frac{1}{2}\right\rfloor+\frac{1}{2}.$$

This completes the proof. □

Emphasizing the dependence of λ_n on w Proposition 3.2 can be re-stated as:

Corollary 3.4. *Write $\lambda_n(w)=\lambda_n(w,\ell,\xi)$. The integer part of λ_n is independent of w : $\lfloor\lambda_n(w)\rfloor=\lfloor\lambda_n(0)\rfloor$ for all $0\le w<1$. Whether the fractional part of λ_n is smaller or larger than $\frac{1}{2}$ is independent of w : If $0\le\langle\lambda_n(0)\rangle<\frac{1}{2}$, then $0\le\langle\lambda_n(w)\rangle<\frac{1}{2}$, for all $0\le w<1$. Similarly, if $\frac{1}{2}\le\langle\lambda_n(0)\rangle<1$, then $\frac{1}{2}\le\langle\lambda_n(w)\rangle<1$, for all $0\le w<1$.*

Proof. Combine Lemma 3.1 and Proposition 3.2. □

To prepare for the proof of Theorem 3.8 we establish the follow form of Proposition 3.2.

Lemma 3.5. *$k\le\lambda_n<k+1$ if and only if*

$$\xi+k(1+\ell)\le n<\xi+(k+1)(1+\ell).$$

Proof. Recall λ_n is the solution to $n-\xi-\ell t=g(t)$. Write $t=s+k$, where $0\le s<1$ and k is an integer. So, using (2.15), equation (2.7) can be re-written as

$$n-\xi-(1+\ell)\,k-\ell s=g(s,w). \tag{3.1}$$

Setting $c=c_{n,k}=n-\xi-(1+\ell)\,k$ equation (3.1) takes the form

$$c-\ell s=g(s,w). \tag{3.2}$$

Since $g(\cdot,w)$ is increasing and maps $[0,1]$ onto $[0,1]$, equation (3.2) has a solution with $0\le s<1$ if and only if

$$0\le c<1+\ell. \tag{3.3}$$

Plugging $c=n-\xi-(1+\ell)\,k$ into (3.3) and solving for n completes the proof. □

The version of Lemma 3.5 we require is:

Lemma 3.6. *If $0\le a<b<1$ and j is an integer, then $j+a\le\lambda_n<j+b$ if and only if*

$$\xi+j\,(1+\ell)+\ell a+g(a,w)\le n<\xi+j\,(1+\ell)+\ell b+g(b,w).$$

Proof. Write (3.2) as $c=\ell s+g(s,w)$. Since the right hand side is an increasing function of s, it follows that (3.2) has a solution with $a\le s<b$ if and only if

$$\ell a+g(a,w)\le c<\ell b+g(b,w).$$

Plugging in $c=n-\xi-(1+\ell)\,j$ and solving for n completes the proof. □

3.1. **Irrational** ℓ

Lemma 3.7. *Suppose ℓ is irrational, then*

$$\lim_{k\to\infty}\frac{\textit{number of eigenvalues in [0,k+1[whose fractional part is in } [a,b[}{\textit{number of eigenvalues in [0,k+1[}}$$
$$=\frac{\ell(b-a)+g(b,w)-g(a,w)}{1+\ell}.$$

Proof. Pick $\delta > 0$ such that $b - a < \delta$ implies $\ell(b-a) + g(b,w) - g(a,w) < 1$,

It follows from Lemma 3.6 that $\lambda_n \in \bigcup_{j=0}^{k} [j+a, j+b[$ if and only if

$$n \in \bigcup_{j=0}^{k} [\xi + j\,(1+\ell) + \ell a + g(a,w), \xi + j\,(1+\ell) + \ell b + g(b,w)[$$
$$= \bigcup_{j=0}^{k} (\xi + j\,(1+\ell) + [\ell a + g(a,w), \ell b + g(b,w)[)\,.$$

Let $\gamma := \ell(b-a) + g(b,w) - g(a,w)$. The interval

$$\xi + j\,(1+\ell) + [\ell a + g(a,w), \ell b + g(b,w)[$$

can be written as a translate of the interval $[0,\gamma[$:

$$\xi + \ell a + g(a,w) + j\,(1+\ell) + [0,\gamma[\,. \tag{3.4}$$

Since $\gamma < 1$, we can consider this as intervals in $\mathbb{T}$ (The Reals modulo the integers). We need $\xi + \ell a + g(a,w) + j(1+\ell) \in [-\gamma, 0[$, so that 0 is in the interval in (3.4). Since ℓ is irrational the points $\xi + \ell a + g(a,w) + j(1+\ell)$ are uniformly distributed in $\mathbb{T}$. Hence

$$\frac{\#\{0 \le j \le k : \xi + \ell a + g(a,w) + j(1+\ell) \in \bigcup_{m=0}^{\infty}[m-\gamma, m[\}}{k+1} \to \gamma \tag{3.5}$$

as $k \to \infty$.

If follows from Lemma 3.5 that $0 \le \lambda_n < k+1$ if and only if

$$\xi \le n < \xi + (k+1)(1+\ell).$$

Hence, using that ℓ is irrational, it follows that the number of eigenvalues in $[0, k+1[$ is $\lfloor \xi + (k+1)(1+\ell) \rfloor$. Consequently, we have the equality

$$\frac{\text{number of eigenvalues in [0,k+1[whose fractional part is in } [a,b[}{\text{number of eigenvalues in [0,k+1[}}$$
$$= \frac{\#\{0 \le j \le k : \xi + \ell a + g(a,w) + j(1+\ell) \in \bigcup_{m=0}^{\infty}[m-\gamma, m[\}}{\left\lfloor \frac{1}{2} + (k+1)(1+\ell) \right\rfloor}$$
$$\to \frac{\gamma}{1+\ell} \text{ as } k \to \infty.$$

The convergence claim follows from (3.5). This completes the proof for $[a,b[$ with $b - a < \delta$.

Let $0 \le a < b < 1$. Then $[a,b[$ can be written as a finite disjoint union of intervals of the form $[c_i, c_{i+1}[$ where $a = c_1 < c_2 < \cdots < c_N = b$ and $c_{i+1} - c_i < \delta$, for $i = 1, 2, \ldots, N-1$. Then, for each i,

$$\frac{\text{number of eigenvalues in [0,k+1[whose fractional part is in } [c_i, c_{i+1}[}{\text{number of eigenvalues in [0,k+1[}}$$
$$\to \frac{\ell(c_{i+1} - c_i) + g(c_{i+1}, w) - g(c_i, w)}{1+\ell} \text{ as } k \to \infty.$$

Summing this over $i = 1, \ldots, N$ gives the desired result. $\square$

By Lemma 3.5 the number of eigenvalues in $[0, k+1[$ is

$$n_k := \lfloor \xi + (k+1)(1+\ell) \rfloor \tag{3.6}$$

and that the number of eigenvalues in $[0,k+1[$ whose fractional part is in $[a,b[$ is

$$\#\{0 \leq j \leq k : \xi + \ell a + g(a,w) + j(1+\ell) \in \bigcup_{m=0}^{\infty} [m-\gamma, m[\}.$$

Thus

$$A([a,b[,n_k) = \#\{0 \leq j \leq k : \xi + \ell a + g(a,w) + j(1+\ell) \in \bigcup_{m=0}^{\infty} [m-\gamma, m[\}.$$

Here $A([a,b[,n)$ is as in Definition 1.1 of [9], hence $A([a,b[,n)$ is the number of eigenvalue λ_j, $j=1,2,\dots,n$ whose fractional part is in $[a,b[$.It follows from Lemma 3.7

$$\lim_{k\to\infty} \frac{A([a,b[,n_k)}{n_k} = \frac{\ell(b-a) + g(b,w) - g(a,w)}{1+\ell}. \tag{3.7}$$

In particular, in the terminology of [9, p. 53],

$$\frac{\ell x + g(x,w)}{1+\ell}$$

is *a distribution function for* λ_n *modulo* 1. Dividing (3.7) by $b-a$ and letting $b \to a$ we get the density at a is

$$\frac{\ell + \partial_t g(a,w)}{1+\ell}.$$

The asymptotic distribution function of λ_n *modulo* 1 is

$$\lim_{n\to\infty} \frac{A([0,x[,n)}{n},$$

see [9].

Theorem 3.8. *If ℓ is irrational, then the asymptotic distribution function of λ_n modulo 1 is*

$$\frac{\ell x + g(x,w)}{1+\ell}.$$

In particular, λ_n is uniformly distributed modulo 1 if and only if $w=0$.

Proof. The function $A(n) = A([a,b[,n)$ is an increasing function of n, and n_k is an increasing function of k. Hence, if $n_k \leq n < n_{k+1}$, then

$$\frac{A(n_k)}{n_{k+1}} \leq \frac{A(n)}{n} \leq \frac{A(n_{k+1})}{n_k}. \tag{3.8}$$

By (3.6) , $\frac{n_{k+1}}{n_k} \to 1$ as $k \to \infty$. Consequently, the first claim follows from (3.8) and Lemma 3.7.

The second claim follows from the fact that $g(x,w) = x$ if and only if $w=0$. □

3.2. **Rational ℓ**

When ℓ is rational the sequence of fractional parts $(\langle\lambda_n\rangle)_{n\in\mathbb{Z}}$ is finite. This follows from (2.7) and the periodicity (2.15) of g.

Theorem 3.9. *If $\ell = \frac{p}{q,}$ where p,q are positive integers, then there is a set $\Gamma_0 = \{\gamma_k \mid 1 \leq k \leq p+q\}$ with $p+q$ elements $\gamma_k = \gamma_k(w,\ell,\xi)$, such that $0 \leq \gamma_k < q$ and*

$$\{\lambda_n(w,\ell,\xi) \mid n \in \mathbb{Z}\} = \Gamma_0 + q\mathbb{Z},$$

where $\Gamma_0 + q\mathbb{Z} := \{\lambda + qk \mid \lambda \in \Gamma_0, k \in \mathbb{Z}\}$.

Proof. By (2.7) λ_n is the solution to

$$n - \xi - \ell\lambda_n = g\left(\lambda_n, w\right).$$

Adding q to both sides and using (2.15) and $\ell q = p$, we arrive at

$$n + p + q - \xi - \ell\left(\lambda_n + q\right) = g\left(\lambda_n + q, w\right).$$

Consequently $\lambda_n + q = \lambda_{n+p+q}$. It follows from Corollary 4.3 that

$$\lim_{N\to\infty} \frac{\#\left\{n \mid 0 \leq \lambda_n < N\right\}}{N} = 1 + \ell.$$

Since, $1 + \ell = \frac{p+q}{q}$, it follows that Γ_0 has $p + q$ elements. $\square$

Corollary 3.10. *If $\ell = \frac{p}{q,}$ where p, q are positive integers, then the set of fractional parts*

$$\Lambda_w := \left\{\langle\lambda_n\rangle \mid n \in \mathbb{Z}\right\} \tag{3.9}$$

has at most $p + q$ elements.

Proof. It follows from Theorem 3.9 that

$$\Lambda_w = \left\{\langle\gamma\rangle \mid \gamma \in \Gamma_0\right\}$$

where the set Γ_0 has $p + q$ elements. Hence, the set $\Lambda_w = \left\{\langle\gamma\rangle \mid \gamma \in \Gamma_0\right\}$ has at most $p + q$ elements. $\square$

Lemma 3.11. *If $w = 0$ and $\ell = \frac{p}{q,}$ where p, q are positive integers with* $\gcd(p, q) = 1$, *then the set of fractional parts $\Lambda_0 = \left\{\langle\lambda_n(0)\rangle : n \in \mathbb{Z}\right\}$ is an arithmetic progression with $p + q$ elements, specifically:*

$$\Lambda_0 = \left\{\frac{\lceil q\xi\rceil - q\xi + k}{p + q} : k = 0, 1, \ldots, p + q - 1\right\}, \tag{3.10}$$

where $\lceil q\xi\rceil$ in the integer satisfying $\lceil q\xi\rceil - 1 < q\xi \leq \lceil q\xi\rceil$.

Proof. By Lemma 3.1 $\lambda_n = \lambda_n(0, \ell, \xi) = \frac{n-\xi}{1+\ell}$. Using $\ell = p/q$, we see that

$$\frac{n - \xi}{1 + \ell} = \frac{qn - q\xi}{p + q}.$$

Since p and q are relatively prime, it follows that $\lambda_n = \frac{qn-q\xi}{p+q}$ and $\lambda_m = \frac{qm-q\xi}{p+q}$ are congruent modulo one if and only if $n - m$ is a multiple of $p + q$. For the same reaon $\lambda_n - \lambda_m$ is an integer multiple of $\frac{1}{p+q}$. The claim follows from this. $\square$

Example 3.12. Let $\ell = \xi = 1$ so that $\Lambda_0 = \left\{0, \frac{1}{2}\right\}$. Note that $g\left(\lambda_n, w\right) = g\left(\lambda_n, 0\right)$ since each eigenvalue is a fixed point of g with respect to w. Thus, $\Lambda_w = \Lambda_0$ is an arithmetic progression for all $0 \leq w < 1$.

4. Bounds on the Eigenvalues

In this section we establishes several different bounds on the eigenvalues, exploring the dependence of the bounds on the parameters ℓ and w.

4.1. Dependence on ℓ

We first establish the dependence of λ_n on ℓ uniformly in w.

Proposition 4.1. *For $n \in \mathbb{Z}$,*

$$\frac{n-\xi}{1+\ell} - \frac{1}{2\,(1+\ell)} < \lambda_n < \frac{n-\xi}{1+\ell} + \frac{1}{2\,(1+\ell)}.$$

Hence, we have the approximation $\lambda_n(w,\ell,\xi) \approx \lambda_n(0,\ell,\xi)$ with error strictly less than $\frac{1}{2(1+\ell)}$.

Proof. Since $g(t,w)$ is increasing as a function of t, it follows from (2.11), (2.14) and (2.15) that

$$\frac{m}{2} \le t < \frac{m+1}{2} \implies \frac{m}{2} \le g(t,w) < \frac{m+1}{2} \tag{4.1}$$

for all $m \in \mathbb{Z} := \{\ldots,-1,0,1,\ldots\}$.

Fix n and pick m such that

$$\frac{m}{2} \le \lambda_n < \frac{m+1}{2}. \tag{4.2}$$

By (4.1)

$$\frac{m}{2} \le g(\lambda_n,w) < \frac{m+1}{2}. \tag{4.3}$$

Plugging (2.7) into (4.3) we get

$$\frac{m}{2} \le n-\xi-\ell\lambda_n < \frac{m+1}{2}. \tag{4.4}$$

Using (4.2) and (4.4) we get

$$|\lambda_n - (n-\xi-\ell\lambda_n)| < \frac{1}{2}.$$

Solving for λ_n establishes the desired inequalities. □

n	λ_n	$\left\lfloor \frac{n-\frac{1}{2}}{1+\ell} \right\rfloor$	$\left\lfloor \frac{n-\frac{1}{2}}{1+\ell} \right\rfloor + 1$	$\frac{n-1}{1+\ell}$	$\frac{n}{1+\ell}$
1	0.139315	0	1	0.	0.414214
2	0.66486	0	1	0.414214	0.828427
3	1.01957	1	2	0.828427	1.24264
4	1.43076	1	2	1.24264	1.65685
5	1.91813	1	2	1.65685	2.07107
6	2.21175	2	3	2.07107	2.48528
7	2.7542	2	3	2.48528	2.89949
8	3.06189	3	4	2.89949	3.31371
9	3.52874	3	4	3.31371	3.72792
10	3.96366	3	4	3.72792	4.14214

TABLE 1. For $\ell = \sqrt{2}, w = 1/2, \xi = 1/2$ the table shows eigenvalues λ_n, $n = 1,2,\ldots,10$ and the corresponding bounds from Proposition 3.2 and Proposition 4.1.

Example 4.2. Let $\xi = \frac{1}{2}$. Table 1 contains the eigenvalues λ_n, $n = 1,\ldots,10$ and the corresponding lower and upper bounds on λ_n from Proposition 3.2 and Proposition 4.1. Examining the line corresponding to $n = 5$ we see that Proposition 4.1 provides a larger lower bound for λ_5 and Proposition 3.2 provides a smaller upper bound. For $n = 8$, this is reversed: Proposition 3.2 provides a larger lower bound and Proposition 4.1 provides a smaller upper bound. In particular, neither proposition implies the other. $\square$

Corollary 4.3. *Let $a \in \mathbb{R}$, then* $\lim_{k\to\infty} \frac{\#\{n\in\mathbb{N}\,:\,a<\lambda_n<a+k\}}{k} = 1+\ell$.

Proof. Let $a \in \mathbb{R}$, and $a < \lambda_n < a + k$. We have from Proposition 4.1,

$$\frac{n-\xi}{1+\ell} - \frac{1}{2\,(1+\ell)} < \lambda_n < \frac{n-\xi}{1+\ell} + \frac{1}{2\,(1+\ell)},$$

thus

$$\frac{n-\xi}{1+\ell} - \frac{1}{2\,(1+\ell)} < a+k \implies n < a(1+\ell) + k(1+\ell) + \frac{1}{2} + \xi.$$

It also follows that

$$a < \frac{n-\xi}{1+\ell} + \frac{1}{2\,(1+\ell)} \implies a(1+\ell) - \frac{1}{2} + \xi < n,$$

thus

$$a(1+\ell) - \frac{1}{2} + \xi < n < a(1+\ell) + k(1+\ell) + \frac{1}{2} + \xi.$$

Hence

$$\frac{k(1+\ell)-1}{k} \le \frac{\#\left\{n\in\mathbb{N}\,:\,a<\lambda_n<a+k\right\}}{k} \le \frac{k(1+\ell)+1}{k},$$

thus taking the limit we arrive at

$$1+\ell \le \lim_{k\to\infty} \frac{\#\left\{n\in\mathbb{N}\,:\,a<\lambda_n<a+k\right\}}{k} \le 1+\ell.$$

Hence

$$\lim_{k\to\infty} \frac{\#\left\{n\in\mathbb{N}\,:\,a<\lambda_n<a+k\right\}}{k} = 1+\ell$$

$\square$

4.2. **Separation of Eigenvalues**

If $0 \le w < 1$, Proposition 4.1 shows that $\lambda_n < \lambda_{n+1}$ for all integers n. But does not give an estimate of the size of $\lambda_{n+1} - \lambda_n$. The purpose of this section is to provide a lower bound $\delta > 0$ for $\lambda_{n+1} - \lambda_n$.

Proposition 4.4. *For all ℓ and all $w < 1$, we have*

$$\frac{1}{g_t(0,w)+\ell} \le \lambda_{n+1} - \lambda_n \tag{4.5}$$

for all integers n and all ξ.

Proof. Subtracting

$$n - \xi - \ell\lambda_n \quad = \quad g\left(\lambda_n, w\right)$$

from

$$(n+1) - \xi - \ell\lambda_{n+1} = g\left(\lambda_{n+1}, w\right)$$

and applying the Mean Value Theorem, we get

$$\begin{aligned} 1 - \ell\left(\lambda_{n+1} - \lambda_n\right) &= g\left(\lambda_{n+1}, w\right) - g\left(\lambda_n, w\right) \\ &= g_t\left(s, w\right)\left(\lambda_{n+1} - \lambda_n\right) \end{aligned}$$

where $\lambda_n < s < \lambda_{n+1}$. Hence,

$$\lambda_{n+1} - \lambda_n = \frac{1}{g_t(s,w)+\ell} \geq \frac{1}{g_t(0,w)+\ell},$$

since $g_t(t,w) \leq g_t(0,w)$ for all t. □

Remark 4.5. (a) It follows from (2.5) that

$$g_t(0,w) = \frac{1-w^2}{(w-1)^2} = \frac{1+w}{1-w}.$$

(b) Since $g'(0,w) \to \infty$ as $w \nearrow 1$ the lower bound in (4.5) goes to zero as $w \nearrow 1$. This is necessary since, when $w = 1$, some eigenvalues may have multiplicity 2, i.e, for some n, we may have $\lambda_{n+1} = \lambda_n$.

(c) Similarly, the lower bound in (4.5) goes to zero as $\ell \to \infty$. It follows from Proposition 4.1 that this is necessary.

(d) Consider rational ℓ and irrational ξ. Then $\{\langle\lambda_n\rangle : n \in \mathbb{Z}\}$ is finite and the fractional parts $\langle\lambda_n(1-)\rangle \neq 0$ for all n. Where $\lambda_n(1-)$ is as in Theorem 4.6. It follows from Theorem 4.6, see Figure 4.1, there is $\delta > 0$, depending on ξ and on ℓ, such that $\delta \leq \lambda_{n+1} - \lambda_n$ for all n and all w. □

4.3. **Dependence on** w

We establish the qualitative behavior of λ_n as a function of w.

Theorem 4.6. *Fix ℓ and ξ. Consider $\lambda_n(w) = \lambda_n(w,\ell,\xi)$ as a function of $0 \leq w < 1$. Recall, $\lambda_n(0) = \frac{n-\xi}{1+\ell}$. Consider the closed interval*

$$I_n := \left[\lambda_n(0) - \tfrac{1}{2(1+\ell)}, \lambda_n(0) + \tfrac{1}{2(1+\ell)}\right]. \tag{4.6}$$

If $\langle\lambda_n(0)\rangle \in \left\{0, \frac{1}{2}\right\}$, let $\lambda_n(1-) := \lambda_n(0)$. If $\langle\lambda_n(0)\rangle \notin \left\{0, \frac{1}{2}\right\}$ there are three cases:

(i) if $I_n \cap \mathbb{Z} = \emptyset$ let $\lambda_n(1-) = \lambda_n(0) + \ell^{-1}\left(\langle\lambda_n(0)\rangle - \frac{1}{2}\right)$,

(ii) if $I_n \cap \mathbb{Z} \neq \emptyset$ and $0 < \langle\lambda_n(0)\rangle < \frac{1}{2}$, let $\lambda_n(1-) := \lfloor\lambda_n(0)\rfloor$, and

(iii) if $I_n \cap \mathbb{Z} \neq \emptyset$ and $\frac{1}{2} < \langle\lambda_n(0)\rangle < 1$, let $\lambda_n(1-) := 1 + \lfloor\lambda_n(0)\rfloor$.

Let J_n be the half-open interval with endpoints $\lambda_n(0)$ and $\lambda_n(1-)$ that contains $\lambda_n(0)$ but not $\lambda_n(1-)$. If $\langle\lambda_n(0)\rangle \in \left\{0, \frac{1}{2}\right\}$, then $\lambda_n(w) = \lambda_n(0)$ for all $0 \leq w < 1$. If $\langle\lambda_n(0)\rangle \notin \left\{0, \frac{1}{2}\right\}$, then $\lambda_n(w)$ is a continuous monotone function mapping the half-open interval $[0,1[$ onto J_n. Furthermore, as $w \nearrow 1$ the eigenvalues $\lambda_n(w) \to \lambda_n(1-)$ uniformly in $n \in \mathbb{Z}$.

Sketch of Proof. We sketch the proof leaving some details for the reader. Recall, $\lambda_n(0) = \frac{n-\xi}{1+\ell}$, by Lemma **??**.

It follows from the fixed point properties (2.11), (2.14) and (2.15) and monotonicity properties of g, see Remark 2.8, that

- (I) If $\left\langle\frac{n-\xi}{1+\ell}\right\rangle \in \left\{0, \frac{1}{2}\right\}$, then $\lambda_n(w) = \frac{n-\xi}{1+\ell}$ for all w in $[0,1)$.
- (II) If $0 < \left\langle\frac{n-\xi}{1+\ell}\right\rangle < \frac{1}{2}$, then $\lambda_n(w)$ is a strictly decreasing function mapping the half-open $[0,1[$ into the half-open interval $\left]\left\lfloor\frac{n-\xi}{1+\ell}\right\rfloor, \frac{n-\xi}{1+\ell}\right]$.
- (III) If $\frac{1}{2} < \left\langle\frac{n-\xi}{1+\ell}\right\rangle < 1$, then $\lambda_n(w)$ is a strictly increasing function mapping the half-open interval $[0,1[$ into the half-open interval $\left[\frac{n-\xi}{1+\ell}, \left\lfloor\frac{n-\xi}{1+\ell}\right\rfloor + 1\right[$.

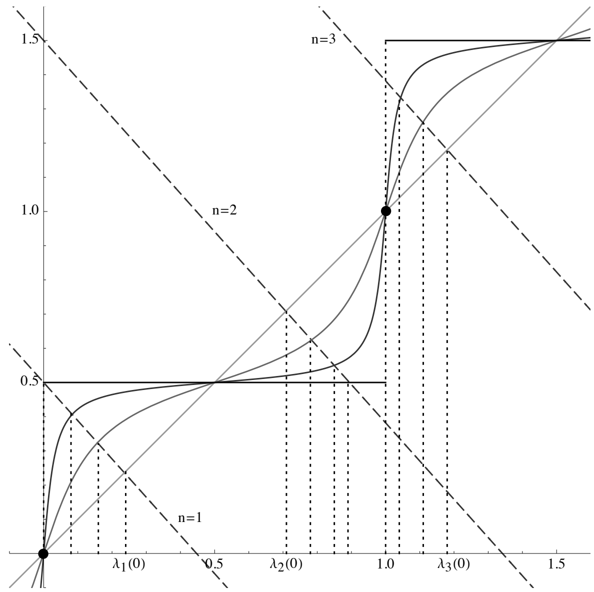


FIGURE 4.1. $g(t,w)$ as a function of t, for $w = 0, 0.5, 0.85, 1$. The lines $n - \xi - \ell t$, $\xi = \frac{1}{2}$,$\ell = \frac{\sqrt{5}}{2}$, $n = 1, 2, 3$. The t–coordinates of the intersections are "dropped" onto the t–axis, illustrating how the eigenvalues λ_n depend on w. This is a combination of Figure 2.2 and Figure 2.1.

See Figure 4.1.

The stated range for $\lambda_n(w)$ is optimal, unless the line $L(t) := n - \xi - \ell t$ passes through one of the horizontal lines in the graph of $g(\cdot, 1)$. Thus the range for $\lambda_n(w)$ stated above is optimal if and only if

$$k - \tfrac{1}{2} \leq n - \xi - \ell k \leq k + \tfrac{1}{2}, \tag{4.7}$$

for some integer k.

Suppose $L(t)$ passes through one of the horizontal line segments in the graph of $g(\cdot, 1)$. Then that line segment is

$$\left\{ \left(t, \left\lfloor \tfrac{n-\xi}{1+\ell} \right\rfloor + \tfrac{1}{2}\right) : \left\lfloor \tfrac{n-\xi}{1+\ell} \right\rfloor < t < \left\lfloor \tfrac{n-\xi}{1+\ell} \right\rfloor + 1 \right\}.$$

Solving $L(t) = \left\lfloor \frac{n-\xi}{1+\ell} \right\rfloor + \frac{1}{2}$ for t, shows that $\lambda_n(1-) = \ell^{-1} \left(n - \xi - \left\lfloor \frac{n-\xi}{1+\ell} \right\rfloor - \frac{1}{2} \right)$.

For $0 \leq w < 1$, let $C_w := \{\{t, g\{t, w\} : t \in \mathbb{R}\}$, $0 \leq w < 1$, of $g(\cdot, w)$. Let C be the graph of curve that is the union of the graph $\{(t, g(t, 1) : t \in \mathbb{R}\}$ of $g(\cdot, 1)$ and the vertical line segments $\{k\} \times \left[k - \frac{1}{2}, k + \frac{1}{2}\right]$. The graphs C_w converge uniformly to the curve C as $w \nearrow 1$. This follows, for example, from Dini's Theorem by rotating the graphs C_w 45 degrees in the clockwise direction, see Figure 4.2. Consequently, $\lambda_n(w)$ converges uniformly (in n) to $\lambda_n(1-)$ as $w \nearrow 1$. A similar argument shows that $w \to \lambda_n(w)$ is continuous (uniformly in n) at any $0 \leq w_0 < 1$. $\square$

When $w = 1$ it follows from Remark 2.4 that the spectrum of P is the union of $\mathbb{Z}$ and $\frac{1-2\xi}{2\ell} + \frac{1}{\ell}\mathbb{Z}$.

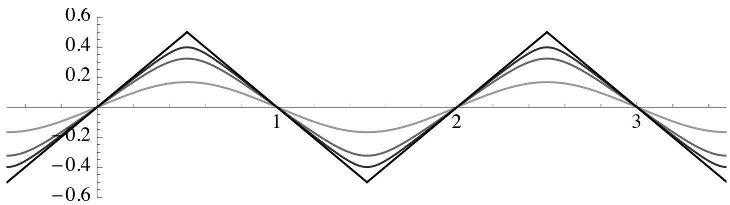


FIGURE 4.2. Essentially this is Figure 2.2 rotated 45 degrees in the clockwise direction. The gaps in the original curve with $w = 1$ have been filled in by straight lines.

Corollary 4.7. *The sequence* $\lambda_n(1-)$, $n \in \mathbb{Z}$ *coincides with the spectrum of* $P_{w=1}$.

Proof. The limit curve $g(t, 1-)$, see Figure 4.1 and Figure 4.2 is the disjoint union of the horizontal line segments

$$H_k := \left\{ \left(t, \frac{2k-1}{2}\right) \mid k \le t < k+1 \right\}, k \in \mathbb{Z}$$

and the vertical line segments

$$V_k := \left\{ (k - \phi, y) \mid k - \frac{1}{2} \le y < k + \frac{1}{2} \right\}, k \in \mathbb{Z}.$$

Consider the lines

$$L_n := \{(t, n - \xi - \ell t) \mid t \in \mathbb{R}\}.$$

It follows from the proof of Theorem 4.6, that the set $\Lambda^- := \{\lambda_n(1-) \mid n \in \mathbb{Z}\}$ is the collection of first coordinates in the intersections

$$L_n \cap H_k, L_n \cap V_k, \quad n, k \in \mathbb{Z}.$$

We must show $\Lambda^- = \Lambda_1 \cup \Lambda_2$, where $\Lambda_1 := \mathbb{Z}$ and $\Lambda_2 := \frac{1-2\xi}{2\ell} + \frac{1}{\ell}\mathbb{Z}$.

Each vertical line segment V_k intersects exactly one of the lines, since the lines L_n and L_{n+1} are separated by a vertical distance of one and the vertical line segments are half-open. It follows that the intersection

$$\left(\bigcup_{k \in \mathbb{Z}} V_k \right) \cap \left(\bigcup_{n \in \mathbb{Z}} L_n \right) = \{(j, y_j) : j \in \mathbb{Z}\}.$$

For some y_j. Hence, the intersections $V_k \cap L_n$ gives us Λ_1.

It remains to show that the set of first coordinates of the intersections $H_k \cap L_n$, $n, k \in \mathbb{Z}$, coincided with Λ_2. If n and k are integers such that $H_k \cap L_n$ is non-empty, then

$$H_k \cap L_n = \left\{ \left(\lambda, \frac{2k-1}{2} \right) \right\},$$

where λ satisfies the equation $n - \xi - \ell\lambda = \frac{2k-1}{2}$. Hence,

$$\lambda = \frac{1-2\xi}{2\ell} + \frac{1}{\ell}(n-k).$$

Consequently, the set of first coordinates of the intersections $H_k \cap L_n$ is a subset of Λ_2. Conversely, let

$$\lambda := \frac{1-2\xi}{2} + m$$

for some integer m. We saw in the proof of Theorem 4.6 that L_n intersects H_k if and only if $k = \left\lfloor \frac{n-\xi}{1+\ell} \right\rfloor$. Hence, to complete the proof, we must show

$$m = n - \left\lfloor \frac{n-\xi}{1+\ell} \right\rfloor$$

has a solution $n = n(m) \in \mathbb{Z}$ for any $m \in \mathbb{Z}$. But this follows from $1 + \ell > 1$. □

Remark 4.8. The proof of Corollary 4.7 shows that the the intersections of the lines L_n with the vertical line segments V_k corresponds to the spectrum of $P_{w=1}$ on $[0, 1]$ and the intersections of the lines L_n with the horizontal line segments H_k corresponds to the spectrum of $P_{w=1}$ on $[1, 1 + \ell]$.

5. Series Expansion of the Eigenvalues

We establish a series expansion $\lambda_n = \sum_{k=1}^{\infty} a_k w^k$ with $|a_k| < 1/\pi k$. Let

$$\widetilde{g}_w(t) := g(t, w) - t \text{ and}$$
$$h_{\ell,w}(t) := (1 + \ell)t + \widetilde{g}_w(t).$$

Lemma 5.1. *$h_{\ell,w}$ is a strictly increasing function mapping $\mathbb{R}$ onto itself and*

$$h_{\ell,w}(t + 1) = 1 + \ell + h_{\ell,w}(t) \tag{5.1}$$

for all $t \in \mathbb{R}$, $0 < \ell$, $0 \le w < 1$.

Proof. It follows from (2.13) that $h_{\ell,w}$ is strictly increasing. By (2.8) $\widetilde{g}_w$ has period one, consequently (5.1) holds. □

Lemma 5.2. *By Lemma 5.1 $h_{\ell,w}$ has an inverse function. Let*

$$p_{\ell,w}(t) := h^{-1}_{\ell,w}(t) - \tfrac{t}{1+\ell}.$$

Then $p_{\ell,w}$ has period $1 + \ell$ and

$$p_{\ell,w}(t) = -\frac{\widetilde{g}_w\left(h^{-1}_{\ell,w}(t)\right)}{1+\ell}$$

for all $t \in \mathbb{R}$, $0 < \ell$, $0 \le w < 1$.

Proof. Fix a real number t. Then

$$\begin{aligned} t &= h^{-1}_{\ell,w}\left(h_{\ell,w}(t)\right) \\ &= \frac{h_{\ell,w}(t)}{1+\ell} + p_{\ell,w}(h_{\ell,w}(t)) \\ &= \frac{(1+\ell)t + \widetilde{g}_w(t)}{1+\ell} + p_{\ell,w}(h_{\ell,w}(t)). \end{aligned}$$

Hence,

$$p_{\ell,w}(h_{\ell,w}(t)) = -\frac{\widetilde{g}_w(t)}{1+\ell}. \tag{5.2}$$

Similarly, using $\widetilde{g}_w$ has period one and (5.1)

$$\begin{aligned} t + 1 &= h^{-1}_{\ell,w}\left(h_{\ell,w}(t+1)\right) \\ &= \frac{h_{\ell,w}(t+1)}{1+\ell} + p_{\ell,w}(h_{\ell,w}(t+1)) \\ &= \frac{(1+\ell)(t+1) + \widetilde{g}_w(t+1)}{1+\ell} + p_{\ell,w}(h_{\ell,w}(t+1)) \end{aligned}$$

$$= \frac{(1+\ell)(t+1) + \widetilde{g}_w(t)}{1+\ell} + p_{\ell,w}(1+\ell+h_{\ell,w}(t)).$$

And therefore,

$$p_{\ell,w}(1+\ell+h_{\ell,w}(t)) = -\frac{\widetilde{g}_w(t)}{1+\ell}. \tag{5.3}$$

Comparing (5.2) and (5.3) and using Lemma 5.1 it follows that $p_{\ell,w}$ has period $1+\ell$. □

Theorem 5.3. *The following expansion*

$$\lambda_n(w,\ell,\xi) = \frac{n-\xi}{1+\ell} + \sum_{k=1}^{\infty} a_k w^k$$

holds, where $a_k = a_k(n,w,\ell,\xi)$ satisfies the estimate

$$|a_k| \leq \frac{1}{\pi k}$$

for all $k \in \mathbb{N}$, $n \in \mathbb{Z}$, $0 \leq w < 1$, $0 < \ell$, and $\xi \in \mathbb{R}$.

Proof. The eigenvalue equation (2.7) can be written as

$$\lambda_n = \lambda_n(w,\ell,\xi) = h_{\ell,w}^{-1}(n-\xi).$$

Hence, it follows from Lemma 5.2 that

$$\lambda_n = \frac{n-\xi}{1+\ell} - \frac{\widetilde{g}_w\left(h_{\ell,w}^{-1}(n-\xi)\right)}{1+\ell}. \tag{5.4}$$

By (2.8)

$$\widetilde{g}_w(t) = \sum_{k=1}^{\infty} w^k \frac{\sin(2\pi k t)}{\pi k}. \tag{5.5}$$

Since $|\sin(t)| \leq 1$, the proof is easily completed. □

Remark 5.4. Equation (5.4) can be written as

$$\lambda_n = \frac{n-\xi}{1+\ell} + p_{\ell,w}(h_{\ell,w}(n-\xi))$$

Hence, Proposition 4.1 essentially amounts to the estimate

$$|p_{\ell,w}(t)| < \frac{1}{2\,(1+\ell)}$$

for all $\ell > 0$, $0 \leq w < 1$, and all $t \in \mathbb{R}$.

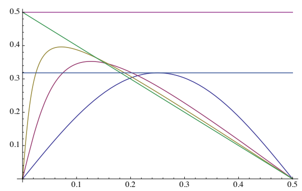
Remark 5.5. Let $\widetilde{h}_w(t) := \widetilde{g}_w(t)/w$. By (5.5), h_w is an odd function with period one. Furthermore, $\widetilde{h}_0(t) = \lim_{w\searrow 0} \widetilde{h}_w(t) = \sin(2\pi t)/\pi$ for all t and $\widetilde{h}_1(t) = \lim_{w\nearrow 1} \widetilde{h}_w(t) = \frac{1}{2} - t$ for $0 \leq t \leq \frac{1}{2}$.

Let

$$\psi(w) := \max_{t\in\mathbb{R}} \left|\widetilde{h}_w(t)\right|.$$

Figure 5.1 shows that ψ is an increasing function of $w \in]0,1]$ with range contained in $]1/\pi, 0.5]$ and $\psi(1) = 0.5$. Leading to an improvement of the bounds in Proposition 4.2 for $w < 1$, in particular, to bounds that depend on w. More precisely, we get

$$\frac{n-\xi}{1+\ell} - \frac{w\psi(w)}{1+\ell} < \lambda_n(w,\ell,\xi) < \frac{n-\xi}{1+\ell} + \frac{w\psi(w)}{1+\ell},$$



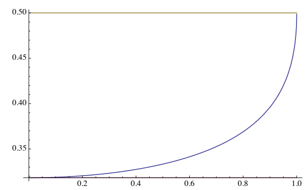


FIGURE 5.1. Plots of $\widetilde{h}_w$ for $w = 0, 0.7, 0.9, 1$ and of ψ.

for all $0 < w \leq 1$, and all n.

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