Design of Thermal Structures using Topology Optimization

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DESIGN OF THERMAL STRUCTURES USING
TOPOLOGY OPTIMIZATION

A dissertation submitted in partial fulfillment of the
requirements for the degree of
Doctor of Philosophy

By

JOSHUA DAVID DEATON
B.S.M.E., Wright State University, 2009

2014
Wright State University
I HEREBY RECOMMEND THAT THE DISSERTATION PREPARED UNDER MY SUPERVISION BY Joshua David Deaton ENTITLED Design of Thermal Structures using Topology Optimization BE ACCEPTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF Doctor of Philosophy

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Deaton, Joshua. Ph.D. in Engineering Program, Wright State University, 2014.
*Design of Thermal Structures using Topology Optimization.*

The design of structures subjected to elevated temperature environments has long been an important area of study in the aerospace industry. This is especially true in the modern day, where new problems related to embedded engine aircraft and high temperature exhaust-washed structures present new structural design challenges not found in past applications. In this work, the response of a class of thermal structures whose responses are characterized by significant amounts of restrained expansion, to which exhaust-washed structures belong, are studied. To address the complex design challenges that become evident in these investigations, structural topology optimization is applied due to its unique ability to identify optimal material layout. Since conventional methods for topology optimization fail to generate effective designs in the presence of thermoelastic effects, new formulations for thermoelastic topology optimization are demonstrated. These include techniques for addressing the amount of reaction loading generated by a structural concept and methods for incorporating stress-based design criteria in topology optimization problems with design-dependent thermal loading. When taken together, the developments in this work provide a design technique in which stresses can be directly treated in thermal structures by identifying the proper arrangement of structural components in a thermal environment.
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Acknowledgments

I would like to thank my advisor Dr. Ramana Grandhi for his mentoring and guidance throughout my graduate studies. The opportunities and experiences I have had working as his graduate student have allowed me to grow as an academic, a researcher, and as a person and instilled the confidence to continue to pursue my goals in the field of engineering and technology.

I would also like to thank the members of my dissertation committee: Dr. Scott Thomas, Dr. Ha-Rok Bae, Dr. Jack McNamara, and Dr. Ed Alyanak. Their insightful comments have no doubt increased the quality of this work. In addition, their sincere encouragement was a source of motivation when my research direction was not always clear.

A special thanks is also due to the Multidisciplinary Science and Technology Center (MSTC) at the Air Force Research Laboratory (AFRL) for sponsoring my graduate studies and providing a valuable forum for me to present my work to aerospace industry and government partners. In addition, I am grateful for my research colleagues at Virginia Tech, including Darcy Allison and Craig Morris, who shared the ups and downs of graduate school alongside me.

The faculty and staff at Wright State University also deserve a great deal of credit for my success. Without the expertise of the faculty and their instruction, I would not have developed the technical aptitude to complete this work and the staff has been extraordinarily helpful in all my dealings. Special thanks are extended to Dr. Nathan Klingbeil for the valuable opportunities outside of research and Alysoun Taylor-Hall who helped me navigate the program requirements and kept me on track.

I also could not have completed this work without my friends, both within the CEPRO research group and outside of WSU. To everyone outside, I look forward to making up for all the invitations I turned down due to “school stuff.” To my colleagues in CEPRO, thank you for the support, discussions, and insight into different cultures.
and ideals. I wish you all the best in your own studies.

Finally, I would like to thank my family for their confidence in me and unwavering support throughout my years in higher education. I will be forever grateful for sacrifices that you have made in order to help me reach this goal.
Chapter 1

Introduction

1.1 Motivation

The motivation for the research presented in this dissertation is twofold. First, a new class of aerospace thermal structures, or structures subjected to thermal environments, related to embedded engine aircraft and sustained hypersonic flight has made evident a lack of appropriate design tools and methodologies. In these cases, the requirements on platform-level capabilities demand improved performance from thermally loaded structures and create new thermoelastic design scenarios that are neither well explored in the literature nor well addressed with existing design tools. In fact, application of conventional design practices may well lead to inherently flawed designs. Second, within the multidisciplinary optimization community, topology optimization has become the most active area of research; however, the majority of applications have been restricted to problems of purely mechanical context. This is despite the fact that the fundamental strengths of topology optimization indicate it would be a valuable design tool for a number of alternative and multiphysics applications, including thermoelasticity. To this end, the research contained in this dissertation aims to investigate and develop topology optimization formulations suit-
able for thermal-structural design environments. These developments have then been demonstrated by application to thermal structures design problems, including engine exhaust-washed structures, to develop structural systems with improved thermoelastic performance.

1.2 Dissertation Organization

The remaining sections of Chapter 1 provide a short historical perspective and literature review on thermal structures for aerospace applications and introduce the design application of interest, which are here referred to as engine exhaust-washed structures. To better understand the unique thermal structures design that must be addressed, Chapter 2 investigates some fundamental thermoelastic responses (using thermal and structural finite element analysis) of both exhaust-washed structures and characteristic thermal structures. In addition, some important observations regarding the nature of geometric nonlinearity in thermoelastic design problems are highlighted. Chapter 3 gives a literature review of topology optimization, which is proposed to address the unique design challenges posed by modern day thermal structures, including methods, basic formulations, and topics relevant to the developments in this work. Chapter 4 presents a formulation for thermoelastic topology optimization, including its novel capabilities, that has been developed for thermal structures design. The application of the formulation to two exhaust-washed structure inspired examples is detailed in Chapter 5 using novel problem formulations. The discussions in Chapter 6 expand the topology optimization framework in Chapters 4 and 5 to include stress-based design criteria. This includes a new technique to efficiently enforce local stress limits in topology optimization, and for the first time, demonstrates the ability for stress-based topology optimization considering thermal stresses. Finally, Chapter 7 concludes the dissertation with summary remarks and guidance for future work, which is related
to incorporating heat transfer analysis into the topology optimization formulation and utilizing the thermoelastic capabilities in multiphysics design problems including buckling and vibration, both of which are also important considerations in exhaust structure design.

1.3 Historical Perspective on Thermal Structures

The design of aerospace structures subjected to elevated temperature environments has been a critical area of research since the advent of supersonic flight in the late 1940’s [1]. As flight speed increased, designers realized that the elevated temperatures resulting from high speed aerodynamic phenomena and their effects on aircraft structural performance could place a “thermal barrier” on supersonic flight. In response, a new area of research emerged, known as thermal structures, to help overcome this barrier with advances in aerospace materials and improved structural designs. The thermal structures field focuses on both the transfer of thermal energy (heat transfer) throughout a structure in addition to the mechanical effects of reduced or elevated temperatures and spatial or temporal temperature gradients on structural components. Fundamentally, these effects are related to two primary considerations. First, the properties of most engineering materials are dependent on temperature. Thus the presence of a thermal environment may reduce structural performance by influencing a components behavior at the material level. Second, elevated or reduced temperatures induce thermal expansion or contraction in structures that, if restrained by boundary fixivity or spatial temperature gradients, leads to thermal stresses, buckling, and other detrimental thermoelastic effects. Since its inception, the thermal structures field has evolved to address these issues alongside varying applications and challenges in the aerospace, defense, energy, and manufacturing industries.

The foundation for the study of thermal structures is well developed in two classi-
cal texts written in the 1950’s. The first, by Boley and Weiner [2], gives a comprehensive mathematical treatment, beginning with the coupled governing energy equation for heat transfer and elastic equilibrium equations, for the case of linear thermoelasticity. The need for a coupled solution is discussed and guidelines for when a weak coupling of the thermal and structural systems may be suitable are provided. Analytical solutions are presented for a variety of structural elements including straight and curved beams, trusses and frames, flat plates and curved shells, and linear thermal buckling for various load and boundary conditions. A limited treatment of inelastic (material nonlinear) thermal structures is also given.

The second text, by Gatewood [3], approaches the topic from a more applied standpoint, especially for aerospace components. Analytical and semi-analytical solutions are provided for basic geometry as in [2], but additional discussion focuses on assembled structures such as skin-stringer combinations and joints in two dimensions. Common to both texts are extensive references to research works that give a flavor of early challenges in the thermal structures field, which at this time stemmed primarily from supersonic flight.

A more recent text, by Thornton [1], highlights the evolution of aerospace thermal structures applications. Presentation spans from the time of [2, 3], through increased supersonic flight speeds in the 1950s, space applications including deployable space structures, atmospheric reentry vehicles, and the Space Shuttle in the 1960s-1970s, to the National Aerospace Plane (NASP) in the late 1980s. In addition, sufficient discussion of mathematical foundations and additional references are provided. Some applications highlighted in [1] along with some other aerospace thermal structures are shown in Figure 1.1.

Thornton [1] also covers two additional topics that are especially important to modern thermal structures problems and only became possible with improvements in computational capacity. The first, geometric nonlinearity in thermal structures, is
addressed with the derivation and presentation of approximate solutions to large displacement beam and shell structures. The second, and perhaps most important topic to all modern day structural analysis, is the introduction of discretized numerical analysis methods, including the finite element method (FEM). A concise discussion of the finite element formulation for quasi-steady (or weakly/sequentially coupled) thermal-structural analysis, including conduction, free convection, radiation, and linear elasticity, is given along with early applications including built-up plate and frame structures.

Advances in finite element analysis techniques gradually increased its acceptance
in the thermal structures community. In recent years, capabilities for multiphysics thermal-structural \textit{analysis}, including chained analysis and mixed-element formulations, have been well matured in commercial finite element programs such as Abaqus, ANSYS, and NASTRAN. In addition, renewed interest in hypersonic flight platforms have driven new developments in thermal structures related to aerothermoelasticity and fluid-structure interaction [4] where the prediction of the elevated temperature environment itself is more challenging than legacy applications.

The introduction of finite element analysis to thermal structures also enabled the application of automated design and optimization methods practiced by the structural and multidisciplinary design optimization (MDO) community. Early works in the structural synthesis of thermal structures include an optimality criteria method for minimum-mass problems with temperature constraints [5], a fully stressed design (FSD) technique for minimum-mass sizing of thermoelastic structures [6], and approximation methods for the combined thermal-structural problem [7, 8]. More recently, improved sensitivity analysis formulations have been developed [9, 10], design-oriented thermostructural analysis frameworks for sizing and shape optimization problems have been demonstrated [11, 12], and limited applications are found in topology optimization primarily for thermal micro-mechanical actuation (as discussed further in Chapter 3). It is important to note that to date the majority of thermoelastic optimization has been only academic work. While some commercial structural optimization software does have the capability to solve simple linear steady-state thermal-structures problems, their automated design capabilities generally lag far behind those for analysis.

In summary, a review of the literature regarding the \textit{design} of thermal structures for elevated temperature applications (dating back to the 1950s) yields two basic design rules: (1) minimize temperatures and gradients and (2) accommodate thermal expansion. Related to the second rule, it is often evident that allowing even small
levels of expansion can altogether eliminate the most damaging thermoelastic effects [13]. The application of this rule is relatively commonplace, with the placement of expansion joints in concrete structures such as bridges and sliding attachments for copper plumbing in addition to aerospace structures subjected to more severe environments. However, one application where these basic rules cannot be readily applied is in the area of embedded engine integration for modern military aircraft. In this case, the configuration-level design requirements necessary for increased performance and tactical capability supersede the conventional thermal structures design wisdom.

1.4 Engine Exhaust-Washed Structures

To meet the growing demands for increased mission capability, combat survivability, and versatility of aerospace systems, future military aircraft will continue to rely on low observable (LO) technology. A critical component of this technology is embedded engine integration, as seen on the current B-2 Spirit stealth bomber and the Efficient Supersonic Air Vehicle (ESAV) concept shown in Figure 1.2. This configuration, in which engines are buried inside the aircraft, allows for a smooth outer mold line (OML), reduced exhaust noise, and cooler exhaust gases, which all reduce the vehicle’s observability by decreasing radar, acoustic, and infrared detectability [14]. In addition, by utilizing a ducted exhaust system to pass exhaust gases to the rear of the aircraft, direct line-of-sight into hot engine components is prevented, denying the enemy a vulnerable infrared target. This is demonstrated in Figure 1.3 with a simplified notional cross-section of an embedded engine configuration with ducted exhaust. While such a configuration affords tremendous tactical capabilities, it also comes with increased structural design complexity. In a legacy aircraft, where engines are located either under the wings or in the aft fuselage, high temperature exhaust gases are expelled directly into the airstream. In the case of embedded engines, the
hot exhaust creates an extreme thermal-structural design environment as it is ducted to the rear of the aircraft. In this environment, damaging effects of elevated temperatures including excessive deformations, thermal stresses, thermal buckling, and creep may occur.

(a) B-2 Spirit stealth bomber  
(b) Efficient Supersonic Air Vehicle (ESAV) concept

Figure 1.2: Military aircraft featuring embedded engine configurations and ducted exhaust.

Figure 1.3: Cross-section of a notional embedded engine configuration showing ducted exhaust path.

The structural components located aft of the engines on an embedded engine aircraft that make up the ducted exhaust path are known as engine exhaust-washed structures (EEWS) [15]. In particular, exhaust-washed structures (EWS) include the ducted nozzle (internal to the aircraft), the aft-deck (exhaust-washed regions on the exterior of the vehicle), and adjoining substructures. These components are subjected to a multidisciplinary environment where both thermal and structural loading produce a combined response. Thermal loads include transient elevated temperature from the
exhaust flow that may be in excess of 1000°F and additional aerodynamic effects, both of which are driven largely by the aircraft’s mission envelope and operating environment. The temperature loading is combined with pressure loading from the exhaust flow, mechanical loading reacted from adjoining airframe structure, and wideband acoustic loading. In this document we restrict consideration primarily to loading of thermal origin.

One of the significant challenges associated with the design of EWS is the dominance of geometric design criteria. In a typical configuration, like that shown in Figure 1.4, the geometric shape of the exhaust flow path is fixed by propulsion system efficiency and observability design considerations. In addition, all components must be smoothly integrated within the aircraft’s OML. This places strict geometric shape and fixivity on structural components that are subject to elevated temperatures. Undoubtedly then, these components are subject to restrained expansion and are susceptible to thermal buckling and thermal stresses. These problems were observed on an exhaust-washed structure on the B-2 bomber known as the aft-deck. In this case, thermal stresses produced cracking and failure in the aft-deck within only a small percentage of its intended life, necessitating costly replacement and retrofitting [16].

Figure 1.4: Conceptual engine exhaust-washed structure (EEWS) configuration located aft of embedded engines and its primary components.
Since future embedded engine concepts share similar configurations to the B-2, it can be expected they will face similar challenges, which are also common to hypersonic flight thermal protection applications. In all cases, structural designers face considerable challenges related to the tradespace between the need to accommodate thermal expansion effects and the configuration constraints related to the vehicle objectives that supersede doing so.

Multidisciplinary and structural design optimization techniques are particularly well suited to addressing this type of design scenario. These techniques can manage multiple competing design responses to arrive at a configuration with improved performance compared to a reference design (which may not represent a feasible solution itself) [17]. As a result, a particular type of structural optimization, called topology optimization, is investigated in this document due to some of its fundamental capabilities, which will be described in detail, that make it attractive for the design problems faced with modern thermal structures.
Chapter 2

Characteristic Responses of Thermal Structures

In this chapter, the fundamental thermal and structural responses that are characteristic of thermal structure design problems of modern interest, including exhaust-washed structures, are presented using finite element analysis results. These responses provide the basic motivation for the application of multidisciplinary topology optimization to resolve the design challenges related to exhaust-washed structures. Both the physical behavior of the systems that must be addressed and several of the technical challenges that must be accommodated in advanced design are demonstrated. In addition, the results of a study regarding the significance of geometric nonlinearity in thermal structures analysis is detailed.

2.1 Exhaust Structure Thermoelastic Response

The thermoelastic response of an EWS system like that shown previously in Figure 1.4 involves the coupled physics of heat transfer and structural mechanics. In each discipline, loading and boundary conditions are governed by the actual vehicle configuration and transient mission envelope. While a variety of exhaust structure
configurations are possible, they share characteristic responses of interest. To demonstrate these responses, an exhaust-washed structure configuration belonging to an Efficient Supersonic Air Vehicle (ESAV) conceptual design, shown in Figure 1.2b, is considered here.

A typical mission for this type of aircraft includes take-off, supersonic cruise, munitions deployment/combat, and landing as shown in Figure 2.1 [18]. Supersonic cruise ranges from Mach 2.0-2.4, which can lead to temperatures due to aerodynamic heating effects of approximately 300°F on external skin panels. In addition, exhaust temperatures from advanced gas turbine engines are approximately 900-1000°F during cruise and exceed 1400°F during high power take-off and combat phases.

Figure 2.1: Typical mission profile for an ESAV aircraft [18].

Figure 2.2 shows the finite element discretization of the EWS system. Details of loading, boundary conditions, and element selection are given in proceeding sections related to heat transfer and structural analysis. Approximate geometry, dimensions, and thicknesses are extracted from an ESAV aircraft model that was originally sized primarily for aeroelastic design criteria including flutter and aerodynamic flight loads. In this configuration, high temperature composite materials are utilized. The aircraft skins and substructure (green/purple/blue shell elements) are constructed from honeycomb sandwich panels consisting of a 0.5 inch titanium honeycomb core and 0.08 inch graphite-bismaleimide (Gr/BMI) facesheets in a symmetric layup. The nozzle structure (red shell elements) exposed directly to the exhaust gases is a CMC material of 0.4 inch thickness and the entire system is assembled using composite pi joints.
(thick black 1-D elements) common to modern composite airframes [19]. In addition, to reduce the effects of radiation from the hot nozzle to surrounding structure, a 0.375 inch layer of aerogel insulation (yellow solid elements) wraps the exterior surface of the nozzle structure.

Figure 2.2: Finite element discretization of an EWS system with outer skins (a) shown and (b) hidden to show internal structure.

Primary loading on the structure includes the temperatures resulting from the thermal environment and pressure loading from the exhaust flow. Heat transfer processes in the structure include transient conduction, radiation both within internal enclosures and externally from the skins to the environment, convective transfer on the skins from aerodynamic flow, and convective ventilation within compartments inside the structure. In addition, acoustic and flight load contributions are present in the combined environment, but are neglected in this analysis. It is also important to note that in reality the EWS is structurally integrated in the aircraft. This is reflected in the local analysis model’s structural boundaries due to the significance of structural fixivity on thermoelastic responses.
2.1.1 Thermal-Structural Analysis Technique

A sequentially coupled analysis technique is employed to study the thermoelastic response of the EWS. In this technique, first a heat transfer analysis (steady-state or transient) analysis is performed to determine nodal temperature distributions throughout the model. These distributions are then mapped to the structural model where additional structural loads and boundary conditions (BCs) are applied. This loosely coupled technique is suitable because in this case the deformations predicted in the structural model do not have a significant effect on heat transfer responses. In the case of transient analysis, temperature distributions from each time-step are mapped to separate load cases in static structural analysis. Dynamic structural effects are not considered. The commercial software MD Nastran is utilized [20].

2.1.2 Heat Transfer Model

The thermal conditions on the EWS model vary throughout different segments of the mission profile. Since initially it was unclear as to how quickly the EWS system would respond to changes in thermal conditions and during which time periods critical structural responses may occur, the entire mission was analyzed. This was done using a nonlinear transient heat transfer solution in MD Nastran. The thermal effects captured in the heat transfer model include:

- high temperature exhaust gases
- aerodynamic effects on skins due to external airflow
- ventilation inside internal bays
- radiative heat loss to the surrounding environment
- radiation within internal bays/enclosures
The true heating effect from high temperature exhaust gases is a product of propulsion emissions and the thermal-fluid characteristics of the resulting exhaust plume. As an alternative to performing high fidelity engine cycle and computational fluid dynamics analysis, the effect of the exhaust gases is approximated using transient temperature boundary conditions applied to the finite element nodes in the nozzle structure. The specified temperatures are based on expected exhaust temperature at different levels of engine power, which vary in between mission segments. The exhaust temperature for a 350 minute mission is given in Figure 2.3a.

Figure 2.3: (a) Exhaust temperature, (b) altitude and Mach for a 350 minute ESAV mission

The effect of aerodynamic heating on skin panels, neglecting effects of localized shocks or stagnation (which are not expected in the EWS region of the aircraft), is approximated using a convection boundary condition, with heat flux into the surface of element $i$ expressed as

$$q_i = h(T_{aw} - T_i)$$

(2.1)

where $q$ is the local heat flux, $h$ is the convection film coefficient, $T_{aw}$ is the adiabatic wall temperature, and $T_i$ is the temperature of element $i$ (taken as the average of nodal values). The adiabatic wall temperature is the temperature a moving fluid across a
surface obtains assuming no heat transfer into the wall. In this study, approximations for \( h \) and \( T_{aw} \), which are typically local quantities on an aircraft, are obtained from the classical boundary layer solution to flow over a flat plate. Studies of this problem [21] indicate the adiabatic wall temperature can be obtained with good accuracy from

\[
T_{aw} = T_\infty \left[ 1 + r \frac{\gamma - 1}{2} M_\infty^2 \right]
\]  

(2.2)

where \( \gamma \) is the ratio of fluid specific heat, \( r \) is the recovery factor, and \( T_\infty \) and \( M_\infty \) are the temperature and Mach number of the free stream flow. The recovery factor is determined from the Prandtl number for turbulent flow as \( r = Pr^{1/3} \). In this study, constant values of \( \gamma = 1.4 \) and \( Pr = 0.71 \) are assumed. \( M_\infty \) is taken as the aircraft Mach and \( T_\infty \) is taken as the temperature of the standard atmosphere at a given altitude. The values of altitude and Mach are given in Figure 2.3b throughout the mission. The convection coefficient can be determined from the Stanton number given by

\[
C_H = \frac{h}{\rho_\infty u_\infty C_p}.
\]  

(2.3)

Using Eckert's reference temperature method [22], the Stanton number for turbulent flow is computed from

\[
C_H = \frac{0.185}{(\log_{10} Re_x^*)^{2.584}} (Pr^*)^{-2/3}
\]  

(2.4)

where \( Re_x^* \) and \( Pr^* \) are evaluated at the reference temperature \( T^* \); that is,

\[
Re_x^* = \frac{\rho^* u_\infty x}{\mu^*} \quad Pr^* = \frac{\mu^* C_p^*}{k^*}
\]  

(2.5)

are evaluated at \( T^* \). The reference temperature is computed as

\[
T^* = T_\infty \left[ 1 + 0.032 M_\infty^2 + 0.58(T_w/T_\infty - 1) \right]
\]  

(2.6)
where the wall temperature $T_w$ is assumed to be 300°F in the model. The $x$-station along the plate is approximated as the distance from the center of the top/bottom skin in the EWS to the leading edge of the ESAV conceptual configuration. As such, different convection coefficients for the top and bottom skins are computed, but are assumed constant over the entire surface of each region. As computed from Equations (2.2)-(2.6) and data in Figure 2.3b, the temporal variation of the free stream and adiabatic wall temperatures and convection coefficients for the top/bottom skins are computed and shown in Figures 2.4a and 2.4b.

![Figure 2.4: (a) Ambient and adiabatic wall temperatures and (b) convection coefficients for the top and bottom skins for a 350 minute ESAV mission](image)

The effect of ventilation on the internal bays of the structure, by using external air for cooling, is also approximated using a convection boundary condition of the form in Equation (2.1). In this case, the convection coefficient $h$ is assumed to be constant throughout the mission with a value of 1.0 Btu/hr $\cdot$ ft$^2 \cdot$ °F and the ambient temperature for convection is again taken as the adiabatic wall temperature.

Radiative heat loss to the environment is modeled using a radiation boundary condition. This condition represents radiation exchange between a surface element $i$
and a black body sink and is given by

\[ q_i = \sigma (\epsilon_i T_i^4 - \alpha_i T_{amb}^4) \quad (2.7) \]

where \( \sigma \) is Boltzmann constant, \( T_i \) is the temperature of element \( i \), \( T_{amb} \) is the temperature of the ambient sink, and \( \epsilon_i \) and \( \alpha_i \) are the emissivity and absorptivity of element \( i \), respectively. In this study, we assume that \( \epsilon_i = \alpha_i \) for all surfaces due to the lack of material and surface data. The ambient sink for external radiation from the top and bottom skins is assumed to be outer space at -60.0°F and earth at 60.0°F, respectively. In addition, internal radiation exchange between surfaces is included in the model as boundary conditions. View factor computations are performed internally by Nastran after specifying the appropriate radiation enclosures.

The thermal properties of the materials and surfaces for the heat transfer analysis are given in Table 2.1 and are assumed constant with respect to temperature. For the honeycomb panels a homogenization method was utilized to determine panel-level properties using the laminate ply and core properties. Finally, transient analysis was conducted using a time step of 10 seconds.

<table>
<thead>
<tr>
<th>CMC Material</th>
<th>Aerogel Insulation</th>
<th>Radiation Emissivity</th>
</tr>
</thead>
<tbody>
<tr>
<td>( k ) (Btu/hr \cdot ft \cdot °F)</td>
<td>0.91</td>
<td>0.0375</td>
</tr>
<tr>
<td>( \rho ) (lbm/ft(^3))</td>
<td>155.5</td>
<td>12.5</td>
</tr>
<tr>
<td>( C_p ) (Btu/lbm)</td>
<td>0.3</td>
<td>0.25</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Laminate Facesheets</th>
<th>Honeycomb Core</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>( k ) (Btu/hr \cdot ft \cdot °F)</td>
<td>6.5</td>
<td>0.5</td>
</tr>
<tr>
<td>( k ) (Btu/hr \cdot ft \cdot °F)</td>
<td>0.5</td>
<td>0.5</td>
</tr>
<tr>
<td>( \rho ) (lbm/ft(^3))</td>
<td>88.1</td>
<td>9.5</td>
</tr>
<tr>
<td>( C_p ) (Btu/lbm)</td>
<td>0.29</td>
<td>0.25</td>
</tr>
</tbody>
</table>

Table 2.1: Material and surface properties for thermal analysis of the EWS system.
2.1.3 Heat Transfer Results

The temperature distribution in the EWS system at 30 minutes into transient analysis is given in Figures 2.5a-c. Note that different portions of the structure are hidden in different sub-figures. At 30 minutes, which corresponds to a time frame of high exhaust temperature during aircraft climb, the spatial distribution of temperature is characterized by extremely high temperatures at the exhaust-washed nozzle surface that are dissipated outward into the attached substructures. The effects of convection and radiation sufficiently cool the substructure and aircraft skins such that temperature levels do not exceed material allowables. We also note that due to the relatively high conductivity of the composite substructure and amount of convective cooling, spatial temperature distributions within individual panels/plates of the built up system are not severe. This distribution of temperature is representative of large portions of the transient analysis time history, with the magnitude of temperatures scaled proportional to the exhaust temperature.

Several nodes are also indicated in Figures 2.5a-c at which transient temperatures are monitored. These locations are described in Figure 2.5d. The transient temperature history at these nodes is plotted in Figure 2.6 to demonstrate the variation of temperature in the structure throughout the entire analysis. In this figure, we note that the structure responds relatively quickly to changes in thermal conditions. This indicates that in future investigations, the temperature distributions at different parts of transient analysis may be captured with sufficient accuracy using steady-state analysis. If this is possible, the application of design optimization methods becomes more feasible from a computational requirement perspective.
(a) Top Skin Hidden, Insulation Shown

(b) Bottom Skin Hidden, Insulation Hidden

(c) Skins and Insulation Hidden

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
</table>

(d) Description of nodes where transient temperatures recorded (see Figure 2.6)

Figure 2.5: Temperature distribution in EEWS at 30 minutes in transient analysis.
Figure 2.6: Transient temperatures at nodes denoted in Figure 2.5: (a) top regions of model, (b) regions near exhaust nozzle, and (c) bottom regions of model.
2.1.4 Structural Model

The primary loading and boundary condition considerations for the EWS structural analysis are:

- Temperature loading from heat transfer analysis
- Pressure loads from exhaust flow
- Stiffness of adjoining structures

The nodal temperature distribution from the heat transfer analysis is mapped onto the structural model to contribute thermal loads. Quasi-steady behavior is assumed such that the distribution for each time step from heat transfer results is utilized in a separate static load case. In addition, pressure loads are applied normal to the inside of the exhaust nozzle surface to represent the forces due to the exhaust flow. The applied pressure load has both spatial and temporal variation. Spatially, the pressure decays linearly along the length of the nozzle from a high pressure value (engine exit) to near zero (atmospheric conditions). The temporal variation of pressure is related to engine power along the mission profile and takes a maximum value at the front of the nozzle of 30 psi during periods of high power, which is also accompanied by high temperature exhaust.

In thermoelasticity, boundary representation is important because it determines the amount of thermal expansion that is externally restrained. In this model, the primary regions of interest include the exhaust nozzle surface and substructures to which it is directly attached. To better capture the appropriate level of internal restraint on these areas, additional surrounding structure, such as the outer aircraft skins, is modeled. Fixed boundaries are applied to this additional structure, which is further from areas of interest. The structural boundary conditions for the model are shown in Figure 2.7 with blue denoting a symmetry condition and black denoting a clamped condition (all degrees of freedom fixed). An alternative would be to perform
sub-component analysis to approximate the stiffness of the adjoining aircraft structure and apply elastic spring elements to structures of interest in the model. This alternative was investigated on reduced complexity models, but proved to give similar results.

Figure 2.7: Boundary conditions on the EWS structural model with blue denoting symmetry over the $x$-$y$ plane and black representing a fixed condition in all nodal degrees of freedom.

In thermal stress analysis, the effect of temperature dependent material properties should be considered. In this study, rather than accounting for dependence at every temperature level, constant material properties, taken at the elevated temperature a particular material is at during the majority of heat transfer analysis, are utilized. This was done due to the lack of a complete set of temperature dependent properties for all materials and the observation from Figures 2.5 and 2.6 that large regions of steady-state behavior exist with relatively uniform spatial gradients in the transient heat transfer results. The structural material properties and their reference temperatures are given in Table 2.2.

Finally, since the potential for large displacements and follower forces does exist, a nonlinear solution sequence in MD Nastran is utilized to ensure that effects of geometric nonlinearity is captured. This solution utilizes incremental load stepping and includes differential terms in the stiffness matrix. Throughout analysis, no issues
related to convergence in the solution sequence was observed.

### 2.1.5 Structural Results

The displacement and stress responses of the EWS structure for the loading at 30 minutes into the mission (the temperatures of which were shown previously in Figure 2.5) are shown in Figure 2.8. Note that in Figures 2.8a and 2.8b the deformation has been magnified by a factor of 10 to more clearly demonstrate the displacement behavior. From Figure 2.8a it is evident that the regions of greatest overall displacement occur within the exhaust-washed nozzle structure, which we recall has attained a temperature of approximately 1500°F at the time shown. This produces a case where the thermal expansion of the inner nozzle structure is restrained by the cooler surrounding structures. This is demonstrated in Figure 2.8b, which shows the x-direction displacement in the structure. We see the magnitude of growth of the inner exhaust structure (over 0.5 inches along the entire length of the nozzle) is much more significant when compared to adjoining substructures and the aircraft skins, which undergoes significantly less growth due to both a lower temperature and boundary restraints. The effect of this restrained expansion is the development of thermal stresses, which are shown in Figures 2.8c and 2.8d. In these figures, the outer skins
have been hidden due to their relatively benign stress levels compared to the nozzle and substructure regions. In addition, stress values shown are those computed for the equivalent shell elements rather than individual composite layers that make up some of the structure. This is done to simply gain an appreciation for the basic structural response and overall stress levels without introducing the complexity of composite laminate failure criteria.

The areas of greatest von Mises stress in the model are observed in corners of the substructure components where they are attached to the nozzle. While these locations are likely those of greatest stress in the substructure due to the bending deformation created by the expanding nozzle, the actual values shown in Figures 2.8c and 2.8d are only an approximation due to idealizations in the finite element model. The exact geometry of the composite pi-joints and fillets in these regions is not reflected in the shell element representation. Nonetheless, overall stress values away from these locations are more indicative of the true scenario. In fact, these stresses reach maximum values of approximately 35-40 ksi throughout the substructure components, which is within the design limits of the Gr/BMI material system [23].

The von Mises stresses in the nozzle structure reach maximum values of approximately 28 ksi, which far exceeds the proportional limit stress, which is commonly used as a design metric for CMC materials [24], of roughly 10 ksi at temperature [25]. The origin of these excessive stresses is demonstrated in Figure 2.9, in which the contour indicates the first (max) principal stress in the isolated nozzle structure. Here we see the nozzle structure may be viewed as an assembly of individual panels separated by the supporting substructures to which they are attached. We see that the restraint placed on the nozzle structural panels by the surrounding substructures leads to out-of-plane deformation. In some panels this deformation is nearly 0.7 inches across approximately a 30 inch span. This is similar behavior to the response of the previously mentioned exhaust-washed structure found on the B-2 Spirit stealth
bomber known as the aft-deck \[15\]. In that instance, excessive out-of-plane deformation of thin exhaust-washed panels formed cracks that eventually caused in-service structural failures.

The stress response generated by the out-of-plane displacement of the nozzle structure is characterized predominately by bending stresses. In each panel/shell, stresses are greatest along the edges at connections to the substructure, which yields a stress profile similar to a beam or plate undergoing cylindrical bending. The slope of deformation along these regions also indicates that the substructure and adjacent plate boundaries do not create a fully clamped condition, but rather that of finite stiffness in both the in-plane direction and rotation over the edges. To more clearly illustrate the stress distribution in some of the panels, Figure 2.10a shows the maximum prin-
Figure 2.9: First principal stress (maximum of either inner or outer shell) for nozzle structure in EWS system. Note deformation is magnified 10 times.

Figure 2.10: (a) Maximum principal stresses in nozzle structure along section lines shown in (b).

principal stress plotted with respect to a section line across the center of each panel as indicated in Figure 2.10b.

We observe again in each panel a stress response that is consistent with a plate or shell undergoing bending. In each case, the greatest stresses occur along panel boundaries on the side opposite the out-of-plane deformation. From these observations, it may be presumed that some of the mechanisms for reducing the stress levels in a panel that is bowing out-of-plane may be effective in the design of built-up EWS structures. One way to accomplish this stress reduction is to somehow reduce the magnitude of
the out-of-plane deformation response. In fact, this idea will be explored in later sections of this document.

Finally, the stresses in various panels in the EWS system, in addition to its first linear buckling factor, are plotted as a function of time in Figure 2.11a. The locations for the panels in Figure 2.11a are given in Figure 2.11b. From these results, it is clear that the most severe thermoelastic responses in each part of the built-up system coincide with periods of highest temperature. This is intuitive because the greatest temperatures naturally lead to the greatest magnitude of thermal expansion. We note that similar to the transient temperature results in Figure 2.6, the responses reach a steady-state relatively quickly after a change in overall temperature state. This is due in large part to the relatively high diffusion properties of the composite materials being utilized. This also indicates that in future studies, with the proper application of loading and boundary conditions, the most severe thermal environment experienced throughout the transient mission may be readily captured using only steady-state heat transfer analysis. In addition, we note from Figure 2.11a that the linear buckling load factor dips below 1.0 during periods of high nozzle temperature. This indicates that buckling is predicted in the structure; however, the accuracy of linear predictions in this environment is subject to question for cases of restrained thermal expansion. During further investigations using a nonlinear buckling analysis procedure outlined by Cook et al [26], no buckling is observed when the effects of geometric nonlinearity are approximated in the linear buckling analysis. This assertion coincides with the results from geometric nonlinear analysis used to predict stresses here. As the structure heats up, smooth equilibrium paths are observed and bifurcation or snap-through is not evident, which would likely be uncovered in the nonlinear solution process.
2.1.6 Summary

In this section, the heat transfer and structural responses of a conceptual EWS configuration have been investigated. Some important conclusions regarding the basic thermoelastic response of this system are as follows:

- The most severe structural responses coincide with periods of highest exhaust nozzle temperature. During other time periods, stress levels are generally proportional to nozzle temperature levels.

- The EWS system responds relatively quickly to temporal changes in thermal conditions. This indicates that the important structural responses may be captured using temperature predictions from only steady-state heat transfer.

- Critical stresses in the exhaust-washed nozzle structure occur near its attachment to substructure and are due to the out-of-plane deformation of the thin shell-like components.

Finally, the nonlinear analysis techniques employed in this section are not desirable when attempting to design structures. This is due to the added cost and complexity when attempting to apply optimization methods using the iterative nonlinear analysis.
Thus, with the findings in mind regarding the behavior of the entire built-up exhaust-washed structure, attention is now focused on the basic physics of the thin shell-like structures from which it is assembled and the necessary analysis capabilities that are required to accurately study them.

2.2 On the Significance of Geometric Nonlinearity for Thermal Structures

In this section, details of a study regarding the significance of geometric nonlinearity in thermoelastic analysis, which has been published in [27], is discussed. Geometric nonlinearity can, in some cases play a significant role in the thermoelastic response of the types of structures considered in this work. The engine exhaust-washed structure presented in the preceding section, as well as other aerospace thermal structures such as thermal protection systems (TPS), can be idealized as a built up assembly of many plate- and shell- like components. When these types of thin-shell structures are subjected to elevated temperatures with sufficient fixivity at their boundaries, they undergo either buckling or bowing due to thermal expansion. Both behaviors lead to out-of-plane deformation with respect to the original geometry. It is this out-of-plane deformation that in some circumstances may exhibit significant nonlinear behavior. In fact, it is likely that the failure of the B-2 aft-deck highlighted earlier was enabled by the lack of nonlinear analysis during its preliminary design [16]. Thus, before proposing design solutions for exhaust-washed structures, a better understanding of the significance of geometric nonlinearity in these components is required. This knowledge lends insight into the analysis fidelity (linear or nonlinear) that is necessary to capture critical physics under different conditions and which ultimately must be transitioned into any structural optimization process.
2.2.1 Section Overview

It is understood that the response of all structural components across any engineering industry includes some amount of nonlinear effect. However, in most cases, the significance of nonlinear effects is small enough such that the nonlinearity may be neglected for the purposes of engineering analysis and design. As a result, structural design practices and computational tools based on linear physics have become commonplace in the aircraft design process. A troublesome consequence arises when structural nonlinearity cannot be neglected, which in the modern day, frequently occurs with innovative aircraft configurations. These configurations promise improved capabilities and performance, but also push the limits of existing design tools, historical insight, and industry “rules of thumb.” In these cases, the potentially detrimental effects of structural nonlinearity, which are not captured by linear analysis-based design tools, often do not become evident until late stages of detailed design, during experimental or prototype testing, or in the worst case, during operation of a vehicle. Structural nonlinearity for thermally loaded structures is investigated and its effect on the behavior of thin plate- and shell-like structures, which are important in aircraft construction is demonstrated. In the discussions that follow, the sources and significance of the nonlinearity is investigated. In addition, based on observations of finite element results, some guidelines regarding when geometric nonlinearity is vital to properly predicting the correct thermoelastic response for a particular class of structures are identified.

2.2.2 Types of Nonlinearity

Material Nonlinearity

The analysis of an aircraft structural component subjected to a thermal environment can include both material and geometric nonlinearity. Material nonlinearity
occurs in the absence of a linear relationship between stress and strain, such as material yielding, and when material properties become time dependent, for example in cases of creep or rate dependent plasticity. In aerospace thermal structures, material nonlinearity also results from the temperature dependence of engineering materials; however, due to the quasi-steady behavior of thermally loaded components, it is easily addressed for the majority of applications [1]. In quasi-steady analysis (also called sequentially coupled or uncoupled thermal-structural analysis) discrete temperature distributions obtained from heat transfer analysis are applied to a structural model to determine its thermoelastic response. For each temperature distribution, a separate structural analysis is performed with the structural properties assigned according to the temperature at each material point. This procedure is appropriate even when linear structural behavior is assumed and does not otherwise require special consideration.

**Geometric Nonlinearity**

Geometric nonlinearity, on the other hand, relates to nonlinearity in the strain-displacement relationship or changes in the boundary conditions or loading of a structure due to its deformation. These effects are apparent in structures which become either more or less stiff as they deform and in loading whose magnitude or direction varies with displacement. In reality, all structures exhibit some form of geometric nonlinear behavior, but in many applications, nonlinear contributions to the overall system response are negligible. When this is true, simplifying assumptions related to the magnitude of structural deformation and strains are used to remove nonlinearity from the mathematical representation of the physical problem and computationally efficient linear analysis methods are applied to predict the structural response. These linear representations are standard in the design of aircraft structures where designers desire rapid predictions of the structural response to parametric variations as well as
in structural optimization where analytical sensitivity information is readily obtained from linear analysis with little to no computational overhead. To date, linear elastic structures have also been utilized to form the basis of design sensitivity analysis formulations for thermoelastic structures [10, 11, 12, 28, 29]. In fact, challenges related to geometric nonlinearity make sensitivity analysis and design optimization with these responses a formidable task even without thermoelastic contributions.

2.2.3 Thermoelastic Analysis

The behavior of a structural component subjected to combined mechanical and thermal loading is governed by the equations of thermoelasticity along with appropriate initial and boundary conditions. In the formulation of these equations, the linearity of the resulting boundary value problem is determined by the choice of stress and strain measures. The development of these equations for both nonlinear and linear analysis in the absence of thermal effects can be found in several continuum mechanics texts [30]. Thus, only the resulting sets of equations with thermal loading, which require including the appropriate thermal strain terms in the constitutive equations are highlighted. In addition, the typical method of solution by the finite element method for each case is briefed. In the proceeding discussion, it is assumed that structural analysis is performed based on a prescribed temperature distribution that does not depend on the deformation state of the structure. This is consistent with the typical uncoupled or quasi-steady formulations discussed earlier. As a result, all material constants are assumed to be taken at the prescribed temperature of a given material point.

2.2.3.1 Geometric Nonlinear Formulation

In nonlinear analysis, an important distinction is drawn between the undeformed configuration $\Omega_o$ of a body and its deformed configuration $\Omega$ after loading. These
states are shown in Figure 2.12.

Using a Lagrangian description, the location of a material point $P_o$ in $\Omega_o$ is given by the position vector $X$. After experiencing a displacement field $u(X)$ the location of the point $P$ in $\Omega$ is given by $x(X)$. The deformation gradient $F$ relates the undeformed and deformed configurations with the derivative of each component of the deformed position $x$ with respect to each component of the reference position $X$. Noting that

$$x = X + u,$$  \hspace{1cm} (2.8)

the deformation gradient is

$$F(X) = I + \frac{\partial u}{\partial X}. $$  \hspace{1cm} (2.9)

The Green-Lagrange strain $E$ is given in terms of the deformation gradient and displacement as

$$E = \frac{1}{2} \left( F^T F - I \right) = \frac{1}{2} \left( \frac{\partial u}{\partial X} + \frac{\partial u^T}{\partial X} + \frac{\partial u^T}{\partial X} \frac{\partial u}{\partial X} \right) $$  \hspace{1cm} (2.10)

where $I$ is the identity matrix. The Green-Lagrange strain is used as the kinematic
(strain-displacement) relationship for geometric nonlinear analysis. The equilibrium equations are given by

\[
\text{div} \left( \mathbf{F}(X) \mathbf{S}(X) \right) + \mathbf{b}(X) = 0 \tag{2.11}
\]

where \( \mathbf{b} \) is the body force vector defined per unit undeformed volume and \( \mathbf{S}(X) \) is the 2nd Piola-Kirchoff stress. We note that this stress measure is energetically conjugate to the Green-Lagrange strain and, unlike the usual Cauchy stress tensor \( \mathbf{\tau} \), is defined on the undeformed configuration of the body. In practice, \( \mathbf{S} \) is used in the solution of the thermoelasticity equations, but the Cauchy stress is determined from

\[
\mathbf{\tau} = \mathbf{FSF}^T \tag{2.12}
\]

and utilized for the engineering stress response. This is done because most engineering design criteria are based on the Cauchy stress and the components of the 2nd Piola-Kirchoff stress lack physical meaning. For an isotropic material that remains elastic, the constitutive equations (stress-strain relationship) in terms of the Green-Lagrange strain and 2nd Piola-Kirchoff stress are given by

\[
\mathbf{E} = \frac{1}{E} \left[ (1 + \nu) \mathbf{S} - \nu \text{tr}(\mathbf{S}) \right] + \alpha T(X) \mathbf{I} \tag{2.13}
\]

where \( \nu \) is the Poisson’s ratio, \( E \) is the elastic modulus, \( \alpha \) is the coefficient of thermal expansion, and \( T(X) \) is a temperature change defined on the undeformed configuration. Finally, the essential and natural boundary conditions are given using prescribed displacements \( \mathbf{u}^p \) and surface tractions \( \mathbf{t}^p \) on the undeformed configuration as

\[
\mathbf{u}(X) = \mathbf{u}^p(X) \tag{2.14}
\]

\[
\mathbf{F}(X) \mathbf{S}(X) \mathbf{n}(X) = \mathbf{t}^p(X) \tag{2.15}
\]
where \( n \) is a unit normal vector for the surface on which \( t^p \) is defined.

### 2.2.3.2 Linear Formulation

The thermoelastic equations for linear analysis are formulated under the assumption that displacements are sufficiently small such that no distinction is necessary between the undeformed and deformed configurations and the deformation gradient in the nonlinear formulation becomes the identity matrix. In addition, it is assumed the displacement gradients are sufficiently small such that their products may be neglected. As a result, the Green-Lagrange strain, given by Equation 2.10, is reduced to the engineering strain \( \epsilon \)

\[
\epsilon = \frac{1}{2} \left( \frac{\partial \mathbf{u}}{\partial \mathbf{X}} + \frac{\partial \mathbf{u}^T}{\partial \mathbf{X}} \right). \tag{2.16}
\]

With no delineation between the body’s configuration before and after displacement, the equilibrium equations can be given directly in terms of the Cauchy stress \( \tau \) and body forces \( b \) as

\[
\text{div} (\tau(\mathbf{X})) + b(\mathbf{X}) = 0 \tag{2.17}
\]

The constitutive equations in Equation 2.13 is utilized again for an elastic material, but is now written in terms of the engineering strain and Cauchy stress by

\[
\epsilon = \frac{1}{E} \left[ (1 + \nu)\sigma - \nu\text{tr}(\tau) \right] + \alpha T(\mathbf{X}) \mathbf{I}. \tag{2.18}
\]

Finally, constraints are specified to complete the boundary value problem by

\[
\mathbf{u}(\mathbf{X}) = \mathbf{u}^p(\mathbf{X}), \tag{2.19}
\]

\[
\tau(\mathbf{X}) \mathbf{n}(\mathbf{X}) = t^p(\mathbf{X}). \tag{2.20}
\]
2.2.3.3 Solution by Finite Element Methods

In structural analysis it is common to recast the elastic equations into a discrete form using variational or Galerkin methods to be solved using finite elements. After doing so, discretizing the body, and summing element contributions, a global set of equations for nonlinear structural finite element analysis may be given by

\[ K(U)U = F_a + F_b(U) + F_{th}(U) \]  \hspace{1cm} (2.21)

where \( K \) is the stiffness matrix, \( F_a \) is the vector of applied loads, \( F_b \) is the body force vector, \( F_{th} \) is the thermal load vector, and \( U \) is the nodal displacement vector. We observe that Equation 2.21 is nonlinear in terms of the unknown displacements \( U \) and the stiffness matrix, body force vector, and thermal load vector are dependent upon the deformation as formulated in Section 2.2.3.1. In practice, incremental iterative techniques based on the Newton-Raphson method are utilized to solve the finite element equations. To account for the dependence of stiffness and load terms on the displacement, the matrices are periodically reformulated throughout the iterative process.

On the other hand, the global equations for a finite element problem formulated under the assumptions in Section 2.2.3.2 for linearized physics are given by

\[ KU = F_a + F_b + F_{th}. \]  \hspace{1cm} (2.22)

We note the system is linear in terms of the displacement vector \( U \), which is determined using standard solution methods for linear systems.

In the investigation that follows, the commercial finite element software MD Nastran is utilized using the appropriate solution procedures for both linear and nonlinear representations.
2.2.4 Beam Strip Model

A parameterized curved beam strip model is first used to demonstrate the significance of geometric nonlinearity induced by thermal loads. This simple 2D model directly represents a semi-infinite cylindrical shell, but also serves as a suitable idealization of the basic components of more complex exhaust-washed structures among other thin-shell thermal structures. A schematic of the model is shown in Figure 2.13. Here, $L$ denotes the span covered by the strip, $\delta$ is a measure of curvature (which is assumed as circular), and $t$ is the thickness of the strip. A rectangular cross section with unit width into the page is also assumed. Since boundary conditions are critical in cases of thermal expansion, linear elastic boundaries are utilized. By varying the values of $K_a$ (axial) and $K_r$ (rotational) stiffness, all cases from fully clamped and simply supported to free expansion can be modeled. In addition, translation of edge nodes in the vertical direction is prevented. Finally, loading consists of a spatially uniform temperature increase $T$.

![Parameterized curved beam strip model](image)

Figure 2.13: Parameterized curved beam strip model (note: curvature is of a circular profile)

In the study of thermal structures, often the magnitude of temperature is misleading with respect to the severity of its structural consequences because it does not reflect the material properties. A more suitable indicator of the severity of a given temperature level for a material is the product of its elastic modulus $E$, coefficient of thermal expansion (CTE) $\alpha$, and the elevated temperature $T$. Examples for various engineering materials at common operational temperatures are given in Table
2.3. We note that with respect to the overall effect of temperature loading, a CMC component with low CTE may experience nearly the same thermoelastic effects as a titanium alloy due to increased operational temperature in cases with sufficient boundary fixivity or spatial temperature gradients. In the following investigations, structural properties are taken as Ti 6242 from Table 2.3, but in an attempt to retain generality, we note here that results were observed to scale rather proportionally with $E\alpha T$ in additional tests with various materials and temperature levels.

<table>
<thead>
<tr>
<th>Material</th>
<th>$T$ (°F)</th>
<th>$E$ (10^6 psi)</th>
<th>$\alpha$ (10^{-6}/°F)</th>
<th>$E\alpha T$ (10^3 psi)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ti 6242 [31]</td>
<td>900.0</td>
<td>12.5</td>
<td>5.5</td>
<td>61.86</td>
</tr>
<tr>
<td>Inconel 718  [31]</td>
<td>1300.0</td>
<td>24.0</td>
<td>8.5</td>
<td>265.2</td>
</tr>
<tr>
<td>Gr-BMI (0/90) [23]</td>
<td>400.0</td>
<td>8.0</td>
<td>12.0</td>
<td>38.40</td>
</tr>
<tr>
<td>CMC [25]</td>
<td>1700.0</td>
<td>14.5</td>
<td>2.5</td>
<td>61.62</td>
</tr>
</tbody>
</table>

Table 2.3: Comparison of engineering material properties at high operating temperature.

The curved strip is modeled in MD Nastran using 250 2-node beam elements, which is a sufficient discretization based on mesh convergence. The model inputs are parameterized using a thickness to span length ratio $t/L$ and a curvature to span length ratio $\delta/L$. In practical applications, the thickness to span length ratio $t/L$ ranges from 0.005 to 0.05 and the curvature to span length ratio $\delta/L$ ranges from 0.0 (perfectly flat) to 0.5 (half circle). Finally, in the discussions that follow, comments regarding the accuracy of linear analysis results assume that the nonlinear response predicts the true structural response.

2.2.5 Deformation Response

The deformation response of the strip model with varying curvature, as predicted by both linear and nonlinear analyses, is given in Figure 2.14. In the plots, three thermal conditions are represented by three $E\alpha T$ values. Boundary conditions are specified using elastic elements with $K_\alpha = K_r = \infty$ (2.14a, 2.14c), which represents a
clamped condition, and $K_a = \infty$, $K_r = 0$ (2.14b, 2.14d), which represents a condition with free rotation over the edge and no translation of edge nodes. The curvature to length span ratios $\delta/L$ of 0.01 (2.14a, 2.14b) and 0.05 (2.14c, 2.14d) are indicated in the figure. The thickness to span length ratio is taken as $t/L = 0.013$. The black curve denotes the undeformed configuration of the strip absent thermal loads and the dashed and solid curves represent deformed shapes from linear and nonlinear analysis, respectively.

From Figure 2.14 it is evident that the effect of geometric nonlinearity in the deformation response, which is observed as the difference between the linear and nonlinear response curves increases with $E\alpha T$ regardless of boundary type and amount of curvature. However, when comparing Figures 2.14a and 2.14b to Figures 2.14c and 2.14d we see that this effect is much less pronounced in the strip of greater curvature. In addition, linear analysis over-predicts the out-of-plane deformation response as the severity of thermal loading increases (note that this observation is not general to all thicknesses as will be shown in forthcoming results). We also note that when using linear analysis, the magnitude of deformation increases monotonically with $E\alpha T$, assuming temperature independent properties. This is not observed in the nonlinear response, where the effects of stress stiffening and follower forces preclude this nonphysical behavior. This is especially evident in the strips of low curvature in Figures 2.14a and 2.14b with free rotation boundaries exhibiting the greatest discrepancy between linear and nonlinear analysis.

### 2.2.6 Effect of Geometric Parameters

To understand the effect of geometric or dimensional parameters on the significance of nonlinearity, a parametric study is now performed by varying the thickness of the strip for different values of curvature. Figure 2.15a shows the vertical displacement measured at the center of the strip (normalized by the span length $L$) and Figure
2.15c shows the maximum stress as a function of thickness $t/L$ for four values of curvature $\delta/L$ and clamped boundaries ($K_a = \infty$, $K_r = \infty$). Figures 2.15b and 2.15d show analogous results for free rotation boundaries ($K_a = \infty$, $K_r = 0$). Here $E\alpha T = 60 \times 10^3$. This level of thermal loading was selected based on deformation plots, which indicated that the difference between linear and nonlinear analysis is most severe at the greatest values of $E\alpha T$. As mentioned earlier, this value corresponds to typical conditions for a high temperature titanium alloy, but the observations here are readily extendable to other material systems.
In all of the plots shown in Figure 2.15 we see that increasing the curvature (denoted by different colored curves) reduces the difference between linear and nonlinear analysis predictions for a large variety of strip configurations. In fact, for curvatures of $\delta/L = 0.150$ (and greater) no appreciable difference in displacements or stresses is evident. This is due to the fact that a curved strip is initially nearer to the thermally loaded equilibrium configuration. As a result, the effect of deformation dependent forces, which are neglected in linear analysis where the loads are applied to only the undeformed structure, is reduced. When utilizing a nonlinear analysis procedure, the...
direction of the loads that result from thermal expansion remains consistent with the deformed configuration via periodic updating.

More interesting than simply the difference between the analyses are the trends that result for parametric variations in thickness $t/L$. In each plot, we see that for strips of larger curvature (blue, red, and cyan curves), the general trend for increasing thickness is similar for both linear and nonlinear analysis. This observation is important when considering the effect of nonlinear analysis in the design and optimization of thermal structures. It implies that for structures whose configuration contains sufficient curvature, linear analysis may contain error, but yields the proper trends in response variability. This aspect is critical when performing trade studies and design space exploration early in a design process. On the other hand, for a curvature ratio $\delta/L = 0.005$ (black curve), which corresponds to a strip with little curvature, linear and nonlinear analysis respond completely different to parametric variation in thickness. This indicates that the linear analysis fails to correctly capture the fundamental physics governing the displacement and stress responses, which we may now assume are considerably nonlinear.

Further investigation into the plots of Figure 2.15 yield a great deal of insight regarding the parametric effect on geometric nonlinearity. For example, for greater curvature ratios, increasing the thickness reduces the discrepancy between linear and nonlinear analysis; however, for nearly flat geometry, the differences persist over an order of magnitude of thickness variation ($t/L$ from 0.005 to 0.05). In addition, instances are observed where the need for nonlinear analysis is not obvious based on conventional structural design “rules of thumb,” which are largely based on mechanical responses without thermoelastic considerations. For example, the occurrence of displacement in a beam or plate structure that approaches or exceeds its thickness is a common indication of the need for nonlinear analysis. In this case, the effects of stress stiffening, or the coupling between transverse displacement and membrane stiffness
that is neglected in the kinematic relations of linear analysis, become significant. In all plots in Figure 2.15 with greater curvatures, we observe cases where displacement is multiple times greater than the strip thickness, but linear analysis predicts the response with excellent accuracy compared to nonlinear analysis. However, for strips of low curvature and high thickness (in excess of $t/L = 0.025$ in Figure 2.15a) we observe a case where geometric nonlinearity is significant, but the displacement predicted by linear analysis is less than 25% of the thickness. This observation indicates that traditional insights for structural analysis may not accurately reflect the need for nonlinearity in thermoelasticity.

### 2.2.7 Effect of Boundaries

To this point, we have only investigated models with simple boundary conditions with fixed degrees of freedom, which represented extreme cases of rigid fixivity for thermal expansion. In reality, the true boundary conditions of thermally restrained structures depend on the stiffness of surrounding components as well as the type of joints. To understand the implications of boundaries with finite stiffness, we now utilize the spring conditions in Figure 2.13. A parametric study similar to the preceding investigations, with varying thickness ratios and multiple curvatures, is performed; however, rather than rigid fixivity, finite values of rotational and axial stiffness are systematically prescribed. To provide relational context to the magnitude of stiffness boundaries, the spring stiffness values $K_a$ and $K_r$ are parameterized as:

$$K_a = k_a \frac{AE}{L}, \quad (2.23)$$

$$K_r = k_r \frac{EI}{L^2}. \quad (2.24)$$

Here, we note two quantities provide estimations for the stiffness of the strip itself. In Equation 2.23, $AE/L$ is analogous to the stiffness of a flat axial bar where $A$ is the
cross section area, $E$ is the elastic modulus, and $L$ is the length. Similarly, in Equation 2.24, $EI/L^2$ is analogous to the bending stiffness of a flat cantilever beam where $I$ equals the moment of inertia. If $K_a$ and $K_r$ are taken as the stiffness of adjoining structures, the parameters $k_a$ and $k_r$ represent an approximation for the ratio between the stiffness of the adjoining structure (or boundary stiffness) and the stiffness of the strip itself in the axial and rotational directions, respectively. It is noted that these quantities hold truly physical meaning only for flat strips; however, they still provide a convenient approximation of the magnitude of boundary stiffness when curvature is not excessive. For clarity, a unit value of $k_a$ or $k_r$ indicates the boundary has approximately equal stiffness to the strip while a large or small (fractional) value asserts that the boundary has much greater or lower stiffness than the strip in that dimension, respectively.

Figure 2.16 shows the maximum stress in the strip as a function of thickness for multiple values of finite rotational stiffness. In the axial direction, zero displacement at the boundaries is prescribed ($K_a = \infty$). We note that the current value of thickness is used in the determination of boundary stiffness in Equations 2.23 and 2.24. Additional studies considering fixed thickness for the boundary stiffness calculation resulted in similar results for trends in stress as a function of thickness. Figure 2.16a shows results for $\delta/L = 0.005$ (nearly flat) and 2.16b shows results for $\delta/L = 0.050$ (modest curvature). Comparing the trends with increasing thickness in each case, we again observe that with greater curvature, linear analysis is better able to capture the trend in parametric stress response. For strips with little curvature, just as in Figure 2.15, the structural response is governed by nonlinear effects. This observation holds for all values of $k_r$ in Figure 2.16a. On the other hand, Figure 2.16b indicates that the stress response for lower values of rotational stiffness $k_r$ is better predicted by linear analysis when modest curvature in the geometry is present. This implies that to some extent, rotational stiffness at boundaries activates some nonlinear effects, as
Figure 2.16: Maximum stress in the beam strip as a function of thickness with finite values of rotational stiffness at boundaries and curvature ratios of (a) 0.005 and (b) 0.050.

exhibited by significant differences around \( k_r = 500 \) and greater in this case. This is likely due to its strong influence on the out-of-plane displacement of the strip.

Figure 2.17 shows the maximum stress results as a function of thickness for multiple values of \( k_a \). Here, zero rotational displacement is prescribed \( (K_r = \infty) \) and results are again given for two values of curvature. Similar to the previous case, we observe that the response of the strip with greater curvature in Figure 2.17b is better captured by linear analysis than that with little curvature in Figure 2.17a. On first inspection, it may appear that in Figure 2.17a, linear analysis captures a trend of decreasing stress with increasing thickness for strips of thickness greater than approximately \( t/L = 0.025 \), but noting the state of stress (indicated by its sign) we conclude otherwise. Here, linear analysis predicts a state of purely compressive stress at much lower thickness values, which again alludes to the dominance of nonlinear effects in the out-of-plane (bending) deformation behavior. Similar to the cases of rotational stiffness, from Figure 2.17b, increasing axial stiffness appears to increase the difference in analysis predictions, with differences becoming significant above
proximately $k_a = 3$. This is again attributed to the contribution of boundary stiffness to the out-of-plane deformation that results. Finally, it may be concluded that while both axial and rotational boundary stiffness does contribute to the nonlinearity, its effect appears to be much less pronounced than changes in the strip curvature. In fact, the inability of linear analysis to capture the proper trends in displacement and stress as observed in Figures 2.16a and 2.17a results from a lack of curvature. In the nearly flat geometries, capturing the effects of deformation on the orientation of thermal stress contributions using iterative updating as done in nonlinear analysis is critical.

### 2.2.8 Summary

In summary, several notable conclusions may be drawn regarding the significance of nonlinearity for thermally restrained structures and the ability of linear analysis to predict thermoelastic behavior. These conclusions have important implications when considering the application of design optimization methodologies, which are generally based on linear analysis models due to computational cost and convenient sensitivity.
analysis. The basic observations are

- increasing temperatures (or increasing $E\alpha T$) increases the effect of geometric nonlinearity,
- the effect of varying thickness on geometric nonlinearity appears to be small compared to other factors,
- increasing the curvature reduces the effect of geometric nonlinearity,
- and increasing both rotational and axial stiffness of boundaries increases the significance of geometric nonlinearity.

Regarding the previous effects on nonlinearity, the curvature (or lack of) is the most important factor. It is suggested that geometric nonlinear analysis be utilized for any thin thermally loaded component whose out-of-plane curvature is less than 5% of the total span covered if boundary conditions are sufficient to generate an out-of-plane response. Failure to do so will likely lead to structural response predictions that are in no way representative of the true nonlinear physics in this domain.

These geometric nonlinear effects are due in large part to the configuration dependence of the thermal loading with additional contributions from stress stiffening behavior. The significance of geometric nonlinearity in predicting the response of these types of thermal structures lead to challenges in the application of structural optimization to improve the designs. As a result, additional research of advanced techniques for thermoelastic optimization is critical to the design of future aircraft with embedded engines and for hypersonic flight thermal protection applications. Due to the similarity of the test structure utilized here to plate and shell-type of structures, it is reasonable to assume that they will behave in much the same way. Thus, care should be taken to investigate geometric nonlinearity in the design of any structure that may undergo out-of-plane deformation due to an elevated temperature.
environment. Without proper understanding of the effects of variability in this design
domain owing to geometric nonlinearity, reduced performance and reliability is likely
to result using conventional design paradigms.

2.3 Challenging Design Responses

In the previous section, investigations were performed to identify characteristics that
play a role in the significance of geometric nonlinearity in the thermoelastic response.
With this understanding we now discuss the behavior of the response itself, rather
than comparisons between the results of different analysis types. Here, we continue
to regard the nonlinear results as the true structural behavior and explore the non-
intuitive effects of parametric variation and the actual implications of nonlinearity in
design responses. We first comment on observations made from the beam strip in the
previous section and conclude with additional results of a three-dimensional curved
shell structure.

2.3.1 Observations from Beam Strip Model

Referring back to Figures 2.16 and 2.17, which give the maximum stress in the strip as
a function of the thickness, we observe interesting parametric stress behavior across
a wide variety of geometries and boundary conditions. Remarkably, in each plot
we observe increases in the maximum stress with increases in thickness, especially
beginning at low values of $t/L$. This behavior is unique to structures with so called
design-dependent loading, which includes temperature loads, and is counter-intuitive
when compared to mechanically loaded structures. It was also specifically noted to
pose significance challenges in the preliminary structural sizing study of a hypersonic
vehicle [32]. One of the most common techniques for reducing stresses in mechanically
loaded structures is to increase their stiffness by increasing size parameters of the
component. In the thin structures considered here, this corresponds to increasing the thickness. Contrary to mechanical loading, the design-dependency of thermal loading results in additional load from the added material that also undergoes thermal expansion. Thus, in some circumstances represented in Figures 2.16 and 2.17, the proper prescription for reducing thermal stresses is actually to reduce the thickness. In practice, this may not always be possible when considering other design requirements related to vibration and dynamic stability, which are also important to the types of thermal structures of general interest in this dissertation. In some cases shown in the same figures, we do see that for higher values of $t/L$ that increasing thickness can lead to stress reduction, particularly in cases with high (or infinite) stiffness in the rotational direction at boundaries. However, we note that if considering an initially thin structure, significant increases in thickness (exceeding 3-5 times the original thickness) are required to realize a reduction in stress. This translates to significant increases in structural weight and also when comparing Figure 2.16a to 2.16b and Figure 2.17a to 2.17b, the effectiveness of this treatment diminishes with added curvature in the geometry.

A final observation of the beam strip results in Figures 2.16 and 2.17 is that various curvatures have an effect on the stress response. This indicates that if the functional design space of a thermal structure includes the possibility for shape changes, while size optimization may not be effective, structural shape optimization may be an effective design approach. Unfortunately, in the design of the exhaust-washed structures of interest in this dissertation, the geometric shape of many parts of the structure are typically fixed by exhaust fluid flow and observability design requirements and eliminate this possibility.
2.3.2 Curved Shell Model

To validate the general observations from the beam strip model, we now briefly investigate the stress response of a curved thin shell subjected to thermal loads. The parameterized model of the shell, which is assumed to have spherical curvature, is given in Figure 2.18a. The blue region denotes the quarter symmetric section of the structure that is utilized for finite element analysis. The finite element model, discretized with 2500 four node quadrilateral plane stress elements, is shown in Figure 2.18b. The boundary conditions explored here are analogous to the axial/in-plane stiffness beam strip study with results in Figure 2.17. Zero rotation and vertical displacement along the edges indicated by red is specified and symmetry conditions are prescribed on the edges highlighted with blue in Figure 2.18b. The in-plane stiffness is again parameterized according to Equation 2.23, but here $A$ is computed for the entire cross-section along the edges of the shell. Once obtained, the total spring stiffness is divided equally among spring elements at each node on the edges denoted by red. The deformed shape and stress response of an example model is shown in Figure 2.19 for both curvatures of $\delta/L$ of 0.005 and 0.050. In this case, $E\alpha T = 18.25$, which corresponds closely to a CMC material at 1000°F, $t/L = 0.02$, and $k_n = 1.0$. We note the characteristic out-of-plane deformation response with respect to the orign-
Figure 2.19: (a) Maximum principal stress in quarter symmetry shell model for $t/L = 0.02$ and $k_a = 1.0$. Note deformation shape is magnified by 10.

Figure 2.20a and 2.20b show the parametric stress response of the shell as a function of thickness for curvature measures of $\delta/L = 0.005$ and 0.050, respectively. Here, in three-dimensions, identical observations to those in the previous section may be made. For the low curvature shell (Figure 2.20a) an increase in thickness again leads to an increase in stress for lower thickness values across a range of boundary stiffnesses. For greater thickness shells, stress reduction via thickness increase is possible, but again when beginning with a relatively low thickness shell, increasing the thickness by factors greater than two is required. For the case of higher curvature...
in Figure 2.20b, we observe no cases where increasing thickness leads to a reduction in maximum tensile stresses.

2.4 Chapter Summary

In this chapter, the characteristic response of exhaust-washed structures, along with the broader class of thin thermal structures with restrained expansion, has been investigated. The fundamental out-of-plane deformation of thin structures, which has been shown to lead to damaging thermal stresses in practical applications, was observed in a conceptual exhaust-washed structure configuration subjected to a transient mission profile. The implications of geometric nonlinearity were also explored and some particular geometric features were identified that help to determine when nonlinear effects will be significant. Finally, the challenging and non-intuitive design space related to thermal stresses was investigated. From this exercise it may be concluded that when attempting to apply design optimization methodologies classes of thermal
structures that experience restrained expansion that basic sizing optimization may not be effective due to design dependency of thermal loads. The stress responses did appear to be influenced by varying the shape (curvature), but within the context of the exhaust-washed structure design space, such a design modification may not be possible. As a result, it is concluded that the design domain for a candidate exhaust-washed structure, or analogous thermal structure such as a TPS panel, must be expanded past modifying size and shape variables of a predefined configuration to include the material layout. The domain should also be extended to include not only the exhaust-washed structure itself, but also adjoining substructures and adjacent regions to achieve the best results due to the influence of boundary attachment. Within the methods of design optimization, this expanded design parameterization can only be achieved using structural topology optimization, which is detailed in the next chapter.
Chapter 3

Literature Review of Topology Optimization

Based on the conclusions in Chapter 2, which indicated that basic sizing and shape optimization may not be sufficient in the design of some thermal structures, it is apparent that a design technique with greater design freedom is desirable. Due to its material layout capabilities, structural topology optimization is a promising alternative. To assess the state-of-the-art for the topology optimization of thermal structures, a review of the literature is performed in this chapter including the current methods, techniques, and applications that are relevant to thermoelastic design.

3.1 Background

Topology optimization is the process of determining the connectivity, shape, and location of voids inside a given solid design domain and is often understood as determining the best material distribution [33]. This allows for greater design freedom when compared to size and shape optimization, which deal with variables such as thicknesses or cross-sectional areas of structural members (sizing) and geometric features (shape) of predefined structural configurations. This is shown in Figure 3.1. As
such, topology optimization has great benefit in early conceptual and preliminary de-
sign phases where changes have a significant impact on final component performance.
Due in large part to its material layout capabilities, topology optimization is also a
promising tool for thermoelastic design, which is fundamentally a problem of deter-
mining optimum structural layout to satisfy functional requirements while properly
managing thermal expansion. It is also important to note that there are two types
of topology optimization: discrete and continuum [34, 35]. In this document, we
investigate only continuum formulations as the discrete methods are not appropriate
for the problems of interest.

Figure 3.1: Three categories of structural optimization. a) Sizing optimization, b)
shape optimization, and c) topology optimization. The initial design parameteriza-
tions are shown on the left with the resulting optimal solutions on the right [33].

3.2 Methods

Several different methods exist for finite element-based topology optimization of con-
tinuum structures, but the common goal is to determine the existence or absence of
material within a given region of a design domain. Usually, “solid/void” designs are
desired because they can be physically realized using isotropic materials. Here, solid
implies existence of material and void implies absence of material. The differences
among methodologies lie in the parameterization of the design space. For example,
some methods explicitly define the design directly on the finite element domain while
others define a design implicitly using a separate function from which the structure is interpreted. The following subsections outline some of the current methods for topology optimization.

### 3.2.1 Homogenization

Practical finite element-based topology optimization of continuum structures began with the landmark work by Bendsøe and Kikuchi, who posed a structural layout problem within the context of homogenization theory [36]. This early technique of the explicit type, known now as the *homogenization method*, is based on the ability to model porous materials, and their associated macro-scale material properties, from a periodic microstructure defined within a unit cell. Two potential microstructures (among many alternatives), a square cell with a rectangular hole and a layered microstructure with two isotropic constituents, along with the relationship to the macro-scale discretized finite element model are shown in Figure 3.2. With the appropriate microstructure definition, and under the assumption of infinitesimally small periodic unit cells, any anisotropic macro-scale material representation can be achieved, including pure solid, pure void, and intermediate (composite/porous) material.

In practice, the parameters corresponding to the microstructure, for example $\mu_1$, $\mu_2$, and $\theta$ in Figure 3.2a, are taken as free variables for design. A single set of microstructure variables may be used for each finite element or a sub-mesh may be utilized for fine structures. The topology optimization problem then becomes to determine the combination of microstructure variables corresponding to the optimal macro-scale distribution of properties that minimizes an objective function. This formulation was successfully applied to various problems throughout the 1990s, but has fallen out of favor in the literature in the last decade due to the development of more efficient methods. These newer methods, which are described in the following sections, can capture the same design features with fewer design variables when com-
pared to homogenization. In addition, many of the common numerical issues found in explicit types of topology parameterizations are more easily circumvented with alternative formulations.

Figure 3.2: Microstructures for 2D continuum topology optimization with the homogenization. (a) Square unit cell with rectangular holes, and (b) layered microstructure with two different isotropic materials [34].

### 3.2.2 Density-based Methods

Currently, the most widely used methods for structural topology optimization are explicit parameterizations that are broadly classified as density-based methods. As with homogenization, these techniques operate on fixed domain of finite elements; however, rather than a set of microstructure properties, each finite element contains only a single design variable. This variable is often understood as the element material density, $\rho_e$. The material properties of each element, for example the elastic modulus or thermal conductivity, are made functions of density design variables by way of an interpolation function. Penalty methods are then utilized to force solutions
to suitable “solid/void” designs. The most well known density-based method is the SIMP method (Solid Isotropic Material with Penalization) and nearly all commercial topology optimization tools utilize a density-based method to the author’s best knowledge.

The fundamental mathematical statement of a density-based topology optimization problem contains an objective function, a set of constraints (that often include limits on material usage), and a discretized representation of the physical system. A general formulation based on linear static finite element analysis may be given as

\[
\begin{align*}
\text{min} : \quad & f(\mathbf{x}, \mathbf{U}) \\
\text{subject to} : \quad & \mathbf{K(x)U} = \mathbf{F(x)} \\
& g_i(\mathbf{x}, \mathbf{U}) \leq 0 \\
& 0 \leq x_e \leq 1
\end{align*}
\]

where \( f \) is the objective function, \( \mathbf{x} \) is the vector of density design variables that are related to \( \rho_e \), \( \mathbf{U} \) is the displacement vector, \( \mathbf{K} \) is the global stiffness matrix, \( \mathbf{F} \) is the force vector, and \( g_i \) are the constraints. We note that the stiffness matrix \( \mathbf{K} \), and sometimes load vector \( \mathbf{F} \) are explicitly dependent upon the density design variables. Within this generalized statement, a number of problems can be formulated considering design responses including compliance, stresses, frequency, displacements, and alternative physics such as eigenvalue problems, fluid flow, and nonlinear systems.

### 3.2.2.1 Density Interpolation/Penalization

A critical aspect of density-based methods is the selection of an appropriate interpolation function and penalization technique to relate the physical quantities of the problem with continuous density design variables. As previously noted, the distributed
function for design is interpreted as the physical density of each finite element, $\rho_e$. The values of density range as $0 \leq \rho_e \leq 1$ or $0 < \rho_{\text{min}} \leq \rho_e \leq 1$ where 0 corresponds to a void element, 1 to a solid element, and $\rho_{\text{min}}$ is the minimum value of density, which is required with some formulations to prevent difficulties associated with zero values. These difficulties include singularity in finite element matrices and issues with the inability of material to reappear in an area with zero density. With the choice of this parameterization comes the need to steer the problem toward a solid/void solution. This is typically accomplished in density-based topology optimization using implicit penalization techniques, the most common of which is the Solid Isotropic Material (originally Microstructure) with Penalization (SIMP) method [37, 38]. In the SIMP method, also referred to as the power law or fictitious material model, density variables are penalized with a basic power law (whose value is finite) and multiplied onto physical quantities such as material stiffness, cost, or conductivity. This is demonstrated in Equation (3.2) where the SIMP method is applied to the elastic modulus of an element.

$$E(\rho_e) = \rho_e^p E_0$$  \hspace{1cm} (3.2)

Here, $E(\rho_e)$ is the scaled modulus, $E_0$ is the modulus of the solid material, and $p$ is a finite penalty parameter. We note that for values of $0 \leq \rho_e \leq 1$ and a positive $p$ (commonly taken as 3), $E$ is bounded between zero at zero density and its solid value $E_0$ when $\rho_e = 1$.

Alternative interpolation schemes have been developed primarily to address deficiencies in the SIMP method for certain classes of problems. One of these is known as the Rational Approximation of Material Properties (RAMP) [39]. A desirable feature of the RAMP model is that, unlike SIMP, it has nonzero sensitivity at zero density. As a result, the RAMP material model has been shown to remedy some numerical difficulties in problems related to very low density values in the presence of design-dependent loading. Another alternative is known as the SINH (pronounced “cinch”)
method, which is based on a hyperbolic sine function [40]. This scheme differs from others in that usually material parameters are penalized, whereas in the SINH formulation the volume is penalized. As such, intermediate density material consumes more volume with respect to its load-carrying capability than solid or void material. A comparison of the SIMP, RAMP, and SINH penalization schemes is shown in Figure 3.3 where \( \rho \) is the physical density, \( p \) is the penalization parameter for SIMP and SINH, \( q \) is the penalization parameter for RAMP, and \( \eta \) is the value of the interpolation. The range of parameters shown is representative of those used in practice, with the actual value depending on the underlying physics of the problem.

\begin{align*}
\text{(a) SIMP: } & \eta(\rho) = \rho^p \\
\text{(b) RAMP: } & \eta(\rho) = \frac{\rho}{1+q(1-\rho)} \\
\text{(c) SINH: } & \eta_1(\rho) = 1 - \frac{\sinh(p(\rho-1))}{\sinh(p)} \\
& \eta_2(\rho) = \rho^p
\end{align*}

Figure 3.3: Interpolation functions for density-based topology optimization. a) SIMP, b) RAMP, and c) SINH.

### 3.2.2.2 Regularization

With the introduction of interpolation functions as described above to parameterize the topology optimization domain, some associated numerical issues must be addressed. These issues are checkerboarding and mesh dependency. Checkerboarding, which refers to the formation of adjacent solid-void elements arranged in a checkerboard pattern, has been shown to be an issue related to common four-node bilinear quadrilateral and eight-node brick finite elements. Apart from dedicated techniques that will be highlighted, higher-order finite elements have been shown to remove
checkerboarding in some cases. Mesh dependency, which concerns the fact that a smaller discretization allows for smaller features to develop in optimum results, must be alleviated to enforce minimum length scale. Examples of topology optimization results that exhibit checkerboarding and mesh dependency are shown in Figures 3.4 and 3.5, respectively. Among the techniques that have been developed to address these features, it is noted that those that prevent mesh dependency will generally also eliminate checkerboards; however, checkerboard-prevention algorithms may not necessarily alleviate mesh dependency.

Figure 3.4: Example of checkerboarding in topology optimization.

Figure 3.5: Example of mesh dependency in topology optimization. Mesh sizes are a) 60 x 20, b) 90 x 30, c) 150 x 50, and d) 300 x 100.

The removal of checkerboards and mesh dependency in topology optimization is known as domain regularization, of which there are two primary methods: constraint methods or filters. Constraint methods utilize localized or global-level constraints that
are added to the optimization problem to control parameters such as the perimeter of structural boundaries. The main drawback of constraint methods is that their application is often problem dependent, with tuning required to achieve desired results. Filtering methods on the other hand are straightforward to implement and add only minor computational expense.

Filtering is applied via direct modification of density variables or sensitivities based on information from a neighborhood of surrounding elements. The basic sensitivity [41] and density filters [42, 43], in addition to more modern projection methods [44, 45], are now highlighted. The sensitivity filter modifies the sensitivities $\frac{\partial f}{\partial x_e}$ as

$$\hat{\frac{\partial f}{\partial x_e}} = \frac{1}{x_e \sum_{i \in N_e} H_{ei} \sum_{i \in N_d} H_{ei} x_i \frac{\partial f}{\partial x_e}}$$  \hspace{1cm} (3.3)

where $N_d$ is the set of designable elements $i$ for which the center-to-center distance $\Delta(e, i)$ to element $e$ is smaller than the filter radius $r_{min}$ and $H_{ei}$ is a weight factor defined as

$$H_{ei} = \max(0, r_{min} - \Delta(e, i)).$$  \hspace{1cm} (3.4)

From this definition, we see that the filtered sensitivity, $\hat{\frac{\partial f}{\partial x_e}}$ is a weighted sum of sensitivities of surrounding elements. The weight for each contribution to the filtered sensitivity at element $e$ decays linearly as the distance increases and is zero for elements whose centroid lies outside the filter radius. This is shown in Figure 3.6. Despite its widespread use, only recently has a rigorous physics-based analogy for the workings of the sensitivity filter been obtained [46]. Prior to this the sensitivity filter was utilized based largely on computational experience that demonstrated its effectiveness [33].

Rather than filtering the sensitivity information, the density filter specifies the element density based on a weighted average of the densities around it. This is given
Figure 3.6: Neighborhood definition for the application of sensitivity and density filters.

as

\[ \rho_e = \frac{1}{\sum_{i \in N_d} H_{ei}} \sum_{i \in N_d} H_{ei} x_i \]  

(3.5)

where we now note the important distinction between the physical density \( \rho \), which defines the actual topology, and the design variable densities \( x \) on which the optimization problem is defined. This distinction is important because after application, the sensitivity of a response \( f \) is explicitly found with respect to the physical density \( \rho_e \) and a chain rule is necessary to find the sensitivity with respect to the design variable density, now denoted as \( x_e \). The required chain rule operation for sensitivity with respect to a density design variable \( x_j \) is given by Equation (3.6) and is easily implemented in practice using matrix operations.

\[ \frac{\partial f}{\partial x_j} = \sum_{e \in N_j} \frac{\partial f}{\partial \rho_e} \frac{\partial \rho_e}{\partial x_j} = \sum_{e \in N_j} \frac{1}{\sum_{i \in N_e} H_{ei}} H_{je} \frac{\partial f}{\partial \rho_e} \]  

(3.6)

This consideration is not necessary for the sensitivity filter because in that case the
design variable and physical densities are equal to one another. Application of either
the sensitivity filter or the density filter will prevent features smaller than the twice the
filter radius \( r_{\text{min}} \) from occurring in the results. Naturally, this prevents checkerboard
patterns since they are a small feature, and also allows the prescription of minimum
size on structural features.

A basic consequence of sensitivity and density filtering is the formation of a gray
transition material between solid and void regions. In many cases, this material can-
not be physically realized in a manufactured component. Recent projection methods
have been developed to address this in cases where crisp boundary definition is im-
portant. Both Heaviside functions [44] and morphology-based operators [45, 47] are
utilized to project filtered densities into 0/1 space.

The Heaviside filter is a modification of the original density filter given by Equation
(3.5) with a Heaviside step function that projects the filtered density (from now on
called the intermediate density and denoted as \( \tilde{x}_e \)) to a physical density. The physical
density \( \rho_e \) equals one if \( \tilde{x}_e > 0 \) and zero if \( x_e = 0 \). To facilitate gradient-based
optimization, which requires continuous functions, a smoothed Heaviside function is
utilized as

\[
\rho_e = 1 - e^{\gamma \tilde{x}_e} + \tilde{x}_e e^{-\gamma}.
\]  

(3.7)

The parameter \( \gamma \geq 0 \) describes the curvature of the projection which is linear at
\( \gamma = 0 \) and approaches a Heaviside step as \( \gamma \) approaches infinity. In most applications,
a continuation scheme is used where \( \gamma \) is initially equal to zero and gradually increased;
however, recently a version that does not require continuation has been demonstrated
[48]. In addition, since density variables are modified by the projection, a chain rule
must be applied to obtain the sensitivity of a response to the intermediate density.
This is given by

\[
\frac{\partial f}{\partial \tilde{x}_e} = \frac{\partial f}{\partial \rho_e} \frac{\partial \rho_e}{\partial \tilde{x}_e}
\]  

(3.8)
where the derivative of the physical density $\rho_e$ with respect to the intermediate density $\tilde{x}_e$ is given by

$$\frac{\partial \rho_e}{\partial \tilde{x}_e} = \gamma e^{-\gamma \tilde{x}_e} + e^{-\gamma}.$$  \hspace{1cm} (3.9)

Figure 3.7 shows characteristic results of filtering and projection. Here filtering has been used to prevent checkerboards and enforce minimum length scale. Note the gray transition material along structural boundaries for (a) sensitivity and (b) density filters. This region is eliminated using the (c) Heaviside projection.

![Figure 3.7: Characteristic topology optimization results using (a) sensitivity filtering, (b) density filtering, and (c) Heaviside projection.](image)

### 3.2.3 Hard-kill Methods

In contrast to the homogenization and density-methods another class of explicit techniques, called *hard-kill* methods, for topology optimization do not relax the discrete
finite-element design domain to a continuous form. They work by gradually removing (or adding) elements to the design domain. The choice of material to be removed or added is based on heuristic criteria, which may or may not be based on sensitivity information. As a result of these heuristic features, the technical rigor of these methods are sometimes called into question as a robust theoretical basis does not exist [49, 50]. The most well known hard-kill methods of topology optimization is Evolutionary Structural Optimization (ESO) [51] and, more recently, Bi-directional Evolutionary Structural Optimization (BESO) [52], which allows for addition of elements.

One of the most attractive features of hard-kill methods such as ESO is the simplicity with which they can be utilized with commercial finite element packages. Often times, the integration of the algorithms with FEA solvers requires only simple pre- or post-processing steps. In addition, hard-kill methods for topology optimization result in a design with crisply defined structural boundaries that are free of intermediate or gray material because finite elements are explicitly define as existent or absent.

3.2.4 Boundary Variation Methods

Boundary variation methods are a most recent development in structural topology optimization with their roots lying in shape optimization techniques. In contrast to the previous methods, they are based on implicit functions that define structural boundaries rather than an explicit parameterization of the design domain. Figure 3.8a shows an explicit representation where the domain, $\Omega$, exists as an explicit parameterization of variables $x$ between 0 and 1. The structural boundary $d\Omega$ then exists at the interface of regions 0 and 1. Figure 3.8b demonstrates an implicit representation where the structural boundary is implicitly specified as a contour line of the field $\Phi$, which is a function of $x$.

Two current boundary variation methods are the level set and the phase-field
methods. These methods produce results in the design domain with crisp and smooth edges that require little post-processing effort to structural features. In addition, these methods are fundamentally different than shape optimization techniques because they allow for not only the movement of structural boundaries, but also the formation, disappearance, and merger of void regions, which defines true topological design.

![Diagram of design domain and boundaries](image)

**Figure 3.8:** (a) Explicit versus (b) implicit representation of a design domain and boundaries.

### 3.2.4.1 Level Set Method

In the level set method, boundaries are represented as the zero level curve (or contour) of a scalar function \( \Phi \) (the level set function) as shown simply for 2D topologies in Figure 3.9. Boundary motion and merging, as well as the formation of new holes, are performed on this the scalar function. The shape of the geometric boundary is modified by controlling the motion of the level set according to the physical problem and optimization conditions [53]. It is also important to note here that while a smooth boundary representation is realized in the design domain as shown in Figure 3.9, most level set formulations rely on finite elements. Thus, boundaries are still represented by a discretized, likely unsmooth, mesh in the analysis domain unless alternative techniques are utilized to represent the geometry in the analysis model, for example extended finite elements [54].

Level sets for moving interface problems in physics were first developed by Osher and Sethian [55] with the fundamental goal of tracking the motion of curves and surfaces and have since been applied in a wide variety of research areas [56, 57].
The level set method was first applied to topology optimization in the early 2000s by Sethian and Wiegmann [58], where it was used to capture the free boundary of a structure in linear elasticity, and Osher and Santosa [59], who combined level sets with a shape sensitivity analysis framework for optimization of structural frequencies.

![Figure 3.9: Level set representations: (a,c) 2D topologies and their (b,d) corresponding level set functions (from [60]).](image)

The modern level set method [61] begins with a structural boundary specified as a level set in implicit form as an iso-surface of a scalar function in 3D as Equation (3.10)

\[ S = \{ x : \Phi(x) = k \} \]  

(3.10)

where \( k \) is the iso-value and is arbitrary, and \( x \) is a point in space on the iso-surface. Structural optimization can be performed by letting the level set model vary in time, yielding Equation (3.11).
\[ S(t) = \{ x(t) : \Phi(x(t), t) = k \} \quad (3.11) \]

Taking the time derivative of Equation (3.11) and applying the chain rule yields the following Hamilton-Jacobi equation

\[
\frac{\partial \Phi(x,t)}{\partial t} + \nabla \Phi(x,t) \frac{dx}{dt} = 0, \quad \Phi(x,0) = \Phi_0(x) \quad (3.12)
\]

which defines an initial value problem for the time dependent function \( \Phi \). In the solution process, let \( \frac{dx}{dt} \) be the movement of a point driven by the objective of optimization such that it can be expressed in terms of the position of \( x \) and the geometry of the surface at that point. The optimal structural boundary then becomes a solution of a partial differential equation on \( \Phi \) given by Equation (3.13)

\[
\frac{\partial \Phi(x)}{\partial t} = -\nabla \Phi(x) \frac{dx}{dt} \equiv -\nabla \Phi(x) \Gamma(x, \Phi), \quad \Phi(x,0) = \Phi_0(x) \quad (3.13)
\]

where \( \Gamma(x, \Phi) \) is the “speed vector” of the level set and depends on the objective of optimization. This vector is obtained as the steepest descent direction of the objective obtained via analytical sensitivity analysis.

The level set method has obvious advantages over density methods because intermediate density material is not utilized in the design domain. However, one drawback of current level set formations is their dependency on the initial design, which is more severe than other methods. However, new developments have made strides to improve this deficiency [62]. Also, level set methods require reinitialization during the process when the level set function becomes too flat or too steep, which adds non-desirable computational complexity and additional tuning parameters to the algorithms.
3.2.4.2 Phase Field Method

The phase-field method for topology optimization is based on theories originally developed as a way to represent the surface dynamics of phase-transition phenomena such as solid-liquid transitions [63]. The methods have also been utilized in a number of different surface dynamics simulations, especially in materials science, including diffusion, solidification, crack-propagation, and multiphase flow. In these theories, a phase field function $\phi$ is specified over the design domain $\Omega$ that is composed of two phases, A and B, which are represented by values $\alpha$ and $\beta$ of $\phi$, respectively, as shown in Figure 3.10. The boundary region between phases is a continuously varying region of thin finite thickness $\xi$.

![Figure 3.10: (a) A 2D domain represented by the phase field function and (b) a 1D illustration of the phase field function (from [64]).](image)

In phase-field topology optimization [64, 65, 66] this region defines structural boundaries and is modified via dynamic evolution of the phase field function $\phi$. A primary difference between the level set and phase-field methods lies in the fact that in the phase-field method the boundary interface between phases is not tracked throughout optimization as is done when using level sets. That is, the governing equations of phase transition are solved over the complete design domain without prior information about the location of the phase interface. In addition, phase-field methods do not require the reinitialization step as do level set functions.
3.2.5 A Biologically-inspired Method

One of the most recent methods for topology optimization is an innovative biologically-inspired layout technique that is capable of generating both discrete and continuum-like structures [67]. In this method, which is based on the cellular division processes of living organisms, topological layout is implicitly governed by a developmental program that when executed completes a sequence of tasks that develop the topology in stages. When driven by a genetic algorithm (GA), the set of rules, called a Lindenmayer or map-L system [68], that defines the tasks of the developmental program become the design variables in the optimization problem. With control over these developmental rules, a diverse set of topological designs can be generated with relatively few design variables.

The key to this method is the $L$ system, which is a type of grammar system originally introduced by biologist Aristid Lindenmayer to model the branched topologies found in plants. A map is defined as finite set of regions with each region bounded by a sequence of edges that intersect at vertices. The maps are analogous to cellular layers, where the regions represent the cells and the edges their walls. By using a series of production rules, an example of which is given in Equation (3.14), which govern the processes that construct the map, interpretations of complex topology can be obtained.

\[ A \rightarrow B[+A]x[-A]B \]
\[ B \rightarrow A \]
\[ x \rightarrow x \]  

(3.14)

Execution of the production rules in Equation (3.14) and the axiom $\omega = ABAB$, which indicates the initial edge labeling, for the first four steps in the process are shown by Figure 3.11.

By utilizing additional rules apart from the simple division process demonstrated
in the figure, more complex features, for example adding radii to edges, can be obtained. In addition, the geometry can be superimposed or stretched onto non-rectangular domains that can also change shape. It is also important to note that the topological layout generated by the map-L system in itself has no physical or structural meaning attached to it. Thus, the geometry must be interpreted into structural elements, which can often be done in a pre-processing step to a finite element analysis.

### 3.3 Literature Formulations & Applications

Topology optimization has been the most active area of research in structural optimization with the number of applications using various methods exploding over the last decade. However, the vast majority of topology optimization applications have been restricted to mechanical design problems concerning stiffness-based design criteria \[70, 71\]. The following subsections include a review of the most common topology optimization problem in addition to alternative and multiphysics topics that are relevant to the work demonstrated and proposed in the remainder of this document.
3.3.1 Minimum Compliance

A well behaved topology optimization problem is best obtained when its objective function is computed as the integral over the designable domain. For purely structural problems, a convenient objective of this form is to minimize the compliance subject to volume constraint. For linear structures, this corresponds to determining the optimal stiffness tensor $E_{ijkl}(x)$, which is assumed to vary spatially. The statement of the minimum compliance problem follows [33].

Introducing the energy bilinear form (i.e., the internal virtual work of an elastic body at the equilibrium $u$ and for an arbitrary virtual displacement $v$)

$$a(u,v) = \int_{\Omega} E_{ijkl}(x) \epsilon_{ij}(u) \epsilon_{kl}(v) d\Omega,$$

(3.15)

with linearized strains

$$\epsilon_{ij}(u) = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right),$$

(3.16)

and the load linear form

$$c(u) = \int_{\Omega} f u d\Omega + \int_{\Gamma} t u d\Gamma,$$

(3.17)

the minimum compliance problem takes the form

$$\min_{u \in U, E} : c(u)$$

subject to : \hspace{1em} $a_E(u,v) = l(v)$, for all $v \in U.$

(3.18)

Here, the equilibrium equation is written in its weak, variational form, with $U$ denoting the space of kinematically admissible displacement fields, $f$ are the body forces, and $t$ the boundary tractions.
Solution of the problem defined in (3.18) by any of the topology optimization methods discussed previously yields a structural topology characterized by maximum global stiffness. Most often, (3.18) is discretized by finite elements and thus fits the general topology optimization problem statement given previously in Equation (3.1). It may be conservatively estimated that greater than 85% of topology optimization work is related to solving this problem. This widespread acceptance is due in large part to two characteristics of the compliance objective: (i) it is self-adjoint and (ii) its sensitivity to material addition is strictly negative. These beneficial characteristics make the problem particularly easy to implement and solve. In engineering practice, the implementation of this problem setup in commercial packages has proven widely successful for purely mechanical applications.

3.3.2 Thermoelastic Structures

In most implementations of compliance-based topology optimization, the body force term in Equation (3.18) is neglected such that the design problem depends only on externally applied, design independent forces. Rodrigues and Hernandes [72] first retained this term by including the effects of a temperature difference in the compliance formulation using the homogenization method of [36]. Using homogenization parameters \( \mu \) and \( \theta \), the compliance objective takes the form

\[
\min_{(0 \leq \mu(x) \leq 1, \theta(x))} \int_\Omega bud\Omega + \int_\Omega \beta^{H}_{ij}(\mu, \theta)\epsilon_{ij}(u)\Delta T + \int_{\Gamma} tudd\Gamma
\]

(3.19)

where we note that both the elastic modulus and the coefficient of thermal expansion have been made functions of \( \mu \) and \( \theta \) by \( \beta_{ij} = E_{ijkl}\alpha_{kl} \). The problem is subjected to the isoperimetric volume constraint

\[
\int_\Omega \mu(x)d\Omega \leq \bar{V}
\]

(3.20)
where \( \overline{V} \) is the allowable amount of volume. One observation from this paper is the dependence of topological results upon the magnitude of the prescribed temperature difference. Figure 3.12 shows results from [72] for the minimum compliance problem of a simple 2D structure subjected to both an externally applied point load and a uniform temperature difference \( \Delta T \). We observe that as the uniform temperature is increased, the resulting topology begins to differ. In addition, as temperature is increased, the compliance of the designs also increases. This is driven by the design dependency of the temperature load contribution in the objective function (3.19).

![Figure 3.12: Minimum compliance results from Rodrigues and Hernandes [72] for various magnitudes of temperature loading.](image)

In fact, it was also observed in [72] that for higher temperatures, not all of the available material is utilized. This behavior is consistent with both observations in Chapter 2 of this document, where it was observed that material addition may increase the severity of some structural responses, in addition to more recent topology optimization work using the level set method [73]. In the case of minimum compliance
topology optimization, depending on the level of temperature loading, adding material may increase compliance.

Jog [74] extended the compliance objective function in (3.19) to the general case of nonlinear thermoelasticity using a density-method with linear penalization and a perimeter constraint for regularization. It was also noted that the compliance objective function would not reduce peak stresses because in the presence of thermal loading stress is not proportional to strain. It is suggested that to do so, another functional would be required that would yield a more uniform (magnitude wise) stress distribution in the structure. Along these lines, recently Pedersen and Pedersen [75] proposed an alternative problem based on recursive iteration to obtain uniform energy density. They properly argue that compliance is a questionable objective for thermoelastic problems, but due so under the incorrect assertion that “with less material that will improve the compliance the strength will normally not be as desirable, i.e., the maximum von Mises stress is increased” and furthermore argue that it is beneficial for strength design to fully utilize all available material. As was clearly demonstrated in Chapter 2, this is not true in many cases. This aspect was not investigated in the paper; as benchmark problems were used to simply demonstrate the effectiveness of the recursive scheme for obtaining a topology. The resulting topologies were not objectively evaluated from a design point of view. In a later publication, this work was expanded by its authors to 3D domains with alternative interpolation schemes [76].

In many of the topology optimization results reported in the previous publications using both homogenization and density-based methods, large regions of intermediate gray material are evident. In fact, these regions are difficult to physically realize in problems of practical importance. In the minimum compliance problem, the material usage constraint in the general formulation must be active to force a crisp black/white design. Thus, without special treatment, it is impossible to ensure that the optimal
topology will not contain gray material when using the homogenization or any density-based approach. Gao and Zhang recently investigated this issue and demonstrated that in some cases the RAMP interpolation scheme was able to alleviate many of the issues related to the formation of gray material when compared to the SIMP scheme [77]. This was attributed to fact that the RAMP interpolation, unlike SIMP, has nonzero sensitivity at zero density as was previously shown in Figure 3.3. Figure 3.13 demonstrates the alleviation of gray material by using RAMP for a benchmark problem from [77]. While this prescription seems effective in the cases presented in the paper, additional numerical tests seem to indicate that while RAMP does provide improved performance, it is not guaranteed to alleviate all gray material many objectives. This problem appears much more dependent upon the relative magnitude of mechanical and thermal loading rather than the interpolation scheme that is utilized.

Li et al. demonstrate a variety of thermoelastic topology optimization problems using the ESO method including design of shell structures [78] and displacement minimization [79]. More recently, Kim et al. used ESO for the design of thermal protection system (TPS) panels for both thermal stress and frequency criteria [80]. It is of merit to note that in their results, intermediate material is not an issue due to the discrete nature of the ESO method, but its heuristic operation and inability to conveniently handle multiple constraints make it an unattractive option for the scope of problems pursued in this dissertation.

In recent years, successful applications of topology optimization with thermal loading in the literature have relied on problems apart from the usual minimum compliance formulation or specific procedures to generate good designs for specific problems of interest. Wang et al. [81] proposed a bi-objective problem for designing thermoelastic structures that have both high stiffness and low thermal expansion in particular directions. The problem statement takes the form:
Figure 3.13: Thermoelastic minimum compliance results from Gao and Zhang [77] where SIMP methods exhibit gray material that is eliminated when using RAMP.

\[ \min : \quad F(x) = wF_1(x) + (1 - w)F_2(x) \]

subject to : \quad material usage constraints \hspace{1cm} (3.21)

\[ KU^S = F^S \]
\[ KU^{Tem} = F^{Tem} \]

where a simple multi-objective setup is utilized with \( w \) being a weight factor between 0 and 1. We note the presence of two finite element systems denoted by superscripts \( S \) and \( Tem \), which correspond to cases of purely structural and purely temperature load, respectively. The first objective in the problem is taken as the compliance of
the structure subjected to only structural loading given by

\[ F_1(x) = \frac{(F^S)^T U^S}{(F^S_0)^T U^S_0} \]  

(3.22)

where \((F^S_0)^T U^S_0\) is the compliance of the initial design and is simply used for normalization. The second objective consists of the magnitude of the summation of nodal displacement components for which displacement should be minimized and is given by

\[ F_2(x) = \frac{\sum_{i=1}^{n} (u_{Tem}^{i})^2}{\sum_{i=1}^{n} (u_{Tem}^{i})^2_0} \]  

(3.23)

where again the term in the denominator is the response at the initial design for normalization. By using the displacement vector from the purely temperature loaded analysis, and selecting degrees of freedom corresponding globally to a particular direction, this objective serves to reduce the thermal expansion of the structure in that direction. A similar formulation was demonstrated by Deng et al. [82] who utilized homogeneous and periodic microstructure design rather than the bi-material formulation of [81].

Yang and Li included temperature loading into the topology optimization for minimum dynamic compliance of bi-material plates [83]. In their implementation, the thermal stress distribution in the structure is first computed via static analysis with a prescribed uniform temperature field. This thermal stress is then taken as a pre-stress by formulating the corresponding differential stiffness and including it in a subsequent transient dynamic analysis with harmonic loading at specified frequencies. Later, they modified the algorithm to include a normal modes analysis, which also includes differential stiffness terms due to thermal stress, to compute resonant frequencies. The dynamic analysis is then performed at frequencies corresponding to resonance behavior [84]. In a more advanced implementation, Stanford and Beran used a similar prescription to include the effects of temperature loading in the topology optimization.
of plates for buckling and flutter speed [85].

Finally in an early yet interesting application, Sigmund and Torquato utilized topology optimization to develop material microstructures with extremal thermal expansion properties [86]. That is, the microstructures developed exhibited large directional expansion, zero isotropic expansion, and even negative isotropic expansion at the macro-scale.

### 3.3.3 Heat Transfer

Topology optimization techniques have also been applied to a number of generalized scalar field problems including heat transfer. In fact, for the simplest case of steady-state heat conduction, it is straightforward to formulate a design problem that is analogous to minimum structural compliance. For a given design domain $\Omega$, the steady-state heat equation with with homogeneous boundary conditions is given by

$$\nabla \cdot (k \nabla T) + f = 0 \text{ in } \Omega$$

$$T = 0 \text{ on } \Gamma_D$$

$$(k \nabla T) \cdot n = 0 \text{ on } \Gamma_N$$

(3.24)

where $T$ is the temperature, $k$ is the heat conduction coefficient, $f$ a volumetric heat source, $n$ an outward normal unit vector, and $\Gamma_D$ and $\Gamma_N$ surfaces upon which Dirichlet and Neumann boundary conditions are applied, respectively. Upon investigation, it becomes obvious that the temperature and volumetric heat source are analogous to displacement and mechanical forces, respectively, in structural mechanics, and the mechanical compliance inspired objective can be conveniently stated as

$$\int_{\Omega} fT d\Omega.$$  (3.25)
Minimizing this functional by varying the conductivity under a material usage constraint physically corresponds to finding the optimal conductivity distribution that produces the least heat when the amount of high conductivity material is limited. This formulation, or slight variations of it, have been extensively studied in the literature (likely due to its well-posed nature like the mechanical compliance formulation) and are claimed to represent the design of an optimal conductive device. It has been demonstrated using density methods (using both finite element [33] and finite difference [87] solution schemes), ESO [88, 89, 90], and level-sets [91, 92]. An example of a problem of this class is given in Figure 3.14. The branched conduction paths to the fixed temperature sink are characteristic of the results obtained using all of the different methods.

Figure 3.14: Topology optimization for an optimal heat conductor. The design domain has adiabatic boundaries all around and is subjected to uniform volumetric heat generation and the temperature condition shown. (from [33])

Of much greater interest than the simple heat conduction literature is the work by Bruns [93] who introduced the physics of free convection for density-based topology optimization. Mathematically, this corresponds to allowing the boundary conditions of the problem in Equation (3.24) to become non-homogeneous as given by

\[(k\nabla T) \cdot n = h(T - T_\infty) \text{ on } \Gamma_h\]  

(3.26)
where \( h \) is the convective film coefficient, \( T_\infty \) is the ambient temperature, and \( \Gamma_h \) are the surfaces upon which convection acts. In topology optimization, applying these conditions is challenging because the structural geometry, including the convection surfaces, constantly varies. To overcome this, Bruns demonstrated that with a finite element-based implementation, the convection conditions can be applied to all elements, including both the faces and edges of 2D elements and all faces of 3D elements. After doing so, both the conductivity \( k \) and convective film coefficient \( h \) are parameterized according to density design variables via interpolation schemes. With this implementation, the convection boundary conditions can be continuously varied from states of existence or absence according to the state of surrounding elements. Bruns also notes that special treatment is required to prevent numerical instabilities at the boundaries between regions of high and low conductivity elements in the presence of convection effects. An example of this phenomena is shown in Figure 3.15. In Figure 3.15a, the finite element model of a cooling fin is shown with black elements representing the high conductivity solid material and white elements representing low conductivity void regions where \( k_2 \ll k_1 \). This representation of the domain is consistent with that found during topology optimization. The entire domain is subjected to uniform surface convection to an ambient temperature of 0.0 and a heat flux boundary is applied to the left edge of the solid fin region.

Figure 3.15b shows the temperature contour plot of the fin examined by FEA using the usual consistent formulation of convection matrices. Note that the temperature denoted in the black region falls below the ambient fluid temperature. The nonphysical oscillations are quite apparent in the temperature profiles plotted at discrete intervals along the fin length in Figure 3.15d. By using a lumped form of the convection matrices, these oscillations are removed in the results as indicated in Figures 3.15c and 3.15e. In similar work, Iga et. al. also included the effects of design-dependent convection using a homogenization approach, but employed a
Figure 3.15: (a) Cooling fin embedded in a fixed domain. Temperature contour plots of the fin by FEA using (b) consistent and (c) lumped convection matrices. Corresponding temperature response profile plots along the length of the fin for (d) consistent and (e) lumped convection matrices.

computationally challenging boundary searching algorithm [94]. We also note that in these works, the motion of fluid in the void region is not modeled. Doing so would require the solution of different governing equations in the solid and void regions and an improved representation of the boundary. While algorithms that are capable of doing this do exist, they are very much in their infancy in the literature [95]. Thus, for practical purposes the representation of convection can only be done as a flux-type boundary condition.

Finally, it is of merit to note that while the effects of radiation to or from an ambient condition is possible using formulation similar to the treatment of convection in [93], no instances of radiation within cavities or enclosures are found in the topology optimization literature. Doing so would likely require a discrete method for topol-
ogy optimization to properly identify the cavities formed inside the design domain. Computing the necessary view factors would also likely be extremely computationally expensive because the number and shape of radiation cavities would change every design iteration.

### 3.3.4 Coupled Thermal-Structures

With examples of thermoelastic topology optimization with prescribed temperature loading and topology optimization of heat transfer problems that yield a temperature distribution, a natural extension is to combine the two. In [96], Cho and Choi formulate such a method using coupled adjoint sensitivities to account for the dependence of temperature loading in the structural analysis on the results of a heat transfer analysis. Li et. al. also demonstrate the coupled solution using the ESO method where coupled sensitivity numbers are derived to drive the evolutionary optimization process [97, 98]. The concept of combining heat transfer and structural topology optimization as is done in these works lends itself well to the design of exhaust-washed structures. With such a capability, topologies may be obtained with efficient conduction pathways and convective cooling to reduce their overall temperature levels. This would lead to reduced thermal expansion and thermal stresses, perhaps with more effect than any pure structurally motivated modification.

A more common application of a coupled thermal-structural topology optimization is the design of thermally compliant mechanisms or microactuators. These components are microscale structures whose thermal expansion characteristics are designed to produced controllable actuation forces or displacement at a particular location. Formulations for density-based topology optimization of these components for one material and two material systems are detailed by Sigmund [99, 100]. An example from [99] is given in Figure 3.16. Here, the design problem is to find the optimal topology in the designable region that when subjected to an elevated temperature
maximizes the work done in the spring. We see that for different stiffness springs, alternative topologies are developed that provide different magnitudes of force and displacement. Other examples are also given in the articles where topology is obtained that directly drives the displacement of a control point or causes its location to pass through prescribed way-points for different temperature loads.

(a) Thermal actuator design domain

(b) Topology optimization

Figure 3.16: (a) Initial design domain for a thermally actuated compliant mechanism along with (b) optimal topologies.

In recent years, similar problems related to thermal microactuator design have been performed using the ESO method [101] and level sets [102]. Time transient effects in heat-transfer analysis have also been considered in the topology optimization for the homogenization of thermally actuated snap-fit mechanisms [103]. In this application, elevated temperature causes thermal expansion in such a way so as to release fixivity between components, which provides for rapid disassembly of recyclable components.
3.3.5 Stress-based Topology Optimization

Ultimately, identifying how to reduce excessive thermal stresses is the most important consideration in most thermal structures design problems. To investigate the feasibility of directly addressing this consideration, the literature related to structural topology optimization using stress-based design criteria is now surveyed.

The treatment of stresses in topology optimization for continuum structures was first investigated in the late 1990s by Duysinx and Bendsoe [104]. Since then, three primary challenges related to the introduction of stress-criteria in topology optimization have been identified. These are (i) the so-called “singularity” phenomenon, (ii) the highly nonlinear behavior of stresses with topology variation, and (iii) the local nature of stresses [33]. After early investigations, stress-based criteria were seemingly abandoned in favor of the well-posed compliance problems in topology optimization literature. However, in the last 4-5 years, stress considerations have enjoyed a resurgence and are currently one of the most active research areas in the field. In addition, a much more conventional structural optimization problem formulation based on minimizing material usage subject to stress constraints in the domain has been adopted in stress-based publications.

The “singularity” problem relates to the fact that an n-dimensional feasible design space contains degenerate subspaces of dimension less than n [105], that is, where some of the n design variables vanish from the problem. Further, the globally optimal design often lies in one of these degenerate subspaces. This is evidenced, for example, in density-based topology optimization for minimum compliance where a majority of elements have density design variable values equal to zero. However, for a number of design responses, asymptotic behavior as variables approach zero prevent a nonlinear programming algorithm from reaching the degenerate subspaces and leads to convergence to undesirable local minima. This occurs in density-based topology optimization where local element stresses increase without bound as density design
variables approach zero. When constraints are placed on an element stress, it then becomes impossible to fully remove that element from the design domain because of constraint violations. This problem was first addressed using an $\epsilon$-relaxation [104], which is derived from an analogous problem in truss optimization where the stress in a truss element (given simply by $\sigma = P/A$) becomes singular as the cross-section area nears zero [106]. In general, a stress constraint in topology optimization may be written as

$$g(\rho) = \sigma(\rho) \leq \bar{\sigma} \text{ for } \rho > 0$$

(3.27)

where $\sigma$ is an elemental stress measure (for example, von Mises stress), $\bar{\sigma}$ is the allowable stress, and the condition $\rho > 0$ indicates the existence of the element. To eliminate the condition $\rho > 0$ from the constraint, the following modified formulation is considered

$$g(\rho) = \rho \left( \frac{\sigma(\rho)}{\bar{\sigma}} - 1 \right) \leq 0.$$  

(3.28)

Also, to prevent singularity using $\epsilon$-relaxation, the constraint is replaced by

$$g(\rho) = \rho \left( \frac{\sigma(\rho)}{\bar{\sigma}} - 1 \right) \leq \epsilon$$

(3.29)

where the $\epsilon$ parameter is given. For $\epsilon = 0$, the original problem with stress constraints arises; however, for any $\epsilon > 0$, the $\epsilon$-relaxed problem is characterized by a design space that is no longer degenerate, that is, the optima are now placed in regions of the design space with a non-zero constraint measure and it becomes possible to reach them using the usual structural optimization algorithms based on the Karush-Kuhn-Tucker (KKT) conditions. To return the problem to its original form, a continuation approach is used to gradually reduce the value of $\epsilon$ during the optimization process as described in [104]. More recently, variants of this approach have been applied successfully by other authors [107, 108, 109, 110, 111, 112].

Recently, Le et. al. [113] introduced another method to address the singularity
phenomena by simply penalizing the stress measure with an interpolation function. Using the stress interpolation function $\eta_\sigma(\rho)$, the stress constraint can be relaxed as

$$g(\rho) = \eta_\sigma(\rho)\sigma(\rho) \leq \sigma_{\text{all}}$$  \hspace{1cm} (3.30)$$

where the relaxed stress is $\sigma_r(\rho) = \eta_\sigma(\rho)\sigma(\rho)$. In the paper, it is demonstrated that by using a SIMP interpolation (i.e. $\eta = \rho^p$) and appropriate selection of penalization parameters for interpolation of stress, stiffness, and volume, degenerate regions of the design space can be reached efficiently. In their work, SIMP parameters of $1/2$, $3$, and $1$ are suggested for stress, stiffness, and volume, respectively, but also note other selections that satisfy a set of necessary criteria are viable.

The highly nonlinear nature of stress constraints occurs because of the high sensitivity of elemental stress values to surrounding variations in topology. This is addressed by ensuring that sensitivity information that is given to the optimization algorithm is numerically consistent with the design problem that is being solved. In [113] it is advised that when applying filtering that only numerically consistent schemes should be employed, such as the density and Heaviside filters, and that those that smear sensitivity information, such as the sensitivity filter should be avoided.

The local nature of stress constraints is the focus of much of the stress-based topology optimization literature in the past 5 years. In a design problem, stresses should satisfy limits at every point in the structure. In sizing and shape optimization, this is easily accomplished using intuition to enforce constraints in critical locations (i.e. in critical elements) based on the geometry of the structure or by using active-set strategies to dynamically select critical stresses throughout the optimization process. However, in topology optimization, it is impossible to identify which elements will have critical stress because the geometrical configuration is not initially known. Thus,
a potential solution is to place stress constraints on all elements $e$ such that

$$g_e(\rho) = \sigma_e(\rho) \leq \bar{\sigma} \text{ for } e = 1, 2, \ldots, n.$$  \hfill (3.31)

While this prescription, known as the *local* method, has been demonstrated in the literature [33, 107, 108, 110], it is not applicable in practical implementation (problems with $> 10,000$ elements and multiple load cases) because the computation of local sensitivities by either the direct or adjoint method is computationally prohibitive. Reducing the computational cost corresponds to reducing the number of constraints and upon first consideration an attractive option is to collect the element stresses into a single maximum stress constraint as:

$$g(\rho) = \max_{e=1,..,n} (\sigma_e(\rho)) \leq \bar{\sigma}. \hfill (3.32)$$

However, the *maximum* operator is not differentiable, which prevents the derivation of analytical sensitivities, and must be smoothed. For this purpose, continuous aggregation functions have been adopted in the literature [33, 111, 112, 114, 115] to combine the local stresses into a single quantity. These techniques are known as *global* methods and variations of the Kresselmeier-Steinhauser (KS) function given by

$$\sigma_{KS} = \frac{1}{p} \ln \left[ \sum_{e=1}^{N_e} \exp \left( \frac{F(\sigma_e)}{\bar{\sigma}} \right) \right] \hfill (3.33)$$

or a basic $p$-norm measure

$$\sigma_{PN} = \left[ \sum_{e=1}^{N_e} \left( \frac{F(\sigma_e)}{\bar{\sigma}} \right)^p \right]^{1/p} \hfill (3.34)$$

are commonly utilized. A drawback to using a *global* method is that the aggregation functions serve as only an approximation to the maximum value in a set and the quality of the approximation generally degrades as the set size is increased. As a result, in
an optimization setting, while the global formulations streamline the problem, they cannot guarantee that the maximum stresses are maintained locally.

Recently, a number of researchers have investigated so called regional (also called block or clustered) stress measures [113, 116, 117, 118]. These techniques operate identically to global measures using aggregation functions, but rather than grouping the entire global domain, they aggregate multiple regions that canvas the entire domain. This results in multiple regionalized constraints that operate on sets of reduced size, and thus benefit from better approximation, at a fraction of the computational expense of local methods (tens of constraints versus tens or hundreds of thousands). A visual comparison of the local, global, and regionalized/block methods for stress constraints is given in Figure 3.17.

![Figure 3.17: Three methods to apply stress constraints in structural topology optimization.](image)

Building on the basic regional methods, a number of researchers have proposed techniques to further improve the approximation of stress constraints with respect to the actual local values. In [113], response information from previous design iterations is used improve local predictions at the current iteration, [118] adaptively sorts the sets of elements to improve the accuracy of each regional measure, and [119] treats potentially active and non-active local constraints separately while updating aggregation metrics throughout optimization. In each of these works, their adaptive mechanisms actually modify the basic optimization problem at various points. As a
result, an entirely new mathematical optimization problem is being solved after these variations occur, which can result in convergence difficulties for many optimization algorithms.

As previously noted, interest in stress-based criteria for topology optimization of late has been high. This has resulted in a number of applications using the methods described previously. Stress-based design criteria are found for functionally graded materials [120], Drucker-Prager stresses that account for different material strength in tension and compression [121], design-dependent self weight loads [122], cases of both compliance and stress [123], and even fatigue [124]. Boundary variation methods have also been demonstrated for stress-based topology optimization including both level set [125, 126, 127] and phase field [128] methods.

3.4 Literature Summary

From the preceding review, it is a clear that a variety of topology optimization methods exist in the literature, with the density-based formulations being the most common and most mature. Limited work related to the topology optimization of thermoeLASTIC problems exists, but a widely accepted objective function and problem setup has yet to be identified as is observed with minimum compliance for mechanical problems. Ultimately, several works question the validity of compliance when considering thermal loads and remark that an alternative formulation is necessary. In addition, in examples available in the literature, the application of thermoelastic topology optimization to structures of practical interest is not observed. In all cases, only simple problems with levels of thermoelastic loading that are comparatively benign to those found in modern aerospace thermal structures have been solved. This is likely due in large part to the issues surrounding the usual minimum compliance formulations that simply cannot reliably produce solutions in the presence of significant design-
dependent loads.

In addition, when considering the thermal structures of interest in this document, we recall that thermal stresses are a primary design consideration. In the literature, methods to treat stress-based criteria have been developed for mechanical loading and may present an effective formulation for thermal structures; however, these techniques have yet to be explored for thermal stresses. Finally, the possibility of addressing the combined thermal-structural environment with coupled heat transfer and structural analysis appears feasible based on literature examples. The method to do so can likely be inspired by the existing work related to the topology optimization of thermal microactuators.

In conclusion, it is evident that topology optimization may be an effective design tool for thermal structures design, but a suitable formulation for practical thermal structures design (including large-scale structures) has yet to be identified and demonstrated. This is especially true for the cases found in exhaust-washed structures design and other thermal structures applications with considerable amounts of restrained expansion. In the following chapters, novel research in the pursuit of an effective topology-based design solution for these cases and new methodologies for the topology optimization of thermoelastic structures is presented and demonstrated.
Chapter 4

Formulation of Thermoelastic Topology Optimization

Based on the review of topology optimization methodologies in Chapter 3, the density-based method has been selected for exploration in this work. Despite the lack of applications to thermal structures, the fundamental methods and algorithms for density-based topology optimization are well established compared to the alternatives. Thus, by using this method, the existing parameterization, filtering, and basic solution techniques can be employed for developing the appropriate topology optimization formulations to address the challenges of thermal structures and exhaust-washed structure design.

In this chapter, a formulation of density-based topology optimization including thermoelastic loading criteria is introduced. The derivation of design responses, including new responses that are not found in the literature, but are important for the thermoelastic problems in this research such as reaction loading, are given. In addition, the analytical sensitivity analysis for these responses with respect to density design variables are provided along with demonstration cases for validation.
4.1 Design Optimization

The topology optimization framework, including thermoelastic effects, that is introduced in this chapter is based on the typical process for structural optimization. A schematic for this process is given in Figure 4.1 and begins with an initial design configuration, an objective function to minimize, and a set of constraints that must be satisfied. The objective function and constraints are evaluated using a suite of analysis tools, which in this dissertation consists of the thermoelastic finite element analysis that is formulated in subsequent sections within this chapter. Sensitivity information regarding the objective and constraints are utilized by a numerical optimization algorithm to propose a new design. This process repeats in an iterative fashion, sometimes using a so-called approximate problem for increased efficiency, until convergence is observed in the objective function.

An important consideration in this process is related to the design dependency of thermal loading that was observed in Chapter 2. Contrary to topology optimization with mechanical loading, which is entirely independent of design variability, this dependency must be consistently captured in the sensitivity analysis. Finally, the mathematical statement of the design optimization problem for which the proceeding formulation is meant to address is given as:

\[
\begin{align*}
\text{min} & \quad f(\rho, U(\rho)) \\
\text{subject to} & \quad K(\rho)U(\rho) = F(\rho) \\
& \quad g_i(\rho, U(\rho)) \leq 0 \\
\text{variables} & \quad 0 < x_e \leq 1 \text{ for } e = 1, 2, ..., N
\end{align*}
\]

We note that this statement is similar to the general form of topology optimization problems given in the literature review by Equation (3.1); however, the parameteriza-
tion is stated in terms of the physical density variables $\rho_e$. The optimization problem is still solved in terms of the design variables $x_e$. We recall that the physical densities are a function of the design variables as $\rho_e(x)$ depending on the filtering or projection that is applied. The reader is referred to Section 3.2.2.2 for the available choices of filters. Throughout the remainder of the dissertation, when example results are given, care will be taken to provide details regarding the selection and application of the filter.

![Figure 4.1: Standard design optimization process utilized in structural optimization.](image)

**4.2 Implementation**

Based on investigations of current commercial optimization software that have topology optimization capabilities, it became evident that no tool was able to appropriately capture the contribution of thermal loading. In some cases (MD NASTRAN and VR&D Genesis), a fatal error occurred when a temperature load was assigned...
in the analysis step from which a topology optimization job was to get its design responses. In other software (Altair OptiStruct), a temperature load could be applied and appeared to be present in the finite element solution; however, based on the behavior of the solution process, it did not appear that the contribution of thermal loading was formulated in such a way to obtain useful results.

As a result, a custom implementation of the topology optimization formulations presented in the remainder of the dissertation was created. This includes the finite element analysis and adjoint sensitivity analysis for the design responses of interest in addition to the algorithms necessary for topology optimization (regularization, filtering, optimizer, etc.). Currently, these formulations have been developed in the engineering programming language MATLAB.

### 4.3 Finite Element Parameterization

Since the density method for topology optimization is selected, the finite element system must first be parameterized to accommodate the density-based design variables that are applied to designable finite elements. Here we develop the parameterization based on a generalized thermoelastic design domain. This domain $\Omega$ is shown for two dimensions in Figure 4.2 and contains fixed displacement boundary conditions, externally applied surface tractions, and a prescribed temperature change (with respect to a reference temperature) that may be uniform or spatially varying. The domain consists of regions of fixed void material, fixed solid material (non-designable), and designable areas whose topology is to be determined from optimization.

The design domain is discretized using $N$ finite elements with $N_d$ designable and $N_{nd}$ nondesign elements. Each designable element is assigned a design variable $x_e$ ranging from $0 < x_e \leq 1$ with $e = 1, 2, ..., N_d$. Together these variables form the design variable vector $\mathbf{x}$. The design variables $\mathbf{x}$ are related to the physical densities
\( \rho \) by the choice of a standard filter as described in Section 3.2.2.2.

Static equilibrium for a finite element representation of the domain, including both mechanical and temperature loading, is given by

\[
\mathbf{K}(\rho) \mathbf{U}(\rho) = \mathbf{F}(\rho)
\]  
(4.2)

where \( \mathbf{K}(\rho) \) is the global stiffness matrix, \( \mathbf{U} \) is the nodal displacement vector, and \( \mathbf{F}(\rho) \) is the nodal load vector. In the general case, \( \mathbf{F} \) consists of a combination of design-independent mechanical loading \( \mathbf{F}^m \) and design-dependent thermal loads \( \mathbf{F}^{th}(\rho) \) as

\[
\mathbf{F}(\rho) = \mathbf{F}^m + \mathbf{F}^{th}(\rho).
\]  
(4.3)

By taking the thermal load vector as a function of the density variables, the appropriate dependency of thermal loading on the design configuration is captured. The stiffness matrix \( \mathbf{K}(\rho) \) is assembled as the summation of element stiffnesses by

\[
\mathbf{K}(\rho) = \sum_{e=1}^{N_d} \mathbf{k}_e(\rho_e)
\]  
(4.4)
where
\[ k_e(\rho_e) = \int_{\Omega_e} B_e^T C_e(\rho_e) B_e d\Omega. \] (4.5)

Here, \( B_e \) is the element strain-displacement matrix, which consists of derivatives of the element shape functions that are independent of topology design variables. \( C_e \) is the element elasticity matrix, which for isotropic materials can be written as a linear function of elastic modulus as
\[ C_e(\rho_e) = E(\rho_e) \bar{C}_e \] (4.6)

where \( \bar{C}_e \) consists of constant terms related to the material constitutive matrix and \( E(\rho_e) \) is the elastic modulus of element \( e \) that is dependent on density variables. For nondesign elements, the previous equations can be utilized by simply assuming \( \rho_e = 1 \). The mechanical load vector \( F^m \) is assembled from externally applied forces on specific degrees of freedom. The thermal load vector \( F^{th}(\rho) \) is parameterized using the thermal stress coefficient (TSC) [72, 77] for topology optimization and described as follows.

The nodal load vector for a designable element \( e \) is given as
\[ f^{th}_e(\rho_e) = \int_{\Omega_e} B_e^T C_e(\rho_e) \epsilon^{th}_e(\rho_e) d\Omega \] (4.7)

where \( \epsilon^{th}_e(\rho_e) \) is the thermal strain vector for the element given by
\[ \epsilon^{th}_e(\rho_e) = \alpha(\rho_e) \Delta T_e \phi^T. \] (4.8)

Here, \( \alpha(\rho_e) \) is the thermal expansion coefficient that is also dependent on element density, \( \Delta T_e \) is the temperature change of element \( e \) (taken here as the average of nodal temperatures), and \( \phi \) is simply the vector \([1 1 0] \) or \([1 1 1 0 0 0]\) for two and three dimensional problems, respectively. Substitution of Equations (4.6) and (4.8)
into (4.7) yields
\[ f_{\text{th}}^e(x_e) = E(\rho_e)\alpha(\rho_e) \int_{\Omega_e} B_e^T \tilde{C} \Delta T_e \phi^T d\Omega \] (4.9)
in which we note that both $E(\rho_e)$ and $\alpha(\rho_e)$ are dependent on density design variables and thus both necessitate material interpolation. To simplify, we combine these parameters into a single thermal load coefficient as
\[ \beta(\rho_e) = E(\rho_e)\alpha(\rho_e). \] (4.10)
This quantity is then treated as an inherent material property and $f_{\text{th}}^e(\rho_e)$ can be rewritten as
\[ f_{\text{th}}^e(\rho_e) = \beta(\rho_e) \int_{\Omega_e} B_e^T \tilde{C} \Delta T_e \phi^T d\Omega. \] (4.11)
Finally, the global thermal load vector is assembled by summing element contributions
\[ \mathbf{F}^{\text{th}}(\rho) = \sum_{e=1}^{N} f_{\text{th}}^e(\rho_e). \] (4.12)
Again, the element thermal load vector for nondesign elements can be obtained from the previous relations by taking $\rho_e = 1$.

### 4.3.1 Interpolation Schemes

It is known that in the presence of design-dependent loading, which includes thermal loads, the SIMP interpolation scheme presents numerical difficulties as discussed in Section 3.3.2. As a result, in the presence of thermal loads, the RAMP model is adopted. Thus, the stiffness and thermal load are interpolated according to
\[ E(\rho_e) = \eta_E(\rho_e) E_o = \frac{\rho_e}{1 + R_E(1 - \rho_e)} E_o \] (4.13)
\[ \beta(\rho_e) = \eta_\beta(\rho_e) E_o = \frac{\rho_e}{1 + R_\beta(1 - \rho_e)} E_o \alpha_o \] (4.14)
where $R_E$ and $R_\beta$ are RAMP penalization parameters. In addition, $E_o$ and $\alpha_o$ are baseline material properties for the elastic modulus and coefficient of thermal expansion, respectively.

### 4.3.2 Adjoint Sensitivity Analysis

In topology optimization, it may be assumed that design responses (both objectives and constraints) take the form of functionals $F$ that depend upon both the density variables $\rho$ and the displacement response vector $U(\rho)$ as $F(\rho, U(\rho))$. Thus, the required gradients of the functional with respect to a density variable $\rho_j$ are obtained by

$$
\frac{dF}{d\rho_j} = \frac{\partial F}{\partial \rho_j} + \frac{\partial F}{\partial U} \frac{dU}{d\rho_j}.
$$

For many responses in structural optimization, the term $\partial F/\partial \rho_j$ is equal to zero; however, for some topology optimization formulations, like those involving the stress-based criteria presented in Chapter 6, this is not true so we retain it for generality.

The derivative of the displacement vector to the physical density $dU/d\rho_j$ is obtained by differentiating the governing equation shown previously in Equation (4.2). Doing so and rearranging yields

$$
\frac{dU}{d\rho_j} = K^{-1} \left( \frac{dF^{th}}{d\rho_j} - \frac{dK}{d\rho_j} U \right).
$$

Substituting the preceding relationship into Equation (4.15) results in

$$
\frac{dF}{d\rho_j} = \frac{\partial F}{\partial \rho_j} + \frac{\partial F}{\partial U} K^{-1} \left( \frac{dF^{th}}{d\rho_j} - \frac{dK}{d\rho_j} U \right),
$$

which gives the sensitivity of a general response functional $F$. Depending on the nature of the design optimization problem, Equation (4.17) can be solved using either the direct or adjoint method. Since topology optimization problems are characterized
by a large number of design variables with comparatively few constraints, it is most efficient to solve (4.17) via the adjoint method by introducing the adjoint variable $\lambda$ as

$$\frac{dF}{d\rho_j} = \frac{\partial F}{\partial \rho_j} + \lambda^T \left( \frac{dF^{th}}{d\rho_j} - \frac{dK}{d\rho_j} U \right).$$

(4.18)

The adjoint variable vector is determined by solving the adjoint system given by

$$K\lambda = \frac{\partial F}{\partial U},$$

(4.19)

which we note takes the same form as the governing state equations and can actually recycle parts of the solution from $K^{-1}$. Thus, by specifying the appropriate chain rule expansions of $\partial F/\partial U$, which generally relate an engineering design quantity to the displacement solution of the finite element analysis, the sensitivity of nearly any design response can be obtained.

### 4.4 Boundary Reaction Response

The reaction loads of a thermally loaded structure are important because they represent loading that is transferred to adjoining structures. In an effort to control this loading in topology optimization, in this section a new reaction load response for topology optimization is proposed. In the design of thermal structures, such a response can help reduce the load from restrained thermal structures that is exerted on surrounding components. In the broader field of topology optimization with purely mechanical loads, this response can be utilized to better control the load paths of a topologically derived structure as well as to appropriately distribute loading across several fasteners such as rivets or bolted connections that are limited in load capacity.

In this section, a technique is described to obtain the reaction load at specified degrees of freedom (or total reaction load summed across several degrees of freedom)
from finite element results. The sensitivity analysis is also derived using the adjoint
method. Finally, the reaction response and sensitivity analysis is validated with
numerical results and a simple minimum compliance topology optimization problem
with constraints on reaction loading is demonstrated.

### 4.4.1 Reaction Response and Reaction Sensitivity

After solution of the finite element system presented in Section 4.3, the reaction load
vector can be obtained from the resulting displacement and system matrices as

$$
R(\rho) = K(\rho)U(\rho) - F(\rho).
$$

(4.20)

Note that $R(\rho)$ contains the reaction loading for all degrees of freedom in the model.
Thus, the total reaction loads on a set of degrees of freedom $k$ can be found from

$$
S(\rho) = \sum_{i \in D} R_i(\rho)
$$

(4.21)

where $D$ is the set of indices corresponding to the degrees of freedom of interest. Note
that for the load at a single degree of freedom, the summation can simply be ignored.

The sensitivity of the reaction load $S$ to physical density variable $\rho_j$ can be ob-
tained from

$$
\frac{dS}{d\rho_j} = \sum_{i \in D} \frac{dR_i}{d\rho_j}
$$

(4.22)

where the sensitivity of the resultant load at a particular degree of freedom $dR_i/d\rho_j$
is developed as follows. We begin by differentiating Equation (4.20) with respect to
$\rho_j$ as

$$
\frac{dR}{d\rho_j} = \frac{d}{d\rho_j} (K(\rho)U(\rho) - F(\rho)) = \frac{dK}{d\rho_j} U + K \frac{dU}{d\rho_j} - \frac{dF^{th}}{d\rho_j}.
$$

(4.23)

The derivative of the displacement vector to the physical density $dU/d\rho_j$ is obtained
by differentiating the equilibrium equation as in Equation (4.16). Substitution into
Equation (4.23) yields

\[
\frac{dR}{d\rho_j} = \frac{dK}{d\rho_j} U + KK^{-1} \left( \frac{dF_{th}}{d\rho_j} - \frac{dK}{d\rho_j} U \right) - \frac{dF^i_{th}}{d\rho_j} 
\] (4.24)

which gives the sensitivity for the entire reaction load vector. The computational requirements can be reduced by only computing the sensitivity for the individual indices of the reaction \(dR_i/d\rho_j\) by

\[
\frac{dR_i}{d\rho_j} = \frac{dK_i}{d\rho_j} U + \lambda^T_i \left( \frac{dF^i_{th}}{d\rho_j} - \frac{dK_i}{d\rho_j} U \right) - \frac{dF^i_{th}}{d\rho_j} 
\] (4.25)

where the adjoint vector \(\lambda_i\) is found from the solution of the adjoint system, which takes the same form as Equation (4.19),

\[
K\lambda_i = K_i. 
\] (4.26)

In the previous relations, \(K_i\) is the column of the global stiffness matrix and \(F^i_{th}\) is the global thermal load entry that corresponds to degree of freedom \(i\). In addition, the adjoint system in Equation (4.26) can be solved for all right hand side terms \(K_i\) in a single matrix operation to further reduce effort.

### 4.4.2 Validation

The performance of the reaction load response and its analytical sensitivity analysis for density variables is validated in two demonstration exercise in this section. In each exercise, the density of a single element is continuously varied from 0 (no material) to 1.0 (solid) material and the effect on the total reaction load \(S\) is computed on a predefined set of finite element degrees of freedom according to Equation (4.21). The sensitivity of \(S\) is also computed according to the adjoint process outlined in Equations (4.22) to (4.26) and validated with finite difference results. The relevant
material properties for the exercise structures are taken as $E = 30 \text{ MPa}$, $\nu = 0.3$, $\alpha = 10.0 \times 10^{-6}/\degree \text{C}$ and element thickness $t = 1\ \text{cm}$. Also, in the first exercise a SIMP interpolation scheme is utilized with $p_E = 3$ and in the second exercise a RAMP interpolation is utilized with $R_E = 8$ and $R_\beta = 0$ due to the presence of thermoelastic effects. No filter is applied to the validation cases, which means that the density variables equal the design variables as $\rho_e = x_e$ for $e = 1, 2, \ldots, N_d$.

![Mechanical Load Reaction Validation Model](image1)

![Thermal Load Reaction Validation Model](image2)

Figure 4.3: Models for validation of the reaction response and sensitivity analysis for (a) mechanical and (b) thermal loads (dimensions in meters).

In the first exercise, a schematic of which is given in Figure 4.3a, consists of a structure modeled with four bilinear quadrilateral plane stress elements subjected to purely mechanical loading. The structure is fixed in $x$-direction (and free to contract in the $y$-direction under loading) at the left edge and a case of pure axial loading is applied to nodes on the right side in the $x$-direction. The density of the gray element in the structure is taken as the variable and the response of interest is the total reaction load $S$ of the upper two nodes on the left side of the structure, denoted by filled circles. By inspection, the total reaction load along all nodes on the left edge is equivalent to the $3\ \text{kN}$ applied load; however, depending on the density value of
the gray element, its load carrying capacity will vary, which leads to variation in the
distribution of reaction loads.

In the second exercise, given by the schematic in Figure 4.3b, the ability to capture
the proper variation and sensitivity of reactions due to thermal effects is validated.
The test structure is composed of two quadrilateral plane stress elements subjected
to an elevated temperature of 500°C. Displacement in the $x$-direction is prevent at
both the left and right edges of the structure (the structure is free to expand in the
$y$-direction). Again, the density of the gray element is taken as the variable and the
total reaction in the $x$-direction at the two nodes on the left edge of the structure is
the response of interest. When the gray element has a density of 1 (solid material),
the domain represents a fixed bar under axial loading, which from simple analytical
relationships will produce a total reaction load of 1.5 kN. However, as the the density
of the gray element is reduced, it contributes less thermal loading and the remaining
element is able to expand. Ultimately, we expect the reaction load to vanish as the
density of the gray element approaches zero because thermal expansion is then freely
accommodated.

The results of the first exercise are shown in Figure 4.4 where Figure 4.4a gives the
reaction load response as a function of density and Figure 4.4b gives the analytical
sensitivity and several points of finite difference validation. We observe in Figure
4.4a that when the density of the variable element equals zero, the total reaction
load in the degrees of freedom of interest (recall these are the solid blue nodes in
Figure 4.3a) equals $-3$ kN, or simply the opposite of the applied load. Intuitively,
this result is expected as the lower node is completely disconnected from the load
bearing body. As the density is increased, the magnitude of the reaction load in
the upper boundary nodes is reduced as the lower node begins to carry a portion of
the applied loads. Note that one might expect the load to be perfectly distributed
across the three boundary nodes when the variable element is solid; however, due to
boundary condition effects in the small model, the center node will always experience slightly greater reaction loading unless the applied loading is prescribed using a non-uniform distribution. Finally, we note that the adjoint sensitivity analysis is in nearly perfect agreement with the finite difference sensitivities in Figure 4.4b. These results indicate the reaction response is behaving appropriately within the physical context of the system for mechanical loading.

Similarly, the results of the second exercise, where loading is of thermal origin, are given in Figures 4.5a and 4.5b. In Figure 4.5a, we observe the boundary reaction is zero when the density of the variable element is zero and begins to increase as a function of density until the expected bounding case of a uniform axial bar with fixed ends and a reaction load of 1.5 kN. The increase in reaction load results from two contributing effects. First, the increasing stiffness of the variable element begins to restrain the expansion of the fixed solid element in the $x$-direction. This results in increasing internal thermal stresses and loads at the boundaries. Second, due to a linear parameterization of the thermal load vector in the finite element analysis (RAMP parameter $R_\beta = 0$), the thermal load contribution of the variable element increases linearly as its density increases. In fact, the slight nonlinearity in Figure 4.5a is due to the nonlinear RAMP interpolation on the element stiffness ($R_E = 8$). From a purely thermoelastic point-of-view, the relative contribution of the thermal load and stiffness parameterizations to the behavior of the reaction load is a telling observation. It is a clear demonstration of the dominance of thermal load contributions compared to the stiffness of the underlying material as the nonlinear parameterization on stiffness has only limited influence in the combined effect in the figure. Finally, as in the purely mechanical load case, the adjoint sensitivity of the reaction load is in perfect agreement with the finite difference validation points, which serves as a strong indicator of the correct performance of the reaction load response as derived here.
Figure 4.4: (a) Reaction load and (b) reaction load sensitivity validation for a case of purely mechanical loading.

### 4.4.3 Reaction Constraint Example

In this section, the effectiveness of the reaction response is demonstrated in a topology optimization example problem. Here, we wish to solve a standard minimum compliance, volume constrained problem with additional constraints on the reaction loading at prescribed locations. The mathematical statement of this optimization problem is given in Equation 4.27 where $S(\rho)$ is the reaction load on a specified set of degrees of freedom that should remain below the value given by $R_0$.

\[
\begin{align*}
\text{min :} & \quad c(\rho) = U(\rho)^T K(\rho) U(\rho) \\
\text{subject to :} & \quad g_1(\rho) = \sum_{e=1}^{N} (\rho_e v_e - V_f v_e) \leq 0 \\
& \quad g_2(\rho, U) = S(\rho, U) \leq R_0 \\
& \quad K(\rho) U(\rho) = F^m \\
\text{variables :} & \quad 0 < x_{min} \leq x_e \leq 1 \text{ for } e = 1, 2, \ldots, N
\end{align*}
\]
Figure 4.5: (a) Reaction load and (b) reaction load sensitivity validation for a case of purely thermal loading.

The topology optimization domain for this problem is given in Figure 4.6a and consists of a 1 meter by 1 meter square domain with a 3000 kN load applied in the center of its top edge. In this case, no thermal loading is present. Boundary conditions of fixed degrees of freedom on the lower corners and central bottom edge of the domain are indicated by the black regions in Figure 4.6a. The structural domain is discretized using a 64 by 64 element mesh of bilinear plane stress quadrilateral finite elements. Figure 4.6b shows the details of boundary condition and load application. We note that the 3000 kN load is distributed across 3 nodes and the boundary conditions consist of fixed translations in both the vertical and horizontal directions at the nodes shown. The mechanical properties for the test case are taken as $E = 210$ GPa, $\nu = 0.30$, and element thickness $t = 5$ cm.

The topology optimization problem is then to fill at most 25% of the domain with solid material such that the compliance is minimized. The reaction constraint $g_2$ from the statement in Equation (4.27) is prescribed on the total reaction load in the vertical direction for the 5 nodes located at the bottom center of the structure. The optimization problem is performed with values of the constraint limit $R_o$ of 0, 750, 1500, 2250, and 3000 kN to demonstrate the ability to control the reaction load. The
MMA optimizer is utilized to solve the optimization problem, which contains 4096 designable elements. In addition, no filtering is applied to the problem.

The density distributions resulting from each optimization case are given in Figure 4.7 along with the compliance value for the corresponding structures. The gray dotted lines in each sub-figure represent the outer edge of the designable region. Based on static equilibrium, we know the total reaction load must be equal to the applied 3000 kN loading. In addition, minimizing compliance in this domain is equivalent to minimizing the displacements at the nodes to which the applied load is prescribed. Thus, it is no surprise that the design which results for a reaction constraint of $R_o = 3000$ kN is one where all of the material is located in the middle of the domain with a structure mimicking an axially loaded bar as shown in Figure 4.7a. As the allowable reaction loading at the central boundary is reduced, some amount of the loading must be transferred to the boundaries at the corners of the domain. We observe in Figures 4.7b, 4.7c, and 4.7d that this is accomplished by a simple three member structure.
with the size of the structural members that connect the applied loads to the corner boundaries depending on the allowable reactions. As the center allowable reaction constraint is tightened, the diagonal connecting members are stiffened to achieve the proper load paths. Finally, when no reaction loading is permitted in the central boundary, the middle member disappears completely, as shown in Figure 4.7e.

(a) $R_o = 3000 \text{ kN}, c = 4617 \text{kN} \cdot \text{m}$  (b) $R_o = 2250 \text{ kN}, c = 4856 \text{kN} \cdot \text{m}$  (c) $R_o = 1500 \text{ kN}, c = 5085 \text{kN} \cdot \text{m}$

(d) $R_o = 750 \text{ kN}, c = 5801 \text{kN} \cdot \text{m}$  (e) $R_o = 0 \text{ kN}, c = 6219 \text{kN} \cdot \text{m}$

Figure 4.7: Topology optimization results for the constrained reaction load example problem.

The iteration history of the objective function (compliance) and the reaction load responses for each case are given in Figure 4.8. We note that a penalty in objective performance is paid by tightening the reaction load constraint due to the mechanics of the problem at hand. Also, we note that in all cases with the exception of $R_o = 2250$ kN the reaction constraint is active. While the reaction load constraint is satisfied
in this case, it is not active due to the limited variability in the design imposed by the finite element discretization. In order to make the constraint active, less material should be utilized in the diagonal connecting members (with the excess material then being placed on the central member). In order to realize this, a finer mesh should be used to accommodate the formation of thinner structural members.

Figure 4.8: Iteration histories for the (a) compliance objective function and (b) reaction load responses for the constrained reaction load example problem.

4.5 Chapter Summary

In this chapter, a formulation suitable for appropriately capturing the design dependency of thermoelastic loading in topology optimization was presented. In addition, a novel reaction load response was presented, validated, and demonstrated that is useful for constraining the reaction loading at structural boundaries in topology optimization. In the next chapter, the topology optimization capabilities presented here are utilized in the development of new topology optimization problem formulations for the design of exhaust-washed structures and other thermal structures subjected to restrained expansion.
Chapter 5

Design of Exhaust-Washed Structures via Topology Optimization

In this chapter, the thermoelastic topology optimization formulation in Chapter 4 is utilized for the design of two characteristic exhaust-washed structures. Most significantly, it is demonstrated that the usual topology optimization problem with a minimum compliance objective is not appropriate in the presence of design-dependent thermoelastic loading and design cases related to exhaust-washed structures. As a result, alternative problem formulations are investigated and a novel formulation utilizing the reaction response in Section 4.4 is proposed and demonstrated to be effective for the class of design problems where the goal is to develop stiffening concepts for pre-existing thermal structures. The goal of the first demonstration problem is to design a stiffening structure that can be attached to the beam strip first investigated in Section 2.2. In the second problem, the design space is expanded to include a larger portion of the exhaust-washed structure and various levels of design freedom are demonstrated including both stiffening structure development as well as the ca-
pacity for complete substructure design via topology optimization.

5.1 Beam Strip Stiffening Application

5.1.1 Introduction

From the findings in Section 2.3, it is obvious that changing only the size of thermally loaded structures may not provide a suitable design solution for thermal stresses. While it was observed that reducing the thickness of a thin thermally restrained shell may decrease thermal stresses, this likely is not possible in practice due to the detrimental effect on other design criteria such as natural frequency. In this section, topology optimization techniques to stiffening thin structures subjected to restrained thermal expansion are investigated. First, the desired structural behavior for achieving a reduction in thermal stresses is demonstrated on the beam strip model introduced previously in Section 2.2.4. Next, three formulations of topology optimization are presented. The first formulation is the basic minimum compliance formulation, which exhibits severe deficiencies in the design cases presented here. The second two formulations are developed with the specific goal of generating topological designs with desirable thermoelastic behavior.

5.1.2 Displacement-stress Relationship

In Section 2.3, it was observed that simply adding structural material in an attempt to stiffen a restrained thermal structure is not only ineffective, but can actually be detrimental to both the primary component and any structures it is affixed to. It was also identified that the tensile stresses in critical locations of the model are a result of the out-of-plane displacement of the structure. A short investigation to identify potential criteria required for an effective stiffening technique is now presented. In this study, a nonlinear solution sequence in MD Nastran is utilized (for clarity, in the
topology optimization demonstration that follows, the custom implementation from
Chapter 4 is used).

Since the fully clamped edge scenario, or an infinite stiffness parameters \( k_a \) and
\( k_r \), seemed to place an upper bound on the detrimental effects of material addition,
we select the fully clamped boundary condition with a thickness of \( t = 0.16 \) inches,
span of \( L = 12 \) inches and curvature measure of \( \delta = 0.5 \) inches (corresponding to
radius of 144 inches). These parameters correspond to a thickness and curvature
ratio of 0.0133 and 0.0417 for reference to geometry in Section 2.2.4. The thermal
load is applied to the model by specifying \( E\alpha T = 60 \times 10^3 \) and the equilibrium state
is determined. A series of enforced displacement conditions are then imposed on the
structure so as to incrementally return it to the undeformed state. At each step of
this process the stress in the beam and the boundary reaction loads are measured
and plotted in Figures 5.1 and 5.2, respectively.

![Figure 5.1: Stress at root of beam as a function of out-of-plane displacement at the
center of beam strip.](image)

In the plots, the displacement is measured at the center of the curved strip model,
which corresponds to the location of maximum out-of-plane displacement. From
Figure 5.1, we observe that significant gains in stress reduction can be obtained by
reducing the out-of-plane displacement. For example, from the deformed state with

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Figure 5.2: (a) Boundary reaction load in horizontal direction and (b) reaction moment as a function of out-of-plane displacement at the center of beam strip.

a center displacement of approximately 0.42 inches a 50% reduction in stress can be achieved by reducing the displacement to between 0.30 and 0.35 inches. Based on further investigations of varying geometry, this general observation can be extended to geometries with any curvature, thickness, and length in addition to responses at different stress levels. However, physically it is impossible to reduce the displacement as is done here, but through this exercise it is readily evident that one mechanism that can reduce thermal stresses induced by restrained thermal expansion is to reduce out-of-plane deformation.

One challenge associated with attempting to reduce stress by displacement reduction is observed in Figure 5.2a. In this plot we note that reducing deformation increases the boundary reaction load significantly. This stems from the reorientation of the thermal loads generated via restrained expansion by the enforced displacement conditions. This reorientation can be thought of as a transition from significant bending effects at greater deformation to significant axial (or “in-plane” type) effects as deformation disappears. This behavior is supported by Figure 5.2b, which shows a reduction in reaction bending moment at the root of the strip as deformation is decreased. In a physical application, this increase in reaction loads corresponds directly
to increases in shear loading into fasteners at structural joints and significantly higher loads that are exerted on adjoining sub- and supporting structures.

With these observations in mind, it becomes clear that to achieve the desired stress reduction without significant reaction increases, the design space must be expanded past modification of the properties of the thin structure itself, which in design optimization would be considered simple sizing optimization. Thus, structural topology optimization is applied to generate a stiffening structure that can simultaneously reduce stress in critical locations and limit reaction loading. Such a solution requires the satisfaction of several competing design criteria and managing both the amount and direction of thermal expansion within the structure.

5.1.3 Functional Topology Design Space

Figure 5.3 shows a section of representative exhaust structure from which the beam strip in the previous demonstration was extracted. Here we see the beam strip structure subjected to thermal loading from hot exhaust gases and its fixivity to substructure. In this configuration, the underside of the exhaust-washed structure consists of open bays that are absent any obstructing substructure or other subsystems. Thus, it is assumed that in this region a stiffening structure can be designed that is affixed to the underside of the exhaust-washed structure to reduce thermal stresses. The proposed design domain for this case is shown in Figure 5.4.

Figure 5.3: Two-dimensional schematic of exhaust-washed structure and substructure region.
This design concept can be envisioned in three-dimensions by referring to the EWS concept configuration previously shown in Figure 1.4. In practice, any open region (or open internal bay) between the exhaust-washed nozzle surface and the outer aircraft skins could be utilized as a topology design space in which to develop a stiffening structure. In fact, topology optimization could potentially be utilized to design all of the structure, including integral supporting and sub-structures, as is done in the second demonstration problem contained in this chapter.

5.1.4 Topology Optimization Problem Formulations

In the following subsections, three topology optimization methods are presented using the two-dimensional strip example. Note that the methods demonstrated can be readily applied to other geometries by following the simple rules specified for how to setup loading conditions and optimization responses for each case.

5.1.4.1 Minimum Compliance with Thermal Loading

The first formulation investigated is the basic minimum compliance (maximum stiffness) objective with a volume fraction constraint. As discussed in the literature review in Section 3.3.2, the validity of the compliance objective for topology optimization with thermal loads has been questioned. It is nonetheless investigated here to demonstrate why it fails to yield useful results for the problems of interest. The
mathematical statement of the topology problem for minimum compliance is given as

\[
\begin{align*}
\min : & \quad c(\rho) = F(\rho)^T U(\rho) = U(\rho)^T K(\rho) U(\rho) \\
\text{subject to} : & \quad g(\rho) = \sum_{e=1}^{N_d} (\rho_e v_e - V_f v_e) \leq 0 \\
& \quad K(\rho) U(\rho) = F^{th}(\rho) \\
\text{variables} : & \quad 0 < x_{\min} \leq x_e \leq 1 \text{ for } e = 1, 2, ..., N_d
\end{align*}
\] (5.1)

where \( c \) is the compliance, \( \rho_e \) is the physical density of element \( e \) (related to the design variable \( \mathbf{x} \) through a filter), \( v_e \) is the volume of element \( e \), \( V_f \) is the allowable volume fraction, \( N_d \) is the total number of designable elements and \( x_{\min} \) is the minimum allowable value of design variables. Note that \( F^{th}(\rho) \) is the thermal load vector resulting from the application of prescribed elevated temperature. Since no external loading is applied, the load vector as defined in Section (4.3) simply becomes \( F(\rho) = F^{th}(\rho) \). The sensitivity of the compliance objective to the physical density \( \rho_j \) is derived as

\[
\frac{dc}{d\rho_j} = \frac{d}{d\rho_j} (F^T U) = \frac{dF^T}{d\rho_j} U + F^T \frac{dU}{d\rho_j}.
\] (5.2)

Substituting \( dU/d\rho_j \) from (4.24) into (5.2) yields,

\[
\frac{dc}{d\rho_j} = \left( \frac{dF^{th}}{d\rho_j} \right)^T U + \left( F^{th} \right)^T \left( \frac{dF^{th}}{d\rho_j} - \frac{dK}{d\rho_j} U \right)
\] (5.3)

where we note the compliance problem is self-adjoint such that \( (F^{th})^T K^{-1} = U^T \). Equation (5.3) then simplifies to

\[
\frac{dc}{d\rho_j} = U^T \left( 2 \frac{dF^{th}}{d\rho_j} - \frac{dK}{d\rho_j} U \right).
\] (5.4)
The sensitivity of the volume constraint is given by

\[
\frac{dg}{d\rho_j} = \frac{d}{d\rho_j} \left( \sum_{e=1}^{N_d} (\rho_e v_e - V_f v_e) \right) = \frac{d}{d\rho_j} \left( \sum_{e=1}^{N_d} \rho_e v_e \right).
\]  \tag{5.5}

Finally, we note that the sensitivity equations in (5.4) and (5.5) are given with respect to physical densities \(\rho_j\). In the optimization, the sensitivity with respect to design variable densities \(x_j\) are required, which take into account the effects of filtering and projection. Here, the Heaviside projection filter is employed so the appropriate chain rule modifications from Section 3.2.2.2 for the density filter (Equation (3.6)) and Heaviside projection (Equations (3.8) and (3.9)) are used to obtain the sensitivity of the previous design responses to the design variables.

Figure 5.5 shows the design domain with thermal loading for the beam strip demonstration case. Here we note loading consists of only a prescribed temperature distribution that may be spatially varying in the general case.

5.1.4.2 Artificial Mechanical Load Method

The second topology optimization formulation takes a different approach to stiffness design for restrained expansion. It was first demonstrated by Haney [16] and attempts to derive a structure using only mechanical loading that behaves favorably in a thermal environment. The mathematical statement here takes the form of a basic
minimum compliance case with mechanical loading:

\[
\begin{align*}
\min & \quad c(\rho) = F(\rho)^T U(\rho) = U(\rho)^T K(\rho) U(\rho) \\
\text{subject to} & \quad g(\rho) = \sum_{e=1}^{N_d} (\rho_e v_e - V_f v_e) \leq 0 \\
& \quad K(\rho) U(\rho) = F^a \\
\text{variables} : & \quad 0 < x_{min} \leq x_e \leq 1 \text{ for } e = 1, 2, ..., N_d
\end{align*}
\]

We note that there is no thermal load and \( F^a \) is an externally applied artificial mechanical load. To determine the application of this load, we recognize the fact that the optimum structure resulting from a problem of the form in (5.6) has a material distribution to best resist the applied load \( F^a \). Simply put, a structure is developed that is resistant to deformation in the direction of the applied load, but has very little stiffness in any other direction. Recalling the potential design mechanism identified in Section 5.1.2, which was to reduce out-of-plane deformation, it follows that this may be accomplished by applying an artificial mechanical load (in the absence of any thermal loading) in the direction we wish to reduce displacement. Logically, it also follows that since the mechanically derived structure has little stiffness in off-load directions, it is rendered incapable of generating large reaction forces because the added stiffening material is not likely aligned in a path to do so. This is in perfect contrast to the case of a pure thickness increase, where added material contributes directly to reaction loads.

While intuitively suitable designs may be generated if the artificial loads are applied in the proper direction, in practice the thermoelastic performance of the resulting designs is sensitive to how the artificial loads are applied to the design model. This limitation was not discussed in prior work regarding this technique [16]. In the beam strip stiffening demonstration example, one may choose to apply artificial loading as a uniform distributed load or fewer discrete loads as shown in Figure 5.6. While it will
be demonstrated that different artificial load configurations yield varying thermoe-elastic performance, this method is still attractive because of its simplicity. In fact, since thermal loading is not taken directly in the topology optimization, commercial software can be utilized for this formulation.

Finally, the sensitivity computations in this problem are identical to those in Equations (5.4) and (5.5) when the design-dependent load vector terms are neglected.

![Beam strip stiffening topology optimization domain with two different load sets for the artificial mechanical load formulation.](image)

**Figure 5.6:** Beam strip stiffening topology optimization domain with two different load sets for the artificial mechanical load formulation.

### 5.1.4.3 Thermoelastic Combination Method

The final optimization formulation is an extension of the previous artificial mechanical load method where the reaction response from Section (4.4) is employed to gain better control of the reaction loading that results from thermal loading at the attachment points of the structure. In formulating the topology optimization problem, the minimum compliance objective function is retained with the compliance determined in the absence of thermal loading and subjected to artificial loading. In addition, a separate thermoelastic analysis is performed where loading consists of the original temperature field. The reaction loading computed in this analysis is then utilized to directly enforce constraints in the optimization problem. Thus, the mathematical statement is given as:
\[ \min \quad c(\rho) = U_1(\rho)^T K(\rho) U_1(\rho) \]

subject to:
\[ g_1(\rho) = \sum_{e=1}^{N_d} (\rho_e v_e - V_f v_e) \leq 0 \]
\[ g_2(\rho, U_2) = S_L(\rho, U_2) \leq R_o \]
\[ g_3(\rho, U_2) = S_R(\rho, U_2) \leq R_o \]  
(5.7)
\[ K(\rho) U_1(\rho) = F^a \]
\[ K(\rho) U_2(\rho) = F^{th}(\rho) \]

variables: \[ 0 < x_{\text{min}} \leq x_e \leq 1 \text{ for } e = 1, 2, ..., N_d \]

In Equation (5.7), we note there are two finite element analyses with \( U_1 \) being the displacement vector for the system with artificial mechanical load \( F^a \) and \( U_2 \) being the displacement vector for the system with thermal load \( F^{th}(\rho) \). The compliance objective is computed from \( U_1 \) and is only dependent on the artificial load. \( S_L \) and \( S_R \) are the total reaction loads summed over the fixed degrees-of-freedom in the \( x \)-direction at the left and right boundaries, respectively. We note that these depend only on the results of the thermoelastic system \( U_2 \).

To summarize, Figure 5.7 shows the topology optimization domain for the beam strip stiffening problem with both the artificial load system and the thermoelastic system. Here the arrangement of artificial load is arbitrary and multiple configurations will be tested.

### 5.1.5 Design Problem Setup

The discretized finite element model for the beam strip demonstration problem corresponding to the design domain shown in Figure 5.4 is given in Figure 5.8. Both the non-design and designable regions are meshed using four-node bilinear quadrilateral...
Figure 5.7: Beam strip stiffening topology optimization domain with two separate finite element analysis required to obtain design responses.

elements (in plane stress). A total of 7800 elements are utilized (7500 designable). The elements are assumed to have unit depth in the plane. Their dimensions are approximately 0.08 by 0.08 inches, but due to the curvature of the top of the design domain they are not perfectly square/rectangular. Finally, boundary conditions consist of clamped conditions applied at the edge of the non-design domain represented by fixed nodal translations. Any boundary that forms inside the the designable region as the topology evolves is assumed to remain traction free. The MMA optimization algorithm [129] is utilized for all cases. Finally, the Heaviside projection filter with a filter radius of 0.16 inches and initial $\gamma = 0$.

5.1.6 Topology Optimization Results

Minimum Compliance with Thermal Load

The result for the minimum compliance topology optimization problem with purely thermal loading outlined in Equation (5.1) is shown in Figure 5.9a, where the dotted
Figure 5.8: Finite element mesh for the beam strip stiffening example problem with non-design elements (dark gray) and designable elements (white).

line indicates the boundary of the designable region. In this case, the RAMP parameters are $R_E = 8$ and $R_\beta = 0$ and the allowable volume fraction is taken as 0.20. The prescribed temperature distribution is taken as $\Delta T = 900^\circ F$ uniformly throughout the structure. We observe that the optimum structure contains only trace amounts of added material near the application of the boundary conditions. This occurs due to the participation of thermal loading directly in the compliance objective. In this scenario, where thermal loads heavily dominate mechanical effects, any material addition, with the exception of in localized regions that may reduce deformation in the non-design domain, actually increases the compliance of the entire structure. This becomes obvious in the iteration history of the compliance and volume shown in Figure 5.9b. It is readily apparent that to achieve a minimum compliance design, as little material as possible should be utilized.

To further demonstrate, the sensitivity of the compliance objective to the element design variables is shown in Figure 5.10. Sensitivities for a design variable field with all elements equal to 0.001, 0.10, and 0.20 are provided. We see in each the majority of elements, with the exception of small localized regions which were retained in the optimum structure in Figure 5.10, have positive gradients. This indicates that the existence of the elements, and the accompanying increased thermal load, serves to only increase compliance.

From these results it becomes evident that in cases with significant thermal loads
Figure 5.9: (a) Resulting topology and (b) iteration history for the minimum compliance with thermal load problem formulation.

and an absence of mechanical effects, the minimum compliance topology optimization formulation is unable to produce suitable designs. In fact, the majority of thermoelastic topology optimization literature, including that highlighted in the literature review in Chapter 3, only investigates cases where the amount of thermal loading is benign compared with mechanical loading and thus these effects go largely unnoticed. With the fictitious mechanical load and thermoelastic combination results that follow, we demonstrate that alternative approaches to the thermoelastic problem can lead to usable results.

**Artificial Mechanical Load**

Figure 5.11 shows the optimum topology designs obtained via the artificial mechanical load method of Equation (5.6) for the two load sets in Figure 5.6. In addition, the results from using two volume fractions, 0.20 and 0.30, are shown. The RAMP parameter for all cases of this method is taken as $R_E = 8$ (note the value of $R_\beta$ does not matter since the thermal loading has been removed).

In contrast to the previous method, it is obvious that potentially useful stiffening structures are obtained. This is expected because a well posed minimum compliance with purely mechanical loading problem was utilized. We note that in each case,
optimum structures span the entire depth of the designable region and contain a lower inverted arch structure. This inverted arch is connected to the upper non-design region at locations where the fictitious mechanical loads are applied. We also note that increasing the allowable volume fraction results in an identical design with thicker members. These results are characteristic of those obtained when using the minimum compliance objective. The iteration history for the compliance objective and volume fraction constraint is given in Figure 5.12 (results shown for volume fraction of 0.20 only). Note the sharp jumps in responses correspond to iterations when the $\gamma$ parameter in the Heaviside filter is increased.

To investigate the thermoelastic performance of each structure, a separate analysis was performed wherein the fictitious loads were removed, and the structures were subjected to a uniform elevated temperature of $\Delta T = 900^\circ F$. The reaction ratio
Figure 5.11: Topology optimization results for volume fractions of 0.20 (left column) and 0.30 (right column) for fictitious load case 1 (a,b) and case 2 (c,d).

Figure 5.12: Iteration histories for the artificial mechanical load method with load cases 1 and 2 and volume fraction of 0.20.

$R/R_o$, which here is taken as the reaction load for a particular design divided by that of the unstiffened non-design domain (i.e., the original strip) was obtained to assess the designs from a reaction increase perspective. $R/R_o$ for the designs in Figure 5.11 are (a) 2.18, (b) 2.32, (c) 1.73, and (d) 1.81. From these results, we observe that in these cases, allowing more material utilization will increase the reactions. It is also apparent that some configurations of fictitious loading may lead to superior designs from a thermoelastic point-of-view. The results obtained from the thermoelastic combination method presented in the next method further investigate this conclusion.
Thermoelastic Combination

Figure 5.13 gives the results for the thermoelastic combination method problem formulation from Equation (5.7) for reaction constraints corresponding to reaction ratios of $R/R_o$ of 1.25, 1.50, and 1.75. The topological designs using both of the fictitious load cases from the previous section are presented and thermal loading again consists of a uniform temperature distribution of $\Delta T = 900^\circ F$. The allowable volume fraction is taken as 0.2 and the RAMP parameters are $R_E = 16$ and $R_\beta = 2$.

Figure 5.13: Topology optimization results for the thermoelastic combination problem with constraints on reaction load corresponding to $R/R_o$ of (a,b) 1.25, (c,d) 1.50, and (e,f) 1.75.

It is observed that for low allowable reaction loading ($R/R_o = 1.25$) in Figure 5.13a and 5.13b that nearly identical designs are obtained for both fictitious load cases. This indicates that by including direct consideration of reaction loading in the design problem, the sensitivity of the results to the application of the fictitious load cases can be removed. It also appears that a structure has been generated in
which the in-plane expansion of the upper non-design domain and the lower arch-like structure, which is now more rounded when compared to previous designs in Figure 5.11, is counteracted by the mechanics of the internal connecting members. This apparent tailoring of thermal expansion allows for the satisfaction of tighter limits on reaction load.

If the reaction constraint is relaxed such that $R/R_o$ is 1.50, we observe from Figure 5.13b and 5.13c that the resulting structures still contain fundamentally the same topology. The designs now contain less complex internal connecting members that in fact, begin to resemble those in Figure 5.11c. Continuing to relax the reaction constraint to allow $R/R_o$ of 1.75 in Figure 5.13e and 5.13f leads to structures that differ for each fictitious load case and begin to resemble those obtained by the previous method. This is to be expected as the allowable reactions are now close to the values observed previously and the significance of reactions to the design problem has been reduced. The primary differences in these results when compared to those in Figure 5.11 is that the internal connecting members appear to be angled slightly more towards the horizontal and the structures do not span the entire depth of the designable region. This demonstrates that as one seeks to obtain stiffening structures that lead to more benign reaction loading, additional information must be directly included in the design problem because it is difficult, if not impossible, to identify which load case will lead to suitable results a priori.

The iteration history of the compliance, volume constraint, and reaction constraint for each case is given in Figure 5.14. Again, large jumps in the plots that occur roughly every 50 iterations are a result of continuation in the Heaviside filter. It is notable from this figure that by introducing thermal loading into the problem by way of the reaction constraint, rather than directly in the compliance objective (as was done with the minimum compliance with thermal load problem) the volume constraint remains active in the final design, which helps lead to a solid/void design. It should also
be commented that convergence is much less smooth in these cases when compared to that observed in Figures 5.11b and 5.12 for the prior two problem formulations. By closely inspecting the numerical behavior of the reaction load constraint, it was observed that it is a highly sensitive quantity in elements with near zero density. In addition, the sensitivity for some elements is not unconditionally positive (increasing density increases the reaction loading) as one might expect. Thus, as the optimization process begins to approach a topology with large regions of void material (nearly zero density), small oscillations appear in convergence. However, as observed in the iteration history, these oscillations do not cause instabilities that impede convergence.

![Iteration history for compliance, volume constraint, and reaction constraint corresponding to the topology thermoelastic combination topology optimization problems](image)

Figure 5.14: Iteration history for compliance, volume constraint, and reaction constraint corresponding to the topology thermoelastic combination topology optimization problems
5.1.7 Qualitative Assessment

In this section the thermoelastic qualities of the structures produced by the fictitious mechanical load and thermoelastic combination methods are studied. Designs are compared in terms of the reaction ratio $R/R_o$ and the displacement ratio $U/U_o$. Similar to the reaction ratio, the displacement ratio is defined as the out-of-plane displacement measured at the top center node of the non-design domain for a particular stiffened design to that of the non-design domain with no stiffening material. These metrics are given in Table 5.1, and also provide insight into the unique mechanics by which they accomplish a reduction in deformation while limiting reaction load.

<table>
<thead>
<tr>
<th>Topology Problem</th>
<th>Figure</th>
<th>Reaction Ratio ($R/R_o$)</th>
<th>Displacement Ratio ($U/U_o$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Artificial load (Case 1), $V_f = 0.20$</td>
<td>Figure 5.11a</td>
<td>2.18</td>
<td>-0.0470</td>
</tr>
<tr>
<td>Artificial load (Case 1), $V_f = 0.30$</td>
<td>Figure 5.11b</td>
<td>2.32</td>
<td>-0.0646</td>
</tr>
<tr>
<td>Artificial load (Case 2), $V_f = 0.20$</td>
<td>Figure 5.11c</td>
<td>1.73</td>
<td>-0.0567</td>
</tr>
<tr>
<td>Artificial load (Case 2), $V_f = 0.30$</td>
<td>Figure 5.11d</td>
<td>1.81</td>
<td>-0.0540</td>
</tr>
<tr>
<td>Thermoelastic Combo (Case 1), $R/R_o = 1.25$</td>
<td>Figure 5.13a</td>
<td>1.25</td>
<td>0.5303</td>
</tr>
<tr>
<td>Thermoelastic Combo (Case 2), $R/R_o = 1.25$</td>
<td>Figure 5.13b</td>
<td>1.25</td>
<td>0.5593</td>
</tr>
<tr>
<td>Thermoelastic Combo (Case 1), $R/R_o = 1.50$</td>
<td>Figure 5.13c</td>
<td>1.50</td>
<td>-0.0364</td>
</tr>
<tr>
<td>Thermoelastic Combo (Case 2), $R/R_o = 1.50$</td>
<td>Figure 5.13d</td>
<td>1.50</td>
<td>-0.0425</td>
</tr>
<tr>
<td>Thermoelastic Combo (Case 1), $R/R_o = 1.75$</td>
<td>Figure 5.13e</td>
<td>1.75</td>
<td>-0.0547</td>
</tr>
<tr>
<td>Thermoelastic Combo (Case 2), $R/R_o = 1.75$</td>
<td>Figure 5.13f</td>
<td>1.75</td>
<td>-0.0630</td>
</tr>
</tbody>
</table>

Table 5.1: Reaction and displacement comparison for all structural topologies for strip stiffening example.

We note from the table that in all cases, the displacement ratio is reduced. This indicates that the minimum compliance problem using the fictitious mechanical load cases does result in structures that are resistant to deformation out-of-plane and satisfy the basic design goals for displacement reduction. This holds true even when the structure is subjected to the elevated temperature environment after having been derived using mechanical loads. As evidenced by negative displacement ratios, for all of the designs, excluding those with the tightest reaction constraints, deformation at the middle of the non-design domain actually occurs downwards. This indicates the me-
mechanics that produce reduction in displacement, when compared to the undeformed case, result in a pull-down effect due to the expansion of the stiffening structure. For the designs produced by the thermoelastic combination method and reaction ratio limited to 1.25, we note the non-design domain still deforms in the positive direction, but only do so with roughly half the magnitude of the unstiffened structure. In practical application, this halving of displacement may result in removing nearly all tensile stresses as observed in the demonstration case in Section 5.1.2. As an added benefit, this stress reduction could come at only a 25% increase in reaction loading. This effect is achieved only by harnessing the potential thermoelastic tailoring capabilities of topology optimization when the proper responses are included in the problem formulation.

5.2 EWS Example Problem

In the previous section, the ability to utilize topology optimization to design potential stiffening layouts for thermally loaded components was investigated. It was demonstrated that to obtain suitable results, alternative formulations of the topology optimization problem were required to circumvent the limitations of the standard minimum compliance formulation with design-dependent thermal loading. In this section, a second demonstration problem is solved via the alternative artificial mechanical load formulation; however, this time the design space is expanded to include greater portions of exhaust-structure. In addition, the resulting stress distributions in the designs obtained via topology optimization are explored.

5.2.1 Introduction

The structure investigated here is taken as a 2D idealization of the conceptual exhaust-washed structure studied in Section 2.1. Figure 5.15 shows this idealization with the
two-dimensional section taken from the upper aft region of the structure. Contrary to the composite materials used in Section 2.1, the properties of the structure are now taken entirely as Ti-6242 [31]. It is assumed that the hot exhaust gases bring the exhaust-washed surface to an elevated temperature of 900°F. Since cooling mechanisms would likely prevent the attached structures from reaching such a temperature, the rest of the structure is assumed to remain at 300°F. The material properties in each region are taken at the prescribed elevated temperature.

Figure 5.15: (a) Reference exhaust-washed structure configuration and (b) idealized 2D section to be designed in this section.

The stress response of the idealized structure subjected to the elevated temperature environment and fixed boundary conditions at its corners is given in Figure 5.16.
Note that the response was obtained by nonlinear finite element analysis using MD NASTRAN and the structure has been discretized using two node beam elements, which is sufficient to capture the stress response based on mesh convergence. The deformation in the figure has also been magnified by a factor of 10. We note that the maximum stress is observed in the exhaust-washed surface structure, just as predicted in the models of Chapter 2, and exceeds 80 ksi. This stress level is greater than the allowable yield stress of approximately 60-70 ksi, which corresponds to that of the titanium alloy at high temperatures. Thus, the goal of topology optimization is to identify an improved structural configuration with a reduced level of thermal stresses.

![Figure 5.16: Stress response of the idealized 2D exhaust-washed structure.](image)

### 5.2.2 Topology Optimization Models

Similar to the problem in the previous section, we assume that open areas located behind the exhaust-washed surface can be utilized as a designable domain. Meshing this region with quadrilateral and triangular plane stress finite elements yields the design model for topology optimization. Figure 5.17 shows the finite element models for distinct design cases, where the gray elements constitute the designable region, the red, yellow, and green elements are taken as beam elements that model the exhaust-washed surface, internal substructure, and outer aircraft skin, respectively. In Figure 5.17a, the existing substructure has been retained. In the results that
follow, two cases of connectivity of the designable region are investigated. In the first case (the results of which will be referred to as design 1), it is assumed that the designable elements may be attached to only the exhaust-washed surface (red). In the second case (design 2), the designable elements are attached to both the exhaust-washed surface (red) and pre-existing internal structure (yellow). In both cases it is important to formulate the topology optimization problem such that elements on either side of a pre-existing substructural member do not share density information through the application of a regularization filter. Here, this is prevented by defining regions over which the filter neighborhood information is computed separately. Figure 5.17b shows the finite element model wherein the existing internal support structures have been removed. In this case (design 3), the topology optimization will be used to design the entire substructure arrangement. In addition, it is assumed that the substructure may attach to all internal surfaces of the domain. In all three problems, there are approximately 34,000 design variables in the optimization problem, which is based on the formulation previously given in Equation (5.6), and the MMA optimizer is utilized. Finally, the artificial loading has been applied normal to the exhaust-washed surface in directions pointing away from the designable region according to the basic rules outlined in the previous example.

5.2.3 Topology Optimization Results

The results of topology optimization for the three design cases are given in Figure 5.18, where the allowable volume fraction has been taken as 8% for design cases 1 and 2 with retained support structure and 16% for design case 3. Figure 5.18a gives the results for design 1, where the designable structure was affixed to only the exhaust-washed surface. As a result, the topology appears quite similar to those in the previous section obtained when stiffening the beam strip model. On the other hand, the material distribution in Figure 5.18b allows for the designable region to
Figure 5.17: Design models for the exhaust-washed structure where (a) the internal structure has been retained and (b) where the entire substructure is to be design via topology optimization.

We see that the topology optimization has taken advantage of this fact. This is to be expected because the resulting structure with increased internal connectivity has significantly increased stiffness to resist the applied artificial loading. Finally, Figure 5.18c gives the material distribution for design 3, which we note was afforded the most design freedom, and has resulted in the most complex substructural topology. In fact, we observe that no vertical members similar to the pre-existing supports are present in this design. It is also noted that material is more concentrated near the left edge of the design domain with structural members of greater size.
5.2.4 Stress Analysis

To assess the true effectiveness of the structural configurations obtained by topology optimization, a nonlinear stress analysis was performed on each of the three design concepts. In this analysis, the designable elements with density less than 0.9 were removed from the model and all elements that remained were assumed to take the properties of perfectly solid material. Just as in Section 5.2.1, a temperature of 900°F was prescribed to the exhaust-washed surface. The rest of the structure, including the new regions from topology optimization, were prescribed an elevated temperature of 300°F. In the results that follow, the deformation has been magnified by a factor of 10 for visual purposes. Also, the representation of designable structure as the pixelated structure from the topology optimization is not ideal for stress resolution with a high degree of accuracy, especially in the connection regions between members. In practice, a CAD model should be created with a smooth representation of the internal geometry that is suitable for developing a new finite element mesh; however, for the purposes of this study, the current representation of the structure is sufficient to draw general conclusions regarding the effect of topological layout on the overall magnitude of stresses.

Figure 5.19 gives the stress distribution found in design 1. We note that the maximum stress still occurs in the exhaust-washed surface in Figure 5.19a, but the overall magnitude has been reduced by approximately 25 ksi when compared to the unstiffened configuration previously shown in Figure 5.16. This is due in large part to the significantly decreased deformation. We also see in the maximum principal stress results shown in Figure 5.19b that a relatively benign stress state exists in the new stiffening structure with the exception of small hotspots that could likely be easily removed by introducing proper fillets in the corners.

Figure 5.20 gives the stress distribution for design 2, in which the designable region was allowed to attach to both the exhaust-washed surface and the existing
support structure. We see clearly in 5.20a that the topology optimized design in fact increases the overall state of the stress in the exhaust-surface and existing structures significantly. This is likely due to the fact that the orientation of the new structural members in the designable domain is such that expansion cannot occur in directions where it will not be restrained and generate additional harmful loading as was observed in design 1. Despite the fact that a completely harmless stress state is observed in Figure 5.20b in the topology derived structure, design 2 has completely failed to achieve the design goal of stress reduction exhaust-washed structure.

Finally, Figure 5.21 shows the stress distribution in design 3, in which the entire substructure layout was determined with topology optimization. In this case, the potential for overall stress reduction is quite surprising. In Figure 5.21a we observe that the harmful tensile stresses in the exhaust-washed surface structure have been altogether eliminated and the stress in the surrounding structures remains well below the allowable limit for tensile yielding. However, it is noted that stress in the exhaust-washed structure takes on a state of nearly pure compression, which is highly resistant to damaging fatigue cracks that have plagued real world exhaust structures, but is also susceptible to local thermal buckling or yielding under compression. Despite that buckling phenomena was not observed in the nonlinear analysis results shown here, it is a design consideration that should not be neglected. We also observe in Figure 5.21b that the new substructure configuration is characterized by a relatively low state of maximum principal thermal stresses.

5.3 Chapter Summary

In this chapter, the application of topology optimization to exhaust-washed structures design problems was demonstrated with the fundamental design goal to reduce thermal stresses. In the first example, a characteristic beam strip structure, the
breakdown of the typical formulation of topology optimization was demonstrated for thermoelastic structures whose thermal loads are significant in comparison to mechanical loading. As a result, alternative problem formulations are required and an extension of the artificial mechanical load method was proposed that includes design constraints to limit the reaction loading to surrounding structures. It was demonstrated that with this technique, appropriate stiffening structure configurations for thermal structures could be generated with limited additional reaction loading that in practical applications must be carried by adjoining components and fasteners.

The second design problem expanded the exhaust-washed structure design model to directly include pre-existing support structures and increased the design space with multiple designable domains. Three design cases, two of which retained pre-existing support structures and one designed completely by topology optimization, were investigated. Results indicate that, in some cases, application of the alternative topology optimization problems yields improved designs from a stress perspective. However, without a direct treatment of stresses, complete control of the physics is not achieved and unacceptable structural layouts can arise. It was also observed in design 3, where topology optimization was used to design the complete substructural topology and connectivity of the exhaust-washed surface, significant tensile stress reduction can be achieved through increased design freedom. The lack of vertical support members, like those in the pre-existing structural layout, indicates that from a purely thermoelastic point-of-view, such an arrangement may be undesirable. However, one should keep in mind that despite the superior stress performance of this design, it may be susceptible to failure from other physics, including vibration or buckling considerations since they were not included in the design problem formulation. In addition, due to added complexity of the structural features, this configuration may not be easily manufacturable using existing industry capabilities.

Overall, the most obvious drawback of the methods, including the newly pro-
posed techniques, utilized to obtain the results in this chapter is the lack of a direct treatment of stresses in the design problem despite the fact that stress reduction was the primary design goal. In an effort to pursue more effective design solutions, the next chapter proposes another alternative formulation for the topology optimization problem wherein the thermal stresses are directly considered in the structural optimization problem. This represents a significant step towards a comprehensive treatment of thermal structures design issues.
Figure 5.18: Density distribution results of exhaust-washed structure topology optimization for the three design problems.
Figure 5.19: Stress response for stiffened EWS structure with retained substructure supports. Designable substructure may attach to only the exhaust-washed surface.
Figure 5.20: Stress response for stiffened EWS structure with retained substructure supports. Designable substructure may attach to both exhaust-washed surface and existing substructure.
Figure 5.21: Stress response for stiffened EWS structure with substructure design completely via topology optimization.
Chapter 6

Stress-based Topology Optimization with Thermal Loads

While the alternative topology optimization formulations developed in the previous chapter are capable of generating structures with desirable thermoelastic performance, a direct treatment of thermal stresses, which represent a primary design consideration for exhaust-washed structures, was not included in the design problem. In this chapter, the application of stress-based design criteria in the topology optimization of thermal structures is investigated and new formulations are proposed to better address thermal structures design issues.

Based on the current state-of-the-art, appropriately considering stresses in topology optimization is a formidable undertaking in itself (even considering only mechanical loading effects) due to the local nature of stresses and the associated computational expense. In addition, the application of stress criteria to thermal structures has yet to be demonstrated in the literature. Further motivation for the consideration of stresses is first discussed, then methods for addressing the fundamental challenges of thermal stress topology optimization are introduced, including an adaptive method that has been developed to ensure local control of stress constraints is maintained.
Note that the proposed method for local stress control is readily applicable to general cases of topology optimization with mechanically induced stresses in addition to those of thermoelastic origin.

### 6.1 A Practical Consideration

Apart from the fact that the minimum compliance formulation with thermal loading cannot consistently produce results for practical thermal load cases, further support for a stress-based design formulation for thermal structures is apparent from purely a structural mechanics point of view. That is, by nature, minimum compliance generates a structure that is resistant to expansion. As is now clear, this stands opposite conventional thermal structures design wisdom. To demonstrate, two identical axial rods subjected to a uniform elevated temperature $\Delta T$ are shown in Figure 6.1.

![Figure 6.1: Axial rods with (a) fixed-free and (b) fixed-fixed end conditions subjected to a uniform elevated temperature $\Delta T$ along with the compliance and thermal stress of each.](image)

(a) Rod A: $C = EA\alpha^2 (\Delta T)^2$, $\sigma = 0$

(b) Rod B: $C = 0$, $\sigma = -E\alpha\Delta T$

Rod A is freely allowed to expand while rod B is rigidly fixed on both ends. We observe that in the case of a free end, the stress equals zero because there is no restrained expansion, but compliance takes a positive value. On the other hand, rod B undergoes no deformation and thus has zero compliance, but is subjected to thermal stresses due to the restrained expansion. From strictly a thermal stress point of view, the behavior of rod A is more desirable than that of rod B; however, a minimum compliance optimization would attempt to reduce the deformation to
reduce compliance, which is diametrically opposed to the basic design prescription for design against thermal stress, which we recall is to simply accommodate thermal expansion [3]. In this simple case, only compressive thermal stresses develop; however, in problems that allow for bending effects, the restrained expansion can lead to tensile stresses as observed previously. This behavior provides the basic motivation for us to seek a topology optimization formulation that can directly capture the stress behavior in the domain. This is in contrast to the alternative formulations of the previous chapter, which while effective for some problems, to not directly depend on stress responses.

6.2 Finite Element Analysis for Stress

The fundamental finite element analysis and parameterization for topology optimization for the thermal stress problem follows that developed in Chapter 4. Once assembled using the parameterization based on density variables at the element level, the finite element problem in Equation (4.2) is solved to determine the global displacement vector. Using local element displacements, for a general continuum element \( e \), the stress vector \( \sigma_e \) can be computed as

\[
\sigma_e = E_o \bar{C}_e B_e U_e - E_o \bar{C}_e \alpha_o \phi^T \Delta T_e,
\]

(6.1)

which we note is only implicitly dependent on the topology and thus is called the solid stress vector. For a two-dimensional continuum element, the stress vector contains two normal stress components, \( \sigma_{x,e} \) and \( \sigma_{y,e} \), in addition to a shear stress \( \tau_{xy,e} \) as

\[
\sigma_e = \begin{bmatrix} \sigma_{x,e} & \sigma_{y,e} & \tau_{xy,e} \end{bmatrix}^T.
\]

(6.2)
The principal stresses and maximum shear stress in element \( e \) can then be computed as

\[
\sigma_{1,e} = \frac{\sigma_{x,e} + \sigma_{y,e}}{2} + \sqrt{\left(\frac{\sigma_{x,e} - \sigma_{y,e}}{2}\right)^2 + \tau_{xy,e}^2},
\]

(6.3)

\[
\sigma_{2,e} = \frac{\sigma_{x,e} + \sigma_{y,e}}{2} - \sqrt{\left(\frac{\sigma_{x,e} - \sigma_{y,e}}{2}\right)^2 + \tau_{xy,e}^2},
\]

(6.4)

\[
\tau_{\text{max},e} = \sqrt{\left(\frac{\sigma_{x,e} - \sigma_{y,e}}{2}\right)^2 + \tau_{xy,e}^2},
\]

(6.5)

which yield the most severe (max/min) normal and shear stress conditions in element \( e \) for the prescribed stress vector. While failure in an element can be based on the principal or maximum shear stresses, in design more failure metrics are commonly utilized to generalize material failure to states of combined normal (tension and compression) and shear stress. For example, let \( F_e^{(g)} \) represent a general failure metric is taken as a function of the element stress tensor as \( F_e^{(g)}(\sigma_e) \), material failure is predicted in element \( e \) if

\[
\frac{F_e^{(g)}}{F} \leq 0
\]

(6.6)

where \( F \) is a limiting value of \( F_e^{(g)} \), for example the yield stress of the material. One of the most common failure metrics for metallic materials is the von Mises failure criterion, which can be determined directly from the element stress vector as

\[
F_e^{(vm)} = \sqrt{\sigma_{x,e}^2 - \sigma_{x,e}\sigma_{y,e} + \sigma_{y,e}^2 + 3\tau_{xy,e}^2}.
\]

(6.7)

Other failure metrics, for example the Tresca (maximum shear) criterion, maximum normal stress (for brittle materials), or the Drucker-Prager criteria (which allows for different tensile and compressive yield strength), can be selected based on knowledge about the nature of material failure.

The foregoing discussions also apply to a variety of different element types. For example, the stress vector for a three-dimensional continuum element contains six
components (three normal stresses and three shear stresses) as

\[
\mathbf{\sigma}_e = \begin{bmatrix}
\sigma_{x,e} & \sigma_{y,e} & \sigma_{z,e} & \tau_{xy,e} & \tau_{yz,e} & \tau_{xz,e}
\end{bmatrix}^T.
\] (6.8)

Three principal stresses and a maximum shear are obtained and various failure metrics can be computed for each element. For non-continuum element types, such as rod and beam elements, alternative stress components and failure metrics can be utilized. Stress relations and suitable failure metrics for most structural or continuum elements are available in most finite element, strength of materials, or applied elasticity texts [26, 130] and can ultimately be treated identically in topology optimization.

### 6.3 Relaxed Stress

As previously noted in Section 3.3.5, one of primary challenges related to including stresses in topology optimization is the singularity phenomena. In this work, the relaxation method of Le et al. [113] is adopted to relax elemental stress values and remove the singularity issues. It is important to note, that to date, this method has not been demonstrated in the presence of design-dependent thermal loading that generates thermal stresses or states of combined stress from both mechanical and thermal loading.

The general elemental failure metric \( F_e^{(g)} \) is multiplied onto an interpolation function for stress \( \eta_\sigma(\rho_e) \) as

\[
F_e^{(r)}(\rho_e) = \eta_F(\rho_e)F_e^{(g)}
\] (6.9)

where \( F_e^{(r)} \) is denoted as the relaxed failure criterion. By ensuring that the solid and void values of the stress interpolation function are \( \eta_F(1) = 1 \) and \( \eta_F(0) = 0 \), the failure metric yields the proper values for solid material and zero for void material. This removes the singularity by allowing for a state of zero stress for void material.
According to the rules for interpolation function selection in [113], to achieve the proper material penalization, it is desirable that intermediate density material should experience a stress state that is more severe than that obtained when a linearly proportional relationship exists between stress and material usage. This is analogous to the desire for intermediate density material to have less stiffness with respect to material whose stiffness is linearly proportional to material usage, which yields the common SIMP penalization of $p = 3$ in minimum compliance problems. Accordingly, based on numerical tests of candidate functions, a stress interpolation function of $\eta_F(\rho_e) = \rho_e^{1/2}$ is selected.

### 6.3.1 Validation

The behavior of the relaxed stress technique is demonstrated using the simple finite element models shown in Figure 6.2. The material properties of the structure are taken as $E = 70$ GPa, $\nu = 0.3$, $\alpha = 25 \times 10^{-6}$ 1/°C, and the thickness is taken as $t = 1$ cm. The model is analyzed using four different loading cases and boundary conditions, which subject it to various cases of (6.2a and 6.2b) mechanical, (6.2c) thermal, and (6.2d) combined stresses.

Figure 6.3 gives the variation for von Mises stress in element 1 as its density is varied from zero (void material) to 1.0 (solid material). In the figure, the dashed lines show the non-relaxed stress computed directly from Equations (6.1) and (6.7), which is the solid stress. The singularity phenomena is readily evident in the solid stress, which increases significantly as the density in element 1 is reduced. However, when the relaxation is applied to the solid stress according to Equation (6.9) with a stress interpolation function of $\eta_F(\rho_e) = \rho_e^{1/2}$, we observe that the singularity in the stress is arrested as density is decreased. Moreover, as the density nears zero, which corresponds to no material, a state of zero stress is achieved. This is realized for all cases of mechanical, thermal, and combined stress states in this example. In addition,
Figure 6.2: Model, boundary conditions, and load cases for the validation of the relaxed stress technique. (a,b) Mechanical loading, (c) Thermal Loading, and (d) Combined Loading.

A penalizing effect in terms of stress is observed for intermediate densities. That is, intermediate density material experiences a more severe stress state than both solid and void material, which is desirable in the formulation of the topology optimization to ensure that the non-physical intermediate density is removed in the final design.

Figure 6.4 demonstrates gives further evidence for the effectiveness of the relaxed stress technique at removing the singularity phenomena in the presence of thermal stresses. Two arbitrary density distributions are given in Figure 6.4a and Figure 6.4d where white regions correspond to void and black regions to solid material. The structure is fixed in all directions along the vertical edges on the left and right side and a mechanical load is applied at the center of the bottom edge. A uniform elevated temperature is also applied to the entire domain. Figures 6.4b and 6.4e show the von Mises solid stress field as computed directly from Equations (6.1) and (6.7) for the
two density distributions. We note that excessively high stresses are apparent for both cases in regions where there is actually no material due to the singularity in the stress function. Figures 6.4c and 6.4f show the relaxed von Mises stress measure that results after application of the stress interpolation function in Equation (6.9). Here we see again that the relaxation is effective at removing the singularities in regions where there is no material, while retaining stress information in solid regions of the structure.

6.4 Scaled Stress Aggregation

Another primary challenge in stress-based topology optimization is effectively capturing the the maximum stress within an evolving design domain at a level of computational expense that is suitable for industrial size topology optimization problems. As discussed previously Chapter 3, the local method, in which stresses are monitored in every element, proves computationally infeasible in topology optimization due to the excessive cost of sensitivity analysis (one adjoint evaluation per element). Global
or aggregation methods can group the stress measure of a number of elements into a single value; however, doing so severely degrades the local resolution of stress. In fact, increasing the number of elements that are grouped together tends to increase the discrepancy between the aggregated prediction and the actual maximum stress within the grouped set such that limits on stress or material failure cannot be reliably enforced.

To overcome the loss of local accuracy associated with aggregation methods while retaining their efficient sensitivity computation, a new scaled aggregation technique to capture the maximum stress within the design domain is proposed. At each iteration in topology optimization, the relaxed failure criterion for stress $F_{e}^{(r)}(\rho_{e})$, from Equation (6.9), is first computed for each element whose stress is of interest. Without loss of generality, a modified $p$-norm function is utilized to aggregate the elemental...
failure metrics as
\[
PN(\rho) = \left[ \sum_{e=1}^{N_e} \left( \frac{F_e^{(r)}(\rho_e)}{\bar{F}} \right)^p \right]^{1/p}
\] (6.10)

where \(N_e\) is the number of elements to be aggregated together and \(p\) is a specified tuning parameter. In general, higher values of \(p\) increase the accuracy of the approximation with \(PN \to \max(F_e/\bar{F})\) for \(p \to \infty\); however, in practice both numerical and optimization convergence issues occur if the value selected for \(p\) is too large. In the optimization literature, when using \(p\)-norm functions, suitable values for \(p\) are generally found in the range from 4 to 10. This limitation on the values of \(p\) ensures that considerable error between \(PN\) and \(\max(F_e/\bar{F})\) is present and we cannot reliably enforce the limit value \(\bar{F}\).

With this in mind, the development of the proposed technique begins by assuming no discrepancy between \(PN\) and \(\max(F_e/\bar{F})\). In this case, material failure is predicted in an element included in the aggregation if \(PN \geq 1\) and a constraint in the optimization problem can be posed as
\[
g(\rho) = PN(\rho) - 1 \leq 0.
\] (6.11)

Considering now, the behavior of a typical gradient-based optimization process. As the objective function converges and constraints are satisfied, converging behavior is also generally observed in all system responses, including elemental stresses. If an aggregation, like that in Equation (6.10), is utilized in the optimization statement, this implies that the values that are grouped together, in this case \(F_e/\bar{F}\) for all \(N_e\), and the aggregation result \(PN\) exhibit convergent behavior. It follows then that the discrepancy between \(PN\) and \(\max(F_e/\bar{F})\) approaches a constant value. Exploiting this feature, the constraint in Equation (6.11) can be rewritten with an adaptive factor \(s^i\) as
\[
g = s^iPN^i - 1 \leq 0.
\] (6.12)
where \( i \) is the optimization iteration number and \( s^i \) is taken as the ratio between the PN value and \( \max(F_e/\bar{\sigma}) \) in the previous optimization iteration. This is simply calculated as

\[
s^i = \frac{\text{PN}^{i-1}}{\max(F_e/\bar{\sigma})^{i-1}}. \tag{6.13}
\]

We note here that by modifying constraints in the optimization problem in this way, a new problem is actually posed in each iteration. As a result, some amount of noise and error is introduced to the process, which may interfere with convergence of some mathematical optimization algorithms. However, as iteration continues, these issues are alleviated as variations in the design are reduced and \( s^i \) approaches a constant value. In addition, as \( s^i \) becomes constant, the constraint in Equation (6.12) is acting to satisfy the prescribed limit stress \( \bar{F} \) with the discrepancy introduced by aggregation effectively removed.

### 6.5 Adaptive Element Grouping

The scaled aggregation technique proposed in Section 6.4 works to ensure that as the optimization problem converges, each constraint that employs an aggregated response function operates approximately on the maximum value contained in the aggregation set. From purely the perspective of capturing the maximum stress in a topologically designable domain, it appears that a single aggregation function could be utilized for all responses to achieve the best computational efficiency. However, as a larger number of responses are aggregated, the resolution of local sensitivity information is reduced. Thus, it is beneficial to employ multiple aggregation functions with each grouping a subset of design responses. Doing means that the computation of the sensitivity of the aggregated response function is more rich in information related to the variability effects of the local responses contained within it. With improved sensitivity information, the numerical optimization algorithm can more effectively traverse
the design space, which has been found to be highly nonlinear when considering stress measures for topology design variables. We note that utilizing more aggregated constraints is accompanied by increased computational cost since each added constraint requires an additional adjoint computation. Thus, in practice it is desirable to use the greatest number of aggregation measures that is feasible within computational and time limits. Note that the limiting case of utilizing an equivalent number of aggregation functions to the number of design responses results in an identical problem to the local method of treating stress constraints, which we know to be ideal, but computationally intractable for real world problems.

The use of multiple aggregation functions in the formulation of stress constraints and the scaled aggregation method from Section 6.4 is given in general for \( m \) total aggregations as

\[
g_m = s^i_m P\mathbf{N}^i_m - 1 \leq 0 \quad (6.14)
\]

where \( P\mathbf{N}^i_m \) is the aggregation function defined over the responses in region \( m \) and \( s^i_m \) is the scaling value found as

\[
s^i_m = \frac{P\mathbf{N}^{i-1}_m}{\max(\mathcal{F}_e/F)^{i-1}_m} \quad (6.15)
\]

using information only within region \( m \). We note that just as in Section 6.4, the scaling value for each region is updated every iteration and no two scaling values are likely to be identical at a given iteration.

When considering multiple aggregation regions, a final factor to address is how to determine the distribution of design responses (or element stresses) into regions. While randomly distributing or grouping elements by spatial proximity into regions are potential approaches, the nature of the stress sensitivities in the optimization problem implies a better alternative and is discussed in detail in [118] and highlighted as follows.
In topology optimization, the stress in an element is likely to be more sensitive when its own density value is low. This occurs because low element density interpolates to low element stiffness. This can cause relatively large displacement changes in response to varying internal forces that result from design variability (anywhere within the domain). In addition, an elements own density has a high sensitivity to its own density due to the nature of the interpolation function on stress, which actually has infinite sensitivity for perfectly void material. Thus, the large displacement changes translate to increased elemental stress sensitivity. An accompanying behavior is that, again in general, very low density elements ($\rho_e < 0.1$) are characterized by low elemental stresses due to the nonlinear stress interpolation. In the optimal design, the combined effect of these observations results in a large number of elements, namely those who represent void material with $\rho_e$ nearly equal to zero, that have low values of stress, but are highly sensitive to variability in the design domain. It is ideal then to ensure that sensitivity information related to the stress response in intermediate or full density elements is not polluted by the sensitivity of elements who have effectively vanished from the design domain when the responses are aggregated together. One method of accomplishing this is by grouping elements into regions based on their stress level. Doing so places all lowly stressed elements into regions with one another and into a constraint that is always satisfied since all values in the set are presumably near zero.

A simple scheme for response sorting by stress level is given as follows. As an example, assume that $N_e$ element stresses are to be grouped into $n$ regions. The first $n-1$ clusters contain $N_m$ elements and the last cluster $n$ holds the remaining elements. After computing the element stress failure metrics $F_e(\rho_e)$ for $e = 1, 2, \ldots, N_e$, they are then sorted in descending order. The regions then can be defined as

$$F_1 \geq F_2 \geq F_3 \geq \ldots \geq F_{\frac{N_e N_m}{n_m}}, \quad \ldots \geq F_{\frac{2 N_e N_m}{n_m}}, \quad \ldots \geq F_{\frac{(n-1) N_e N_m}{n_m}}, \quad \ldots \geq F_{\frac{N_e N_m}{n_m}}.$$  \hspace{1cm} (6.16)
It is important to note that in [118], the authors did not utilize a scaled stress measure as proposed in Section 6.4 and explored the sorting of stresses in an attempt to better control local stress limits since aggregation functions are more accurate for sets of values with nearly the same value. Since the stresses change throughout optimization, resorting and regrouping of the elements periodically was necessary and the quality of results was dependent upon the choice of resorting frequency (or the number of iterations between resorting and updating the aggregation regions). At this point the proper treatment of regionalizing and resorting is unclear when the technique is combined with a scaled stress measure since, just as with stress scaling, when the elements are resorted, the mathematical optimization problem is perturbed, which may cause numerical convergence issues. In the first demonstration case in this chapter, the effect of resorting is investigated to address this concern.

6.6 Sensitivity Analysis Derivation

The adjoint sensitivity for the aggregated stress measure containing stresses in continuum elements, or those whose stress is found using Equation (6.1), is now derived with respect to the physical density variables. In the sensitivity analysis implementation, care should be taken to properly treat the stress in elements that are designable (dependent on density) compared to elements that remain solid. In the derivation that follows, this consideration is noted where appropriate.

We begin by differentiating the scaled aggregation constraint for region $m$ in Equation (6.14) (note that the superscript $i$ that gives the iteration number has been removed for clearer presentation) with respect to a physical density variable $\rho_j$

$$\frac{dg_m}{d\rho_j} = s_m \frac{dPN_m}{d\rho_j}. \quad (6.17)$$

The derivative of the aggregation function $PN_m$ with respect to the density variable
where $N_m$ is the number of elements in region $m$. In Equation (6.18) the first partial derivative term $\partial P_N / \partial F_e^{(r)}$ is obtained by differentiating Equation (6.10) with respect to the relaxed elemental failure metric $F_e^{(r)}$, which yields

$$
\frac{\partial P_N}{\partial F_e^{(r)}} = \left[ \sum_{e=1}^{N_m} \left( \frac{F_e^{(r)}(\rho_e)}{F} \right)^p \right] \left( \frac{F_e^{(r)}(\rho_e)}{F} \right)^{p-1}.
$$

Next we recall that $F_e^{(r)}$ is the product of the stress interpolation function $\eta_F(\rho_e)$ and, in the most general case, a solid element failure criterion $F_e^{(g)}(\sigma_e)$ as given in Equation (6.9). Differentiating this relation yields

$$
\frac{dF_e^{(r)}}{d\rho_j} = \left[ \frac{\partial F_e^{(r)}(\rho_e)}{\partial \eta_F} \frac{d\eta_F}{d\rho_j} + \left( \frac{\partial F_e^{(r)}}{\partial F_e^{(g)}} \right) \left( \frac{\partial F_e^{(g)}}{\partial \sigma_e} \right)^T \frac{d\sigma_e}{d\rho_j} \right]
$$

where the derivative term $d\sigma_e/d\rho_j$ is the derivative of stress tensor of element $e$ with respect to the density variable. This is computed by differentiating Equation (6.1), which gives

$$
\frac{d\sigma_e}{d\rho_j} = E_0 \bar{C}_e B_e \frac{dU_e}{d\rho_j} = E_0 \bar{C}_e B_e \left( \frac{\partial U_e}{\partial U} \right)^T \frac{dU}{d\rho_j}
$$

where $\partial U_e / \partial U$ forms a transformation from local element degrees of freedom to the global degrees of freedom. Substituting Equation (6.21) into (6.20) and (6.20) into (6.18) yields
\[
\frac{d\text{PN}_m}{d\rho_j} = \sum_{e=1}^{N_m} \left[ \frac{\partial \text{PN}}{\partial F_j^{(g)}} \left( F_j^{(g)} \frac{d\eta}{d\rho_j} + \eta F_{j^{(g)}} \left( \frac{\partial F_j^{(g)}}{\partial \sigma_e} \right)^T E_o \tilde{C}_e B_e \left( \frac{\partial U_e}{\partial U} \right)^T \frac{d\text{U}}{d\rho_j} \right) \right] + \sum_{e=1}^{N_m} \left[ \frac{\partial \text{PN}}{\partial F_e^{(g)}} \frac{d\eta}{d\rho_j} \right] \\
+ \sum_{e=1}^{N_m} \left[ \frac{\partial \text{PN}}{\partial F_e^{(g)}} \eta F_{j^{(g)}} \left( \frac{\partial F_e^{(g)}}{\partial \sigma_e} \right)^T E_o \tilde{C}_e B_e \left( \frac{\partial U_e}{\partial U} \right)^T \frac{d\text{U}}{d\rho_j} \right]. 
\]

(6.22)

We note that the \(d\eta_{\sigma}/d\rho_j\) term in the first summation in Equation (6.22) is nonzero only for \(e = j\). Thus, the summation can be ignored and the equation can be reduced to

\[
\frac{d\text{PN}_m}{d\rho_j} = \frac{\partial \text{PN}}{\partial F_j^{(g)}} F_j^{(g)} \frac{d\eta}{d\rho_j} + \sum_{e=1}^{N_m} \left[ \frac{\partial \text{PN}}{\partial F_e^{(g)}} \eta F_{j^{(g)}} \left( \frac{\partial F_e^{(g)}}{\partial \sigma_e} \right)^T E_o \tilde{C}_e B_e \left( \frac{\partial U_e}{\partial U} \right)^T \frac{d\text{U}}{d\rho_j} \right]. 
\]

(6.23)

Along with \(d\text{U}/d\rho_j\) given previously in Equation (4.16), the adjoint variable \(\lambda\) is introduced to Equation (6.23), which then simplifies to

\[
\frac{d\text{PN}_m}{d\rho_j} = \frac{\partial \text{PN}}{\partial F_j^{(g)}} F_j^{(g)} \frac{d\eta_{\sigma}}{d\rho_j} + \lambda^T \left( \frac{d\text{F}^{th}}{d\rho_j} - \frac{d\text{K}}{d\rho_j} \text{U} \right) 
\]

(6.24)

and the adjoint variable is determine by the solution of the adjoint problem given by

\[
\text{K}\lambda = \sum_{e=1}^{N_m} \left[ \frac{\partial \text{PN}}{\partial F_e^{(g)}} \eta F_{j^{(g)}} \left( \frac{\partial F_e^{(g)}}{\partial \sigma_e} \right)^T E_o \tilde{C}_e B_e \left( \frac{\partial U_e}{\partial U} \right)^T \frac{d\text{U}}{d\rho_j} \right]. 
\]

(6.25)

Note that Equations (6.24) and (6.25) give the sensitivity using a general stress failure criteria. The term \(\partial F_j^{(g)}/\partial \sigma_e\) varies according to how the failure criteria is calculated using the components of the element stress tensor. As an example, the the sensitivity using the von Mises failure criterion, which is used in the example problems, is given in the following subsection.
6.6.1 von Mises Stress

When the von Mises stress is computed according to Equation (6.7), the derivatives of the von Mises failure criterion with respect to the elements of the stress tensor are

\[
\frac{\partial F_e^{(vm)}}{\partial \sigma} = \begin{bmatrix}
\frac{\partial F_e^{(vm)}}{\partial \sigma_{x,e}} \\
\frac{\partial F_e^{(vm)}}{\partial \sigma_{y,e}} \\
\frac{\partial F_e^{(vm)}}{\partial \tau_{xy,e}}
\end{bmatrix}
\]

(6.26)

where

\[
\frac{\partial F_e^{(vm)}}{\partial \sigma_{x,e}} = \frac{1}{2F_e^{(vm)}} (2\sigma_{x,e} - \sigma_{y,e}),
\]

(6.27)

\[
\frac{\partial F_e^{(vm)}}{\partial \sigma_{y,e}} = \frac{1}{2F_e^{(vm)}} (2\sigma_{y,e} - \sigma_{x,e}),
\]

(6.28)

\[
\frac{\partial F_e^{(vm)}}{\partial \tau_{xy,e}} = \frac{3\tau_{xy,e}}{F_e^{(vm)}}.
\]

(6.29)

6.6.2 Validation

Using the models and analysis cases previously given in Figure 6.2 the analytical sensitivity analysis formulation for the aggregated measure given by Equations (6.18) to (6.29) is validated against finite difference sensitivities. Here, the sensitivity of the aggregated measure with respect to the physical density of element 1 where four different regions of elements are defined according to Table 6.1. For each region described in Table 6.1 (note some regions contain only a single element), the aggregated measure is computed according to (6.10) using the von Mises failure criterion and \(\bar{F} = 350\) MPa. Thus, we note that by testing these regions, various cases including designable elements, nondesign elements, and regions of mixed elements are evaluated.

The sensitivity of the aggregated measures computed using regions 1 to 4 from Table 6.1 are shown in Figures 6.5a to 6.5d, respectively. In each plot, different loading and boundary conditions are indicated by different colors with the analytical
Table 6.1: Region definitions for sensitivity analysis validation of the aggregated stress measure.

<table>
<thead>
<tr>
<th>Region Number</th>
<th>Elements</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>1 and 2</td>
</tr>
<tr>
<td>4</td>
<td>1 to 4</td>
</tr>
</tbody>
</table>

sensitivity given by solid lines and finite difference validation points indicated by filled dots. Finally, the variation of the sensitivity with respect to the density of element 1 is plotted (with the density along the horizontal axis) to demonstrate the nonlinear dependence of the stress sensitivities on density. From all the plots in Figure 6.5 it is observed that the finite difference points lie exactly on the analytical sensitivity curves, which indicates the accuracy of adjoint formulation.

In addition, we recall that regions 1 and 2 consist of only a single designable and non-designable element, respectively. Observing the sensitivity of each in Figures 6.5a and 6.5b, we observe significantly different behavior. As stated previously, the stress in an element whose density is near zero has a high sensitivity to variations in its own density, which is clearly indicated in Figure 6.5a. When these two elements are combined into a single measure in region 3 (Figure 6.5c), we observe features from both Figure 6.5a and 6.5b including the high sensitivity at low densities. These features are persistent even when additional non-design elements are added to make up region 4 (Figure 6.5d).
6.7 Problem Formulations

With the derivation of an effective stress response, a number of topology optimization problems can be formulated that incorporate stress-based design criteria to best suit a particular design issue. In the demonstration problems that follow, two stress-based topology optimization formulations are demonstrated. In the first formulation, the
The objective is to minimize the material usage with constraints to limit the maximum value of stress in the structure. The mathematical statement of this formulation is given as

\[
\begin{align*}
\text{min} : & \quad V(\rho) = \sum_{e=1}^{N_d} (\rho_e v_e) \\
\text{subject to} : & \quad g_m(\rho, U) = s^i_m PN(\rho, U)^i_m - 1 \leq 0 \text{ for } m = 1, 2, \ldots, n \quad (6.30) \\
& \quad K(\rho)U = F^m + F^{th}(\rho) \\
\text{variables} : & \quad 0 < x_{\text{min}} \leq x_e \leq 1 \text{ for } e = 1, 2, \ldots, N_d
\end{align*}
\]

where \( n \) is the number of stress regions that are utilized. In addition, various criteria \( F_e^{(r)} \) to measure stress failure can be utilized in the computation of the \( PN^i_m \) aggregation measures under the restriction that the failure metrics be formulated to only return positive values once normalized by the limiting value \( \bar{F} \). For example, failure due to purely compressive stress that is of negative magnitude should be divided by a negative value.

The second formulation is an extension of the minimum compliance formulation. This formulation is useful, especially for mechanical structures, where a stiff structure is desired within stress limits. By including stress constraints, the geometric features that create stress concentrations that are characteristic of pure stiffness-based topology optimization can be avoided. This formulation is mathematically stated by

\[
\begin{align*}
\text{min} : & \quad c(\rho) = U(\rho)^T K(\rho) U(\rho) \\
\text{subject to} : & \quad g_1(\rho) = \sum_{e=1}^{N_d} (\rho_e v_e - V_f v_e) \leq 0 \\
& \quad g_m(\rho, U) = s^i_m PN(\rho, U)^i_m - 1 \leq 0 \text{ for } m = 1, 2, \ldots, n \quad (6.31) \\
& \quad K(\rho)U(\rho) = F^m + F^{th}(\rho) \\
\text{variables} : & \quad 0 < x_{\text{min}} \leq x_e \leq 1 \text{ for } e = 1, 2, \ldots, N_d
\end{align*}
\]
In addition, the stress measure could be utilized to augment nearly any structural or multidisciplinary topology optimization problem. For example, the artificial mechanical load and thermoelastic combination methods demonstrated in Chapter 5 could be augmented by stress constraints to directly treat the thermal stresses.

6.8 Demonstration Problems

6.8.1 L-shaped Bracket

The L-shaped bracket is a popular test example for topology optimization with stress criteria in the literature. This is because the design domain contains a re-entrant corner that causes an initial stress singularity that is not removed when the problem is solved using the typical minimum compliance solution strategy. This problem is used here to demonstrate the effectiveness of the stress-constrained formulation using the scaled stress measure. It should be noted that for a stress-based topology optimization to be considered successful, the maximum stress in the structure should remain at or below a prescribed limit that is based on realistic failure criteria. As previously stated retaining local control of stresses is one of the main challenges related to the use of aggregation functions for stress constraints and is the focus of the proposed algorithm for scaling aggregation measures.

The design domain for the L-bracket is given in Figure 6.6. The structure is fixed in all degrees of freedom along its top edge and a 750 N concentrated load is applied as indicated. To prevent an unresolvable stress singularity that would impede the optimization due to the point load, a 2 by 3 region of elements are taken as non-designable and excluded from the stress constraints. The thickness of the structure is assumed to be 1 mm and the material properties are taken as those of 7075-T6 aluminum (shown in Table 6.2) at room temperature. The domain is discretized using 6400 quadrilateral plane stress finite elements. The von Mises
failure criterion given in Equation (6.7) is utilized for stress failure. The minimum material, stress-constrained topology optimization problem is solved according to the statement in Equation (6.30) with a total of \( m = 10 \) stress aggregation regions and the penalization parameter in the \( p\)-norm functions is taken as \( p = 8 \). In the results that follow, various settings of the resorting frequency are tested for both the SIMP and RAMP interpolation schemes and all design variables are initialized with value of 0.5. Finally, the optimization problems are solved using the MMA algorithm; however, the asymptote procedure in the algorithm has been altered take more conservative step sizes in the highly nonlinear design domain.

![Design domain, boundary conditions, and loading for the L-bracket example problem.](image)

**Figure 6.6:** Design domain, boundary conditions, and loading for the L-bracket example problem.

<table>
<thead>
<tr>
<th>Property</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Elastic Modulus, ( E_0 ) (GPa)</td>
<td>68.9</td>
</tr>
<tr>
<td>Poisson’s Ratio, ( \nu )</td>
<td>0.33</td>
</tr>
<tr>
<td>Coefficient of Thermal Expansion, ( \alpha_0 )( \times 10^{-6}/^\circ \text{C} )</td>
<td>23.5</td>
</tr>
<tr>
<td>Yield Strength, ( F ) (MPa)</td>
<td>275.0</td>
</tr>
</tbody>
</table>

**Table 6.2:** Material Properties of 7075-T6 Aluminum [31].
6.8.1.1 Room Temperature Results

Figure 6.7 shows the resulting density and von Mises stress distributions obtained for utilizing a SIMP interpolation scheme with penalization parameter of 3 and various frequencies of stress region resorting. Figure 6.7a gives results for resorting stress aggregations every iteration, 6.7b every 20 iterations, and 6.7c for no constraint resorting. We observe in each design that the fundamental topological layout (number of holes and members) is identical, with slightly variations in the shape of the structure. However, an important observation related to assessing the effectiveness of the scaled stress measure is that all designs satisfy the limit on maximum stress of 275.0 MPa (within convergence tolerance of the optimizer) with similar material usage of approximately 27-29%. Upon investigating the structures from a mechanical design point of view, it appears that the structure in 6.7c is the best design as it contains the greatest radius around the initial stress singularity. This indicates that when utilizing a scaled stress measure, the aggregation functions should not otherwise be modified with any resorting procedures. In this case, it is likely that the local resolution of stress information in high stress areas is lost in additional numerical noise introduced by the discrete changes in the optimization problem during resorting. This assertion is supported by the iteration history plots given in Figures 6.9a, 6.9c, and 6.9e for resorting every iteration, resorting every 20 iterations, and no resorting, respectively. Convergence in Figures 6.9a and 6.9c, which both contain resorting contain considerable amounts of noise. In fact, large jumps in the responses are readily evident every 20 iterations when resorting occurs in Figure 6.9c. However, when no resorting is utilized, convergence is much smoother as observed in Figure 6.9e.

A similar set of results to those in Figure 6.7 are given in Figure 6.8; however, this time the RAMP interpolation scheme is utilized for material stiffness with a RAMP parameter of 8. Again, the effects of resorting are demonstrated with Figure 6.8a showing results for resorting every iteration, Figure 6.8b resorting every 20 it-
erations, and 6.8c with no resorting. In these results, as opposed to those using the SIMP interpolation, altogether different topologies are obtained; however, we note that in each design, limiting the maximum stress to under 275.0 MPa is still achieved by the scaled stress measure. In the RAMP designs, it is also apparent that not all of the intermediate density material is removed in these cases and trace amounts of gray material remain. This creates higher stresses in these areas when compared to surrounding truly void material since the relaxation on element stresses only effectively zeros the stresses for nearly zero density. After careful consideration, it may be presumed that this feature is a result of the fundamental difference between the SIMP and RAMP schemes. Recalling that a limitation of the SIMP interpolation is that is has zero sensitivity at zero density, which RAMP does not (refer to Figure 3.3). Thus, as material is removed and becomes void, it is much more difficult for it to reappear when utilizing SIMP than RAMP (which is a critical drawback of SIMP for design-dependent loading). When combined with the scaled stress measure, which imparts a small perturbation on the mathematical optimization problem every iteration, the relative insensitivity of SIMP proves to be an advantage. In the RAMP interpolation, the increased sensitivity of responses in low density regions means that the optimization algorithm reacts more strongly to perturbed problem. This conclusion is supported by the iteration history plots using the RAMP interpolation in Figures 6.9b, 6.9d, and 6.9f. Compared to the SIMP convergence, even when no region resorting is utilized in Figure 6.9f, considerably more oscillation is evident in the maximum stress convergence, which results from a stronger response to the stress scaling process.
Figure 6.7: Minimum volume, stress constrained optimization results for L-Bracket using SIMP interpolation with (a) resorting every iteration, (b) every 20 iterations, and (c) no resorting.
Figure 6.8: Minimum volume, stress constrained optimization results for L-Bracket using RAMP interpolation with (a) resorting every iteration, (b) every 20 iterations, and (c) no resorting.
Figure 6.9: Iteration history for (a,c,e) SIMP and (b,d,f) RAMP interpolation and (a,b) region resorting every iteration, (c,d) every 20 iterations, and (e,f) no region resorting.
6.8.1.2 Thermal Loading

To investigate the effectiveness of the scaled stress measure in the presence of thermal loading, the L-shaped bracket is now subjected to a uniform elevated temperature of $T = 20^\circ C$. The applied mechanical load of 750 N and boundary conditions remain unchanged. Physically, $20^\circ C$ does not appear to be a significant source of thermal loading; however, considering the two-dimensional context of the problem, where all strain energy is contained in only in-plane degrees-of-freedom, it actually represents a considerable load. To demonstrate this, the optimal design obtained using no region resorting and the RAMP interpolation scheme, previously shown in Figure 6.9f, is subjected to the new combined loading environment. The resulting stress distribution is shown in Figure 6.10. In the figure, all regions that appear as deep red exceed the previous allowable stress of 275.0 MPa and the maximum stress in the domain is now 346.8 MPa, which represents an increase of approximately 25%. We note that the elevated stresses occur directly within the connecting structural members. This implies that while such a material layout carries mechanically-induced stresses very well, it may not be appropriate when additional thermal effects are considered.

As a result, the stress-constrained topology optimization problem (limiting maximum stress to 275 MPa) is re-solved with the combined thermal and mechanical loads. The RAMP interpolation scheme is utilized here despite the superior performance of the SIMP scheme in the last section with purely mechanical loads because it is well understood that SIMP does not perform well in the presence of design-dependent loading. The RAMP penalization parameters are taken as 8 for stiffness and 0 for the thermal load. In addition, based on the results of the last section, no region resorting is utilized. The resulting density and stress distributions are given in Figure 6.11a and the iteration history for volume usage and maximum stress are shown in Figure 6.11b.

In Figure 6.11a, we note that a different topological layout is obtained when the
effects of the elevated temperature are considered. In this layout, as evidenced by reduced thermal stresses, the material has been redistributed such that structural members do not restrain the thermal expansion of one another. Thus, the stress limit of 275 MPa is satisfied within the tolerances of the optimizer and a nearly identical amount of material has been utilized as the purely mechanical load case.

Investigating the iteration history in Figure 6.11b, the absence of gray material and significant oscillations in convergence compared to previous results in Figure 6.8c is readily evident. This is attributed to the presence of the thermal loading, which due to its design-dependency, serves as a penalty on the reappearance of material. This effectively damps the optimizer’s reaction to the scaling stress measures between each iteration.

### 6.8.1.3 Effectiveness of Scaled Stress Measure

To demonstrate the effectiveness of the scaled stress measure in enforcing local limits on maximum stress, the iteration history of the components used to compute it are shown in Figures 6.12a and 6.12b. The results correspond to the designs in Figures 6.8c (mechanical load only) and 6.11a (combined thermal and mechanical load), respectively, which were obtained using RAMP interpolation and no region resorting.
Figure 6.11: Topology optimization results for L-Bracket subjected to combined thermal and mechanical loading: (a) density and stress distributions and (b) iteration history.

In the optimization problem, we recall that 10 constraints were utilized based on 10 aggregated regions; however, for demonstration, only results the first constraint $g_1$ are shown. In the figure, the blue curve denotes the value of $\text{PN}^i_1$ directly computed from the elements in its region according to Equation (6.10). The red dots denote the actual maximum value of the stress failure criteria (normalized by the limit $\bar{F}$) within region 1 in each iteration. The green curve denotes the scaling parameter $s^i_1$ that is computed at each iteration using the value of $\text{PN}^{i-1}_1$ and the maximum value from the previous iteration as stated in Equation (6.15). The black curve then gives...
the product $s_i^j PN_i^j$, which is the scaled aggregation measure that approximately gives the maximum stress value in the region. This quantity is then constrained to limit maximum stresses according to Equation (6.14). We observe in both cases, the approximation of the maximum value within the region rapidly approaches the actual maximum value within only a small number of iterations and local limits on stress are enforced effectively.

Figure 6.12: Iteration history for components of scaled stress measure in constraint 1 for (a) room temperature design and (b) combined mechanical and thermal load design.

6.8.1.4 Problem Summary

In this demonstration case, the common L-shaped bracket problem was utilized to demonstrate a number of important features of the stress-based topology optimization formulation proposed in this chapter. First, it was demonstrated that when utilizing multiple regionalized stress constraints and employing the scaled stress measure formulated in Section (6.4), the resorting process on the domains utilized in previous works should not be performed because it appears unnecessary and can cause numerical issues in convergence. Next, it was shown that while in cases of purely mechanical loading, the SIMP scheme has superior performance when compared to
RAMP, when thermal effects are added, the issues related to the increased sensitivity of RAMP causing excessive gray material were alleviated. This is important because SIMP cannot be reliably utilized with design-dependent loading without modification. Finally, in all the cases investigated, the proposed scaled stress measure was shown to be effective at enforcing stress limits for both pure mechanical loading and combined mechanical/thermal loading.

6.8.2 Bi-clamped Thermoelastic Domain

The bi-clamped beam problem is a demonstration example that has been used in a limited number of publications regarding thermoelastic topology optimization to demonstrate the behavior of the compliance objective in the presence of thermal loading. It was first introduced by Rodrigues and Fernandes in 1995 [72]. In this section, we apply the stress-based formulations to investigate their behavior with combined thermal and mechanical loading. In addition, the conventional minimum compliance problem is solved for comparison since, in this case, the formulation is able to produce solutions. Figure 6.13 shows the design domain for the structure. It is discretized with four node bi-linear quadrilateral plane stress elements with 60 elements in the horizontal direction and 40 elements in the vertical direction. Black elements in the figure denote non-design regions in the model while gray elements give the designable domain. A mechanical load of $F = 150$ kN is applied to the center of the bottom edge along with a uniform temperature increase of $\Delta T = 25^\circ C$. Similar to the L-shaped bracket, intuitively this temperature level does not seem considerable, but considering the 2D nature of the problem, it in fact represents a significant amount of thermally induced strain energy in the structure. A small non-design region near the application of the mechanical point load is included to prevent a geometrical stress singularity. The material properties are taken as those of 4340 steel at room temperature and are given in Table 6.3 and a thickness of 1 cm is
Three different topology optimization problems are solved using the bi-clamped beam domain. First, the minimum volume, stress constrained problem is solved as stated in Equation (6.30) using $m = 10$ stress regions and the von Mises failure criterion. The $p$-norm parameter in the aggregated stress measures is taken as 8 and based on the results of the L-Bracket problem, no region resorting is performed. After obtaining the stress constrained design, another optimization is performed using the standard minimum compliance, volume constrained formulation where the volume limit is taken as the volume achieved in the stress constrained problem. Finally, the minimum compliance problem with volume and stress constraints stated in Equation (6.31) is solved using the constraint limits from the previous problems. In each formulation, the RAMP interpolation model is utilized with a penalization parameter of $R_E = 8$ on the material stiffness and $R_\beta = 0$ on the thermal load and the standard
density filter is employed with a filter radius of 0.02 to enforce minimum length scale.

6.8.2.1 Results

Figure (6.14) shows the density and stress distributions and Figure (6.15) gives the iteration history for topology optimization of the bi-clamped domain with each of the three formulations.

Inspecting results of the stress-constrained problem in Figure (6.14a), we observe that the stress limit of 400 MPa is satisfied with a material usage of only 14.6%. In addition, the structure is uniformly stressed, which implies it is an effective layout for carrying the combined thermal and mechanical loading. In Figure (6.14b), we observe that a significantly different topology is obtained using the minimum compliance formulation even when the material usage is restricted to near that of the stress constrained design (15%). A nonuniform stress distribution is also readily evident with significantly higher stresses (note the plots do not use the same color scale in Figure (6.14)). In fact, the maximum stress observed in the minimum compliance design is 596.6 MPa, or nearly a 50% increase, compared to the stress-constrained structure. The areas of greatest stress occur adjacent to the non-design elements near the applied loading and at the lower left and right corners of the domain, which from a mechanical design perspective, are expected locations. These comparatively increased stresses are likely due in large part to the internal deformation caused by thermal expansion. Compared to the stress-constrained design, where no structural member lies in an orientation directly between the restrained boundaries and expansion can occur with limited bending deformation in the direction of the applied load, expansion in the minimum compliance design is more restrained. This restraint translates into significant internal thermal loading and increased stress levels that cannot be addressed by simply minimizing compliance. This is supported by the results in Figure (6.14c), where the stress constraints have been introduced to the
minimum compliance problem. In this formulation, results are effectively identical to the purely stress-constrained design, because stress information, which with design-dependent types of loads is not consistent with compliance as with mechanical loading, is directly available to the design optimization problem.

### 6.8.2.2 Effectiveness of Scaled Stress Measure

To reaffirm the effectiveness of the scaled stress measure, its components are plotted for the stress constraint corresponding to region 1 in Figure (6.16). Similar to the L-shaped bracket problem in Section (6.8.1.3), the blue curve denotes the value of $PN_1^i$ computed from the elements in its region according to Equation (6.10). The red dots give the maximum value of the stress failure criteria (normalized by the limit $F$) within region 1 in each iteration. The green curve denotes the scaling parameter $s_1^i$ that is computed at each iteration using the value of $PN_1^{i-1}$ and the maximum value from the previous iteration as stated in Equation (6.15). The black curve then gives the product $s_1^iPN_1^i$, which is the scaled aggregation measure that approximately gives the maximum stress value in the region. This quantity is then constrained to limit maximum stresses according to Equation (6.14). Just as in the previous problem, the approximation of the maximum value within the region quickly reaches the actual maximum value within only a small number of iterations and the local stress limits are satisfied.

### 6.8.2.3 Problem Summary

In this problem, the bi-clamped beam domain was optimized using the (i) minimum volume, stress constrained, (ii) minimum compliance, volume constrained, and (iii) minimum compliance, volume and stress constrained formulations. It is well understood that in cases of purely mechanical loading, the minimum compliance formulation generally produces designs with suitable stress distributions (with the exception
of geometric singularities) in terms of overall material layout with effective load paths. However, in this demonstration case, results demonstrated the limitation of the minimum compliance objective when considering design-dependent thermal loading. In this case, it was shown that by directly enforcing limits on stresses, even within a minimum compliance formulation, alternative material layouts are generated that better limit the detrimental effects of thermal expansion.
(a) Minimum Volume, Stress Constrained Design: $V_f = 0.146, F_{\text{max}} = 400.5 \, \text{MPa}, c = 315.4 \, \text{N} \cdot \text{m}$

(b) Minimum Compliance, Volume Constrained Design: $V_f = 0.15, F_{\text{max}} = 596.6 \, \text{MPa}, c = 310.5 \, \text{N} \cdot \text{m}$

(c) Minimum Compliance, Volume and Stress Constrained Design: $V_f = 0.144, F_{\text{max}} = 400.5 \, \text{MPa}, c = 311.4 \, \text{N} \cdot \text{m}$

Figure 6.14: Density and stress distributions for bi-clamped domain topology optimization with (a) stress-constrained formulation, (b) minimum compliance formulation, and (c) minimum compliance formulation with stress constraints.
Figure 6.15: Iteration history for bi-clamped domain topology optimization with (a) stress-constrained formulation, (b) minimum compliance formulation, and (c) minimum compliance formulation with stress constraints.
Figure 6.16: Iteration history for scaled stress measure used in constraint 1 in bi-clamped domain topology optimization.
6.8.3 Piston Design

In this example, the internal structure of an aluminum piston from an internal combustion engine is designed by topology optimization. The finite element model for the piston is given in Figure 6.17. Since a detailed representation of the attachment of the piston to the connecting rod is not desired, the model is reduced using axisymmetric symmetry to a planar problem. In this model, the piston crown is assumed to be flat and the ring notches are ignored. A uniform pressure load of 1500 psi is applied to the top surface of the piston, which is consistent with the cylinder pressures during the combustion stroke. Although the thermal conditions in an engine are in general transient, a static temperature distribution is assumed for this example with a maximum temperature of 600°F on the piston crown. The temperature is assumed constant in the $x$-direction and varies in the $y$-direction as shown in Figure 6.17. In the finite element model the nodes that lie along the purple line on the left edge of the structure are fixed in the $x$-direction and the nodes along the green lines are fixed in the $y$-direction. The elements colored in gray are assumed to be designable. Temperature dependent elastic modulus and coefficient of thermal expansion are taken as those of Aluminum 7075-T6 from Reference [31]. In addition, a significant knockdown on the yield strength is applied as Al 7075-T6 loses anywhere from 60-80% of its tensile strength at these elevated temperatures depending on its exposure time according to [31]. Thus, the stress limit in this problem is taken as 20 ksi. Additional comments regarding the physical context of this problem and the assumptions are given in Section 6.8.3.3.

6.8.3.1 Topology Optimization Problem

The design goals of the piston are to develop the stiffest piston possible such that the combined stresses resulting from the combustion pressure and the thermal environment remain below 20 ksi. To obtain a stiff design, a minimum compliance objective
Figure 6.17: Finite element model with boundary conditions and loading along with design domain for piston design problem.

is utilized using only the combustion pressure loads and a volume fraction constraint. Stress constraints are then added to the structure subjected to the combined thermal and pressure loads. The mathematical statement of this problem is given as

$$\begin{align*}
\min : & \quad c(\rho) = U_1(\rho)^T K(\rho) U_1(\rho) \\
\text{subject to} : & \quad g_1(\rho) = \sum_{e=1}^{N_d} (\rho_e v_e - V_f v_e) \leq 0 \\
& \quad g_m(\rho, U_2) = s_m PN(\rho, U_2)^i - 1 \leq 0 \text{ for } m = 1, 2, ..., n \\
& \quad K(\rho)U_1(\rho) = F^p \\
& \quad K(\rho)U_2(\rho) = F^p + F^{th}(\rho) \\
\text{variables} : & \quad 0 < x_{\min} \leq x_e \leq 1 \text{ for } e = 1, 2, ..., N_d
\end{align*}$$

where we note that two finite element analysis cases are utilized, similar to the thermoelastic combination method developed in Chapter 5. A total of $n = 9$ regions for stress constraints are utilized with no region resorting and the von Mises stress
criterion is employed. All elements, with the exception of those colored in red in Figure 6.17, are included in the stress constraints. These elements have been omitted since they are not representative of the actual attachment of the piston to the connecting rod and utilized primarily to enforce boundary conditions. A volume fraction of $V_f = 0.20$ is used and RAMP interpolation is taken for both stiffness and thermal loading with penalization parameters of $R_E = 8$ and $R_\beta = 8$. Finally, the density filter is utilized with a radius of 0.1, which is approximately two times the element size to prevent checkerboarding effects.

### 6.8.3.2 Results

The results of the topology optimization are shown in Figure 6.18. We note from the density distribution in Figure 6.18a that a structure is obtained that connects the piston crown directly to the portion of the inner model that is supported in the vertical direction. In addition, no material connects to the piston skirt since it experiences little loading and is free to expand. Figure 6.18b gives the stress distribution in the topology optimization model from which we observe that the stress limit of 20 ksi has been achieved throughout the entire domain.

In order to better assess the topologically derived structural concept, a physical idealization is taken from the density distribution as shown in Figure 6.19a. This interpretation is utilized to create a new model of the piston structure, which can be analyzed using a mesh that is better suited to capturing stresses. The new finite element mesh for the interpreted design is given in Figure 6.19b. Figure 6.20 gives the stress distribution obtained when subjecting the interpreted structure to identical loading conditions and effectively identical boundary conditions as the topology optimization model. Comparing the stress distribution in Figure 6.20 to that in Figure 6.18b, we observe that in most parts of the model, the lower quality representation of the structure in topology optimization predicts a similar distribution of stresses with
Figure 6.18: (a) Density and (b) stress distribution obtained from topology optimization for the piston structure.

areas of higher stresses in similar locations (although in some areas stresses slightly greater than 20 ksi are observed). The exception to this is the upper right region of the piston crown where significantly higher stresses are observed in the interpreted design mesh that are not effectively captured by the topology optimization analysis. This is likely due to the relative density of the meshes in this region. In the interpreted design mesh, elements that are roughly four times smaller are utilized and better capture the bending stresses that are apparent in this region of the structure. A refined mesh in the topology optimization would likely remedy this. Overall however, the similarity of the results gives confidence in the effectiveness of the stress-based topology optimization.

6.8.3.3 Contextual Notes

It is duly noted here that the assumptions limit the physical context of the piston design problem. The first assumption of an axisymmetric geometry is a severe limitation as the internal structure of a piston is not symmetric. In addition, the assumptions
regarding the static temperature distribution in the structure are not realistic as well. In fact, due to these assumptions, a configuration has been obtained that does not resemble those used in actual piston design. Despite these limitation, the simplified problem still serves to demonstrate the effectiveness of the stress-based formulation for a prescribed temperature distribution.

To properly conduct a piston topology optimization problem, a 3D model is required with the correct boundary conditions for wrist pin attachment in addition to ring slots and proper ring loading. Even more important is the need for a better representation of the thermal environment. In an internal combustion piston, oil from the crank case that is splashed onto the under-side of the piston is utilized for cooling and the actual temperature distribution experienced by the structure is both transient and highly dependent on the topological configuration. The transient loading can easily be capturing using multiple load cases in the optimization and the piston cooling should be captured in a heat transfer analysis. By doing so, a topology it
Figure 6.20: von Mises stress (psi) distribution for the piston configuration as interpreted from the topology optimization results.

is likely that a topology will be obtained that more closely resembles those used in practice due to their improved cooling capabilities.
Chapter 7

Summary Remarks and Recommendations

In the present day, modern applications in aerospace thermal structures, including exhaust-washed structures for embedded engine aircraft, have presented design scenarios that are not well addressed by conventional design methods and computational tools. This concern was demonstrated in Chapter 2, which investigated the responses of a unique class of thermal structures that is characterized by significant amounts of restrained thermal expansion. From this, the need for a more advanced design approach that considers the layout of structural material within the design domain, rather than simply varying the size and shape of pre-existing configurations, was observed. Structural topology optimization was proposed to address this need and a thorough review of the relevant methods and applications in the literature was given in Chapter 3. An effective formulation for thermoelastic topology, including the necessary parameterization of design-dependent thermoelastic loads, was developed in Chapter 4. Using optimization problem statements based on this formulation, topology optimization was applied in the design of exhaust-washed structural systems in Chapter 5. Finally, in Chapter 6 the topology optimization formulation was extended
to include stress-based design criteria. This entailed the development of new methods to effectively parameterize thermal stresses. Ultimately this, for the first time, provides a capability to directly treat thermal stresses in topology optimization, which by nature of its material layout capabilities, is a promising tool for the design of countless thermal structures not just for aerospace applications, but also challenging problems in the automotive and energy industries as well.

As the body of work in any dissertation cannot represent a comprehensive treatment of the entire subject of focus, the following sections highlight the specific contributions to the thermal structures and multidisciplinary design optimization fields that have been made in this research as well as some recommendations and direction for future research.

### 7.1 Research Contributions

In this dissertation, a number of contributions have been made that (i) further the understanding of a unique class of thermal structures that are important to modern applications and (ii) advance the state-of-the-art in structural topology optimization. The details of some key contributions are enumerated as follows:

1. Based on computational investigations, fundamental criteria that contribute to the significance of geometric nonlinearity in thermally loaded structures were identified. The study of these criteria resulted in a set of guidelines for when geometric nonlinear analysis should be utilized. This knowledge is important from a design point of view because it indicates under what circumstances current design tools, which are generally based on linear analysis, can be effectively employed and when more advanced formulations that consider the nonlinear effects of stress stiffening and thermoelastic follower forces are required. These investigations were presented in Sections 2.2 and 2.3 and published in [27].
2. A novel reaction load response for topology optimization was formulated, including analytical sensitivity analysis, and demonstrated. With this constraint, the amount of loading that is imposed on a specified set of nodal degrees of freedom in the finite element model can be limited within the topology optimization problem. Such a constraint is especially important in the design of thermally loaded structures because material addition in an elevated temperature environment results in increases in reaction loading, which in real world problems corresponds to increased loading on adjoining structures. Limiting this quantity in the topology optimization problems allows for the evolution of structures whose local design variation does not considerably effect surrounding structures from a load transfer point of view. In addition, for mechanically loaded structures, this constraint can be utilized to limit the loading that is carried by individual fasteners and can provide greater control of load transfer paths in the topological design.

3. The conventional minimum compliance topology optimization problem that has enjoyed considerable success in both academic research and industrial applications for mechanical problems was demonstrated to suffer from severe deficiencies in the presence of design-dependent thermoelastic loading. As a result, new topology optimization problem formulations were proposed based on concepts of fictitious mechanical loads, different combinations of pure mechanical and thermal loading conditions, and stress criteria. In application examples, these formulations were shown to produce designs with desirable thermal structures characteristics and improved thermoelastic performance. These alternative formulations were discussed in Chapters 4 and 5 and published in [70].

4. One of the primary challenges in stress-based topology optimization is the local nature of stress constraints and the inability to precisely control maximum stress
levels when using computationally efficient aggregation functions. To this end, a scaled aggregation measure for stress was proposed. This technique, in which stress and aggregation information from prior design iterations is utilized in addition to the current iteration response information, demonstrated the ability to reliably satisfy local limits on stress at a computational cost that is amenable to large scale topology optimization problems.

5. To date, stress-based topology optimization has only been applied to problems of purely mechanical loading in the literature. Here, a formulation for considering thermal stresses in topology optimization was proposed using a stress interpolation function to relax the so-called singularity phenomena and a scaled aggregation technique to treat local stress limits. With this formulation, thermal stresses, which represent the primary design concern for the thermal structures of interest, can be directly addressed within the topology optimization problem. This represents a significant improvement when compared to the existing techniques, which are either impotent for practical levels of thermal loading or require the solution of alternative optimization problems to obtain desirable thermoelastic performance, for topology optimization of thermoelastic domains.

In addition to these contributions, this research also produced a survey of topology optimization, given in Reference [95], that was published and has to date enjoyed positive feedback and acceptance in the international MDO community.

### 7.2 Recommendations

**Coupled Thermal-Structural Topology Optimization**

As stated in Section 1.3, the design of thermal structures is best addressed by following two fundamental rules: (1) minimize temperatures and temperature gradients
and (2) accommodate thermal expansion. In this dissertation, the design problem was approached from purely a structural point of view with prescribed temperature distributions and, as a result, only rule (2) was truly considered. Thus, including the design-dependent physics of heat transfer in the topology optimization of thermal structures is an important consideration. In most thermal structures problems, temperature loads are a result of heat flux across boundary surfaces that is conducted throughout the structure. This leads to spatially varying distributions that are heavily influenced by density variations during the course of the topology optimization process. Referring to the beam strip stiffening example, Figure 7.1 shows a theoretical EEWS design case where heat transfer effects are considered. Here, we see that the effect of hot exhaust gas is more appropriately represented as a flux boundary condition, from which thermal energy is conducted through the structure. As the topology evolves, the surfaces of void regions also develop flux boundaries to represent active or convective cooling sources.

Figure 7.1: Example of heat transfer effects to be included in topology optimization.

While the prescribed temperature assumptions represent a first step to developing effective thermoelastic topology optimization capabilities, the assertion that including design-dependent heat transfer is of benefit is straightforward. Not only does this provide a better representation of the true loading environment, but it also affords additional freedom to the topology optimization, which may enable better designs, as the optimization problem could then accommodate design rule (1). When heating
is represented as a heat flux, a structure that is more effectively cooled by efficient conduction paths and convective and radiative heat loss will likely posses superior thermoelastic performance when compared to one that maintains higher temperatures due to limited cooling. This possibility is unlocked by making the temperature distribution sensitive to the topology design process. A finite element formulation that could be utilized for the basis of a combined heat transfer and structural topology optimization tool along with the adjoint sensitivity analysis is briefly derived in the following subsections.

**Finite Element Formulation**

The steady-state, linear thermoelastic scenario can be analyzed using the finite element systems

\[
K_t(\rho)T(x) = F_t(\rho) \tag{7.1}
\]

\[
K(\rho)U(\rho) = F^m + F^{th}(T(\rho), \rho) \tag{7.2}
\]

where Equations (7.1) and (7.2) represent the heat transfer and structural systems, respectively. Here \( K_t \) is the conductivity matrix, \( T \) is the nodal temperature vector, and \( F_t \) is the load vector for heat transfer. Variables in Equation (7.2) remain consistent with definitions in Chapter 4; however, we note now that the thermal load vector in (7.2) is dependent on the temperature result from Equation (7.1).

Topology optimization for the coupled system is accomplished by parameterizing both finite element systems with density variables. Assuming conduction and free convection through a film coefficient are included in the heat transfer analysis, both the thermal conductivity and convection film coefficient must be interpolated. The
element conductivity matrix for heat conduction is given by

\[ k_{c,e} = \int_{\Omega_e} B^T k B d\Omega_e, \quad (7.3) \]

and can be parameterized using a conventional interpolation scheme applied to the thermal conductivity \( k \). The convection contributions to the global conductivity matrix \( K_t \) and load vector \( F_t \), given at the element level by

\[ k_{h,e} = \int_{S_e} h N^T N dS_e \] \[ f_{h,e} = \int_{S_e} h T_e N dS_e, \quad (7.5) \]

will require more advanced interpolation of the convection coefficient \( h \). In the preceding relationships, \( \Omega_e \) is the volume domain of element \( e \), \( S_e \) is the surface on which convection is applied at \( e \), and \( N \) is the vector of shape functions, whose spatial derivatives are contained in \( B \). Finally, the structural system is parameterized according to Chapter 4.

**Sensitivity Analysis**

The sensitivity analysis for the coupled heat transfer and structural problem is accomplished using a coupled adjoint solution. Assuming that topology optimization objectives and constraints are computed from displacement results obtained from Equation (7.2), the sensitivity analysis is formulated by first taking the derivative of a general function \( f(U(\rho)) \) with respect to the physical density \( \rho_j \) as

\[ \frac{df}{d\rho_j} = \frac{\partial f}{\partial \rho_j} + \frac{df}{dU} \frac{dU}{d\rho_j}. \quad (7.6) \]
Differentiating the structural system in Equation (7.2), the sensitivity of the displacement vector to the density is obtained as

\[
\frac{dU}{d\rho_j} = K^{-1} \left( \frac{dF_{th}}{d\rho_j} - \frac{dK}{d\rho_j} U \right). \tag{7.7}
\]

Substituting (7.7) into (7.6) yields

\[
\frac{df}{d\rho_j} = \frac{\partial f}{\partial \rho_j} + \frac{\partial f}{\partial U} K^{-1} \left( \frac{dF_{th}}{d\rho_j} - \frac{dK}{d\rho_j} U \right) = \frac{\partial f}{\partial \rho_j} + \lambda_1^T \left( \frac{dF_{th}}{d\rho_j} - \frac{dK}{d\rho_j} U \right) \tag{7.8}
\]

in which the adjoint vector \( \lambda_1 \) has been introduced and is determined by solving the adjoint system

\[
K\lambda_1 = \frac{\partial f}{\partial U}. \tag{7.9}
\]

From (7.2), we recall that the load vector \( F_{th} \) is a function of the temperature distribution \( T \), which is no longer assumed as design independent. Thus, the sensitivity of the thermal load vector in the structural analysis is expanded as

\[
\frac{dF_{th}}{d\rho_j} = \frac{\partial F_{th}}{\partial \rho_j} + \frac{\partial F_{th}}{\partial T} \frac{dT}{d\rho_j}, \tag{7.10}
\]

where the partial derivative term \( \frac{\partial F_{th}}{\partial \rho_j} \) captures the explicit sensitivity of the interpolation used in the thermal load and the terms \( (\frac{\partial F_{th}}{\partial T})(\frac{dT}{d\rho_j}) \) capture the implicit sensitivity of the temperature distribution \( T \) to the design variable \( \rho_j \).

Including Equation (7.10) into (7.8) gives

\[
\frac{df}{d\rho_j} = \frac{\partial f}{\partial \rho_j} + \lambda_1^T \left( \frac{\partial F_{th}}{\partial \rho_j} + \frac{\partial F_{th}}{\partial T} \frac{dT}{d\rho_j} - \frac{dK}{d\rho_j} U \right), \tag{7.11}
\]

\[
= \frac{\partial f}{\partial \rho_j} + \lambda_1^T \left( \frac{\partial F_{th}}{\partial \rho_j} - \frac{dK}{d\rho_j} U \right) + \lambda_1^T \frac{\partial F_{th}}{\partial T} \frac{dT}{d\rho_j} \tag{7.12}
\]

which now necessitates computation of the derivative of the temperature vector to variable \( \rho_i \). This is again accomplished by first differentiating the state equation, in
this case for the heat transfer analysis, in (7.1) yielding

\[
d\frac{\mathbf{T}}{\rho_j} = \mathbf{K}_t^{-1} \left( \frac{d\mathbf{F}_t}{d\rho_j} - \frac{d\mathbf{K}_t}{d\rho_j} \mathbf{T} \right). \tag{7.13}
\]

Here, for generality, the design dependency of the load vector for heat transfer \( F_t(\rho) \) is retained, which may not occur in all cases. Now, substituting Equations (7.13) into (7.8) yields

\[
\frac{df}{d\rho_j} = \frac{\partial f}{\partial \rho_j} + \lambda_1^T \left( \frac{\partial \mathbf{F}^{th}}{\partial \rho_j} - \frac{d\mathbf{K}}{d\rho_j} \mathbf{U} \right) + \lambda_1^T \frac{\partial \mathbf{F}^{th}}{\partial \mathbf{T}} \mathbf{K}_t^{-1} \left( \frac{d\mathbf{F}_t}{d\rho_j} - \frac{d\mathbf{K}_t}{d\rho_j} \mathbf{T} \right), \tag{7.14}
\]

to which a second adjoint vector is introduced after simplification as

\[
\frac{df}{d\rho_j} = \frac{\partial f}{\partial \rho_j} + \lambda_1^T \left( \frac{\partial \mathbf{F}^{th}}{\partial \rho_j} - \frac{d\mathbf{K}}{d\rho_j} \mathbf{U} \right) + \lambda_2^T \left( \frac{d\mathbf{F}_t}{d\rho_j} - \frac{d\mathbf{K}_t}{d\rho_j} \mathbf{T} \right). \tag{7.15}
\]

The additional adjoint vector \( \lambda_2 \) is determined from

\[
\mathbf{K}_t \lambda_2 = \frac{\partial \mathbf{F}^{th}}{\partial \mathbf{T}} \lambda_1. \tag{7.16}
\]

In summary, obtaining the sensitivity of a response for topology optimization from the sequentially coupled heat transfer and structural analysis is a multistage process. First the finite element solution to the heat transfer problem in Equation (7.1) is obtained, which is utilized to solve for the displacement response of the structure in Equation (7.2). Next, the appropriate adjoint problem in Equation (7.9) is formulated, with the adjoint load term \( \partial f/\partial \mathbf{U} \) depending on the form of response \( f \), and the adjoint vector \( \lambda_1 \) is obtained. These results are then used in the formation of the second adjoint problem in Equation (7.16). Here it is important to note that if element average temperatures are utilized, the matrix \( \partial \mathbf{F}^{th}/\partial \mathbf{T} \) is a sparse matrix with the number of nonzero terms in each column equal to the number of nodes in
each element and the system can be assembled quickly. Solution of the second adjoint system yields all the necessary information required for substitution into Equation (7.15) to obtain the final sensitivity of the response $f$ to a design variable $\rho_j$. The solution of Equation (7.15) also benefits in efficiency because it can be solved at the element matrix level (and also in parallel), rather than the system matrix level because the partial derivative with respect to a design variable for an element yields zero sensitivity for all global induces not in that element. The solution of the additional adjoint problems has modest computational cost because the decomposed stiffness and conductivity matrices are available from the finite element solution. As a result, just as with topology optimization based on single physics analysis, the process is tractable for a reasonable number of design responses. Finally, we recall that the physical density $\rho$, which the preceding sensitivities have been derived with respect to, is either equal to the design variable density $x$ or a function of the design variable density as $\rho(x)$ depending on the filter. With the appropriate chain rules from Section (3.2.2), the sensitivity of the function to the density design variables $df/dx_i$, can be obtained.

**Geometric Nonlinearity in Topology Optimization**

Despite the focus on geometric nonlinearity in the discussions of Section 2.2, the topology optimization formulations and results presented in this work were primarily based on design formulations employing linear analysis. Note that despite this, care was taken to evaluate the performance of the resulting designs using nonlinear analysis prior to drawing any conclusions. This was done as a result of two primary causes. First, early investigations into incorporating geometric nonlinearity in the topology optimization faced significant computational hurdles related to the behavior of low density regions, which have corresponding low stiffness, in the nonlinear solver in the presence of thermoelastic effects. While these issues have been treated with
some success in the topology optimization literature for mechanical problems, a solution has yet to be identified for the numerical challenges that arise in the nonlinear analysis procedure when design-dependent loading sources are present. Second, after obtaining candidate designs via the topology optimization techniques proposed in this dissertation, comparison of linear and nonlinear analysis on the final designs indicate that configurations are produced in which nonlinearity is not significant. Thus, while it appears the topology optimization techniques produce final designs that are free of considerable nonlinearity, the effect of utilizing a true geometrically nonlinear analysis in the topology optimization, along with the potential for realizing further performance gains, for general thermal structures remains to be seen.

**Multidisciplinary Responses**

While the focus of this dissertation remained fixed on thermal stresses, most thermal structures must satisfy a number of additional design criteria. For example, the vibration and dynamic characteristics of aircraft engine exhaust-washed structures are also important design considerations due to additional wide-band acoustic loading from the exhaust flow. In addition, thermal buckling is another important factor that should be considered. As a result, the topology optimization problem should be expanded to include other functional design criteria including frequency responses and bifurcation buckling. Taking this one step further, the effect of the thermal environment on these additional responses should be rigorously treated. As an example, the fundamental frequency of a structure at room temperature will differ from that at an elevated temperature due to the effect on material properties and the pre-stressed condition created in the presence of restrained expansion. In topology optimization, these additional effects must be appropriately parameterized with the density variables and will require further investigation to identify the appropriate interpolation functions to ensure a well-posed problem.
Three-dimensional Geometry

In this work, only two-dimensional topology optimization was demonstrated due to computational limits. However, every method and technique that was formulated and presented herein is readily extendable to three-dimensions with the introduction of three-dimensional continuum finite elements. Doing so with linear and quadratic tetrahedral and hexagonal elements will significantly increase the range of applications for the proposed methods. In addition, effective use of parallel processing in the solution of adjoint sensitivities and element stress recovery would reduce the computational cost associated with three-dimensional topology optimization.

Effective Optimization Algorithms

In the course of this work, challenges were often encountered regarding the choice of the appropriate optimization algorithm. In topology optimization, gradient-based algorithms based on dual-primal methods with approximate sub-problems are the most widely used, including the MMA algorithm (and the more robust GCMMA) employed here. However, depending on the level of design-dependent thermal loading, a troublesome behavior was frequently observed when the approximate problem was unconditionally non-conservative with respect to the true response values at a proposed design point. In many cases, this occurred regardless of the size of the move limits imposed on the sub-problem. Often this impeded convergence to a feasible optimum. The source of this behavior appears to lie in the choice of the approximations utilized by the moving asymptote algorithms, which consider gradient and response information only at the current design point and are limited in the degree of nonlinearity they can accurately capture. Within the class of MMA algorithms, the quality of the sub-problem could be improved by incorporating multi-point approximations that utilize gradient and response values from previous iterations and can achieve a greater degree of conservativeness for highly nonlinear responses like those found in
the stress-based topology optimization problems. Alternatively, other gradient-based optimizers (for constrained problems) can be investigated, for example the commercial tool SNOPT, which is based on sequential quadratic programming (SQP) with efficient techniques to formulate the second-order terms in the Hessian.
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