2017

Bistatic SAR Polar Format Image Formation: Distortion Correction and Scene Size Limits

Davin Mao
Wright State University

Follow this and additional works at: https://corescholar.libraries.wright.edu/etd_all

Part of the Electrical and Computer Engineering Commons

Repository Citation
https://corescholar.libraries.wright.edu/etd_all/1781

This Thesis is brought to you for free and open access by the Theses and Dissertations at CORE Scholar. It has been accepted for inclusion in Browse all Theses and Dissertations by an authorized administrator of CORE Scholar. For more information, please contact corescholar@www.libraries.wright.edu.
Bistatic SAR Polar Format Image Formation: Distortion Correction and Scene Size Limits

A thesis submitted in partial fulfillment of the requirements for the degree of Master of Science in Electrical Engineering

by

Davin Mao
B.S. Physics, University of Portland, 2013

2017
Wright State University
I HEREBY RECOMMEND THAT THE THESIS PREPARED UNDER MY SUPERVISION BY Davin Mao ENTITLED Bistatic SAR Polar Format Image Formation: Distortion Correction and Scene Size Limits BE ACCEPTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF Master of Science in Electrical Engineering.

___________________________________________
Brian D. Rigling, Ph.D.
Thesis Director

___________________________________________
Brian D. Rigling, Ph.D.
Chair, Department of Electrical Engineering

Committee on
Final Examination

___________________________________________
Brian D. Rigling, Ph.D.

___________________________________________
Michael A. Saville, Ph.D., P.E.

___________________________________________
Joshua N. Ash, Ph.D.

___________________________________________
Robert E.W. Fyffe, Ph.D.
Vice President for Research and
Dean of the Graduate School
ABSTRACT

Mao, Davin. M.S.E.E., Department of Electrical Engineering, Wright State University, 2017. 
*Bistatic SAR Polar Format Image Formation: Distortion Correction and Scene Size Limits.*

The polar format algorithm (PFA) for bistatic synthetic aperture radar (SAR) image formation offers the compromise between image quality and computational complexity afforded by PFA, while enabling the geometric flexibility of a bistatic collection scenario. The use of the far-field approximation (FFA), which enables the use of the two-dimensional (2D) fast Fourier transform (FFT) in PFA, introduces spatially-varying distortion and defocus effects causing geometric warping and blurring in the resulting image. In this thesis, the residual phase errors due to the FFA are analyzed by decomposing the residual phase errors in the time dimension into their constant, linear, and quadratic Taylor series components. Based on the analysis, a 2D interpolation-based distortion correction technique is developed, and accurate scene size limits are derived for the corrected image to mitigate the effects of defocus. The phase error analysis is conducted with respect to arbitrary transmitter and receiver trajectories, and examples are demonstrated for both the ideal linear and ideal circular flight geometries using a point target scene simulation.
Copyright by
Davin Mao 2017
# Table of Contents

List of Figures ................................................................. vii  
Acknowledgment ............................................................... viii 
Dedication ................................................................. ix 

## 1 Introduction
1.1 Bistatic Synthetic Aperture Radar ........................................ 1  
1.2 SAR Image Formation Algorithms ........................................ 2  
1.3 Wavefront Curvature Correction ......................................... 3  
1.4 Contribution ............................................................... 5  
1.5 Thesis Outline ............................................................. 5  

## 2 Geometric Definition
2.1 Arbitrary Flight Geometry ................................................... 7  
2.2 Linear Flight Geometry .................................................. 9  
2.3 Circular Flight Geometry ............................................... 10  

## 3 Analysis of Phase History
3.1 Phase History Model ....................................................... 14  
3.2 Residual Phase Error Analysis ........................................... 16  
3.2.1 Taylor Expansion of Bistatic Differential Range ................. 18  
3.2.2 Taylor Expansion of Bistatic PFA Differential Range ............ 19  
3.3 PFA Distortion Prediction & Correction ................................ 20  
3.4 Scene Size Limits ......................................................... 23  

## 4 Application to Ideal Flight Geometries
4.1 Circular Flight Geometry .................................................. 25  
4.2 Linear Flight Geometry ................................................... 30  

## 5 Comparison to Previous Results
5.1 Previous Scene Size Limits for Bistatic PFA ............................ 37  
5.2 Comparison to Monostatic Results ...................................... 40  
5.3 Defocus Correction and the Bistatic Look Angle ....................... 41
List of Figures

4.1 Point target scene and circular flight geometry. 26
4.2 PFA image, circular flight geometry. 27
4.3 Distorted coordinate grid, circular flight geometry 29
4.4 PFA image with distortion correction, circular flight geometry 29
4.5 Residual quadratic phase error, circular flight geometry 30
4.6 Corrected PFA image with scene size limits, circular flight geometry 31
4.7 Point target scene and linear flight geometry. 32
4.8 PFA image, linear flight geometry. 32
4.9 Distorted coordinate grid, linear flight geometry. 34
4.10 PFA image with distortion correction, linear flight geometry. 35
4.11 Residual quadratic phase error, linear flight geometry. 36
4.12 Corrected PFA image with scene size limits, linear flight geometry. 36

5.1 Corrected PFA image with old and new scene size limits, circular geometry. 39
5.2 Corrected PFA image with old and new scene size limits, linear geometry. 40
5.3 Cartesian coordinate system rotated by arbitrary angle $\phi$. 44
Acknowledgement

I would first like to thank my thesis advisor, Dr. Brian Rigling, for the guidance and mentorship he has provided since this thesis was initiated two years ago. Despite his many endeavors, he still made the time to coach me through mine. I would also like to thank the other members of my thesis committee, Dr. Josh Ash and Dr. Mike Saville, for their support. I would also like to thank Dr. LeRoy Gorham whose dissertation work laid the foundation for this thesis.

At the Air Force Research Laboratory Sensors Directorate, I would like to thank my supervisors, Michael McConkey and Captain Jeffrey Nishida, my technical advisor, Dr. Braham Himed, and my mentors, Chuck Berdanier, Chuck Mott, and Dr. Justin Metcalf, for their enduring support and encouragement over the last two years.

Lastly, I would like to thank my physics professors at the University of Portland, Dr. Shannon Mayer, Dr. Osiel Bonfim, Dr. Tamar More, Dr. Mark Utlaut, and Dr. Maximilian Schlosshauer for imbuing me with a love of learning that fueled me through graduate school.
Dedication

This thesis is dedicated to my parents Chhommony and Mardine Mao. When faced with adversity, I persisted with the knowledge that my hardships paled in comparison to the trials they faced afford me the opportunities that I am privileged with today.
Chapter 1

Introduction

1.1 Bistatic Synthetic Aperture Radar

Spotlight-mode synthetic aperture radar (SAR) is an image formation technique in which the radar host platform traverses a planned trajectory while uniformly transmitting and receiving radio frequency pulses to and from a scene of interest. By coherently integrating pulses collected from different aspect angles with respect to the scene of interest, the traditional limits on cross-range resolution of the fixed aperture radar may be extended. In the bistatic regime, the SAR transmitter and receiver are located on two separate platforms and are allowed to traverse independent trajectories, affording several advantages over the monostatic SAR case in which the SAR transmitter and receiver are collocated. First, only one of either the transmitter or receiver are required to traverse the synthetic aperture allowing for flexibility in choice of different combinations of existing ground-based, airborne, or spaceborne radar systems for SAR imaging. Second, only one of either the transmitter or receiver is required to have a velocity component perpendicular to the look angle with respect to the imaged scene. This provides operational advantages in that an aircraft mounted receiver operating in the forward-looking mode may generate SAR images as it
travels covertly towards the scene of interest, while the aircraft mounted illuminator flies
the synthetic aperture at a safe distance [1, 2]. For ease of narration, the language in this
thesis assumes the transmitter and receiver are both located on airborne systems, and that
their corresponding synthetic apertures are traversed via flight.

1.2 SAR Image Formation Algorithms

Image formation using phase history data collected from the SAR collection geometry at-
tempts to estimate the complex surface reflectivity function of the spotlighted scene. The
bistatic SAR matched filter (MF), derived as the maximum likelihood estimator in the pres-
ence of Gaussian measurement errors [3], is considered to be the optimal solution which
maximizes signal-to-noise ratio [4]. To estimate the reflection coefficient at a single pixel
in the reconstructed image, the contribution of the signal phase from that pixel is estimated
for each data sample. To reconstruct an $N \times N$ pixel image from collected phase history
data containing $N$ slow time and $N$ fast time samples, the MF requires computations on
the order of $O(N^4)$ making it impractical for tactical implementation.

The computational burden of the SAR MF has motivated the development of several
suboptimal, but computationally permissible SAR algorithms. The backprojection algo-
rithm (BPA) [5, 6] efficiently implements the MF by calculating the individual contribution
of each pulse and interpolating that contribution to the image grid achieving $O(N^3)$ compu-
tational complexity without loss of image quality. While this is a significant improvement
over the MF, the large computation cost becomes prohibitive for data sets containing large
numbers of pulses. Various fast implementations of BPA have been developed such as the
fast-factorized BPA [7] and its bistatic implementation [8, 9] which achieve logarithmic
computational complexity by subaperturing the data. The pulse-by-pulse processing used
by BPA makes it highly parallelizable and amenable to GPU implementation [10].

The polar format algorithm (PFA) for spotlight-mode SAR [11, 12, 13], and its bistatic
implementation [2, 3, 14, 15] are computationally efficient, suboptimal SAR imaging algorithms that implement an approximation of the SAR MF with $O(N^2 \log_2 N)$ complexity. This is achieved by interpolating the polar formatted phase history data to a uniformly sampled rectilinear grid enabling the use of the computationally efficient two-dimensional (2D) fast Fourier transform (FFT). The 2D FFT is facilitated by a far-field approximation (FFA), a spatial first-order Taylor series approximation of the differential range [3, 16]. While the approximation is valid for small scenes, high frequency pulses, and collection geometries in which the SAR is adequately far from the target scene, its use introduces spatially-varying distortion and defocus effects in the SAR image [11].

A thorough comparison of the computational cost of image formation algorithms is conducted in [17], but ultimately the choice of algorithm will depend on systems engineering factors such as hardware architecture and data collection parameters. An exhaustive comparison of image formation algorithms is outside the scope of this thesis. Instead, the bistatic PFA will be examined in detail with respect to understanding and mitigating the effects of errors due to wavefront curvature.

### 1.3 Wavefront Curvature Correction

Extensive research has been conducted to understand and correct the errors due to wavefront curvature in other contexts. It was shown that the phase error due to the FFA could be decomposed into linear and quadratic components by performing a 2D Taylor series expansion on the phase error in the frequency domain [11]. Geometric distortion, which causes keystone warping of the rectangular image, was attributed to the linear components of the error. The quadratic component of the phase error caused defocusing in the image, described in [11] as “an astigmatic focus error and will cause degradation in resolution.” The author’s phase analysis led to a method to undistort the image, however defocus correction was not presented. Rather, the author proposed to mitigate the effect of defocus
by restricting the scene radius using the constraint that the quadratic phase error (QPE) be less than $\pi/2$. Extending the phase analysis of [11], a space-variant post filtering (SVPF) technique was developed in [18, 19] which corrected for the defocus due to QPE. Having compensated for the QPE, the scene size restriction was subsequently derived by restricting the uncompensated cubic phase error to be less than $\pi/8$. The larger scene size limit after post filtering was demonstrated in [20]. Various versions of the SVPF have been presented in the literature including an interferometric SAR implementation [21], an improved SVPF for arbitrary geometries [22], and the SVPF for PFA extended to the bistatic geometry [23].

An alternative phase analysis technique was used in [16], wherein a three-dimensional (3D) spatial-domain Taylor series expansion was performed on the differential range. The authors showed that the FFA discarded all but the linear term of the expansion, and that the acceptable scene radius could be derived by bounding the QPE term. Further research was conducted by the authors of [24] to define scene size limits for the specific application of automatic target recognition by looking at the statistical separability of point targets in the presence of geometric distortion and defocus. In [3], the authors extended the analysis of PFA into the bistatic geometry, resulting in a bistatic PFA that efficiently utilized the collected data. Again, they derived an expression for the allowable scene radius by bounding the discarded quadratic phase term of the Taylor series expansion of the bistatic differential range.

In [25, 26], the authors showed that performing a Taylor expansion on the differential range in the slow time domain could be used to develop a method to predict the distortion in a PFA image. Their phase analysis also resulted in analytic functions that predicted the space-variant residual QPE allowing them to derive scene size limits that were arbitrary in geometry, a departure from previous results which suggest bounding the scene to a circular region of focus. By exploiting a symmetry in the residual QPE function for SAR phase history data collected from a circular flight trajectory, the authors of [27] developed a fast and efficient PFA exclusive to the circular geometry. The phase analysis and distortion
correction were extended to the bistatic PFA geometry in [28].

Several implementations of bistatic PFA have been published in the open literature, many presenting solutions for wavefront curvature correction for specific collection geometries. For example, the authors of [29] implement SVPF for the bistatic case in which the receiver is stationary. The authors of [30, 31] also implement the one-stationary bistatic case, but with a spaceborne transmitter. The forward-looking bistatic SAR geometry is presented in [32], and the circular geometry is presented in [14, 33].

1.4 Contribution

In this thesis, the phase analysis for monostatic PFA presented by Gorham [25] is extended to the bistatic geometry for arbitrary broadside flight trajectories. Following Gorham’s methodology, the phase analysis is used to predict distortion and defocus in the scene reconstructed using bistatic PFA, allowing for distortion correction using bilinear interpolation. The defocus prediction is used to define accurate scene size limits for the image. The results are verified with a point target scene simulation for the ideal linear and circular bistatic collection geometries using parameters analogous to the circular and linear monostatic geometries simulated by Gorham.

1.5 Thesis Outline

The remainder of this thesis will be structured as follows. In Chapter 2, the geometric framework for the phase analysis is presented in terms of arbitrary broadside flight trajectories. The parameterized equations for the ideal circular and linear flight trajectories are also defined. In Chapter 3, the bistatic phase history model is stated and analysis of the phase error is performed following Gorham’s method. Based on the phase analysis, Gorham’s method for distortion correction, defocus mapping, and scene size limit derivations are developed for arbitrary bistatic geometries. In Chapter 4, the arbitrary expressions
derived in Chapter 3 are applied to the ideal circular and linear flight geometries introduced in Chapter 2, and the results are verified with a simulated point target scene. In Chapter 5, the new distortion correction functions and scene size limits for bistatic geometries are compared to results presented by previous studies. The thesis is summarized in Chapter 6, and recommendations for future work are presented.
Chapter 2

Geometric Definition

In this chapter, the geometric framework for bistatic SAR is presented. The coordinate system is defined and the arbitrary flight geometry is introduced. The parameterization of the ideal circular and ideal linear flight geometries are also introduced.

2.1 Arbitrary Flight Geometry

The analysis in this thesis builds upon the geometric framework and signal model for bistatic SAR phase history data developed in [3]. Their notation is repeated here for convenience. The Cartesian coordinates of the transmitter and receiver as they traverse the synthetic aperture are denoted by

\[ \gamma_t(\tau) = [x_t(\tau), y_t(\tau), z_t(\tau)]^T \]  (2.1)

and

\[ \gamma_r(\tau) = [x_r(\tau), y_r(\tau), z_r(\tau)]^T, \]  (2.2)
respectively. These coordinates are defined with respect to the origin which is chosen to be the center of the imaged scene. The variable $\tau$ denotes the time dimension throughout the collection and is supported on the interval [-1,1]. The distances from the origin to the transmitter and receiver are defined as

$$r_t(\tau) = \|\gamma_t(\tau)\| = \sqrt{x_t(\tau)^2 + y_t(\tau)^2 + z_t(\tau)^2}$$  \hspace{1cm} (2.3)$$

and

$$r_r(\tau) = \|\gamma_r(\tau)\| = \sqrt{x_r(\tau)^2 + y_r(\tau)^2 + z_r(\tau)^2},$$  \hspace{1cm} (2.4)$$

respectively. The location of a stationary point target in the scene is given by

$$p = [x, y, z]^T.$$  \hspace{1cm} (2.5)$$

The distance from the point target to the transmitter is defined as

$$r_{pt} = \|p - \gamma_t\| = \sqrt{(x - x_t)^2 + (y - y_t)^2 + (z - z_t)^2}$$  \hspace{1cm} (2.6)$$

and to the receiver,

$$r_{pr} = \|p - \gamma_r\| = \sqrt{(x - x_r)^2 + (y - y_r)^2 + (z - z_r)^2},$$  \hspace{1cm} (2.7)$$

where the time dependency is omitted for compactness.
2.2 Linear Flight Geometry

The coordinates of the transmitter and receiver as they traverse an ideal linear flight trajectory are parameterized by the following equations:

\[
\begin{align*}
\dot{x}_t(\tau) &= \ddot{x}_t \\
\ddot{x}_t(\tau) &= 0 \\
\dddot{x}_t(\tau) &= 0
\end{align*}
\]
\[
\begin{align*}
\dot{y}_t(\tau) &= \frac{L_T}{2} \tau \\
\ddot{y}_t(\tau) &= \frac{L_T}{2} \\
\dddot{y}_t(\tau) &= 0
\end{align*}
\]
\[
\begin{align*}
\dot{z}_t(\tau) &= \ddot{z}_t \\
\ddot{z}_t(\tau) &= 0 \\
\dddot{z}_t(\tau) &= 0
\end{align*}
\] (2.8)

\[
\begin{align*}
\dot{x}_r(\tau) &= \bar{x}_r + v_x \tau \\
\ddot{x}_r(\tau) &= v_x \\
\dddot{x}_r(\tau) &= 0
\end{align*}
\]
\[
\begin{align*}
\dot{y}_r(\tau) &= \bar{y}_r + v_y \tau \\
\ddot{y}_r(\tau) &= v_y \\
\dddot{y}_r(\tau) &= 0 \\
\dddot{z}_r(\tau) &= 0
\end{align*}
\] (2.9)

where \(L_T\) is the length of the transmitter’s flight trajectory, and \(v_x\) and \(v_y\) are the \(x\) and \(y\) components of the receiver’s velocity, respectively. The \(\dot{x}\) and \(\ddot{x}\) notation is used to denote the first and second partial derivatives of \(x\) with respect to \(\tau\). For mathematical convenience and consistency with [25], the transmitter’s flight path is intentionally chosen to be centered on the \(x\) axis at time \(\tau = 0\), however it is noted that the selection of how to orient the Cartesian axes is completely arbitrary and may be changed via a rotation of coordinate system about the scene center. Further discussion on selection of axis orientation is explored in Chapter 5. The receiver’s flight path is centered on an arbitrary point \((\bar{x}_r, \bar{y}_r, \bar{z}_r)\). Both the transmitter’s and receiver’s flight paths are constant in elevation throughout the collection period. The receiver’s coordinates are intentionally defined with respect to the velocity components for ease of differentiation, however the length of the receiver’s synthetic aperture may be recovered using

\[
L_R = 2\sqrt{\bar{v}_x^2 + \bar{v}_y^2},
\] (2.10)
where $T$ is the duration of the collection period. The bistatic angle, defined in this thesis as the azimuth angle between the center of the transmitter’s and receiver’s synthetic apertures, may be calculated using

$$\beta = \arctan \frac{y_{r}}{x_{r}},$$

for a receiver with aperture center located in the first or fourth quadrant of the defined coordinate system. For the remainder of this thesis, “bistatic angle” will refer to the azimuth component of the true bistatic angle which is defined as the time-varying angle between vectors $\gamma_t$ and $\gamma_r$.

For simplicity of image formation, it is desired for the transmitter and receiver to travel with the same angular progression. The fact that $L_T$ is a physical parameter forces it to be positive resulting in a counterclockwise (with respect to the right handed coordinate system) transmitter flight trajectory. Lastly, a broadside collection geometry with respect to the scene center is assumed. The parameterization of the transmitter’s flight trajectory forces a broadside collection, however to ensure the receiver’s flight trajectory is also counterclockwise and broadside, the following relationships must be satisfied:

$$v_x = -\frac{L_R}{2} \sin \beta \quad v_y = \frac{L_R}{2} \cos \beta.$$  \hspace{1cm} (2.12)

This thesis does not consider the squinted geometry.

### 2.3 Circular Flight Geometry

Due to the nature of circular trajectories, it is more convenient to define them with respect to a spherical coordinate system. The ideal circular flight path is parameterized by the
The following equations:

\begin{align*}
    r_t(\tau) &= \bar{r}_t & \theta_t(\tau) &= \bar{\theta}_t & \psi_t(\tau) &= \frac{\Psi_T}{2} \tau \\
    \dot{r}_t(\tau) &= 0 & \dot{\theta}_t(\tau) &= 0 & \dot{\psi}_t(\tau) &= \frac{\Psi_T}{2} \\
    \ddot{r}_t(\tau) &= 0 & \ddot{\theta}_t(\tau) &= 0 & \ddot{\psi}_t(\tau) &= 0 \\
    r_r(\tau) &= \bar{r}_r & \theta_r(\tau) &= \bar{\theta}_r & \psi_r(\tau) &= \frac{\Psi_R}{2} \tau + \beta \\
    \dot{r}_r(\tau) &= 0 & \dot{\theta}_r(\tau) &= 0 & \dot{\psi}_r(\tau) &= \frac{\Psi_R}{2} \\
    \ddot{r}_r(\tau) &= 0 & \ddot{\theta}_r(\tau) &= 0 & \ddot{\psi}_r(\tau) &= 0
\end{align*}

(2.13)

(2.14)

The variables \( r_t \) and \( r_r \) represent the distance from the scene center to the transmitter and receiver, respectively, and are constants over the collection period. The variables \( \theta_t \) and \( \theta_r \) represent the elevation angles of the transmitter and receiver, respectively, and are also constants over the collection period implying that the two platforms are flying at constant elevation. The variables \( \psi_t \) and \( \psi_r \) represent the azimuth angles of the transmitter and receiver, respectively, and are defined with respect to \( \Psi_T \) and \( \Psi_R \), the total azimuth angles traversed by the transmitter and receiver over the duration of the collection. The transmitter’s flight path is chosen such that it is centered over the \( x \) axis. The receiver’s flight path is chosen such that the azimuthal angle between the center of the transmitter’s and receiver’s synthetic apertures is \( \beta \), the bistatic angle. Again, the \( \dot{r} \) and \( \ddot{r} \) notation is used to represent the first and second derivative of \( r \) with respect to \( \tau \). Since the phase analysis is derived in Cartesian coordinates, the following spherical-to-Cartesian transformation equations are
used, generally stated as

\[
x(\tau) = r(\tau) \cos \theta(\tau) \cos \psi(\tau)
\]
\[
y(\tau) = r(\tau) \cos \theta(\tau) \sin \psi(\tau)
\] (2.15)
\[
z(\tau) = r(\tau) \sin \theta(\tau).
\]

The first and second derivatives of (2.15) are evaluated for the ideal circular flight trajectories parameterized by (2.13) and (2.14) to find the Cartesian parameterization of the circular flight paths. They are

\[
x_t(\tau) = \bar{r}_t \cos \bar{\theta}_t \cos \left( \frac{\Psi_T}{2} \tau \right)
\] (2.16)
\[
\dot{x}_t(\tau) = -\frac{\Psi_T}{2} \bar{r}_t \cos \bar{\theta}_t \sin \left( \frac{\Psi_T}{2} \tau \right)
\]
\[
\ddot{x}_t(\tau) = -\frac{\Psi_T^2}{4} \bar{r}_t \cos \bar{\theta}_t \cos \left( \frac{\Psi_T}{2} \tau \right)
\]
\[
y_t(\tau) = \bar{r}_t \cos \bar{\theta}_t \sin \left( \frac{\Psi_T}{2} \tau \right)
\]
\[
\dot{y}_t(\tau) = \frac{\Psi_T}{2} \bar{r}_t \cos \bar{\theta}_t \cos \left( \frac{\Psi_T}{2} \tau \right)
\]
\[
\ddot{y}_t(\tau) = -\frac{\Psi_T^2}{4} \bar{r}_t \cos \bar{\theta}_t \sin \left( \frac{\Psi_T}{2} \tau \right)
\]
\[
z_t(\tau) = \bar{r}_t \sin \bar{\theta}_t
\]
\[
\dot{z}_t(\tau) = 0
\]
\[
\ddot{z}_t(\tau) = 0,
\]

for the transmitter, and as follows for the receiver:

\[
x_r(\tau) = \bar{r}_r \cos \bar{\theta}_r \cos \left( \frac{\Psi_R}{2} \tau + \beta \right)
\] (2.17)
\[ \dot{x}_r(\tau) = -\frac{\Psi_R}{2} \bar{r}_r \cos \bar{\theta}_r \sin \left( \frac{\Psi_R}{2} \tau + \beta \right) \]
\[ \ddot{x}_r(\tau) = -\frac{\Psi_R^2}{4} \bar{r}_r \cos \bar{\theta}_r \cos \left( \frac{\Psi_R}{2} \tau + \beta \right) \]
\[ y_r(\tau) = \bar{r}_r \cos \bar{\theta}_r \sin \left( \frac{\Psi_R}{2} \tau + \beta \right) \]
\[ \dot{y}_r(\tau) = \frac{\Psi_R}{2} \bar{r}_r \cos \bar{\theta}_r \cos \left( \frac{\Psi_R}{2} \tau + \beta \right) \]
\[ \ddot{y}_r(\tau) = -\frac{\Psi_R^2}{4} \bar{r}_r \cos \bar{\theta}_r \sin \left( \frac{\Psi_R}{2} \tau + \beta \right) \]
\[ z_r(\tau) = \bar{r}_r \sin \bar{\theta}_r \]
\[ \dot{z}_r(\tau) = 0 \]
\[ \ddot{z}_r(\tau) = 0. \]

Since \( \Psi_T \) and \( \Psi_R \) are physical parameters and therefore positive, the equations defined in this section describe a broadside collection in which both transmitter and receiver platforms travel counterclockwise in perfect circles centered on a point \( \bar{z}_t \) and \( \bar{z}_r \) above the origin.
Chapter 3

Analysis of Phase History

In this chapter, the signal model for bistatic SAR phase history data is introduced using the geometric framework defined in Chapter 2. Gorham’s method for the residual phase error analysis is developed for the arbitrary bistatic flight geometry. The phase analysis is then used to predict the distortion of an image formed using PFA enabling distortion correction based on 2D linear interpolation. Finally, analysis of the residual QPE is used to define accurate scene size limits within the coordinate system of the undistorted image.

3.1 Phase History Model

In the bistatic collection scenario, the SAR transmitter and receiver are located on two different platforms. As the SAR transmitter platform traverses the synthetic aperture, it periodically transmits electromagnetic pulses at the scene of interest. The transmitted pulse is assumed to be a chirped waveform with bandwidth $B$. As the receiver traverses its own synthetic aperture, it collects the reflected pulses and employs dechirp-on-receive process-
The phase history model given by (3.1) corresponds to an empty scene containing a single
isotropic point target located at $p$. The variable $A$ represents the complex-valued reflection
coefficient of the point target. The sampled frequency bin $f_k$ is defined on the interval
$[f_0, f_0 + B]$ where $f_0$ is the carrier frequency. The variable $f_k$ is discretized uniformly over
$N_k$ fast time samples by the analog-to-digital converter such that $f_k = [f_0, f_1, \ldots, f_{N_k-1}]$.
The time variable $\tau$ is uniformly discretized into $N_p$ slow time samples that correspond
to the times at which each pulse is transmitted such that $\tau_n = [\tau_0, \tau_1, \ldots, \tau_{N_p-1}]$. In this
thesis, a pulse is assumed to be transmitted and received at the same $\tau_n$. The model also
assumes perfect knowledge of transmitter and receiver location relative to the scene center
and perfect motion compensation of the phase history data to the scene center.

The phase of the signal is temporally dependent only on the bistatic differential range
term $\Delta R$, defined as

$$\Delta R(\tau_n) = r_{pt}(\tau_n) - r_t(\tau_n) + r_{pr}(\tau_n) - r_r(\tau_n),$$ (3.2)

which represents the difference between the path length from the transmitter to a point in
the scene to the receiver and the path length from the transmitter to the scene center to the
receiver.

A scene containing multiple point targets is represented as the superposition of (3.1)
for each point target given by

$$S(f_k, \tau_n) = A \exp\left(\frac{-j2\pi f_k \Delta R(\tau_n)}{c}\right),$$ (3.3)

where $N_m$ is the number of point targets in the scene, $A_m$ is the scattering coefficient of
the $m^{th}$ target, and $\Delta R_m$ is the time dependent differential range to the location of the $m^{th}$ target.

### 3.2 Residual Phase Error Analysis

In this section, the phase analysis technique developed in [25] is used to analyze the bistatic signal model described in the previous section. The approach is to decompose the MF phase into its constant, linear, and quadratic components using a one-dimensional (1D) Taylor series expansion in the time domain. The FFA is also introduced and used to express the “PFA phase.” The PFA phase is similarly decomposed into its constant, linear, and quadratic terms using a 1D Taylor series expansion in the time domain. Finally, the residual phase errors are calculated by comparing the individual components of the exact phase to the corresponding components of the PFA phase. The phase analysis is derived with the assumption that a single point target exists in the PFA reconstructed scene, however since the coordinates of the point target are expressed arbitrarily as $(x, y)$, the geometric arguments may be applied to the whole image by simply reapplying the analysis with the assumption that a point target with reflection coefficient equal to the complex pixel value exists at the center of each pixel.

While collected phase history data is discretized, for the purposes of this analysis, continuous time $\tau$ is assumed with the understanding that the equations that define the flight trajectories are continuous functions.

To image data modeled by (3.3), the phase term of the MF kernel [3] is defined as

$$\Phi(\tau) = \frac{-2\pi \Delta R(\tau)}{\lambda},$$

(3.4)

where $\lambda$ is the wavelength of the transmitted signal. A temporal Taylor series expansion on
the phase about $\tau = 0$ is performed such that

$$\Phi(\tau) = \Phi_0 + \Phi_1 \tau + \Phi_2 \tau^2 + \ldots, \quad (3.5)$$

where

$$\Phi_0 = \frac{-2\pi \Delta R(\tau)}{\lambda} \bigg|_{\tau=0} \quad (3.6)$$

$$\Phi_1 = \frac{-2\pi \Delta \dot{R}(\tau)}{\lambda} \bigg|_{\tau=0} \quad (3.7)$$

$$\Phi_2 = \frac{-\pi \Delta \ddot{R}(\tau)}{\lambda} \bigg|_{\tau=0} \quad (3.8)$$

are the constant, linear, and quadratic coefficients of the Taylor series. The $\Delta \dot{R}$ and $\Delta \ddot{R}$ notation is used to denote the first and second partial derivatives of $\Delta R$ with respect to $\tau$.

The PFA uses an approximation of the differential range that is the linear term of the spatial Taylor expansion of $\Delta R$ about the point $[x, y, z] = (0, 0, 0)$ [3], given by

$$\Delta \hat{R} = \frac{1}{r_t} [x_t \tilde{x} + y_t \tilde{y} + z_t \tilde{z}] - \frac{1}{r_r} [x_r \tilde{x} + y_r \tilde{y} + z_r \tilde{z}], \quad (3.9)$$

where the time dependency is omitted for compactness. This approximation is commonly referred to as the “far-field approximation” because it approximates spherical waves as plane waves. The variables $\tilde{x}, \tilde{y},$ and $\tilde{z}$ are used to differentiate between the PFA differential range and the exact differential range, and represent the distorted coordinates a target located at actual coordinates $[x, y, z]$ appears in the PFA image. Analogous to (3.4), the phase of the PFA kernel is given similarly by

$$\Phi(\tau) = \frac{-2\pi \Delta \hat{R}(\tau)}{\lambda}. \quad (3.10)$$
Again, a temporal Taylor series expansion is performed on the PFA phase such that

\[ \hat{\Phi}(\tau) = \hat{\Phi}_0 + \hat{\Phi}_1 \tau + \hat{\Phi}_2 \tau^2 + \ldots, \]  

(3.11)

where

\[ \hat{\Phi}_0 = \frac{-2\pi \Delta \hat{R}(\tau)}{\lambda} \bigg|_{\tau=0} \]  

(3.12)

\[ \hat{\Phi}_1 = \frac{-2\pi \Delta \hat{\dot{R}}(\tau)}{\lambda} \bigg|_{\tau=0} \]  

(3.13)

\[ \hat{\Phi}_2 = \frac{-\pi \Delta \hat{\ddot{R}}(\tau)}{\lambda} \bigg|_{\tau=0} \]  

(3.14)

are the constant, linear, and quadratic coefficients of the PFA kernel. The residual phase error introduced by using the FFA is defined as:

\[ \tilde{\Phi}_0 = \Phi_0 - \hat{\Phi}_0, \]  

(3.15)

\[ \tilde{\Phi}_1 = \Phi_1 - \hat{\Phi}_1, \]  

(3.16)

\[ \tilde{\Phi}_2 = \Phi_2 - \hat{\Phi}_2. \]  

(3.17)

Note that in (3.15)-(3.17), the constant terms introduced by the Taylor series expansion are equal and the analysis is reduced to a comparison of the exact differential range and the PFA differential range. For this reason, it is necessary to decompose the differential range terms to proceed with the analysis.

### 3.2.1 Taylor Expansion of Bistatic Differential Range

Since the only term in (3.4) with time dependence is \( \Delta \hat{R}(\tau) \), it is necessary to find the constant, linear, and quadratic terms of its Taylor series expansion which requires calculating the first and second derivatives of \( \Delta \hat{R} \) with respect to time. A thorough Taylor expa-
tion of the bistatic differential range in the presence of motion measurement errors may be found in the appendices of [3, 15]. Setting the motion measurement errors to zero, the first derivative is given by

\[
\Delta \dot{R} = -\frac{1}{r_{pt}}[(x - x_t)\dot{x}_t + (y - y_t)\dot{y}_t + (z - z_t)\dot{z}_t] \\
- \frac{1}{r_t}[x_t\ddot{x}_t + y_t\ddot{y}_t + z_t\ddot{z}_t] \\
- \frac{1}{r_{pr}}[(x - x_r)\dot{x}_r + (y - y_r)\dot{y}_r + (z - z_r)\dot{z}_r] \\
- \frac{1}{r_r}[x_r\dot{x}_r + y_r\dot{y}_r + z_r\dot{z}_r],
\]

and the second derivative is given by

\[
\Delta \ddot{R} = -\frac{1}{r_{pt}}[(x - x_t)\dddot{x}_t - \dot{x}_t^2 + (y - y_t)\dddot{y}_t - \dot{y}_t^2 + (z - z_t)\dddot{z}_t - \dot{z}_t^2] \\
- \frac{1}{r_t}[x_t\dddot{x}_t + \dot{x}_t^2 + y_t\dddot{y}_t + \dot{y}_t^2 + z_t\dddot{z}_t + \dot{z}_t^2 - \dddot{r}_t^2] \\
- \frac{1}{r_{pr}}[(x - x_r)\dddot{x}_r - \dot{x}_r^2 + (y - y_r)\dddot{y}_r - \dot{y}_r^2 + (z - z_r)\dddot{z}_r - \dot{z}_r^2 + \dddot{r}_r^2] \\
- \frac{1}{r_r}[x_r\dddot{x}_r + \dot{x}_r^2 + y_r\dddot{y}_r + \dot{y}_r^2 + z_r\dddot{z}_r + \dot{z}_r^2 - \dddot{r}_r^2].
\]

In both the first and second derivative, the \(\tau\)-dependence of the subscripted variables is suppressed for compactness.

### 3.2.2 Taylor Expansion of Bistatic PFA Differential Range

Like (3.4), the only term with time dependence in (3.10) is \(\Delta \dot{R}(\tau)\), therefore it is necessary to define the first and second derivatives of \(\Delta \dot{R}\) with respect to time. A thorough Taylor expansion of the FFA to the monostatic differential range is presented in [25, 26]. The
bistatic PFA differential range is composed of the sum of the transmitter’s PFA differential range and receiver’s PFA differential range, and both are identical in form to the monostatic PFA differential range. Since differentiation is distributive over addition, the first and second derivatives of the bistatic PFA differential range manifest as the sum of two identical terms, one corresponding to the transmitter, and the other to the receiver. The first derivative is given by

$$\Delta \dot{R} = \frac{\dddot{x}}{r_t^3}(x_t r_t - r_t \dddot{x}_t) + \frac{\dddot{y}}{r_t^3}(y_t r_t - r_t \dddot{y}_t) + \frac{\dddot{z}}{r_t^3}(z_t r_t - r_t \dddot{z}_t) \quad (3.20)$$

$$+ \frac{\dddot{x}}{r_r^3}(x_r \dddot{r}_r - r_r \dddot{x}_r) + \frac{\dddot{y}}{r_r^3}(y_r \dddot{r}_r - r_r \dddot{y}_r) + \frac{\dddot{z}}{r_r^3}(z_r \dddot{r}_r - r_r \dddot{z}_r).$$

The second derivative is given by

$$\Delta \dddot{R} = \frac{\dddot{x}}{r_t^3}(x_t r_t \dddot{r}_t - r_t^2 \dddot{x}_t - 2x_t r_t^2 \dddot{r}_r + 2r_t x_t \dddot{r}_r) \quad (3.21)$$

$$+ \frac{\dddot{y}}{r_t^3}(y_t r_t \dddot{r}_t - r_t^2 \dddot{y}_t - 2y_t r_t^2 \dddot{y}_r + 2r_t y_t \dddot{y}_r)$$

$$+ \frac{\dddot{z}}{r_t^3}(z_t r_t \dddot{r}_t - r_t^2 \dddot{z}_t - 2z_t r_t^2 \dddot{z}_r + 2r_t z_t \dddot{z}_r)$$

$$+ \frac{\dddot{x}}{r_r^3}(x_r \dddot{r}_r \dddot{r}_r - r_r^2 \dddot{x}_r - 2x_r r_r^2 \dddot{x}_r + 2r_r x_r \dddot{x}_r)$$

$$+ \frac{\dddot{y}}{r_r^3}(y_r \dddot{r}_r \dddot{r}_r - r_r^2 \dddot{y}_r - 2y_r r_r^2 \dddot{y}_r + 2r_r y_r \dddot{y}_r)$$

$$+ \frac{\dddot{z}}{r_r^3}(z_r \dddot{r}_r \dddot{r}_r - r_r^2 \dddot{z}_r - 2z_r r_r^2 \dddot{z}_r + 2r_r z_r \dddot{z}_r).$$

Again, the $\tau$-dependence of the subscripted variables is suppressed for compactness in both the first and second derivatives.

### 3.3 PFA Distortion Prediction & Correction

When the PFA differential range was introduced in (3.9), it was defined with respect to a distorted coordinate system $\tilde{\tilde{(\dot{x}, \ddot{y}, \dddot{z})}}$. It was also asserted that $\tilde{\tilde{(\dot{x}, \ddot{y}, 0)}}$ represents the
location a target located at actual coordinates \((x, y, 0)\) appears in the ground plane PFA image. In order to predict the geometric distortion of a single point target, it is desirable to solve for \((\bar{x}, \bar{y})\) as a function of \((x, y)\). Ground plane imaging is assumed and therefore the \(z\) and \(\bar{z}\) components of the point target’s actual and distorted position are assumed to be 0 and are neglected in this analysis. Distortion effects in a PFA image are caused primarily by the linear component of the phase error \([11]\). Therefore, in order to quantify the distortion effects, (3.15) and (3.16) are evaluated to 0 such that

\[
\tilde{\Phi}_0 = \Phi_0 - \hat{\Phi}_0 = 0 \quad \hat{\Phi}_0 = \Phi_0
\]

(3.22)

\[
\tilde{\Phi}_1 = \Phi_1 - \hat{\Phi}_1 = 0 \quad \hat{\Phi}_1 = \Phi_1
\]

(3.23)

which amounts to equating the constant and linear components of the true differential range and PFA differential range. In doing so, the relationship between the true position \((x, y)\) and distorted position \((\bar{x}, \bar{y})\) can be extracted to predict the extent of the distortion.

With \(z = \bar{z} = 0\), holding \(x\) and \(y\) constant for a single given pixel in the image, grouping coefficients for \(\bar{x}\) and \(\bar{y}\) together, and simplifying, (3.22) evaluates to

\[
\begin{align*}
\left[-\frac{x_t}{r_t} - \frac{x_r}{r_r}\right] |_{\tau=0} \bar{x} + \left[-\frac{y_t}{r_t} - \frac{y_r}{r_r}\right] |_{\tau=0} \bar{y} \\
= [r_{pt} - r_t + r_{pr} - r_r] |_{\tau=0}.
\end{align*}
\]

(3.24)

Similarly, (3.23) evaluates to

\[
\begin{align*}
\left[\frac{x_t\dot{x}_t - r_t\dot{x}_t}{r_t^2} + \frac{x_r\dot{x}_r - r_r\dot{x}_r}{r_r^2}\right] |_{\tau=0} \bar{x} \\
+ \left[\frac{y_t\dot{y}_t - r_t\dot{y}_t}{r_t^2} + \frac{y_r\dot{y}_r - r_r\dot{y}_r}{r_r^2}\right] |_{\tau=0} \bar{y} \\
= [\dot{r}_{pt} - \dot{r}_t + \dot{r}_{pr} - \dot{r}_r] |_{\tau=0}.
\end{align*}
\]

(3.25)

Since the Taylor series requires that the coefficients evaluate the differentiated variable
about the point of expansion, \( \tau = 0 \), (3.24) and (3.25) reduce to two equations that are linear in \( \tilde{x} \) and \( \tilde{y} \) of the form

\[
A\tilde{x} + B\tilde{y} = C \\
D\tilde{x} + E\tilde{y} = F,
\]

(3.26)

where the coefficients are defined as

\[
A = \left[ -\frac{x_t}{r_t} - \frac{x_r}{r_r} \right]_{\tau=0}
\]

(3.27)

\[
B = \left[ -\frac{y_t}{r_t} - \frac{y_r}{r_r} \right]_{\tau=0}
\]

(3.28)

\[
C = [r_{pt} - r_t + r_{pr} - r_r]_{\tau=0}
\]

(3.29)

\[
D = \left[ \frac{x_t\dot{r}_t - r_t\dot{x}_t}{r_t^2} + \frac{x_r\dot{r}_r - r_r\dot{x}_r}{r_r^2} \right]_{\tau=0}
\]

(3.30)

\[
E = \left[ \frac{y_t\dot{r}_t - r_t\dot{y}_t}{r_t^2} + \frac{y_r\dot{r}_r - r_r\dot{y}_r}{r_r^2} \right]_{\tau=0}
\]

(3.31)

\[
F = [\dot{r}_{pt} - \dot{r}_t + \dot{r}_{pr} - \dot{r}_r]_{\tau=0}
\]

(3.32)

and are all constant with respect to time. Note that \( C \) and \( F \) are simply \( \Delta R(\tau = 0) \) and \( \Delta \dot{R}(\tau = 0) \), respectively, and are defined in (3.2) and (3.18). The fact that the collection time is supported on \( \tau = [-1,1] \), with \( \tau = 0 \) corresponding to the synthetic aperture center, suggests that the only information required to quantify the distortion in a PFA image is position and velocity of the transmitter and receiver at aperture center with respect to the image origin. This fact is important because it suggests that a parameterization of the synthetic aperture’s trajectories are not necessary to correction for distortion. In practice, an onboard inertial navigation system may report position and velocity information absent knowledge of the total flight trajectory.

Solving for \( \tilde{x} \) and \( \tilde{y} \) is accomplished using the general solution to a system of two
linear equations:

\[ \tilde{x}(x, y) = C - \frac{BF}{E} \frac{A - \frac{DB}{F}}{ } \]  
\[ \tilde{y}(x, y) = C - \frac{AF}{D} \frac{B - \frac{AE}{D}}{ } \]  

(3.33)

The \( \tilde{x}(x, y) \) and \( \tilde{y}(x, y) \) notation implies that for a given collection geometry, the distorted coordinates may be expressed as a function of the pixel coordinates, reinforcing the assertion that a target located at actual coordinates \( p = (x, y, 0) \) is mapped to distorted coordinates \( \tilde{p} = (\tilde{x}, \tilde{y}, 0) \) in the PFA image. Using the mapping functions calculated in (3.33), the distortion of a single point target may be predicted.

The distortion may be calculated for every pixel location in the scene to predict the overall distortion of the image. In order to correct for the distortion of the entire image, a 2D interpolation of the image is performed from the actual coordinates \( x \) and \( y \) to the distorted coordinates \( \tilde{x} \) and \( \tilde{y} \) yielding an undistorted image.

### 3.4 Scene Size Limits

While the method described in the previous section corrects for the image distortion, it does not correct for defocus effects introduced by QPE. In order to quantify the defocus effects, it is necessary to solve for (3.17), restated as

\[ \tilde{\Phi}_2(x, y) = \Phi_2(x, y) - \hat{\Phi}_2(\tilde{x}(x, y), \tilde{y}(x, y)), \]  

(3.34)

with \( (x, y) \) dependency added to emphasize the fact that the residual QPE will be calculated for the undistorted image. The dependency of \( \tilde{\Phi}_2 \) on \( \tilde{x}(x, y) \) and \( \tilde{y}(x, y) \) may be satisfied using (3.33). Using the definitions for \( \Phi_2 \) and \( \hat{\Phi}_2 \) given by (3.8) and (3.14), respectively, (3.34) is calculated for every pixel location resulting in a mapping of the residual QPE over the entire undistorted image. The allowable scene size can be determined by defining an
allowable residual QPE threshold, $\Phi_{max}$, and solving for pixels that satisfy the inequality, expressed as

$$|\tilde{\Phi}_2| < \tilde{\Phi}_{max}.$$  \hspace{1cm} (3.35)

A common threshold that is used to bound the focused region of the image is $\tilde{\Phi}_{max} = \pi/2$ [11], however others use a stricter threshold of $\tilde{\Phi}_{max} = \pi/4$ [13].
Chapter 4

Application to Ideal Flight Geometries

In this chapter, parametric solutions for (3.27)-(3.32) are derived for the parameterized linear and circular flight geometries described in Chapter 2. To illustrate the distortion correction effects, the phase history data corresponding to a point target scene is simulated using (3.3). The scene is composed of 361 isotropic point scatterers of equal reflectivity arranged in a 19 by 19 square grid regularly spaced at 5 m intervals. The grid is located on the \( xy \) ground plane such that the center point of the grid is located at the origin of the coordinate system. A normalized SAR image of the scene is formed using bistatic PFA. The distortion correction is applied using 2D linear interpolation. Lastly, the residual QPE is plotted in the undistorted coordinate system and scene size limits are defined by bounding the residual QPE at \( \Phi_{max} = \pi/2 \) and \( \Phi_{max} = \pi/4 \).

4.1 Circular Flight Geometry

The ideal circular flight trajectories of the transmitter and receiver used in the point target scene simulation are parameterized by (2.13) and (2.14) with slant ranges \( \bar{r}_t = \bar{r}_r = 106.6 \) m, elevation angles \( \bar{\theta}_t = \bar{\theta}_r = 45^\circ \), aperture arcs \( \Psi_T = \Psi_R = 8.6^\circ \), and azimuthal bistatic angle \( \beta = 30^\circ \). The point target scene and the 2D ground plane projection of the transmitter
and receiver synthetic apertures are shown in Fig. 4.1 where each blue x represents the location of an isotropic point scatterer. The flight trajectory of the transmitter is represented by the green arc, and the flight trajectory of the receiver is represented by the red arc. The simulation was conducted with center frequency $f_0 = 10$ GHz with bandwidth $B = 150$ MHz. The parameters for the simulation were chosen specifically to severely stress the FFA and accentuate the distortion and defocus effects.

A SAR image reconstruction of the point target scene using PFA is shown in Fig. 4.2. Notice that the rectangular grid of point targets appears warped in the PFA image. Additionally, point targets near the edges of the grid experience severe defocusing relative to points closer to the center.

To correct the distortion in the PFA image, it is first necessary to solve for $\tilde{x}$ and $\tilde{y}$. The coefficients (3.27)-(3.32) are calculated using the parametric equations for the transmitter.
Figure 4.2: PFA image, circular flight geometry.

and receiver locations defined in (2.16) and (2.17) resulting in

\[ A = -[\cos \tilde{\theta}_t + \cos \tilde{\theta}_r \cos \beta] \]  
\[ B = -\cos \tilde{\theta}_r \sin \beta \]  
\[ C = r_{pt0} - r_t + r_{pr0} - r_r \]  
\[ D = \frac{\Psi_R}{2} \cos \tilde{\theta}_r \sin \beta \]  
\[ E = -\frac{\Psi_T}{2} \cos \tilde{\theta}_t - \frac{\Psi_R}{2} \cos \tilde{\theta}_r \cos \beta \]  
\[ F = -\frac{\Psi_T r_t \cos \tilde{\theta}_t}{2r_{pt0}} y - \frac{\Psi_R r_r \cos \tilde{\theta}_r}{2r_{pr0}} (y \cos \beta - x \sin \beta) \]
where

\[ r_{pt0} = \sqrt{(x - x_t(0))^2 + y^2 + (z - z_t(0))^2} \]  \hspace{1cm} (4.7)

\[ r_{pt0} = \sqrt{(x - \bar{r}_t \cos \hat{\theta}_t)^2 + y^2 + (z - \bar{r}_t \sin \hat{\theta}_t)^2} \]

and

\[ r_{pr0} = \sqrt{(x - x_r(0))^2 + (y - y_r(0))^2 + (z - z_r(0))^2} \]  \hspace{1cm} (4.8)

\[ r_{pr0} = \sqrt{(x - \bar{r}_r \cos \hat{\theta}_r \cos \beta)^2 + (y - \bar{r}_r \cos \hat{\theta}_r \sin \beta)^2 + (z - \bar{r}_r \sin \hat{\theta}_r)^2} \]

are the distances from a pixel at \((x, y)\) to the center of the transmitter’s and receiver’s synthetic apertures, respectively, and are found by evaluating (2.6) and (2.7) at time \(\tau = 0\).

Note that the chosen geometry removes the \(y_t\) dependency in (4.7). Solving for \(\tilde{x}\) and \(\tilde{y}\) is accomplished using (3.33). Calculating (3.33) for each \((x, y)\) pair that defines each pixel in the image results in the distorted grid corresponding to the 140 m by 120 m imaged scene shown in Fig. 4.3. Finally, the distorted image is interpolated to the distorted coordinates resulting in the undistorted PFA image shown in Fig. 4.4.

While the corrected PFA reconstruction of the point target grid shown in Fig. 4.4 appears to be free of distortion, it still suffers from severe defocusing. To mitigate the effects of defocus, the useful image is restricted to regions that are well-focused. To explicitly define the well-focused regions, it is first necessary to calculate the residual QPE defined in (3.34) for each pixel in the undistorted image. The intensity plot of the residual QPE in the undistorted coordinate system is shown in Fig. 4.5. Once the residual QPE has been calculated for each pixel in the image, the scene size limit is defined by bounding the region where the residual QPE satisfies the inequality defined in (3.35). In Fig. 4.5, the inner white contour bounds the region in which the residual QPE is less than \(\Phi_{max} = \pi/4\) and the outer white contour bounds the region in which the residual QPE is less than \(\Phi_{max} = \pi/2\).
Figure 4.3: Distorted coordinate grid, circular flight geometry

Figure 4.4: PFA image with distortion correction, circular flight geometry
These same contours are displayed over the corrected PFA image in Fig. 4.6.

Figure 4.5: Residual quadratic phase error, circular flight geometry.

4.2 Linear Flight Geometry

The ideal linear flight trajectories of the transmitter and receiver used in the point target scene simulation are parameterized by (2.8) and (2.9) with transmitter coordinates at aperture center \((\bar{x}_t, 0, \bar{z}_t) = (75, 0, 75)\) m, transmitter aperture length \(L_T = 11.27\) m, receiver coordinates at aperture center \((\bar{x}_r, \bar{y}_r, \bar{z}_r) = (64.95, 37.5, 75)\) m, and receiver velocity components \((v_x, v_y) = (-3.97, 6.88)\) m/s corresponding to a broadside receiver aperture with length \(L_R = 11.27\) m and an azimuthal bistatic angle of \(\beta = 30^\circ\). The point target scene and the 2D ground plane projection of the transmitter and receiver synthetic apertures are shown in Fig.4.7 where each blue x represents the location of an isotropic point scatterer.
The flight trajectory of the transmitter is represented by the green line, and the flight trajectory of the receiver is represented by the red line. The simulation was conducted with center frequency $f_0 = 10$ GHz with bandwidth $B = 150$ MHz. The parameters for the simulation were chosen specifically to stress the FFA and accentuate the distortion and defocus effects.

A SAR image reconstruction of the point target scene using PFA is shown in Fig. 4.8. Notice that the rectangular grid of point targets appears warped in the PFA image. Additionally, point targets near the edges of the grid experience severe defocusing relative to points closer to the center.

To correct the distortion in the PFA image, it is first necessary to calculate $\tilde{x}$ and $\tilde{y}$. The coefficients (3.27)-(3.32) are calculated using the parametric equations for the transmitter
Figure 4.7: Point target scene and linear flight geometry.

Figure 4.8: PFA image, linear flight geometry.
and receiver locations defined in (2.8) and (2.9) resulting in

\begin{align*}
A &= -\frac{\bar{x}_t - \bar{x}_r}{r_{t0} - r_{r0}} \quad (4.9) \\
B &= -\frac{\bar{y}_t - \bar{y}_r}{r_{t0} - r_{r0}} \quad (4.10) \\
C &= r_{pt0} - r_{t0} + r_{pr0} - r_{r0} \quad (4.11) \\
D &= -\frac{v_x}{r_{r0}} \quad (4.12) \\
E &= -\frac{L_T}{2r_{t0}} - \frac{v_y}{r_{r0}} \quad (4.13) \\
F &= -\frac{L_T y}{2r_{pt0}} - \frac{1}{r_{pr0}} [(x - \bar{x}_r)v_x + (y - \bar{y}_r)v_y] \quad (4.14)
\end{align*}

where

\[ r_{t0} = \sqrt{\bar{x}_t^2 + \bar{z}_t^2} \]

and

\[ r_{r0} = \sqrt{\bar{x}_r^2 + \bar{y}_r^2 + \bar{z}_r^2} \quad (4.15) \]

are the distances from the origin to the center of the transmitter and receiver synthetic apertures, respectively. The variables \( r_{pt0} \) and \( r_{pr0} \) are defined in (4.7) and (4.8), but are calculated with respect to the linear flight trajectory parameters. Note that for the chosen geometry, \( y_t(\tau = 0) = 0 \). Solving for \( \tilde{x} \) and \( \tilde{y} \) is accomplished using (3.33). Calculating (3.33) for each \((x, y)\) pair that defines each pixel in the image results in the distorted grid corresponding to the 140 m by 120 m imaged scene shown in Fig. 4.9. Notice that the distorted grid for the linear geometry is identical to the distorted grid shown in Fig. 4.3 corresponding to the circular geometry. The chosen simulation parameters result in identical values for position and velocity at aperture center in both geometries, further reinforcing that distortion may be corrected using this knowledge alone. Finally, the distorted image is interpolated to the distorted coordinates resulting in the undistorted PFA image shown in
While the corrected PFA reconstruction of the point target grid shown in Fig. 4.10 appears to be free of distortion, it still suffers from severe defocusing. To mitigate the effects of defocus, the useful image is restricted to regions that are well-focused. To explicitly define the well-focused regions, it is first necessary to calculate the residual QPE defined in (3.34) for each pixel in the undistorted image. The intensity plot of the residual QPE in the undistorted coordinate system is shown in Fig. 4.11. Once the residual QPE has been calculated for each pixel in the image, the scene size limit is defined by bounding the region where the residual QPE satisfies the inequality defined in (3.35). In Fig. 4.11, the inner white contour bounds the region in which the residual QPE is less than $\Phi_{max} = \pi/4$ and the outer white contour bounds the region in which the residual QPE is less than $\Phi_{max} = \pi/2$. These same contours are displayed over the corrected PFA image in
Fig. 4.12. Notice that while the distortion of the linear geometry simulation is identical to distortion in the circular geometry, the defocus functions vary drastically. This is attributed to non-zero acceleration components in the circular geometry at aperture center that do not exist in the linear geometry. It is also interesting to note that while the scene size limits corresponding to the parameters used in the simulated bistatic circular geometry enclose a simply-connected space about the origin, the contours bounding the scene size limits corresponding to the simulated linear flight geometry contain a “hole of defocus” within the outer the scene limits. In Fig. 4.11, this hole is seen as the light blue region (corresponding to a residual QPE of $3\pi/4$) centered approximately about the point $(50, 10)$. In Fig. 4.12, the point targets located within the hole are noticeably defocused.
Figure 4.11: Residual quadratic phase error, linear flight geometry.

Figure 4.12: Corrected PFA image with scene size limits, linear flight geometry.
Chapter 5

Comparison to Previous Results

In this chapter, the new results presented in Chapter 4 are compared to the previous results documented in the open literature. In particular, new scene size limits and an analysis of the progression from the monostatic to bistatic scenarios are examined. Finally, the bistatic look angle is revisited as a framework for the development of a defocus correction algorithm.

5.1 Previous Scene Size Limits for Bistatic PFA

According to [3], the maximum allowable scene radius for bistatic PFA image without additional compensation for errors due to wavefront curvature is

$$r_{\text{max}} = \sqrt{2\lambda \left( \frac{L_t^2}{r_t^3} + \frac{L_r^2}{r_r^3} \right)^{-1/2}},$$

(5.1)

where $\lambda$ is the wavelength of the transmitted signal, $r_t$ and $r_r$ are the distances from the scene center to the synthetic aperture center of the transmitter and receiver, respectively, and $L_t$ and $L_r$ are the lengths of the transmitter and receiver synthetic apertures, respectively. The maximum allowable radius defined in (5.1) was derived by restricting the magnitude
of the quadratic phase term neglected by the FFA to be less than $\pi/2$. Repeating their derivation using the more restrictive $\pi/4$ criterion for a focused image yields

$$r_{\text{max}} = \sqrt[4]{\frac{L_t^2}{r_t^2} + \frac{L_r^2}{r_r^2}}^{-1/2}. \quad (5.2)$$

These results are specific to the broadside linear flight geometry in which the transmitter’s and receiver’s velocity vectors are parallel, and they predict circular regions of focus without accounting for distortion caused by the residual constant and linear phase errors.

Figure 5.1 shows the PFA image of the point target scene with distortion correction applied formed using a circular collection geometry. The inner and outer solid white contours correspond to regions where the residual QPE is less than $\pi/4$ and $\pi/2$, respectively. The allowable scene radii defined by (5.1) and (5.2) are drawn as dashed contours, with the inner dashed circle corresponding to the more restrictive $\tilde{\Phi}_{\text{max}} = \pi/4$ criterion. For this particular geometry, the newly derived scene size limits based on the residual QPE allows for up to 3 to 4 times as much space to be imaged compared to the previous result. The new boundaries more accurately represent regions of focus. For example, a point target located at $(0, 35)$ appears to be focused, but is well beyond the allowable scene size radius defined by (5.1), however it falls within the region of focus defined by the new scene size limit.

Figure 5.2 shows the PFA image of the point target scene with distortion correction applied formed using a linear collection geometry. The inner and outer solid white contours correspond to regions where the residual QPE is less than $\pi/4$ and $\pi/2$, respectively. The allowable scene radii defined by (5.1) and (5.2) are drawn as dashed contours, with the inner dashed circle corresponding to the more restrictive $\tilde{\Phi}_{\text{max}} = \pi/4$ criterion. For this particular geometry, the newly derived scene size limits based on the residual QPE more accurately represents regions of focus. For example, a point target located at $(-45, 35)$ appears to be focused, but is well beyond the allowable scene size radius defined by (5.1), however it falls within the region of focus defined by the new scene size limit. The region
adjacent to the point target at \((-45, 35)\) immediately outside of the white contours shows evidence of defocusing effects, further illustrating the predictive power and accuracy of this novel phase analysis.

The parameters that define the circular and linear flight trajectories simulated in Chapter 4 result in exactly the same scene size radius as predicted by (5.1) and (5.2), despite their very different geometries. While the circular and linear flight geometry simulations share the same radar center frequency, flight elevation, transmitter and receiver slant ranges at aperture center, transmitter and receiver aperture lengths, and azimuthal bistatic angular separation, their space-variant defocus functions vary greatly due to acceleration components that exist in the circular geometry that are zero in the linear, constant-velocity geometry.
5.2 Comparison to Monostatic Results

In addition to creating an image with obvious distortion and defocus effects, the parameters that define the circular and linear flight trajectories simulated in Chapter 4 were chosen for ease of comparison to the monostatic results presented in [25]. In both geometries, if the bistatic angle were reduced to zero, the simulation would collapse to the monostatic case simulated in [25]. Qualitatively, the shapes of the bistatic defocus contours agree with the monostatic results, however two characteristic differences are observed when switching from the monostatic to the bistatic scenario.

The first observation is that the allowable scene sizes become more restrictive as the bistatic angle increases. The results presented in [28] use simulation parameters for the bistatic circular flight geometry that are identical to the parameters used in this thesis, but with a smaller bistatic angle, $\beta = 15^\circ$. Consequently, their results predict a smaller region of focus than the monostatic case [25], but a larger region of focus than the results
presented in this thesis where the bistatic angle is doubled, $\beta = 30^\circ$. In comparing the scene size limits for a circular collection geometry simulated using $\beta = 0^\circ$ (monostatic), $\beta = 15^\circ$, and $\beta = 30^\circ$, it becomes evident that as the bistatic angle grows, the allowable scene size limit shrinks.

The second observation is that the bistatic contours are rotated versions of the monostatic contours. The amount of rotation for the circular geometry appears to be half the bistatic angle. This observation is consistent and more apparent when comparing the monostatic and bistatic defocus contours for the linear geometry. It is important to note that given that the slant ranges $r_t$ and $r_r$ are equal in the simulations presented in this thesis, the bisector of the bistatic angle is a specific case of the bistatic look angle defined in [3, 35] (in their notation) as

$$\phi_b = \arctan \frac{f_y(f_i, \tau_{kB})}{f_x(f_i, \tau_{kB})} = \arctan \frac{\sin \phi_t(0) \cos \theta_t(0) + \sin \phi_r(0) \cos \theta_r(0)}{\cos \phi_t(0) \cos \theta_t(0) + \cos \phi_r(0) \cos \theta_r(0)}.$$

5.3 Defocus Correction and the Bistatic Look Angle

While a defocus correction algorithm is outside of the scope of the research presented in this thesis, it is mentioned in this section because it is the ultimate goal of PFA phase analysis. The results presented in [27] extends the monostatic phase error analysis by proposing a computationally efficient PFA with defocus correction for the circular flight geometry. By noticing that the defocus contours in the distorted domain vary predominantly as a function of the range dimension, a defocus correction was applied by complex multiplication of a 1D phase correction factor in the azimuth compressed domain. With regards to the two observations noted in the previous section, the defocus contours are scaled and rotated versions of the monostatic case in the undistorted domain so it stands to reason that defocus contours in the distorted domain for the bistatic circular geometry will exhibit similar
symmetries to the monostatic counterpart motivating an extension of [27] into the bistatic geometry. Two significant barriers to defocus correction for the bistatic circular geometry exist.

To apply the defocus correction presented in [27], it is first necessary to solve for the residual QPE in terms of the distorted coordinates expressed as

$$\tilde{\Phi}_2(\tilde{x}, \tilde{y}) = \Phi_2(x(\tilde{x}, \tilde{y}), y(\tilde{x}, \tilde{y})) - \hat{\Phi}_2(\tilde{x}, \tilde{y}).$$

(5.4)

In this thesis, expressions for $\tilde{x}(x, y)$ and $\tilde{y}(x, y)$ are derived, however to solve for $\Phi_2(\tilde{x}, \tilde{y})$, it is necessary to find the inverse functions $x(\tilde{x}, \tilde{y})$ and $y(\tilde{x}, \tilde{y})$. While finding the inverse relationship was a matter of algebra in the monostatic case, the fact that the true bistatic differential range is defined as a sum of two Euclidean distances makes it nearly impossible to solve for a closed form solution.

The second challenge to defocus correction in the bistatic case is the lack of a well-defined range and cross-range dimension. In both the linear and circular bistatic geometries, the defocus functions appear to be symmetric about a line that appears to be the bisector of the bistatic angle. In the monostatic case, this line of symmetry is colinear with the $x$ axis and represents the range dimension. Since the defocus correction was applied along the range dimension, it is desirable to define an equivalent range dimension corresponding to the bistatic case. As previously stated, the bisector to the bistatic angle is the specific case of the bistatic look angle when the transmitter’s and receiver’s slant ranges are equal, restated in the notation of this thesis as

$$\phi_b = \arctan \left( \frac{\sin \psi_t(0) \cos \theta_t(0) + \sin \psi_r(0) \cos \theta_r(0)}{\cos \psi_t(0) \cos \theta_t(0) + \cos \psi_r(0) \cos \theta_r(0)} \right).$$

(5.5)

To further analyze the line of symmetry, equation (3.26) is reexamined within the context of coordinate rotation. Suppose it is desired that a rotated coordinate system be defined
such that the first equation in (3.26) is reduced to

\[ A' \tilde{x} = C', \quad (5.6) \]

effectively decoupling distortion in the $\tilde{x}$ dimension from $\tilde{y}$. The apostrophe notation is used to define the coefficients as a function of a rotated coordinate system. This condition is satisfied when

\[ B' = \left[ \begin{array}{c} -\frac{y'_t}{r'_t} - \frac{y'_r}{r'_r} \end{array} \right] \bigg|_{\tau=0} = 0, \quad (5.7) \]

otherwise stated as

\[ \frac{y'_t(0)}{r'_t(0)} = -\frac{y'_r(0)}{r'_r(0)}. \quad (5.8) \]

Recalling that the trigonometric interpretation of the general ratio $y/r$ represents the sin of the angle, (5.8) may be interpreted as the condition of rotation such that the magnitudes of the sin functions to the center of the transmitter’s and receiver’s flight trajectories are equal.

To proceed, the rotated coordinate system is introduced with respect to arbitrary rotation angle $\phi$, as shown in Fig. 5.3 For simplicity, only the 2D rotation is considered. The newly rotated coordinates as a function of the old coordinate system may be expressed as

\[ x' = x \cos \phi + y \sin \phi \quad (5.9) \]
\[ y' = -x \sin \phi + y \cos \phi. \]

Recalling that the spherical variable $r$ is rotation invariant, substituting (5.9) into the condition of rotation imposed by (5.8) and performing cross multiplication yields the following relationship

\[ -r_t(-x_t \sin \phi + y_t \cos \phi) = r_r(-x_r \sin \phi + y_t \cos \phi), \quad (5.10) \]

where the time dependency is omitted, and all variables are assumed to represent locations
Figure 5.3: Cartesian coordinate system rotated by arbitrary angle $\phi$. 
at aperture center. Algebraically rearranging the terms in (5.10) to solve for $\phi$ results in

$$\frac{y_r r_t + y_t r_r}{x_r r_t + x_t r_r} = \tan \phi$$  \hspace{1cm} (5.11)

As an intermediary step, both the numerator and denominator of (5.11) are multiplied by $1/r_t r_r$ to yield

$$\frac{y_r}{r_r} + \frac{y_t}{r_t} = \tan \phi.$$  \hspace{1cm} (5.12)

Using the Cartesian-to-spherical transformation equations defined in (2.15), $\phi$ may be expressed as a function of spherical coordinates as

$$\phi = \arctan \frac{\sin \psi_t \cos \theta_t + \sin \psi_r \cos \theta_r}{\cos \psi_t \cos \theta_t + \cos \psi_r \cos \theta_r},$$  \hspace{1cm} (5.13)

which agrees exactly with the definition of the bistatic look angle given in (5.5).
Chapter 6

Conclusion

6.1 Summary

This thesis applied the novel PFA phase analysis method presented in [25] to the bistatic PFA framework. The exact phase, based on the differential range, and the PFA phase, based on the FFA to the differential range, were decomposed in the slow time dimension to isolate their constant, linear, and quadratic Taylor series coefficients. The components of the residual phase error were defined as the difference between corresponding Taylor coefficients in the exact phase and the PFA phase. Analysis of the constant and linear residual phase errors resulted in analytic expressions that predicted the distortion in a PFA image, which were subsequently undistorted by applying bilinear interpolation. The residual QPE was mapped over the undistorted image, and a scene size limit was defined by imposing a restriction on the residual QPE. The analytic expressions derived in this paper were verified using simulations with parameterized circular and linear flight trajectories justifying significantly larger scene sizes when compared with previous results.
6.2 Recommendations and Future Work

While this thesis performs phase error analysis, corrects geometric distortion, and maps defocus for bistatic PFA, it falls short of defocus correction which is the ultimate goal for SAR imaging using PFA. Future work may be directed at implementing defocus correction within the framework of the coordinate system rotated by the bistatic look angle. If an analytic solution for defocus correction is mathematically impossible to derive for arbitrary geometries, an investigation into certain ideal flight geometries such as stationary/circular or forward-looking/circular may yield a case specific solution. Additionally, numerical methods may be used to solve and apply the defocus correction when algebraic methods fail.

Another limitation of this thesis is that it assumes a broadside collection geometry. Future work may be directed towards reaccomplishing the derivations presented in this thesis using the general form of the bistatic phase history model which accounts for antenna squint.

Lastly, the simulations conducted in this thesis are relatively low fidelity and do not account for important systems engineering factors such as antenna beam pattern, motion measurement error, noise in the scene, scene topography, and behavior of non-isotropic scatterers. Any of the neglected factors listed may be the focus of additional investigation. All of these factors at minimum would need to be considered if the results presented in this thesis are to be applied to real data.
Bibliography


[16] B. D. Rigling and R. L. Moses, “Taylor expansion of the differential range for mono-
static SAR,” *IEEE Transactions on Aerospace and Electronic Systems*, vol. 41, no. 1,
pp. 60–64, Jan 2005.

[17] B. Rigling, W. Garber, R. Hawley, and L. Gorham, “Wide-area, persistent SAR imag-
ing: Algorithm tradeoffs,” *IEEE Aerospace and Electronic Systems Magazine*, vol. 29,

for wavefront curvature correction in polar-formatted spotlight-mode SAR images us-
ing space-variant post-filtering,” in *Proceedings of International Conference on Image

polar-formatted spotlight-mode SAR imagery,” Sandia National Laboratories, Albu-

with postfiltering,” *IEEE Transactions on Aerospace and Electronic Systems*, vol. 49,

SAR images for wavefront curvature correction and interferometric processing,” in
*Geoscience and Remote Sensing Symposium, 2002. IGARSS ‘02. 2002 IEEE Interna-

[22] X. Mao, D. Zhu, and Z. Zhu, “Polar format algorithm wavefront curvature compensa-
tion under arbitrary radar flight path,” in *Proceedings of 2011 IEEE CIE International


