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An Engage or Retreat Differential Game with Two Targets

A Thesis submitted in partial fulfillment
of the requirements for the degree of
Master of Science in Electrical Engineering

by

Bikash Shrestha
B.E., Tribhuvan University, Nepal, 2015

2017
Wright State University

Wright State University
GRADUATE SCHOOL

24 July 2017

I HEREBY RECOMMEND THAT THE THESIS PREPARED UNDER MY SUPERVISION BY Bikash Shrestha ENTITLED An Engage or Retreat Differential Game with Two Targets BE ACCEPTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF Master of Science in Electrical Engineering.

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ABSTRACT

Shrestha, Bikash. M.S.E.E , Department of Electrical Engineering, Wright State University, 2017.
An Engage or Retreat Differential Game with Two Targets.

This thesis develops the equilibrium solution for a two-target engage or retreat differential game. In this game, the attacking player is modeled as a massless particle moving with simple motion about an infinite, obstacle-free plane. The opposing player, referred to as the defender, is tasked with the protection of two high-value targets. The mobile attacker must choose to either engage one of the high-value targets or retreat across a predefined boundary. Simultaneously, the defensive player must choose whether to minimize or maximize the attacker's integral utility in an effort to persuade the attacker to choose retreat from certain initial conditions. It is shown that the solution to the game can be constructed in terms of two related optimization problems referred to as the Game of Engagement and Optimal Constrained Retreat. The optimality conditions of the game are developed and numerical solutions are presented for several illustrative examples.

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Dedicated to:

To my Family

List of Symbols

Chapter 2

$\hat{\psi}$	Global Angular Position
ψ	Relative Angular Position
α	Angular Position Between High Value Targets and Attacker
d	Relative Distance
u_A	Attacker's Control
u_D	Defender's Control
v_p	Attacker's Heading Speed
ψ	Attacker's Heading Angle
c_1	Cost Imposed By Defeneder On Attacker
c_2	Time Lapse During Engagement

Chapter 3

λ	Adjoint variables
$\phi_E(\mathbf{x}_f)$	Capture Condition For Engagement
$\phi_R(\mathbf{x}_f)$	Capture Condition For Retreat
ν	Lagrange multiplier
$V_E(\mathbf{x})$	Value Function For Engagement
$V_R(\mathbf{x})$	Value Function For Retreat
H_E	Hamiltonian For Engagement
H_R	Hamiltonian For Retreat

Introduction

There are many situations in which the protection of high-value assets from attack is an important but challenging task. This task can be made all the more difficult when the vulnerable assets are static and unable to withdraw from a threat. As an example, consider the protection of important buildings such as embassies that could represent potential targets for terrorists or insurgences. Since these types of targets are unable to withdraw to safer locations by their very nature, one must instead utilize defensive assets to neutralize a potential threat if it appears or present such a credible threat that the potential attackers choose to never engage the target to begin with. Through the strategic use of defensive assets, the potential cost of engaging a target can offset any benefit or reward the attacker receives from successfully engaging the target. When this occurs, retreat may become a more attractive option for the attacker, and the defender achieves its goal of protecting the high-value asset.

The defender's manipulation of the attacker's cost represents a form of indirect control that the defender can leverage to manipulate the attacker's behavior. In order to successfully protect the high-value targets, the defender must make the option of retreating more attractive from the attacker's perspective than engaging the target. Therefore, it may sometimes be beneficial for the defender to cooperate with the attacker by maximizing its utility when it retreats. However, care must be taken to ensure the attacker does not move into a region in which engagement becomes optimal. As a result, there may exist regions in which the defender maximizes the attacker's utility, but the attacker must follow a retreat

trajectory around the target in order to avoid regions in which engagement becomes a more attractive option.

Game theory provides an ideal environment to analyze the conflicting goals of the attacker and defender and their interactions within the system. Differential Game theory was first introduced by Rufus Issacs [7] and has since been applied to a wide range of attacker-defender scenarios. An active target defense scenario was examined in [2, 3], in which a defensive team consisting of a mobile high-value target and mobile protective agent maneuvered such that the defensive agent intercepted a mobile target as far from the high-value target as possible. In [6], a two-team differential game was presented in which one team consists of a single mobile attacker, and the defending team is composed of a mobile high-value target and mobile defensive agent. The attacker strives to get as close to the high-value target as possible before being intercepted by the mobile defensive agent. These previous games always assume the attacker must engage the targets and the defenders must minimize the attacker's utility.

In this thesis, we present a two-player, differential game where one player represents an attacker which chooses to terminate the game either by engaging one of two high value targets or crossing a defined retreat boundary. The opposing player represents a defender tasked with defending two static, high-value targets. The defender does not possess direct control on the dynamics of the system, but instead manipulates the utility function in an effort to persuade the attacker to choose retreat over engagement. This particular problem is an extension of a previous game presented in [5], in which only one target was considered. The game is solved using the general solution technique developed in [4]. Using this technique, the solution for the game is obtained by solving two related optimization problems called *Game of Engagement (GoE)* and *Optimal Constrained Retreat (OCR)*. In the GoE, it is assumed that the attacker strives to engage any of the static high value targets while the defender aims to minimize the attacker's utility function. In the OCR, it is assumed that the attacker intends to retreat while the static defensive team responds to cooperate

the attacker's intention by maximizing its utility function. An inequality constrained is imposed to restrict the attacker from moving into regions in which engagement becomes optimal. This creates some constrained retreat trajectories which we will refer to as *escort trajectories*. It was shown in [4] that for a given initial state, the equilibrium solution to the Engage or Retreat game is provided by the solution of either the GoE or OCR.

The thesis is organized as follows. Section 2 provides a description of the overall Engage or Retreat Game under consideration. The optimality conditions for the equilibrium solutions are developed in Section 3. In order to solve the game, we develop the optimality conditions for the Game of Engagement and Optimal Constrained Retreat in Section 3.1 and Section 3.2 respectively. The overall solution of the game is developed in Section 3.3. Several numerical examples are presented in Section 4, and concluding remarks are provided in Section 5.

Game Description

This work investigates a two player, engage or retreat differential game, in which one player represents a mobile attacker and the opposing player represents a defender protecting two static, high-value targets. The attacker is modeled as a massless particle that chooses to terminate the game either by capturing any one of the two high-value targets or retreating to a defined retreat surface. The defender's sole goal here, is to protect the high-value targets, and the defender attempts to persuade the attacker into retreat by manipulating the integral cost within the attacker's utility function. This chapter describes the game under consideration. In Section 2.1, the system dynamics are described using two equivalent coordinate systems. The player utility functions are described in Section 2.2, and the game is formally defined in Section 2.3 .

2.1 System Description

To analyze this game, two different, but equivalent, coordinate systems are used. The first coordinate system is referred to as the global coordinate system. Using this coordinate system, the position of the attacker is defined by a pair of Cartesian coordinates: $\hat{\mathbf{x}} := (x_A, y_A)$. The position of the high-value Target 1 and Target 2 are defined by their respective pair of Cartesian coordinates: $\mathbf{x}_1 := (x_1, y_1)$ and $\mathbf{x}_2 = (x_2, y_2)$. Since the targets are static, their positions are not considered to be components of the state, but simply parameters of one instantiation of the game. The heading of the attacker is defined by the

angle $\hat{\psi}$, which is measured counter-clockwise from the x-axis and represents the Attacker's control variable $\hat{u}_A := \hat{\psi}$. The Defender's control $\hat{u}_D := \theta$ does not influence the state of the system, but manipulates the integral cost of the Attacker, which will be discussed in detail in Section 2.2. The global coordinate system is graphically depicted in Figure 2.1. The system dynamics, $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \hat{u}_A, \hat{u}_D)$, are described by a system of two ordinary differential equations:

$$\dot{x}_A = v_p \cos(\hat{\psi}) \quad (2.1)$$

$$\dot{y}_A = v_p \sin(\hat{\psi}), \quad (2.2)$$

where v_p represents the Attacker's constant speed.

In order to simplify later analysis of the optimality conditions, a relative coordinate system is also used. For the relative coordinates, the state of the system is defined as $\mathbf{x} := (d, \alpha)$. The state component d represents the distance between the Attacker and the nearest high-value target, and angle α represents the angle measured counterclockwise from between the vector $\overleftarrow{AT_i}$ and the x-axis. The Attacker's relative heading, ψ , is measured counterclockwise from $\overleftarrow{AT_i}$: $u_A = \psi$, and the Defender's control remains unchanged: $u_D = \theta$. The relative coordinate system is visually depicted in Figure 2.2.

The global and relative coordinates are related through the following equations

$$x_A = d \cos(\alpha) + x_i \quad (2.3)$$

$$y_A = d \sin(\alpha) + y_i \quad (2.4)$$

$$\hat{\psi} = \psi + \alpha \quad (2.5)$$

where

$$i = \underset{i}{\operatorname{argmin}} \sqrt{(x_A - x_i)^2 + (y_A - y_i)^2}. \quad (2.6)$$

Using the coordinate relationships (2.3)-(2.5) and the global system dynamics (2.1)-(2.2), we can compute system dynamics using the relative coordinate system:

$$\dot{d} = v_p \cos(\psi) \quad (2.7)$$

$$\dot{\alpha} = \frac{1}{d_i} v_p \sin(\psi). \quad (2.8)$$

Figure 2.1: Global Coordinates

Figure 2.2: Relative Coordinates

The Attacker may choose to terminate the game by either capturing one of the high-value targets or retreating across a defined retreat boundary. The terminal time, t_f , is defined as the moment, the state of the system satisfies either the engagement condition,

$$\Gamma_E(\mathbf{x}) = \sqrt{(x_A - x_i)^2 + (y_A - y_i)^2} - d_c = 0 \quad (2.9)$$

or the retreat condition,

$$\Gamma_R(\mathbf{x}) = y_A - y_r = 0 \quad (2.10)$$

We define the engagement surface $\mathbf{X}_E$ and retreat surface $\mathbf{X}_R$, as the collection of state values that satisfy their respective termination conditions:

$$\mathbf{X}_R = \{\mathbf{x} \in \mathbb{R}^2 \mid \Gamma(\mathbf{x}_R) = 0\}$$

$$\mathbf{X}_E = \{\mathbf{x} \in \mathbb{R}^2 \mid \Gamma(\mathbf{x}_E) = 0\}.$$

2.2 Player Utilities

The Attacker's utility, $U_A(u_A(t), u_D(t); \mathbf{x}_0)$, consists of a terminal value function and an integral cost function incurred throughout the game. The Defender's utility depends solely on the terminal condition of the game, $U_D(u_A(t), u_D(t); \mathbf{x}_0)$. The two utility functions are defined as

$$U_A(u_A(t), u_D(t); \mathbf{x}_0) := \phi_A(\mathbf{x}_f) - \int_{t_0}^{t_f} C(u_D(t))dt \quad (2.11)$$

$$U_D(u_A(t), u_D(t); \mathbf{x}_0) := \phi_D(\mathbf{x}_f) \quad (2.12)$$

where $C(u_D(t)) = c_1 u_D + c_2$ is the instantaneous cost function integrated throughout the game period. The constant c_1 represents the maximum damage that may be inflicted by the Defender and the constant c_2 represents a time or energy penalty. The terminal value functions $\phi_A(\mathbf{x}_f)$ and $\phi_D(\mathbf{x}_f)$ determine the Attacker and Defender's respective reward or penalty dependent on how the game ends. The terminal value functions are defined as

$$\phi_A(\mathbf{x}_f) = \begin{cases} a_1 & \mathbf{x}_f \in \mathbf{X}_E \\ 0 & \mathbf{x}_f \in \mathbf{X}_R \end{cases} \quad (2.13)$$

$$\phi_D(\mathbf{x}_f) = \begin{cases} -b_1 & \mathbf{x}_f \in \mathbf{X}_E \\ 0 & \mathbf{x}_f \in \mathbf{X}_R \end{cases}, \quad (2.14)$$

where the constant $a_1 > 0$ represents the Attacker's reward for terminating the game in engagement, and the constant $b_1 > 0$ represents the Defender's penalty. For this game, there is no penalty for either the Attacker and Defender for terminating the game in retreat.

2.3 Game Definition

Using the player utility functions, (2.11) - (2.12), we can define a differential game in which each player attempts to maximize their respective utilities:

$$U_A^*(u_D(t); \mathbf{x}_0) = \max_{u_A(t)} U_A(u_A(t), u_D(t); \mathbf{x}_0) \quad (2.15)$$

$$U_D^*(u_A(t); \mathbf{x}_0) = \max_{u_D(t)} U_D(u_A(t), u_D(t); \mathbf{x}_0) \quad (2.16)$$

The Nash Equilibrium solution to this game is the pair of equilibrium open-loop strategies, $u_A^*(t; \mathbf{x}_0)$ and $u_D^*(t; \mathbf{x}_0)$, and resulting equilibrium utility values, $U_A^*(\mathbf{x}_0)$ and $U_D^*(\mathbf{x}_0)$, that satisfy the following Nash Equilibrium conditions

$$U_A(u_A^*(t), u_D^*(t); \mathbf{x}_0) = U_A^*(\mathbf{x}_0) \geq U_A(u_A(t), u_D^*(t); \mathbf{x}_0) \quad (2.17)$$

$$U_D(u_A^*(t), u_D^*(t); \mathbf{x}_0) = U_D^*(\mathbf{x}_0) \geq U_D(u_A^*(t), u_D(t); \mathbf{x}_0) \quad (2.18)$$

In Chapter 3, we develop the optimality conditions used to determine if this condition can be met and what the resulting equilibrium strategies are for each of the players.

Game Solution

The overall solution to the Engage or Retreat game will be constructed from the solutions of two related optimization problems. We will pose and solve the Game of Engagement in Section 3.1. Using the resulting value function, we will pose an Optimal Constrained Retreat problem in Section 3.2. Using the solutions of these problems, we will pose the overall solution of the Engage or Retreat game in Section 3.3

3.1 Game of Engagement

In the Game of Engagement, it is assumed that the Attacker chooses to terminate the game on the engagement surface, $\mathbf{X}_E$. It is also assumed the Defender attempts to minimize the Attacker's utility over the course of the game.

3.1.1 Game Definition

Using the system dynamics (2.1)-(2.2) and the Attacker's utility (2.11) we can pose a zero-sum differential game:

$$V_E(\mathbf{x}_0) = \min_{u_D(t)} \max_{u_A(t)} U_A(u_A(t), u_D(t); \mathbf{x}_0), \quad (3.1)$$

with the constraint $\Gamma_E(\mathbf{x}_f) = 0$.

The function $V_E(\mathbf{x}_0)$ represents the equilibrium value of the game starting at $\mathbf{x}_0$ when the Attacker and Defender implement their respective equilibrium open-loop strategies $u_A^{E*}(t, \mathbf{x}_0), u_D^{E*}(t, \mathbf{x}_0)$, where

$$u_A^{E*}(t; \mathbf{x}_0), u_D^{E*}(t; \mathbf{x}_0) = \arg \min_{u_D(t)} \max_{u_A(t)} U_A(u_A(t), u_D(t); \mathbf{x}_0). \quad (3.2)$$

This formulation represents a standard zero-sum, pursuit evasion differential game, which can be solved using standard techniques [1].

3.1.2 Optimality Condition

We begin developing the optimality conditions for the game of engagement by constructing the Hamiltonian. As we have previously set the terminal reward (2.13) for ending the game in engagement as constant, the Hamiltonian is known to be zero along the optimal trajectories:

$$\begin{aligned} H_E &= \boldsymbol{\lambda}_E^T \mathbf{f}(\mathbf{x}, u_A, u_D) - C(\mathbf{x}, u_A, u_D) \\ &= \lambda_{x_E} v_p \cos \hat{\psi} + \lambda_{y_E} v_p \sin \hat{\psi} - (c_1 u_D + c_2) = 0. \end{aligned} \quad (3.3)$$

The vector $\boldsymbol{\lambda}_E$ consists of the adjoint variables conjugate to the kinematic equations and also represents the gradient of the value function:

$$\begin{aligned} \boldsymbol{\lambda}_E &= (\lambda_{x_E}, \lambda_{y_E})^T \\ &= \left(\frac{\partial V_E}{\partial x}, \frac{\partial V_E}{\partial y} \right)^T. \end{aligned} \quad (3.4)$$

The optimal control strategies for each of the players are found by maximizing and

minimizing the Hamiltonian (3.3) accordingly:

$$\cos(\hat{\psi}^{E*}) = \frac{\lambda_{x_E}}{\sqrt{\lambda_{x_E}^2 + \lambda_{y_i}^2}} \quad (3.5)$$

$$\sin(\hat{\psi}^{E*}) = \frac{\lambda_{y_E}}{\sqrt{\lambda_{x_E}^2 + \lambda_{y_E}^2}} \quad (3.6)$$

$$v_p^{E*} = \bar{v}_p \quad \theta^{E*} = c_1 \quad (3.7)$$

In order for us to solve the game of engagement, we need to know the trajectories of the attacker along the engagement path and for this we need to know the values of adjoint variables. These adjoint variable components can be found by taking the partial derivatives of the Hamiltonian (3.3) with respect to each of the state components, and is given as:

$$\dot{\lambda}_x = -\frac{\partial H}{\partial x} = 0 \quad \dot{\lambda}_y = -\frac{\partial H}{\partial y} = 0 \quad (3.8)$$

We see that the time derivative is zero for both λ_x and λ_y , which implies that the adjoint variables are constant over the course of the game.

Using the capture condition (2.9) and the terminal reward function (2.13), an adjointed terminal value function is constructed:

$$\Phi_E(\mathbf{x}_f) = \alpha_1 + \nu_E \left(\sqrt{(x_A - x_{T_i})_{t_f}^2 + (y_A - y_{T_i})_{t_f}^2} - d_c \right), \quad (3.9)$$

where the term ν_E is a Lagrange multiplier. The partial derivative of this terminal function will provide us with the terminal adjoint variables:

$$\lambda_x(t_f) = \frac{\partial \Phi_E}{\partial x_f} = \nu_E \frac{(x_{Af} - x_{Tf})}{\sqrt{(x_{Af} - x_{Tf})^2 + (y_{Af} - y_{Tf})^2}} \quad (3.10)$$

$$\lambda_y(t_f) = \frac{\partial \Phi_E}{\partial y_f} = \nu_E \frac{(y_{Af} - y_{Tf})}{\sqrt{(x_{Af} - x_{Tf})^2 + (y_{Af} - y_{Tf})^2}}. \quad (3.11)$$

Since the terminal attacker reward (2.13) is constant, the Hamiltonian will always equal zero. Hence, now we can solve for the value of ν_E with the help of optimal control (3.5)-(3.6) and evaluating it at terminal time $t = t_f$.

$$\begin{aligned}
& \lambda_x(t_f)v_p \cos(\psi^{E*}) + \lambda_y(t_f)v_p \sin(\psi^{E*}) - (c_1 + c_2) = 0 \\
& \lambda_x(t_f)v_p \frac{\lambda_x(t_f)}{\sqrt{\lambda_x(t_f)^2 + \lambda_y(t_f)^2}} + \lambda_y(t_f)v_p \frac{\lambda_y(t_f)}{\sqrt{\lambda_x(t_f)^2 + \lambda_y(t_f)^2}} = (c_1 + c_2) \\
& v_p \frac{\lambda_x^2(t_f) + \lambda_y^2(t_f)}{\sqrt{\lambda_x(t_f)^2 + \lambda_y(t_f)^2}} = (c_1 + c_2) \\
& v_p \sqrt{\lambda_x^2(t_f) + \lambda_y^2(t_f)} = (c_1 + c_2) \\
& v_p |\nu_E| = (c_1 + c_2) \\
& \Rightarrow \nu_E = \pm \frac{(c_1 + c_2)}{v_p}
\end{aligned}$$

Looking at the Attacker's equilibrium control strategy (3.5)-(3.6), we see that a positive value for ν_E would imply an Attacker control, which is pointing away from the high-value target, and a negative value for ν_E implies a terminal control pointing towards the high-value target. Therefore, only the negative value provides a feasible solution:

$$\nu_E = -\frac{(c_1 + c_2)}{v_p}. \quad (3.12)$$

Substituting the solution into the terminal adjoint variable expressions (3.10)-(3.11) provides

$$\lambda_{x_i}(t_f) = -\frac{(x_A - x_{T_i})(c_1 + c_2)}{v_p d_c} \quad (3.13)$$

$$\lambda_{y_i}(t_f) = -\frac{(y_A - y_{T_i})(c_1 + c_2)}{v_p d_c} \quad (3.14)$$

3.1.3 Equilibrium Solution

By combining the control strategies, system dynamics and adjoint equations we can calculate the complete solution for the Game of Engagement for a given initial condition. The solution is stated in Theorem 1 given below

Theorem 1. *Assume an initial state $\mathbf{x}_0$ for the Game of Engagement. The equilibrium control strategies are given as:*

$$\cos(\psi^{E*}; \mathbf{x}_0) = -\frac{x_A - x_{T_i}}{\sqrt{(x_A - x_{T_i})^2 + (y_A - y_{T_i})^2}} \quad (3.15)$$

$$\sin(\psi^{E*}; \mathbf{x}_0) = -\frac{y_A - y_{T_i}}{\sqrt{(x_A - x_{T_i})^2 + (y_A - y_{T_i})^2}}, \quad (3.16)$$

where $i = \arg \min_i \sqrt{(x_{A0} - x_{T_i})^2 + (y_{A0} - y_{T_i})^2}$. The resulting state trajectories, terminal time, and value function when the equilibrium control is implemented are:

$$x_A^{E*}(t; \mathbf{x}_0) = x_{A0} + v_p \cos(\psi^{E*})t \quad (3.17)$$

$$y_A^{E*}(t; \mathbf{x}_0) = y_{A0} + v_p \sin(\psi^{E*})t \quad (3.18)$$

$$t_f^{E*}(\mathbf{x}_0) = \frac{\left(\sqrt{(x_A - x_{T_i})^2 + (y_A - y_{T_i})^2} - d_c \right)}{v_p} \quad (3.19)$$

$$V_E(\mathbf{x}_0) = \alpha_1 - \frac{(c_1 + c_2)}{v_p} \left(\sqrt{(x_A - x_{T_i})^2 + (y_A - y_{T_i})^2} - d_c \right). \quad (3.20)$$

Proof. From the equation (3.8) we observe that the change in the adjoint variables is always zero, which implies that the adjoint variables from equations (3.13) and (3.14) are constant over the course of the game. Combining this with attacker's optimal control (3.5) and (3.6) we see that the equilibrium control is also constant over the course of the game. Integrating

forward in time, we can express the Attacker's state trajectory in terms of the equilibrium control:

$$x^{E*}(t; \mathbf{x}_0) = x_{A0} + v_p \cos(\psi^{E*})t \quad (3.21)$$

$$y^{E*}(t; \mathbf{x}_0) = y_{A0} + v_p \sin(\psi^{E*})t \quad (3.22)$$

Evaluating the equilibrium trajectories (3.21)-(3.22) at terminal time and combining with the terminal engagement condition (2.9) assuming that Target j was captured and equilibrium control strategy (3.15)-(3.16) provides a system of four nonlinear algebraic equations with three unknowns $(t_f^*, \psi^*, x_{Af}, y_{Af})$. Solving these equations provides the equilibrium control in terms of the initial condition (3.15)-(3.16) and the terminal time t_f^* . We can then use the terminal time to calculate the equilibrium utility.

This solution provides the equilibrium solution assuming target i was captured. However, a unique solution can be generated for both $i = 1$ and $i = 2$. Since the attacker is capable of capturing either target of its choice, it should choose the target that provides the maximum utility. This is equivalent to capturing the target with minimum distance from the Attacker's initial condition.

□

There exists a subset of the admissible state space in which engaging either target yields the same utility for Player A. This subset forms a type of singular surface known as a dispersal surface. On this surface, choosing to terminate the game in engagement of either Target 1 or Target 2 provides an equally optimal solution. The surface is formally defined in Theorem 2.

Theorem 2. *The surface*

$$\mathbf{S}_D := \left\{ \mathbf{x} \in \mathbf{R}_A \mid \sqrt{(x_A - x_1)^2 + (y_A - y_1)^2} = \sqrt{(x_A - x_2)^2 + (y_A - y_2)^2} \right\} \quad (3.23)$$

represents a dispersal surface with respect to the Attacker's control within the Game of Engagement.

Proof. Assume that $\mathbf{x}_0 \in \mathbf{S}_D$. We begin by analyzing the value function (3.20) for engaging either target:

$$V_{E1}(\mathbf{x}_0) = \alpha_1 - \frac{(c_1 + c_2)}{v_p} \left(\sqrt{(x_A - x_{T_1})^2 + (y_A - y_{T_1})^2} - d_c \right) \quad (3.24)$$

$$V_{E2}(\mathbf{x}_0) = \alpha_1 - \frac{(c_1 + c_2)}{v_p} \left(\sqrt{(x_A - x_{T_2})^2 + (y_A - y_{T_2})^2} - d_c \right) \quad (3.25)$$

Substituting definition of $\mathbf{S}_D$ into V_{E1} we see that engaging either target provides equal utility:

$$V_{E1}(\mathbf{x}_0) = \alpha_1 - \frac{(c_1 + c_2)}{v_p} \left(\sqrt{(x_A - x_{T_1})^2 + (y_A - y_{T_1})^2} - d_c \right) \quad (3.26)$$

$$= \alpha_1 - \frac{(c_1 + c_2)}{v_p} \left(\sqrt{(x_A - x_{T_2})^2 + (y_A - y_{T_2})^2} - d_c \right) \quad (3.27)$$

$$= V_{E2}(\mathbf{x}_0) \quad (3.28)$$

Therefore, both solutions are in equilibrium and the surface $\mathbf{S}_D$ represents a dispersal surface. □

3.2 Optimal Constrained Retreat

Using the solution for the Game of Engagement presented in Theorem 1 , we can now pose the Optimal Constrained Retreat problem. Here, the attacker strives to reach the retreat zone by maximizing its utility function. At the same time, the key role of Defender is to cooperate and also maximize the Attacker's utility in an effort to make retreat as attractive as possible. Using the same dynamics and Player A utility function, we can pose the optimal constrained retreat problem as

$$V_R(\mathbf{x}_0) = \max_{u_A(t)} \max_{u_D(t)} U_A(u_A(t), u_D(t), \mathbf{x}_A) \quad (3.29)$$

with the terminal constraint $\Gamma_R(\mathbf{x}_f) = 0$, which represents the termination condition in retreat.

Here, the defending player will be trying to maximize the utility function of mobile attacking player. So, there might be possibility that the attacker might try to switch to engagement if the engagement utility (3.20) exceeds the value of retreat. In order to avoid such situation from occurring, the high value target impose a restriction in the attacker's utility function so that it would be bound to choose retreat over engagement:

$$V_R(\mathbf{x}(t)) - V_E(\mathbf{x}(t)) \geq 0 \quad \forall t \in [t_0, t_f] \quad (3.30)$$

We can now convert this value function into a state inequality constraint by adding an additional state variable, $c(t)$ with a time derivative given as:

$$\dot{c} = C(u_D(t)) \quad (3.31)$$

This state variable represents the remaining integral cost for the rest of the game, and it has the terminal constraint $c(t_f) = 0$. This means the game would terminate as soon as the

attacker reaches the retreat surface so the cost at that particular terminal time will be zero.

The value function for retreat consists of the terminal value function and the integral cost for terminating the game in retreat:

$$\begin{aligned}
V_R(\mathbf{x}) &= \phi_A(\mathbf{x}(t_f)) - \int_0^{t_f} C(\mathbf{x}(t), u_A(t), u_D(t)) dt \\
&= \phi_A(\mathbf{x}(t_f)) - \cancel{c(t_f)}^0 + c(t_0) \\
&= \phi_A(\mathbf{x}(t_f)) + c(t_0)
\end{aligned} \tag{3.32}$$

From Equations (3.30) and (3.32), we can restate the value function constraint as a state variable constraint,

$$g(\mathbf{x}) = \phi_A(\mathbf{x}(t_f)) + c(t_0) - V_E(\mathbf{x}(t_f)) \geq 0 \tag{3.33}$$

We now need to determine the effect of this constraint on optimal controls of the Attacker and the Defender. However, $g(\mathbf{x})$ is not an explicit function of control. Thus, $g(x)$ is successively differentiated with respect to time until its dependency on control variables (u_A or u_D) shows up.

$$\begin{aligned}
h(\mathbf{x}) &= \frac{\partial g(\mathbf{x})}{\partial t} \\
&= \frac{\partial(\phi_A(\mathbf{x}(t_f)) + c(t_0) - V_E(\mathbf{x}(t_f)))}{\partial t} \\
&= (u_D c_1 + c_2) + \frac{(c_1 + c_2) [(x_A - x_{T_i}) \cos(\psi) + (y_A - y_{T_i}) \sin(\psi)]}{\sqrt{(x_A - x_{T_i})^2 + (y_A - y_{T_i})^2}}
\end{aligned} \tag{3.34}$$

where $i = \arg \min_i \sqrt{(x_A - x_i)^2 + (y_A - y_i)^2}$.

3.2.1 Optimality Condition

Just like the game of engagement, we start calculating the solution for the Optimal Constrained Retreat (OCR) by constructing the Hamiltonian. Also for this game, the terminal reward for completing the game is constant i.e zero. So, the Hamiltonian is known to be zero along its optimal trajectories. The Hamiltonian of the OCR consists of the system dynamics, utility function of attacker and the control equality constraints and additional state value:

$$\begin{aligned}
 H_R &= \boldsymbol{\lambda}_{\mathbf{x}_R}^T \mathbf{f}(\mathbf{x}, u_A, u_B) - C(\mathbf{x}, u_A, u_B) + \mu h(\mathbf{x}) \\
 &= \lambda_{x_R} v_p \cos(\psi) + \lambda_{y_R} v_p \sin(\psi) + \lambda_c (u_D c_1 + c_2) \\
 &\quad - (u_D c_1 + c_2) + \mu [u_D c_1 + c_2 + \kappa]
 \end{aligned} \tag{3.35}$$

where,

$$\kappa = \frac{(c_1 + c_2)(x_A - x_{Ti}) \cos(\psi) + (y_A - y_{Ti}) \sin(\psi)}{\sqrt{(x_A - x_{Ti})^2 + (y_A - y_{Ti})^2}}$$

The additional adjoint variable μ is a scalar and satisfies

$$\begin{aligned}
 \mu(t) &= 0 & \text{when } g(x) > 0 \\
 \mu(t) &\leq 0 & \text{when } g(x) = 0
 \end{aligned} \tag{3.36}$$

The Hamiltonian can be rearranged to factor out the Attacker's control

$$H_R = \cos(\psi) \varrho_1 + \sin(\psi) \varrho_2 + (u_D c_1 + c_2)(\lambda_c + \mu - 1) \tag{3.37}$$

where,

$$\begin{aligned}\varrho_1 &= \lambda_{xR}v_p + \mu \frac{(c_1 + c_2)(x_A - x_{Ti})}{\sqrt{(x_A - x_{Ti})^2 + (y_A - y_{Ti})^2}} \\ \varrho_2 &= \lambda_{yR}v_p + \mu \frac{(c_1 + c_2)(y_A - y_{Ti})}{\sqrt{(x_A - x_{Ti})^2 + (y_A - y_{Ti})^2}}\end{aligned}$$

The equilibrium control for the Attacker is found by maximizing the Hamiltonian:

$$\cos(\psi^{R*}) = \frac{\varrho_1}{\sqrt{\varrho_1^2 + \varrho_2^2}}, \quad \sin(\psi^{R*}) = \frac{\varrho_2}{\sqrt{\varrho_1^2 + \varrho_2^2}} \quad (3.38)$$

The optimal value for u_D^{R*} ranges from 0 to 1 and is determined by the value $(\lambda_c + \mu - 1)$.

$$u_D^{R*} = \begin{cases} 0, & (\lambda_c + \mu - 1) \leq 0 \\ 1, & (\lambda_c + \mu - 1) > 0 \end{cases} \quad (3.39)$$

The adjoint variables gives the data on, how the attacker is approaching any of the high value targets. And these are obtained through the partial derivative of Hamiltonian (3.37) with respect to each of the state variables used i.e.

$$\begin{aligned}\dot{\lambda}_{xR} &= -\frac{\partial H_R}{\partial x} = -\mu(c_1 + c_2)\frac{\varpi_1}{\varrho}, \\ \dot{\lambda}_{yR} &= -\frac{\partial H_R}{\partial y} = -\mu(c_1 + c_2)\frac{\varpi_2}{\varrho} \\ \dot{\lambda}_c &= -\frac{\partial H_R}{\partial c} = 0\end{aligned} \quad (3.40)$$

where,

$$\begin{aligned}\varpi_1 &= \left[\frac{(y_A - y_{T_i})^2 \cos(\psi) - (y_A - y_{T_i})(x_A - x_{T_i}) \sin(\psi)}{\sqrt{(x_A - x_{T_i})^2 + (y_A - y_{T_i})^2}} \right] \\ \varpi_2 &= \left[\frac{(x_A - x_{T_i})^2 \sin(\psi) - (y_A - y_{T_i})(x_A - x_{T_i}) \cos(\psi)}{\sqrt{(x_A - x_{T_i})^2 + (y_A - y_{T_i})^2}} \right] \\ \varrho &= \sqrt{(x_A - x_{T_i})^2 + (y_A - y_{T_i})^2}\end{aligned}$$

In order to calculate the terminal values of the adjoint variables we need to construct a terminal value function like the one below:

$$\Phi_R(\mathbf{x}_f) = \phi_A(\mathbf{x}_f) + \nu_R(y - y_R) \quad (3.41)$$

Here, $\Phi_R(\mathbf{x}_f)$ is the terminal retreat value function and $\psi_A(\mathbf{x}_f)$ is the terminal penalty to attacker for considering the retreat which in this case is zero as given in equation (2.14). Now, the terminal adjoint values are determined by taking the partial derivative of terminal value function (3.41):

$$\lambda_{x_R}(t_f) = \frac{\partial \Phi_R(\mathbf{x}_f)}{\partial x} = 0 \quad (3.42)$$

$$\lambda_{y_R}(t_f) = \frac{\partial \Phi_R(\mathbf{x}_f)}{\partial y} = \nu_R \quad (3.43)$$

$$\lambda_c(t_f) = \frac{\partial \Phi_R(\mathbf{x}_f)}{\partial c} = 0 \quad (3.44)$$

Substituting the above terminal adjoint variables in the Hamiltonian (3.35) and evaluating at terminal time, we will be able to get the value of Lagrange multiplier ν_R . Now, solving the unconstrained Hamiltonian equation (3.35) at $t = t_f$ and equating it with our

terminal adjoint variables (3.42)-(3.44) we get,

$$\vec{H}_R^0 = \vec{X}_{xR}^0(t_f) \bar{v}_p \cos(\psi) + \vec{X}_{yR}^{\nu_R}(t_f) \bar{v}_p \sin(\psi) + \left(\vec{X}_c^0(t_f) - 1 \right) c_2$$

On solving it:

$$|\nu_R| = \frac{c_2}{v_p} \quad (3.45)$$

Though the theoretical value of ν_R can be both positive or negative, we take a negative value in practice, since we are assuming the attacker is moving from retreat region to its original position. Hence,

$$\nu_R = -\frac{c_2}{v_p} \quad (3.46)$$

Our terminal adjoint variables will be

$$\lambda_{xR}(t_f) = \frac{\partial \Phi_R(\mathbf{x}_f)}{\partial x} = 0 \quad (3.47)$$

$$\lambda_{yR}(t_f) = \frac{\partial \Phi_R(\mathbf{x}_f)}{\partial y} = -\frac{c_2}{v_p} \quad (3.48)$$

$$\lambda_c(t_f) = \frac{\partial \Phi_R(\mathbf{x}_f)}{\partial c} = 0 \quad (3.49)$$

The addition of the state constraint (3.31), introduces internal jump conditions on the adjoint variables that occur as the state enters or exits the constraint. These jump conditions satisfy

$$\boldsymbol{\lambda}^T(t_j+) = \boldsymbol{\lambda}^T(t_j-) + \pi_1 \frac{\delta g}{\delta \mathbf{x}(t_j)} + \pi_2 \frac{\delta h}{\delta \mathbf{x}(t_j)}, \quad (3.50)$$

where t_j is the time that the trajectory activates the state constraint.

3.2.2 Optimal Solution

The state constraint and the resulting jump conditions create a piecewise equilibrium solution, which consists of multiple constrained and unconstrained segments. In order to generate a complete trajectory we must develop these trajectory segments piece by piece working backwards from the terminal surface.

We begin solving an unconstrained arc backward from the terminal surface. Since the constraint is not active $\mu = 0$ and our Hamiltonian will be:

$$H = \lambda_{x_R} v_p \cos(\psi) + \lambda_{y_R} v_p \sin(\psi) + \lambda_c (u_D^0 c_1 + c_2) = 0 \quad (3.51)$$

Also since we are dealing with the retreat region, the value of u_D will be zero since the high value targets are maximizing the attacker's utility function i.e $u_D = 0$ giving us the equation:

$$H = \lambda_{x_R} v_p \cos(\psi) + \lambda_{y_R} v_p \sin(\psi) + \lambda_c c_2 = 0$$

The optimal control is obtained as:

$$\cos(\psi^{R*}) = \frac{\lambda_{x_R}}{\sqrt{\lambda_{x_R}^2 + \lambda_{y_R}^2}} \quad , \quad \sin(\psi^{R*}) = \frac{\lambda_{y_R}}{\sqrt{\lambda_{x_R}^2 + \lambda_{y_R}^2}} \quad (3.52)$$

The optimal controls at the terminal time is given as:

$$\begin{aligned} \cos \psi^{R*}(t_f) &= \frac{\lambda_{x_R}(t_f)}{\sqrt{\lambda_{x_R}^2(t_f) + \lambda_{y_R}^2(t_f)}} \\ \sin \psi^{R*}(t_f) &= \frac{\lambda_{y_R}(t_f)}{\sqrt{\lambda_{x_R}^2(t_f) + \lambda_{y_R}^2(t_f)}} \end{aligned}$$

We know the terminal values of our adjoint variables from equations (3.47) ,(3.48)

and (3.49) as:

$$\lambda_{xR}(t_f) = 0, \quad \lambda_{yR}(t_f) = -\frac{c_2}{v_A}, \quad \lambda_c(t_f) = 0 \quad (3.53)$$

Now, using the information from equation (3.52) and (3.53), the optimal control for unconstrained region will become:

$$\cos(\psi^{R*})(t_f) = 0 \quad \sin(\psi^{R*})(t_f) = -1 \quad (3.54)$$

This will also change the kinematic equations for unconstrained region of x_A and y_A as:

$$\dot{x}_A = v_p \overset{0}{\cancel{\cos}}(\psi^{R*})(t_f) = 0 \quad (3.55)$$

$$\dot{y}_A = v_p \overset{-1}{\cancel{\sin}}(\psi^{R*})(t_f) = -v_A \quad (3.56)$$

Integrating the dynamics provide the optimal trajectory segment

$$x(t) = x_f \quad (3.57)$$

$$y(t) = y_f - v_A t + y_c, \quad (3.58)$$

where y_c is unknown constant of integration. Along this unconstrained segment, the value of the optimal constrained retreat is given as:

$$V_R(\mathbf{x}) = -\frac{c_2}{v_A}(y_A - y_r). \quad (3.59)$$

In order to develop the next optimal segment we must calculate the tangency point, $\mathbf{x}_T$ in which the trajectory leaves the constrained arc. This point must satisfy the above

retreat boundary as well as the state constraint (3.30), and the control constraint (3.38). Substituting the unconstrained retreat value and Game of Engagement value function into the state constraint provides

$$\begin{aligned}
g(\mathbf{x}) &= V_R(\mathbf{x}) - V_E(\mathbf{x}) = 0 \\
&= -\frac{c_2}{v_p}(y_A - y_r) - a_1 + \frac{c_1 + c_2}{v_A}(\kappa - d_c) \\
&= c_2(y_A - y_r) - a_1 v_p + (c_1 + c_2)(\kappa - d_c) = 0
\end{aligned} \tag{3.60}$$

The control equality constraint which was obtained by differentiating state inequality constraint with respect to time is:

$$\begin{aligned}
h(\mathbf{x}) &= g'(\mathbf{x}) = \dot{V}_R(\mathbf{x}) - \dot{V}_E(\mathbf{x}) \\
&= -\frac{c_2}{v_p}\dot{y}_A + \frac{c_1 + c_2}{v_p} \left[\frac{\dot{x}_A(x_A - x_{T_i}) + \dot{y}_A(y_A - y_{T_i})}{\sqrt{(x_A - x_{T_i})^2 + (y_A - y_{T_i})^2}} \right]
\end{aligned} \tag{3.61}$$

From equations (3.55) and (3.56), equation (3.61) will change to:

$$\begin{aligned}
h(\mathbf{x}) &= -\frac{c_2}{v_p} * (-v_p) + \frac{c_1 + c_2}{v_A} \left(\frac{-v_p(y_A - y_{T_i})}{\sqrt{(x_A - x_{T_i})^2 + (y_A - y_{T_i})^2}} \right) \\
&= c_2 - \frac{(c_1 + c_2)(y_A - y_{T_i})}{\sqrt{(x_A - x_{T_i})^2 + (y_A - y_{T_i})^2}} = 0 \\
&= \frac{(c_1 + c_2)(y_A - y_{T_i})}{\sqrt{(x_A - x_{T_i})^2 + (y_A - y_{T_i})^2}} = c_2 \\
\Rightarrow \sqrt{(x_A - x_{T_i})^2 + (y_A - y_{T_i})^2} &= \frac{(y_A - y_{T_i})(c_1 + c_2)}{c_2}
\end{aligned} \tag{3.62}$$

Now, implementing equation (3.62) into (3.60):

$$-c_2(y_A - y_r) - \xi + \frac{(c_1 + c_2)^2(y_A - y_{T_i})}{c_2} = 0$$

where,

$$\kappa = \sqrt{(x_A - x_{T_i})^2 + (y_A - y_{T_i})^2}$$

$$\xi = a_1 v_p + (c_1 + c_2) \kappa$$

Then,

$$\frac{(c_1 + c_2)^2}{c_2} y_A - \frac{c(c_1 + c_2)^2}{c_2} y_{T_i} - c_2 y_A + c_2 y_r - \xi = 0$$

$$y_A \left[\frac{(c_1 + c_2)^2 - c_2^2}{c_2} \right] = \frac{(c_1 + c_2)^2}{c_2} y_{T_i} + \xi - c_2 y_r$$

$$y_A = \frac{c_2 \left[\frac{(c_1 + c_2)^2}{c_2} y_{T_i} + \xi - c_2 y_r \right]}{(c_1 + c_2)^2 - c_2^2}$$

The y component of the tangency point then satisfies:

$$y_T = \frac{(c_1 + c_2)^2 y_{T_i} + c_2 a_1 v_p + c_2 (c_1 + c_2) d_c - c_2^2 d_c y_r}{(c_1 + c_2)^2 - c_2^2} \quad (3.63)$$

Using the information of equation (3.63) and implementing it in equation (3.62), we can compute the x component of the tangency point:

$$\sqrt{(x_A - x_{T_i})^2 + (y_A - y_{T_i})^2} = \frac{(y_A - y_{T_i})(c_1 + c_2)}{c_2}$$

$$(x_A - x_{T_i})^2 + (y_A - y_{T_i})^2 = \frac{(y_A - y_{T_i})^2 (c_1 + c_2)^2}{c_2^2}$$

$$(y_A - y_{T_i})^2 \left[\frac{(c_1 + c_2)^2}{c_2^2} - 1 \right] = (x_A - x_{T_i})^2$$

$$x_T = (y_A - y_{T_i}) \frac{\sqrt{(c_1 + c_2)^2 - c_2^2}}{c_2} + x_{T_i} \quad (3.64)$$

Along the constrained trajectory the adjoint variable corresponding to the state constraint μ is nonzero, which ensures the equilibrium Attacker control does not violate the derivative constraint $h(\mathbf{x}) = 0$. We can solve for the constrained value of μ in terms of the state and adjoint variables by substituting the optimal control strategies into the derivative constraint:

$$h(\mathbf{x}) = \dot{V}_R(\mathbf{x}) - \dot{V}_E(\mathbf{x}) \quad (3.65)$$

$$= c_2 - \frac{1}{c_1 + c_2} \left[\frac{x - x_{T_i}}{\sqrt{(x - x_{T_i})^2 + (y - y_{T_i})^2}} \cos(\psi) + \frac{y - y_{T_i}}{\sqrt{(x - x_{T_i})^2 + (y - y_{T_i})^2}} \sin(\psi) \right] = 0 \quad (3.66)$$

Solving for the control provides:

$$\tan(\psi) = \tan \left(\alpha + \cos^{-1} \left(\frac{c_1}{c_1 + c_2} \right) \right), \quad (3.67)$$

where $\tan \alpha = (y - y_{T_i}) / (x - x_{T_i})$. Substituting the equilibrium control strategies for ψ yields

$$\frac{\lambda_{x_R} v_p \sqrt{(x_A - x_{T_i})^2 + (y_A - y_{T_i})^2} + \mu(c_1 + c_2)(x_A - x_{T_i})}{\lambda_{y_R} v_p \sqrt{(x_A - x_{T_i})^2 + (y_A - y_{T_i})^2} + \mu(c_1 + c_2)(y_A - y_{T_i})} = \tan \left[\alpha + \cos^{-1} \left(\frac{c_1}{c_1 + c_2} \right) \right] \quad (3.68)$$

We can now solve directly for μ along the constrained trajectory.

Additionally, the constrained optimal attacker control ψ is provided by (3.67). Using the constrained control, adjoint equations, dynamic, and optimal μ , we can integrated backwards until the state enters the constrained arc at time t_1 . This process is repeated along each constrained and unconstrained segment of the optimal trajectory in order to construct

the complete optimal solution for the given initial condition $\mathbf{x}_0$. Due to the nonlinear and coupled nature of the state and adjoint equations, it is difficult to analytically calculate the constrained trajectories. As a result, these segments are numerically integrated.

3.3 Engage or Retreat Solution

Using the solutions of the Game of Engagement and the Optimal Constrained retreat, we can develop the overall solution to the Engage or Retreat Game [4]. It is possible that a solution may not exist to the OCR for some values of $\mathbf{x}_0$. Therefore, we can divide the admissible state space into two regions: a region that contains states that possess a solution to the OCR and a region whose states do not possess a solution to the OCR. They are formally defined as follows.

Definition 1.

$$\mathbf{R}_R := \{\mathbf{x} \in \mathbf{R}_A | \exists u_A^{R*}(t; \mathbf{x}), u_D^{R*}(t; \mathbf{x})\} \quad (3.69)$$

$$\mathbf{R}_E := \mathbf{R}_A \setminus \mathbf{R}_R \quad (3.70)$$

The overall solution to the ERG can be found for a particular $\mathbf{x}_0$ by identifying which region the initial state belongs to and then implementing the control strategies defined in the following theorem.

Theorem 3. *Suppose that regions $\mathbf{R}_E$ and $\mathbf{R}_R$ exist as defined by Definition 1. Additionally, suppose that $V_E(\mathbf{x}) \leq V_R(\mathbf{x})$ along the boundary of $\mathbf{R}_R$. The following control strategies*

constitute a Nash Equilibrium for the Engage or Retreat Game defined in (3.69) and (3.70).

$$u_A^*(t; \mathbf{x}_0) = \begin{cases} u_A^{R*}(t; \mathbf{x}_0) & \mathbf{x}_0 \in \mathbf{R}_R \\ u_A^{E*}(t; \mathbf{x}_0) & \mathbf{x}_0 \in \mathbf{R}_E \end{cases} \quad (3.71)$$

$$u_D^*(t; \mathbf{x}_0) = \begin{cases} u_D^{R*}(t; \mathbf{x}_0) & \mathbf{x}_0 \in \mathbf{R}_R \\ u_D^{E*}(t; \mathbf{x}_0) & \mathbf{x}_0 \in \mathbf{R}_E \end{cases} \quad (3.72)$$

The resulting Nash Equilibrium utilities for each player are

$$U_A^*(\mathbf{x}_0) = \begin{cases} V_R(\mathbf{x}_0) & \mathbf{x}_0 \in \mathbf{R}_R \\ V_E(\mathbf{x}_0) & \mathbf{x}_0 \in \mathbf{R}_E \end{cases} \quad (3.73)$$

and

$$U_D^*(\mathbf{x}_0) = \begin{cases} -b_1 & \mathbf{x}_0 \in \mathbf{R}_R \\ 0 & \mathbf{x}_0 \in \mathbf{R}_E \end{cases} . \quad (3.74)$$

Numerical Examples

As we have discussed, each of the optimization problems, Game of Engagement and Optimal Constrained Retreat, represents one individual game. Thus, the results are expressed in terms of numerical solutions for each of the game separately and also as a complete solution.

4.1 Game of Engagement

In this following section, we have examined the equilibrium trajectories and the value function of the game using the given parameters.

$$\bar{v}_p = 1, \quad a_1 = 1, \quad c_1 = 1, \quad c_2 = 1, \quad d_c = 1, \quad y_R = -7$$

Plots

To get the plots for Game of Engagement, a series of procedure has been taken. First of all, we choose a single high value target and a single attacker position, then the trajectories were drawn accordingly showing how the attacker approaches the high value target. After having one successful capture, we then considered two high value targets. The dispersal surface represents the state in the game when the attacker gains equal value function for engaging both the high value targets from different initial position .

We have designed the game in such a way that, whenever the attacker is at any position, based on the value function and the distance, it will approach the target nearer to it. The dispersal line separates each of the high value target in a minimal distance or the one which it can have maximum value function with. For this thesis work, we have considered a single attacker and two high value targets, but eventually, this can be used for any number of high value targets. The only thing is there will be complication in trajectories and dispersal surface.

Here, Figure 4.1 represents the analytical solution for two high value targets and a single attacker. Here, in this figure, the blue and the red lines represent the engagement trajectories for either of the high value targets. Whenever the attacker reaches the capture distance of any of the high value target the game will terminate. For the sake of illustration, we have shown the trajectories for both of the high value targets. Figure 4.2 shows the trajectories of our adjoint variables for the above multi-player game. It suggests how the deviation of trajectories will let the attacker to get into the capture zone of either high value target.

Figure 4.3 gives the visualization for the engagement to one particular high value target given several initial positions of attacker. If the attacker is in a position like the way in the figure, then it should definitely engage target 1. The black dotted line represents the dispersal surface. Similarly, Figure 4.4 represents the engagement trajectories for target 2 for various initial positions of attacker. Figure 4.5 shows the general engagement approach by the attacker to either of the targets. Since there is no change in the derivative term of adjoint variables, the plots for the adjoint variables show the trajectory with some constant value. Figure 4.6 and Figure 4.7 represent the plot for the value function of engagement to high value target 1 and 2 respectively. Since we are evaluating this as a single game, we

Figure 4.1: Analytical Solution

Figure 4.2: Adjoint Trajectories.

Figure 4.3: Approaching Target 1

Figure 4.4: Approaching Target 2

Figure 4.5: Engage Trajectories

Figure 4.6: Value Plot for Engaging Target 1

Figure 4.7: Value Plot for Engaging Target 2

Figure 4.8: Combined Value Function

need to get one value function for our game of engagement. Here, we have said the attacker is trying to maximize its utility function, so, we need to get the maximum value between these two value functions. i.e

$$V_E(\mathbf{x}_0) = \arg \max (V_{E_1}(\mathbf{x}_0), V_{E_2}(\mathbf{x}_0))$$

Figure 4.8 represents the combined maximum value function for engaging either of the high value target given any initial position of attacker.

4.2 Optimal Constrained Retreat

The game can be divided into numerous conditions depending upon the orientation of the high value targets. For the retreat game we have designed three possible solution based on the different position of high value targets.

- When two targets are significantly far from each other.
- When two targets are nearer and formed a combined solution.
- A complicated solution.

Case 1

In this game, the targets are farther from each other, so there won't be any intersection between the trajectories of attacker while engaging any of the high value targets. And they behave as an independent high value targets. The solution to this game is provided in just like the one, illustrated by the author Dr. Fuchs [5], in his earlier paper. Here, this game gives us the brief idea about the overall game solution, when two high value targets are significantly farther from each other with no intersection in between. The solution to this game is described below with the respective plots.

In this following plot, the retreat surface is illustrated as red dashed line at $y=-7$, and the capture is depicted as $\sqrt{(x_A - x_{T_i})^2 + (y_A - y_{T_i})^2} - 1 = 0$. Figure 4.9 is the complete solution for both the Game of Engagement and Optimal Constrained Retreat. Figure 4.10 shows the game for just the engagement section, Figure 4.11 represents the game for Optimal Control Retreat. Figure 4.12 shows the jump trajectories that took place throughout the game. Figure 4.13 represents the single game from time t_o to t_f . The red dashed line in this figure represents the escort region which the defender uses to escort the attacker to retreat surface. The solid black line that terminates the surface just before the constrained trajectory marked with red dashed line indicates the tangency coordinates (x_2, y_2) and $(-x_2, y_2)$. Figure 4.14 gives the plot for unconstrained region for this game. Figure 4.15-4.18 represents the plot for constraint (μ), optimal control, value function and adjoint variables simultaneously.

The value function plot Figure 4.17 help us know about the overall game. Looking at the plot, we can figure it out, that for any initial condition where the attacker lies within the optimal constrained region, it is always better for it to retreat rather than engage and which is the objective of this thesis too. This value function plot Figure 4.17 corresponds to the game shown in Figure 4.13, where attacker starts from any one initial position.

Figure 4.9: Complete Solution

Figure 4.10: Game of Engagement.

Figure 4.11: Retreat Trajectories

Figure 4.12: Jump Trajectories

Figure 4.13: Solution with one trajectory

Figure 4.14: Unconstrained Trajectory

Figure 4.15: Constrained Plot

Figure 4.16: Optimal Control Plot

Figure 4.17: Value Function Plot

Figure 4.18: Adjoint Variables Plot

Case 2:

In this section, we have illustrated the solution for the game where the two high value targets are nearer to each other and they form intersection around the edges.

Figure 4.19 is the complete solution to our game at this circumstances. It comprises both the trajectory for game of Engagement and Optimal Constrained Retreat. The red lines indicate the trajectories when the attacker involves in the Engagement phenomenon. The black lines are the trajectories for unconstrained retreat game. The blue line just across the engagement trajectory is the region of constrained retreat. From this point the high value targets attempt to escort the attacker until it hits the tangency point and, from that point the attacker makes its way safely to the retreat zone. The blue dotted line represents the dispersal surface for the attacker. This is the state at which the attacker gains equal value function for engaging either of the high value targets. The wider black line below the engagement surface represents the surface where the value function for engaging is equal to the value

function for retreat and it forms the upper bound for the unconstrained trajectories below it. Similarly, the line dissipating from the constrained region are the jump trajectories as a certain discontinuity occurs in the game due to the movement of attacker from one region to other. They all terminate in a plane where retreating to either side presents equal value function.

Figure 4.20 represents the game of engagement. Figure 4.21 shows the complete game for constrained retreat. Figure 4.22 shows the trajectories after a discontinuity from the constrained region to unconstrained one. And these are the jump trajectories from the constrained regions. The adjoint variables λ_d and λ_α are dependent on the constraint value μ so, it states a jump from the constrained region to unconstrained region. Here, we can see the actual operation of the game. Figure 4.24 depicts the unconstrained region throughout the game. Figure 4.25-4.28 are the figures representing the plot for value function, constraint, optimal control and adjoint variables respectively.

Figure 4.23 gives the plot for the overall game with single trajectory. In this figure, the constrained trajectory marked with red dash line represents the escort region. In this region, the corresponding high value targets cooperate with the attacker to maximize its utility function if the attacker marches into the retreat region. However, if the attacker tries to jump into engagement, the defender switches its strategy to minimize the attacker's utility function which provides a worst utility value for the attacker. So, the defender successfully escorts the attacker in the constrained region till it hits the tangency point and after that the attacker itself can get into the retreat boundary as it will achieve a high value function on retreating from this position compared to engagement.

Figure 4.19: Complete Solution

Figure 4.20: Game of Engagement.

Figure 4.21: Retreat Trajectories

Figure 4.22: Jump Trajectories

Figure 4.23: Solution with One Trajectory

Figure 4.24: Unconstrained Plot

Figure 4.25: Value Function Plot

Figure 4.26: Constraint Plot

Figure 4.27: Optimal Control Plot

Figure 4.28: Adjoint Variables Plot

Case 3:

In this circumstance, there will be the possibility of more than one constrained regions. Basically, it all depend upon the positioning of the high value targets with reference to that of attacker. Here, in this condition, the the two high value targets are in position such that they are one above another. This part of game has a complex solution. Because of the complex nature, the game was solved using the global coordinate system.

We have the following figures to describe this situations:

Here, Figure 4.29 is the representation for the overall game which includes both the optimization problems i.e *Game of Engagement* and *Optimal Constrained Retreat*. Here, the red dash line depicts the trajectories for the Game of Engagement. The black dashed line are the trajectories for the unconstrained regions. The slanted trajectories marked as blue and black dashed line are the jump trajectories in this game.

Figure 4.30 represents the global engagement solution of the game. It shows the respective trajectories for each of the high value targets from different initial position of the attacker. The blue dotted line represents the dispersal surface from where it has the equal probability of engaging any of the high value targets. Figure 4.31 is the solution for the optimal constrained retreat. Here, we can see how the attacker approaches the retreat boundary from different initial conditions. Figure 4.32 represents the jump trajectories that were observed on this case. Here, we can see two different types of jump trajectories, one is from the constrained region of one target to that of other, while the other one represents the overall jump solution from a position where it has an equal value for retreating from either side of the high value target at higher position.

Figure 4.34 shows the unconstrained trajectories for the overall optimal constrained retreat game. Figure 4.35-4.38 depicts the plots of constraint value (μ), value function, adjoint trajectories and optimal control simultaneously. We can see the trajectories are not continuous in Figure 4.35 and Figure 4.37. These are due to the jump that took place while moving from the unconstrained regions to constrained and vice versa. Figure 4.36 shows the relation of values for engagement and retreat. As we can see, around the constrained region, the value function for both of the optimization problems are equal while value function of Engagement is always less than that of Retreat in all other initial position. So, its always beneficial for attacker to retreat on every initial position other than constrained region.

Figure 4.33 represents the overall game of retreat under one initial position of attacker. It shows the different steps that were used in formulating the game. First, it starts at any initial condition. Then, the optimal trajectories arrive at the the first constrained region tangentially and move along the surface until it hits the first tangency point. After this, the attacker again continues to move tangentially along the surface till it hits the final tangency point. And from that position, the attacker eventually go to terminate the game in the retreat surface.

Figure 4.29: Complete Solution

Figure 4.30: Game of Engagement.

Figure 4.31: Retreat Trajectories

Figure 4.32: Jump Trajectories

Figure 4.33: Single Solution

Figure 4.34: Unconstrained Regions

Figure 4.35: Constraint Plot

Figure 4.36: Value Plot

Figure 4.37: Adjoint Trajectories

Figure 4.38: Control Plot

Conclusion

In this thesis, we solved an engage or retreat Differential Game for a single attacker and two high value targets in terms of two related optimization problems. The equilibrium solutions obtained so far shows that for some initial conditions, it is optimal for the defensive team to cooperate with the attacker to make retreat an attractive option. But at the same time, the defender should be aware of attacker's intention as it might switch to engagement under some optimal condition. So, in order to maintain this cooperation, we have imposed a value function constraint on attacker's utility. This produces regions of constrained retreat depending upon the orientation of high value targets, which we refer to as *escort* regions. These are the regions, where, the defender cooperates with the attacker to escort them safely into retreat surface so that both the attacker and defender will have a win-win situation. To be so, the attacker should also follow the escort trajectory and get into the retreat surface.

Even though, we have demonstrated that the strategies presented above are in equilibrium state, the attacker's strategy is in a weak equilibrium at the constraint region. Since, at this region, the value function for either engaging or retreating are same, the attacker could eventually switch back to engagement and receive the same utility as retreat. So, we are looking forward to develop the modified defender strategy which could prevent the attacker from engaging at constraint regions. As an extension to this work, we can work to solve the game with mobile defender where both the high value targets will be in motion.

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