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Null Values and Null Vectors of Matrix Pencils and their Applications in Linear System Theory

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NULL VALUES AND NULL VECTORS OF MATRIX PENCILS AND THEIR APPLICATIONS IN LINEAR SYSTEM THEORY

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I HEREBY RECOMMEND THAT THE THESIS PREPARED UNDER MY SUPERVISION BY Neel Dalwadi ENTITLED Null Values and Null Vectors of Matrix Pencils and their Applications in Linear System Theory BE ACCEPTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF Master of Science in Engineering.

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ABSTRACT

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Considerable literature exists in linear algebra to solve the generalized eigenvalue, eigenvector problem $(F - \lambda G)\mathbf{v} = \mathbf{0}$ where F, $G \in \mathbb{R}^{s \times s}$, are square matrices. However, a number of applications lend themselves to the case where F, $G \in \mathbb{R}^{s \times t}$, and $s \neq t$. The existing methods cannot be used for such non-square cases. This research explores structural decomposition of a matrix pencil $(F - \lambda G)$, $s \neq t$ to compute finite values of λ for which $\text{rank}(F - \lambda G) < \min(s, t)$. Moreover, from the decomposition of the matrix pencil, information about the order of λ at infinity, the Kronecker row and column indices of a matrix pencil can also be extracted. Equally important is the computation of non-zero vectors $\mathbf{w} \in \mathbb{R}^{1 \times s}$ and $\mathbf{v} \in \mathbb{R}^{t \times 1}$ corresponding to each finite value of λ, such that $\mathbf{w}(F - \lambda G) = \mathbf{0}$ and $(F - \lambda G)\mathbf{v} = \mathbf{0}$. Algorithms are developed for the computation of λ, $\mathbf{w}$, and $\mathbf{v}$ using numerically efficient techniques. Proposed algorithms are applied to problems encountered in system theory and illustrated by means of numerical examples.

Contents

1 Acknowledgement

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2 Introduction

Several applications lend themselves naturally to a differential-algebraic relationship of the form

$$F\mathbf{v} = \lambda G\mathbf{v}, \tag{2.1}$$

where F, $G \in \mathbb{R}^{s\times t}$, $\mathbf{v} \in \mathbb{R}^{t\times 1}$, and λ is the differential operator d/dt for the continuous time system and the forward shift operator for the discrete time system. Application where the above differential-algebraic relation is square i.e. $s = t$, includes:

- Econometric systems, where systems are almost always of the form $\lambda G\mathbf{v} = F\mathbf{v} + M\mathbf{u}$, where λ is a forward shift operator. One such system is observed, when representing a mathematical model of a multisector economy using the Leontief model. The Leontief model of a multisector economy has the following form, [7], [8]

  $$\mathbf{v}(k) = H\mathbf{v}(k) + G[\mathbf{v}(k+1) - \mathbf{v}(k)] + \mathbf{d}(k). \tag{2.2}$$

  At any time k, vector $\mathbf{v}(k) \in \mathbb{R}^{n\times 1}$ represents the levels of production in a sector, vector $\mathbf{d}(k) \in \mathbb{R}^{n\times 1}$ represents the amount of production as per current demand, matrix $H \in \mathbb{R}^{n\times n}$ represents the Leontief input-output matrix, $H\mathbf{v}(k)$ indicates the amount required as an input for current production, matrix $G \in \mathbb{R}^{n\times n}$ represents the capital coefficient matrix, and $G[\mathbf{v}(k+1)-\mathbf{v}(k)]$ indicates amount required for capacity expansion to satisfy the future level of production $\mathbf{v}(k+1)$. Rearranging, the Leontief model can be written as

  $$G\mathbf{v}(k+1) = [I - H + G]\mathbf{v}(k) - \mathbf{d}(k). \tag{2.3}$$

  Defining $F \triangleq [I - H + G]$ then,

  $$G\mathbf{v}(k+1) = F\mathbf{v}(k) - \mathbf{d}(k), \tag{2.4}$$

which is of the desired form for finding the solution of the system. If matrix G is non-singular, the solution for the above system is well defined. However, if G is singular, standard methods for state space modeling do not work.

In equation (2.1), when F, G are square matrices, i.e. $s = t$ and $\text{rank}(G) = s$, the relation is purely a differential one. This relation is known as a generalized eigenvalue and generalized eigenvector problem, which is the problem of computing a scalar λ and a non-zero vector $\mathbf{v}$, such that it satisfies

$$(F - \lambda G)\mathbf{v} = \mathbf{0}, \quad \mathbf{v} \neq \mathbf{0}, \tag{2.5}$$

where $\lambda \in \mathbb{C}$ is called a generalized eigenvalue, $\mathbf{v} \in \mathbb{R}^{s \times 1}$ is called the (right) generalized eigenvector corresponding to the generalized eigenvalue λ, and the matrix pair $(F - \lambda G)$ is called a matrix pencil. If $\text{rank}(F - \lambda G) < s$, $\forall \lambda \in \mathbb{C}$, then the matrix pencil is said to be degenerate, otherwise it is said to be non-degenerate. For the rest of the thesis, matrix pencils are assumed to be non-degenerate. For $s = t$ and $\text{rank}(G) < s$, equation (2.1) becomes a differential-algebraic relation, and the generalized eigenvalues of (2.5) can be a set of finite values of λ, an empty set, or a set of λ at infinity. Generalized eigenvalues of the matrix pencil $(F - \lambda G)$ can be computed using the QZ algorithm [4], where orthogonal transformation matrices $Q \in \mathbb{C}^{s \times s}$ and $Z \in \mathbb{C}^{s \times s}$ are computed such that matrices $Q^T F Z$ and $Q^T G Z$ are quasi-triangular matrices. If the diagonal blocks of these quasi-triangular matrices are $\mathbb{R}^{1 \times 1}$ blocks then the ratio f_{ii}/g_{ii} yields generalized eigenvalues and if the diagonal blocks are $\mathbb{R}^{2 \times 2}$ then using the quadratic formula, complex generalized eigenvalues can be determined. The QZ algorithm is only applied on a square matrix pencil, but when the matrix pencils are non-square it cannot be applied.

Next we look at cases where $s \neq t$. Some applications where non-square matrix pencils are encountered include:

- PID control, where the system of the form

$$\begin{aligned} \lambda G\mathbf{v} &= F\mathbf{v} + M\mathbf{u}, \\ \mathbf{y} &= C\mathbf{v}, \end{aligned} \tag{2.6}$$

  having $\mathbf{v} \in \mathbb{R}^{n\times 1}$ as the state vector, $\mathbf{u} \in \mathbb{R}^{m\times 1}$ as the input vector, $\mathbf{y} \in \mathbb{R}^{p\times 1}$ as the output vector, G, $F \in \mathbb{R}^{n\times n}$ are respectively the descriptor and the state matrix, $M \in \mathbb{R}^{n\times m}$ as the input matrix, and $C \in \mathbb{R}^{p\times n}$ as the output matrix. When state derivative feedback $\mathbf{u} = \lambda K\mathbf{v}$ (see [32]) is applied to state equation in (2.6),

$$\lambda G\mathbf{v} = F\mathbf{v} + \lambda MK\mathbf{v}, \tag{2.7}$$

  which leads to

$$\lambda(G - MK)\mathbf{v} = F\mathbf{v}, \tag{2.8}$$

  where $K \in \mathbb{R}^{m\times n}$. The resulting implicit system is square, but the matrix pencil of the system is non-square

$$\begin{bmatrix} F - \lambda(MK - G) \\ C \end{bmatrix} \mathbf{x} = \begin{bmatrix} \mathbf{0} \\ \mathbf{y} \end{bmatrix}. \tag{2.9}$$

  The case when the matrix G is singular lies within the framework of models being explored in this thesis. In this case, the resulting compensated system has infinite frequency modes corresponding to $\lambda = \infty$.

- Interconnected systems, where the natural description of the equations of sets of systems may lead to equations of type (2.6), [7], [19]. Often, a large interconnected system S is regarded as a collection of sub-systems S_i. Where individual sub-system S_i is of the form

$$\begin{aligned} \lambda G_i\mathbf{v}_i &= F_i\mathbf{v}_i + M_i\mathbf{u}_i, \\ \mathbf{y}_i &= C_i\mathbf{v}_i + D_i\mathbf{u}_i, \end{aligned} \tag{2.10}$$

  with $\mathbf{v}_i \in \mathbb{R}^{n\times 1}$ as the state vector, $\mathbf{u}_i \in \mathbb{R}^{m\times 1}$ as the input vector, $\mathbf{y}_i \in \mathbb{R}^{p\times 1}$ as the output vector, G_i, $F_i \in \mathbb{R}^{n\times n}$ are respectively the descriptor matrix and the state matrix, $M_i \in \mathbb{R}^{n\times m}$ as the input matrix, $C_i \in \mathbb{R}^{p\times n}$ as the output matrix, and $D_i \in \mathbb{R}^{p\times m}$ as the direct feed through matrix. Under the relation of interconnections of these sub-systems with overall input $\mathbf{u}$ and output $\mathbf{y}$ defined as

$$\begin{aligned} \mathbf{u}_i &= -\sum_{t=1}^{N} T^t_{(i)}\mathbf{y}_t + K_i\mathbf{u}, \\ \mathbf{y} &= \sum_{i=1}^{N} L_i\mathbf{y}_i, \end{aligned} \tag{2.11}$$

the interconnected system can be represented as a non-square matrix pencil. This is achieved by redefining the state vector

$$\begin{bmatrix} \lambda G_s - F_s & M_s & O & O \\ -C_s & D_s & I & O \\ O & -I & T & K \\ O & O & -L & O \end{bmatrix} \begin{bmatrix} \mathbf{v}_s \\ -\mathbf{u}_s \\ \mathbf{y}_s \\ -\mathbf{u} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ -\mathbf{y} \end{bmatrix}, \tag{2.12}$$

where $\mathbf{v}_s = [\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_N]^T$, $\mathbf{u}_s = [\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_N]^T$, $\mathbf{y}_s = [\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_N]^T$, $F_s = \text{diag}[F_1, F_2, \ldots, F_N]$, $G_s = \text{diag}[G_1, G_2, \ldots, G_N]$, $M_s = \text{diag}[M_1, M_2, \ldots, M_N]$, $C_s = \text{diag}[C_1, C_2, \ldots, C_N]$, $W_s = \text{diag}[W_1, W_2, \ldots, W_N]$,

$$T = \begin{bmatrix} T_1^{(1)} & T_1^{(2)} & \ldots & T_1^{(N)} \\ T_2^{(1)} & T_2^{(2)} & \ldots & T_2^{(N)} \\ \vdots & \vdots & & \vdots \\ T_N^{(1)} & T_N^{(2)} & \ldots & T_N^{(N)} \end{bmatrix},$$

$K = [K_1, K_2, \ldots, K_N]^T$, and $L = [L_1, L_2, \ldots, L_N]$.

For $s \neq t$, the computation of $\lambda = \{\lambda \in \mathbb{C} : \text{rank}(F - \lambda G) < \min(s,t)\}$ needs special consideration. Several methods exist for computation of such λ for a non-square matrix pencil. The method presented in [14], reduces the computation of λ for a non-square matrix pencil to that of a square matrix pencil. This reduction is accomplished by decomposing a non-square matrix pencil $(F - \lambda G)$ to

$$\begin{bmatrix} F_{11} & 0 & 0 & 0 & 0 \\ 0 & 0 & I_l & 0 & 0 \\ 0 & I_k & 0 & 0 & 0 \\ 0 & 0 & 0 & I_j & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} - \lambda \begin{bmatrix} G_{11} & G_{12} & 0 & 0 & 0 \\ G_{21} & G_{22} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{2.13}$$

Here, $(F_{11} - \lambda G_{11})$ is a square matrix pencil and provides information about the finite λ which satisfies the rank condition $\mathrm{rank}(F - \lambda G) < \min(s, t)$. However, this method does not provide any information regarding the values of λ at infinity and the singularities of the matrix pencil, which may be important in the application like PID control as detailed above. In [27] and the references therein, treats the problem as an optimization problem. A minimal perturbation approach is applied on matrix pencil $(F - \lambda G)$, F, $G \in \mathbb{R}^{s \times t}$, extracted from some matrix $H \in \mathbb{R}^{(s+1)\times t}$ to formulate an optimization problem as

$$\begin{aligned} &\min_{\{F_0, G_0 \{\lambda_k, v_k\}_{k=1}^{p}\}} \|F_0 - F\|_F^2 + \|G_0 - G\|_F^2, \\ &\text{subject to:} \\ &\left\{ \begin{aligned} &(F_0 - \lambda_k G_0) v_k = 0 \\ &\|v_k\|_2 = 1 \end{aligned} \right\}, k = 1, \dots, p. \end{aligned} \tag{2.14}$$

Where the matrix F is a result of removing the first row of the matrix H and matrix G is a result of removing the last row of the matrix H. The finite λ_k of $F_0 - \lambda_k G_0$ are known, using this information finite λ of a matrix pencil $(F - \lambda G)$ are computed. The optimization process fails to estimate the finite λ when subjected to strong perturbations. Moreover, this method can only be applied to specially structured matrix pencils and does not provide a basis for computation of λ for general matrix pencil. The basic idea followed in [34], is to determine λ by computing pseudoinverse of a matrix and transform the matrix pencil into a standard eigenvalue problem form

$$\begin{aligned} G^{+} F \mathbf{v} &= \lambda G^{+} G \mathbf{v}, \\ G^{+} F \mathbf{v} &= \lambda I \mathbf{v}, \end{aligned} \tag{2.15}$$

where $G^{+} = V_0 \Sigma^{-1} U_0^H$, $\Sigma = \mathrm{diag}[\sigma_1, \sigma_2, \dots, \sigma_M]$, $\sigma_i, i = 1, \dots, N$ are singular values of G, $V_0 = [\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_M]$ and $U_0 = [\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_M]$, $\mathbf{v}_i$ and $\mathbf{u}_i$ are the singular vectors, the superscript $+$ represents the pseudoinverse and superscript H represents the conjugate transpose. This method provides an estimate of λ rather than the actual value, as it is characterized by singular value decomposition. Since values of λ obtained are the estimate of actual values, the exact non-zero vectors satisfying $\mathbf{w}(F - \lambda G) = \mathbf{0}$ and $(F - \lambda G)\mathbf{v}$ cannot be determined. Another well know approach is to compute a Kronecker canonical form of a matrix pencil. It is shown in [30], that it is possible to compute orthogonal transformation matrices $Q \in \mathbb{R}^{s \times s}$ and $Z \in \mathbb{R}^{t \times t}$, such that the matrix pencil is decomposed to the Kronecker canonical form, from which finite λ can be determined. Kronecker canonical form of a matrix pencils has the following form

$$Q^T(F - \lambda G)Z = \left[\begin{array}{c|c|c} F^r - \lambda G^r & X & X \\ \hline O & F^f - \lambda G^f & X \\ \hline O & O & F^c - \lambda G^c \end{array}\right], \tag{2.16}$$

where

- matrix pencil $(F^f - \lambda G^f)$ contains the information regarding finite elementary divisors,
- matrix pencil $(F^r - \lambda G^r)$ and $(F^c - \lambda G^c)$ contains information regarding the Kronecker row, column indices and order of infinite elementary divisors, respectively.

the above method indicates that the problem of finding the eigenstructure of first order matrix pencil $(F - \lambda G)$ is reduced to computation of Kronecker canonical form.

Existing methods as discussed above mainly focus on square matrix pencils, or specially structured matrix pencil, or computes an estimation of the actual λ. Hence, these methods do not provide a general framework for non-square case, or cannot provide complete information about singularities and the non-zero vectors. In this research, we provide a comprehensive computational method for the computation of scalar λ, non-zero vectors $\mathbf{w}$ and $\mathbf{v}$ for a non-square matrix pencil $(F - \lambda G)$, F, $G \in \mathbb{R}^{s \times t}$, such that $\text{rank}(F - \lambda G) < min(s, t)$, $\mathbf{w}(F - \lambda G) = \mathbf{0}$ and $(F - \lambda G)\mathbf{v}$. The proposed framework provides a comprehensive computation approach which addresses the issues noted above for both square as well as non-square matrix pencils. In the later chapter, it is also shown how this framework is used to handle higher order matrix pencils, supported by a practical example.

The following discussion provides the basic computational principle implemented in this research. For all cases when the matrix G is singular, it is always possible to compress the rows and columns of G as

$$Q^T(F-\lambda G)Z = \begin{bmatrix} F_{11}-\lambda G_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix}, \tag{2.17}$$

where $F_{11},\ G_{11} \in \mathbb{R}^{n\times n},\ F_{12} \in \mathbb{R}^{n\times m},\ F_{21} \in \mathbb{R}^{p\times n},\ F_{22} \in \mathbb{R}^{p\times m},\ s = n+p$, and $t = n+m$. Here, the matrix pencil $(F_{11}-\lambda G_{11})$ is non-degenerate. The equation (2.5) can be rewritten as

$$\begin{bmatrix} F_{11}-\lambda G_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \end{bmatrix}, \tag{2.18}$$

$$\begin{aligned} (F_{11}-\lambda G_{11})\mathbf{x}_1 + F_{12}\mathbf{x}_2 &= \mathbf{0}, \\ F_{21}\mathbf{x}_1 + F_{22}\mathbf{x}_2 &= \mathbf{0}. \end{aligned} \tag{2.19}$$

Equation (2.19) shows that the pure differential relation gets converted to differential-algebraic relation when matrix G is singular. For simplicity of notation, we define

$$E \triangleq G_{11},\ \ A \triangleq F_{11},\ \ B \triangleq F_{12},\ \ C \triangleq F_{21},\ \ D \triangleq F_{22}, \tag{2.20}$$

then

$$\begin{bmatrix} F_{11}-\lambda G_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} = \begin{bmatrix} A-\lambda E & B \\ C & D \end{bmatrix}. \tag{2.21}$$

The value of λ for which the matrix pencil (2.21) loses its maximum rank, is of considerable interest, i.e., λ such that,

$$\text{rank} \begin{bmatrix} A - \lambda E & B \\ C & D \end{bmatrix} < n + \min(m, p). \tag{2.22}$$

The computational principle is to reduce the non-square matrix pencil $(F - \lambda G)$ to a strictly equivalent pencil in (2.22) where the embedded sub-pencil $(A - \lambda E)$ is square and non-singular. Then, λ satisfying the rank condition in (2.22) can be computed. The reduction process implemented to extract a square and non-singular sub-pencil from non-square, singular matrix pencil is based on the algorithms developed in [29], where algorithms are developed to compute the zero of a generalized state space model. Similar computation for a standard state space system $E = I_n$ is developed in [1].

2.1 Thesis outline

- Chapter 3: This chapter discusses mathematical concepts utilized to facilitate computation of null values and null vectors. The topics included are the theory of matrix pencil; QR and QZ matrix decompositions; QR algorithm, QZ algorithm to iteratively compute eigenvalues and generalized eigenvalues; back substitution to solve a system algebraic equations in an echelon form; and discussion about the generalized state space representation of a linear system.

- Chapter 4: This chapter provides the formal definition and characterization of *null values* (λ) and *null vectors* ($\mathbf{v}$ and $\mathbf{w}$) of a non-square matrix pencil, which satisfy the following conditions:

$$\begin{aligned} (F - \lambda G)\mathbf{v} = \mathbf{0}, &\quad \mathbf{v} \neq \mathbf{0}, \\ \mathbf{w}(F - \lambda G) = \mathbf{0}, &\quad \mathbf{w} \neq \mathbf{0}. \end{aligned} \tag{2.23}$$

  Computational algorithms are developed to compute *null values* and *null vectors* of a non-square matrix pencil. Further, it is shown how *null values* of a non-square matrix pencil are related with the transmission zeros of a generalized state space system [29].

- Chapter 5: This chapter explores applications of presented algorithms in linear systems. These include computation of zeros of systems with input derivatives (leading to a non-standard state space representation), computation of zero-directions of linear state space systems.

3 Mathematical Preliminaries

3.1 Matrix Pencils

A matrix pair expressed as

$$F - \lambda G, \tag{3.1}$$

where matrices F, $G \in \mathbb{R}^{s\times t}$ and $\lambda \in \mathbb{C}$, is known as a matrix pencil. The matrices F, G may or may not have full row/column rank. If $s = t$, then there exists a non-zero vector $\mathbf{v} \in \mathbb{R}^{s\times 1}$ corresponding to the scalar λ which satisfies

$$(F - \lambda G)\mathbf{v} = \mathbf{0}, \tag{3.2}$$

then the scalar λ is known as the generalized eigenvalue and the non-zero vector $\mathbf{v}$ is known as the (right) generalized eigenvector of the matrix pencil.

Consider a square matrix pencil $(F - \lambda G)$, F, $G \in \mathbb{R}^{s\times s}$, then the generalized eigenvalues can be computed by finding roots of the polynomial defined by

$$\det(F - \lambda G), \quad \lambda \in \mathbb{C}. \tag{3.3}$$

Note that, if $\det(F - \lambda G) = 0$ uniformly, then the matrix pencil is said to be degenerate. In the sequel, we assume the matrix pencils are non-degenerate.

Corresponding to each finite generalized eigenvalue of the matrix pencil $(F - \lambda G)$, F, $G \in \mathbb{R}^{s\times s}$, there exists a right generalized eigenvector as defined in (3.2). Similarly,

$$\mathbf{w}(F - \lambda G) = \mathbf{0}, \tag{3.4}$$

where $\mathbf{w} \in \mathbb{R}^{1\times s}$, is a non-zero vector known as the left generalized eigenvector. The left generalized eigenvector can also be found by computing the right generalized eivenvector of the transposed matrix pencil as

$$(F - \lambda G)^T \mathbf{w}^T = \mathbf{0}. \tag{3.5}$$

Based on the dimensions and properties of the matrices in the matrix pencil $(F - \lambda G)$, the following definitions hold true [12], [13]:

Definition 3.1 (Non-Degenerate Matrix Pencil)**.** A matrix pencil $(F - \lambda G)$ is called *non-degenerate* if,

- $\det(F - \lambda G) \neq 0$, for atleast one value of $\lambda \in \mathbb{C}$.

Definition 3.2 (Degenerate Matrix Pencil)**.** A matrix pencil $(F - \lambda G)$ is called *degenerate* if,

- $\det(F - \lambda G) = 0, \forall \lambda \in \mathbb{C}$.

Definition 3.3 (Strict Equivalence of Matrix Pencils)**.** Two matrix pencils $(F - \lambda G)$ and $(F_1 - \lambda G_1)$, F, F_1, G, $G_1 \in \mathbb{R}^{s \times t}$, related by orthogonal transformation matrices $Q \in \mathbb{R}^{s \times s}$ and $Z \in \mathbb{R}^{t \times t}$,

$$Q^T (F - \lambda G) Z = F_1 - \lambda G_1, \tag{3.6}$$

are said to be *strictly equivalent.*

Strictly equivalent matrix pencils $(F - \lambda G)$ and $(F_1 - \lambda G_1)$, have the same finite λ such that

$$\operatorname{rank}(F - \lambda G) = \operatorname{rank}(F_1 - \lambda G_1). \tag{3.7}$$

They also have same order of λ at infinity and Kronecker row and column indices. Kronecker row and column indices are discussed later in this chapter.

3.2 Numerical Linear Algebra

QR Decomposition [13], [20], [33]

Given a matrix $A \in \mathbb{R}^{s\times t}$, the QR decomposition computes an orthogonal matrix $Q \in \mathbb{R}^{s\times s}$ such that $Q^T A = R$. If $s = t$ and $\mathrm{rank}(A) = s$, then the matrix R is an upper triangular matrix.

$$R = \begin{bmatrix} \times & \times & \times & \dots & \times & \times \\ 0 & \times & \times & \dots & \times & \times \\ 0 & 0 & \times & \dots & \times & \times \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \times & \times \\ 0 & 0 & 0 & \dots & 0 & \times \end{bmatrix}. \tag{3.8}$$

If $s > t$ and $\mathrm{rank}(A) = t$, then the matrix R will have the following structure

$$R = \left[\begin{array}{c} R_1 \\ \hline O \end{array}\right] = \left[\begin{array}{cccccc} \times & \times & \times & \dots & \times & \times \\ 0 & \times & \times & \dots & \times & \times \\ 0 & 0 & \times & \dots & \times & \times \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \times & \times \\ 0 & 0 & 0 & \dots & 0 & \times \\ \hline 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0 \end{array}\right], \tag{3.9}$$

where matrix $R_1 \in \mathbb{R}^{t\times t}$ is an upper triangular matrix and matrix $O \in \mathbb{R}^{(s-t)\times t}$ is a zero matrix.

If $s > t$ and $\mathrm{rank}(A) = r < t$, then the matrix R will have the following structure

$$
R = \left[\begin{array}{c} R_1 \\ \hline O \end{array}\right] = \left[\begin{array}{cccccc}
\times & \times & \times & \dots & \times & \times \\
0 & \times & \times & \dots & \times & \times \\
0 & 0 & \times & \dots & \times & \times \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \dots & \times & \times \\
\hline
0 & 0 & 0 & \dots & 0 & 0 \\
0 & 0 & 0 & \dots & 0 & 0 \\
\vdots & \vdots & \vdots & & \vdots & \vdots \\
0 & 0 & 0 & \dots & 0 & 0
\end{array}\right], \tag{3.10}
$$

where matrix $R_1 \in \mathbb{R}^{r \times t}$ is a trapezoidal matrix and matrix $O \in \mathbb{R}^{(s-r) \times t}$ is a zero matrix. The orthogonal transformation matrix Q can be used to compress the rows of a matrix and the transformation is called *row compression*.

It is also possible to transform a matrix $A \in \mathbb{R}^{s \times t}$, $s < t$, to a column compressed matrix using transformation of the form $AZ = R$. Let $\operatorname{rank}(A) = s$, then the matrix R will have the following structure

$$
R = \left[\begin{array}{c|c} O & R_1 \end{array}\right] = \left[\begin{array}{ccc|cccccc}
0 & \dots & 0 & \times & \times & \dots & \times & \times & \times \\
0 & \dots & 0 & 0 & \times & \dots & \times & \times & \times \\
\vdots & & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & \dots & 0 & 0 & 0 & \dots & \times & \times & \times \\
0 & \dots & 0 & 0 & 0 & \dots & 0 & \times & \times \\
0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & \times
\end{array}\right], \tag{3.11}
$$

where matrix $R_1 \in \mathbb{R}^{s \times s}$ is a upper triangular matrix and matrix $O \in \mathbb{R}^{s \times (t-s)}$ is a zero matrix.

If $s < t$ and $\operatorname{rank}(A) = r < s$, then the matrix R will have the following structure

$$
R = \left[\begin{array}{c|c} O & R_1 \end{array}\right] = \left[\begin{array}{cccc|ccccc}
0 & \dots & 0 & 0 & \times & \dots & \times & \times & \times \\
0 & \dots & 0 & 0 & \times & \dots & \times & \times & \times \\
\vdots & & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & \dots & 0 & 0 & 0 & \dots & \times & \times & \times \\
0 & \dots & 0 & 0 & 0 & \dots & 0 & \times & \times \\
0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & \times
\end{array}\right], \tag{3.12}
$$

where matrix $R_1 \in \mathbb{R}^{s \times r}$ is a upper trapezoidal matrix and matrix $O \in \mathbb{R}^{s \times (t-r)}$ is a zero matrix. The transformation with the orthogonal transformation matrix Z, which compresses the columns of a matrix is called *column compression*.

QZ Decomposition [13]

Given a matrix pencil $(F - \lambda G)$, $F, G \in \mathbb{R}^{s\times s}$, there exist orthogonal transformation matrices $Q \in \mathbb{R}^{s\times s}$ and $Z \in \mathbb{R}^{s\times s}$, such that $Q^T FZ \in \mathbb{R}^{s\times s}$ is an upper Hessenberg matrix and $Q^T GZ \in \mathbb{R}^{s\times s}$ is an upper triangular matrix. The transformed matrices $Q^T FZ$ and $Q^T GZ$ have the following structure

$$Q^T FZ = \begin{bmatrix} \times & \times & \dots & \times & \times & \times \\ \times & \times & \dots & \times & \times & \times \\ 0 & \times & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \times & \times & \times \\ 0 & 0 & \dots & \times & \times & \times \\ 0 & 0 & \dots & 0 & \times & \times \end{bmatrix}, \quad Q^T GZ = \begin{bmatrix} \times & \times & \dots & \times & \times & \times \\ 0 & \times & \dots & \times & \times & \times \\ 0 & 0 & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & & \times & \times & \times \\ 0 & 0 & \dots & 0 & \times & \times \\ 0 & 0 & \dots & 0 & 0 & \times \end{bmatrix}. \tag{3.13}$$

The upper Hessenberg matrix $Q^T FZ$ obtained may have all its sub-diagonal elements as non-zero. Such a matrix is known as an unreduced Hessenberg matrix. Moreover, if $Q^T FZ$ has at least one sub-diagonal element as zero, then it is known as a reduced Hessenberg matrix.

QR Algorithm [13], [10], [20]

The QR algorithm is an iterative algorithm, which makes use of the QR decomposition in each step to compute all eigenvalues of a matrix. Given a matrix $A \in \mathbb{R}^{s\times s}$, the QR algorithm can be divided into the following steps:

- **Step 1:** Matrix A is reduced to an upper Hessenberg matrix $A^{(0)} \in \mathbb{R}^{s\times s}$.

$$A^{(0)} = \begin{bmatrix} \times & \times & \dots & \times & \times & \times \\ \times & \times & \dots & \times & \times & \times \\ 0 & \times & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \times & \times & \times \\ 0 & 0 & \dots & \times & \times & \times \\ 0 & 0 & \dots & 0 & \times & \times \end{bmatrix}. \tag{3.14}$$

- **Step 2:** This is an iterative step, which overwrites the upper Hessenberg matrix $A^{(0)}$ with $A^{(i+1)} = (R^{(i)}Q^{(i)} + \mu^{(i)}I)$, where $Q^{(i)}R^{(i)} = (A^{(i)} - \mu^{(i)}I)$ is the QR decomposition of the matrix $A^{(i)}$ with a single shift $\mu^{(i)}$. The shift is a scalar, which takes the value $\mu^{(i)} = A^{(i)}_{ss}$. This iteration is carried out until the matrix $A^{(i+1)}$ converges to an upper quasi-triangular matrix. Note that there are considerably more advanced methods using implicit and double implicit shifts to accelerate the convergence.

- At the end of the process QR algorithm produces the transformation matrix $Q \in \mathbb{R}^{s\times s}$, such that the transformation $Q^T AQ$ on the original matrix A yields

$$Q^T AQ = \begin{bmatrix} A_{11} & A_{12} & \dots & A_{1s} \\ 0 & A_{22} & \dots & A_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & A_{kk} \end{bmatrix}. \tag{3.15}$$

  Sub-matrix A_{ii}, $\sum_1^k i = s$, is either a $\mathbb{R}^{1\times 1}$, corresponding to the real eigenvalues or a $\mathbb{R}^{2\times 2}$, corresponding to the complex conjugate eigenvalues of the matrix A. The resulting matrix $Q^T AQ$ is called a real Schur matrix.

QZ Algorithm [13], [4]

The QZ algorithm computes the generalized eigenvalues of a matrix pencil $(F - \lambda G)$, F, $G \in \mathbb{R}^{s\times s}$. Given a matrix pencil $(F - \lambda G)$, the QZ algorithm can be divided into the following steps:

- **Step 1:** The QZ decomposition is applied on the matrices F and G, which overwrites the matrix F with an upper Hessenberg matrix $Q^T FZ$ and the matrix

G with an upper triangular matrix $Q^T GZ$. Both the transformation matrices $Q \in \mathbb{R}^{s\times s}$ and $Z \in \mathbb{R}^{s\times s}$ are orthogonal.

- **Step 2:** If the matrix F obtained from the previous step is an unreduced upper Hessenberg matrix, then the matrix F is overwritten with an upper Hessenberg matrix $Q^T FZ$, and G with an upper triangular matrix $Q^T GZ$. Both transformation matrices $Q \in \mathbb{R}^{s\times s}$ and $Z \in \mathbb{R}^{s\times s}$ are orthogonal.

  If matrix F is a reduced Hessenberg matrix, then the matrices F and G can be partitioned as

$$F = \begin{bmatrix} F_1 & F_2 \\ O & F_4 \end{bmatrix}, \quad G = \begin{bmatrix} G_1 & G_2 \\ O & G_4 \end{bmatrix}, \tag{3.16}$$

  where the sub-matrices F_1 and F_4 are both unreduced Hessenberg matrix. The sub-matrices G_1 and G_4 are both upper triangular matrix. The matrix pencils $(F_1 - \lambda G_1)$ and $(F_4 - \lambda G_4)$ are treated as individual matrix pencil and operated with the steps 2 and 3, to determine the generalized eigenvalues of the matrix pencil $(F - \lambda G)$.

- **Step 3:** The orthogonal matrices $Q \in \mathbb{R}^{s\times s}$ and $Z \in \mathbb{R}^{s\times s}$ are computed such that $Q^T FZ$ is an upper quasi-triangular matrix and $Q^T GZ$ is an upper triangular matrix.

$$Q^T FZ = \begin{bmatrix} F_{11} & F_{12} & \dots & F_{1k} \\ 0 & F_{22} & \dots & F_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & F_{kk} \end{bmatrix}, \quad Q^T GZ = \begin{bmatrix} G_{11} & G_{12} & \dots & G_{1k} \\ 0 & G_{22} & \dots & G_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & G_{kk} \end{bmatrix}. \tag{3.17}$$

  The matrices F_{ii} and G_{ii}, $\sum_1^k i = s$, are either a $\mathbb{R}^{1\times 1}$, corresponding to the real generalized eigenvalues or a $\mathbb{R}^{2\times 2}$, corresponding to the complex conjugate generalized eigenvalues.

Row Echelon Form of a Matrix

As an illustration, given a matrix $A \in \mathbb{R}^{3\times 5}$ and $\text{rank}(A) = 2$, there exists an invertible transformation matrix $T \in \mathbb{R}^{3\times 3}$, such that TA can be reduced to a form

$$TA = \begin{bmatrix} \otimes & \times & \times & \times & \times \\ 0 & 0 & \otimes & \times & \times \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{3.18}$$

This form is called a row echelon form of the matrix. $\otimes$ are pivot elements, the first non-zero element in a row. If the matrix A is a full row rank matrix, then its row echelon form may have a form

$$TA = \begin{bmatrix} \otimes & \times & \times & \times & \times \\ 0 & 0 & \otimes & \times & \times \\ 0 & 0 & 0 & \otimes & \times \end{bmatrix}. \tag{3.19}$$

Back Substitution [5], [13]

Let the matrix equation

$$A\mathbf{x} = \mathbf{b}, \tag{3.20}$$

represent a system of linear equations.

- The matrix $A \in \mathbb{R}^{s\times s}$ consist of coefficients of a system of linear equations, is a non-singular and an upper triangular matrix.
- The vector $\mathbf{x} \in \mathbb{R}^{s\times 1}$ consist of unknown variables.
- The vector $\mathbf{b} \in \mathbb{R}^{s\times 1}$ consist of constant terms.

The elements of vector $\mathbf{x}$ can be determined using the back substitution algorithm.

Algorithm: Back Substitution

for $i = s, \ldots, 1$

$x_i = b_i$

for $j = i+1, \ldots, s$

$x_i := x_i - a_{ij} x_j$

end

$x_i := x_i / a_{ii}$

end

3.3 Structural Decomposition of Matrix Pencils

For any arbitrary matrix pencil $(F - \lambda G)$, F, $G \in \mathbb{R}^{s \times t}$, there exist orthogonal transformation matrices $Q \in \mathbb{R}^{s \times s}$ and $Z \in \mathbb{R}^{t \times t}$, such that it yields a block triangular form, [30].

$$Q^T (F - \lambda G) Z = \left[\begin{array}{c|c|c} F^r - \lambda G^r & X & X \\ \hline O & F^f - \lambda G^f & X \\ \hline O & O & F^c - \lambda G^c \end{array} \right]. \tag{3.21}$$

- The matrices F^f, $G^f \in \mathbb{R}^{g \times g}$ and G^f is invertible. The finite generalized eigenvalues can be computed using the QZ algorithm on the matrix pencil $(F - \lambda G)$.

- The matrix pencil $(F^c - \lambda G^c)$ has the following structure

$$\begin{bmatrix} F^c_{l,l} & F^c_{l,l-1} - \lambda G^c_{l,l-1} & \ldots & F^c_{l,2} - \lambda G^c_{l,2} & F^c_{l,1} - \lambda G^c_{l,1} \\ O & F^c_{l-1,l-1} & \ldots & F^c_{l-1,2} - \lambda G^c_{l-1,2} & F^c_{l-1,1} - \lambda G^c_{l-1,1} \\ O & O & \ldots & F^c_{l-2,2} - \lambda G^c_{l-2,2} & F^c_{l-2,1} - \lambda G^c_{l-2,1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ O & O & & F^c_{2,2} & F^c_{2,1} - \lambda G^c_{2,1} \\ O & O & \ldots & O & F^c_{1,1} \end{bmatrix} \tag{3.22}$$

  where the matrices $F^c_{i,i}$ has full column rank μ_i and matrices $G^c_{i+1,i}$ has full row rank τ_{i+1}. There are $d_i = \mu_i - \tau_{i+1}$ infinite elementary divisor of degree i and $(i = 1, \ldots, l)$, $r_i = \tau_i - \mu_i$ Kronecker row indices of size $(i-1)$, $(i = 1, \ldots, l)$.

- The matrix pencil $(F^r - \lambda G^r)$ has the following structure

$$\begin{bmatrix} F^r_{1,1} & F^r_{1,2} - \lambda G^r_{1,2} & \dots & F^r_{1,k-1} - \lambda G^r_{1,k-1} & F^r_{1,k} - \lambda G^r_{1,k} \\ O & F^r_{2,2} & \dots & F^r_{2,k-1} - \lambda G^r_{2,k-1} & F^r_{2,k} - \lambda G^r_{2,k} \\ O & O & \dots & F^r_{k-2,k-2} - \lambda G^r_{k-2,k-2} & F^r_{k-2,k} - \lambda G^r_{k-2,k} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ O & O & & F^r_{k-1,k-1} & F^r_{k-1,k-1} - \lambda G^r_{k-1,k-1} \\ O & O & \dots & O & F^r_{k,k} \end{bmatrix} \tag{3.23}$$

where the matrices $F^r_{i,i}$ has full row rank $\hat{\mu}_i$ and matrices $G^r_{i+1,i}$ has full column rank $\hat{\tau}_{i+1}$. There are $c_i = \hat{\tau}_i - \hat{\mu}_i$ Kronecker column indices of size $(i-1)$, $(i = 1, \dots, k)$.

3.4 Linear Systems

Generalized State Space Representation

The generalized state space representation is a system of first order differential equations, in terms of internal variables called state variables, along with a set of algebraic equations that combine state variables into physical outputs variables.

For continuous time systems the generalized state space is represented as

$$\begin{aligned} E\dot{\mathbf{x}}(t) &= A\mathbf{x}(t) + B\mathbf{u}(t), \\ \mathbf{y}(t) &= C\mathbf{x}(t) + D\mathbf{u}(t). \end{aligned} \tag{3.24}$$

For discrete time systems the generalized state space is represented as

$$\begin{aligned} E\mathbf{x}(k+1) &= A\mathbf{x}(k) + B\mathbf{u}(k), \\ \mathbf{y}(k) &= C\mathbf{x}(k) + D\mathbf{u}(k). \end{aligned} \tag{3.25}$$

Define λ as the differential operator d/dt for continuous time system (3.24) and the forward shift operator for discrete time system (3.25). Using these notation we can generalize the equations (3.24) and (3.25) as

$$\begin{aligned} \lambda E\mathbf{x} &= A\mathbf{x} + B\mathbf{u}, \\ \mathbf{y} &= C\mathbf{x} + D\mathbf{u}. \end{aligned} \tag{3.26}$$

- $\mathbf{x} \in \mathbb{R}^{n\times 1}$ is called the state vector.
- $\mathbf{y} \in \mathbb{R}^{p\times 1}$ is called the output vector.
- $\mathbf{u} \in \mathbb{R}^{m\times 1}$ is called the input vector.
- $E \in \mathbb{R}^{n\times n}$ is called the descriptor matrix.
- $A \in \mathbb{R}^{n\times n}$ is called the state matrix.
- $B \in \mathbb{R}^{n\times m}$ is called the input matrix.
- $C \in \mathbb{R}^{p\times n}$ is called the output matrix.
- $D \in \mathbb{R}^{p\times m}$ is called the direct feed through matrix.

The system in (3.26) can be compactly represented as 5-tuple (E, A, B, C, D).

Definition 3.4 (Square and Non-Square System)**.** If $m = p$, the system is called a *square system*. However if $m \neq p$, then the system is called a *non-square system*.

Definition 3.5 (Singular and Non-Singular System)**.** If the matrix E is singular, then the system is called a *singular system*. However, if the matrix E is non-singular, then the system is called a *non-singular system*.

The generalized state space system (3.26), can also be represented as

$$\begin{bmatrix} A - \lambda E & B \\ C & D \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{u} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{y} \end{bmatrix}, \tag{3.27}$$

then, the matrix

$$\begin{bmatrix} A-\lambda E & B \\ C & D \end{bmatrix}, \tag{3.28}$$

is called the system matrix. The generalized state space system can now be related to the matrix pencil $(F-\lambda G)$ by defining matrices F and G as

$$F \triangleq \begin{bmatrix} A & B \\ C & D \end{bmatrix} \quad \text{and} \quad G \triangleq \begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix}, \tag{3.29}$$

where F, $G \in \mathbb{R}^{(n+p)\times(n+m)}$, leading to the matrix pencil

$$F-\lambda G = \begin{bmatrix} A-\lambda E & B \\ C & D \end{bmatrix}, \tag{3.30}$$

with F, $G \in \mathbb{R}^{s\times t}$ and $s = n+p$, $t = n+m$.

Zeros of System [11], [17], [3]

Consider a special case of the matrix pencil $(F-\lambda G)$ representing a system

$$F = \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix}, \quad G = \begin{bmatrix} G_{11} & 0 \\ 0 & 0 \end{bmatrix}, \tag{3.31}$$

where $F_{11}, G_{11} \in \mathbb{R}^{n\times n}$, $F_{12} \in \mathbb{R}^{n\times m}$, $F_{21} \in \mathbb{R}^{p\times n}$, $F_{22} \in \mathbb{R}^{p\times m}$ and

$$F-\lambda G = \begin{bmatrix} F_{11}-\lambda G_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix}, \tag{3.32}$$

representations in (3.31) and (3.29) are identical if we let

$$G_{11} = E, \;\; F_{11} = A, \;\; F_{12} = B, \;\; F_{21} = C, \;\; F_{22} = D. \tag{3.33}$$

using these notations, we can define zeros of a system as:

Definition 3.6 (Input Decoupling Zero)**.** Values of λ for which

$$\operatorname{rank}\begin{bmatrix} A-\lambda E & B \end{bmatrix} < n, \tag{3.34}$$

are called the *input decoupling zeros* of the system.

Definition 3.7 (Output Decoupling Zero)**.** Values of λ for which

$$\operatorname{rank} \begin{bmatrix} A - \lambda E \\ C \end{bmatrix} < n, \tag{3.35}$$

are called the *output decoupling zeros* of the system.

Definition 3.8 (Transmission Zero)**.** If the 5-tuple (E, A, B, C, D) does not have any input and output decoupling zeros, then values of λ for which

$$\operatorname{rank} \begin{bmatrix} A - \lambda E & B \\ C & D \end{bmatrix} < n + \min(m, p), \tag{3.36}$$

are called the *transmission zeros* of the system.

4 Main Results

This chapter provides details on definition, characterization and computation of *null values* and *null vectors* of a non-square matrix pencil $(F - \lambda G)$, F, $G \in \mathbb{R}^{s\times t}$.

4.1 Characterization of Null Values

Definition 4.1 (Null Values)**.** Values of $\lambda \in \mathbb{C}$ of a matrix pencil $(F - \lambda G)$, F, $G \in \mathbb{R}^{s\times t}$ for which

$$\operatorname{rank}(F - \lambda G) < \min(s, t), \tag{4.1}$$

are *null values* of the matrix pencil.

Let two matrix pencil $(F - \lambda G)$ and $(F_1 - \lambda G_1)$ be strictly equivalent such that $F_1 = Q^T F Z$ and $G_1 = Q^T G Z$, where the matrices $Q \in \mathbb{R}^{s\times s}$ and $Z \in \mathbb{R}^{t\times t}$ are orthogonal. If there exist finite null values λ of the matrix pencil $(F - \lambda G)$, then following is true:

$$\operatorname{rank}(F - \lambda G) = \operatorname{rank}(F_1 - \lambda G_1) < \min(s, t). \tag{4.2}$$

It is possible to partition a matrix pencil $(F - \lambda G)$ by compressing the rows and columns of matrix G as

$$Q^T(F - \lambda G)Z = \begin{bmatrix} F_{11} - \lambda G_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix}, \tag{4.3}$$

where $F_{11}, G_{11} \in \mathbb{R}^{n\times n}$, $F_{12} \in \mathbb{R}^{n\times m}$, $F_{21} \in \mathbb{R}^{p\times n}$, $F_{22} \in \mathbb{R}^{p\times m}$, $s = n+p$ and $t = n+m$. The matrix G_{11} can possibly be singular. Define

$$E \triangleq G_{11},\ \ A \triangleq F_{11},\ \ B \triangleq F_{12},\ \ C \triangleq F_{21},\ \ D \triangleq F_{22}. \tag{4.4}$$

then the rank condition stated in (4.2) can be rewritten as

$$\operatorname{rank}\begin{bmatrix} A-\lambda E & B \\ C & D \end{bmatrix} < n + \min(m,p), \tag{4.5}$$

where matrix pencil $(A-\lambda E)$ is non-degenerate. Based on the definition of null value and the relation of a matrix pencil with generalized state space system, the following can be stated:

- *Null values* of the matrix pencil (4.5) are the values of $\lambda \in \mathbb{C}$ for which the matrix pencil $(F - \lambda G)$ has rank less than $(\min(s,t))$.
- The matrices of a matrix pencil as defined in (4.4), the null values are same as the transmission zeros of a 5-tuple (E, A, B, C, D) described as

$$\begin{aligned} \lambda E\mathbf{x} &= A\mathbf{x} + B\mathbf{u}, \\ \mathbf{y} &= C\mathbf{x} + D\mathbf{u}. \end{aligned} \tag{4.6}$$

4.2 Characterization of Null Vectors

Definition 4.2 (Left Null Vector)**.** Corresponding to each finite null value $\lambda \in \mathbb{C}$ of the matrix pencil $(F-\lambda G)$, F, $G \in \mathbb{R}^{s\times t}$, there exists a left non-zero vector satisfying

$$\mathbf{w}(F - \lambda G) = \mathbf{0}. \tag{4.7}$$

This non-zero vector $\mathbf{w} \in \mathbb{R}^{1\times s}$ is *left null vector.*

Definition 4.3 (Right Null Vector)**.** Corresponding to each finite null value $\lambda \in \mathbb{C}$ of the matrix pencil $(F - \lambda G)$, F, $G \in \mathbb{R}^{s\times t}$, there exists a right non-zero vector satisfying

$$(F - \lambda G)\mathbf{v} = \mathbf{0}. \tag{4.8}$$

This non-zero vector $\mathbf{v} \in \mathbb{R}^{t\times 1}$ is *right null vector.*

4.3 Computation of Null Values

This section presents algorithms to compute finite null values of a non-square matrix pencil. The computation is based on the algorithms developed in [29], which computes transmission zeros of a 5-tuple (E, A, B, C, D).

Based on the relation established in (4.3) and (4.4), a matrix pencil $(F - \lambda G)$, F, $G \in \mathbb{R}^{s\times t}$ can be represented as

$$\begin{bmatrix} A - \lambda E & B \\ C & D \end{bmatrix} \triangleq \begin{bmatrix} F_{11} - \lambda G_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix}, \tag{4.9}$$

where A, $E \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$, and $D \in \mathbb{R}^{p\times m}$. The matrix E may be singular. Moreover, n, m, and p could be the dimensions based physical characteristics of the underlying physical problem. Details on algorithms to compute null values of a matrix pencil are provided next.

Compressed Matrix Pencil

First the the rows and columns of the matrix E are compressed using orthogonal transformation matrices $U \in \mathbb{R}^{n\times n}$ and $V \in \mathbb{R}^{n\times n}$ such that

$$U^T E V = \left[\begin{array}{c|c} E_{11} & O \\ \hline O & O \end{array}\right], \tag{4.10}$$

where, the sub-matrix E_{11} is invertible. By system equivalence transformation the matrix pencil is partitioned as

$$\left[\begin{array}{c|c} U^T & O \\ \hline O & I \end{array}\right] \left[\begin{array}{c|c} A - \lambda E & B \\ \hline C & D \end{array}\right] \left[\begin{array}{c|c} V & O \\ \hline O & I \end{array}\right] = \left[\begin{array}{c:c|c} A_{11} - \lambda E_{11} & A_{12} & B_1 \\ \hdashline A_{21} & A_{22} & B_2 \\ \hline C_1 & C_2 & D \end{array}\right]. \tag{4.11}$$

Define,

$$\begin{aligned} \tilde{E} &\triangleq E_{11}, \quad \tilde{A} \triangleq A_{11}, \quad \tilde{B} \triangleq \begin{bmatrix} A_{12} & B_1 \end{bmatrix}, \\ \tilde{C} &\triangleq \begin{bmatrix} A_{21} \\ C_1 \end{bmatrix}, \quad \tilde{D} \triangleq \begin{bmatrix} A_{22} & B_2 \\ C_2 & D \end{bmatrix}. \end{aligned} \tag{4.12}$$

If $\text{rank}(E) = r$, then $\tilde{E}$, $\tilde{A} \in \mathbb{R}^{r \times r}$, $\tilde{B} \in \mathbb{R}^{r \times (n-r+m)}$, $\tilde{C} \in \mathbb{R}^{(n-r+p) \times r}$, and $\tilde{D} \in \mathbb{R}^{(n-r+p) \times (n-r+m)}$. The resulting matrix pencil

$$\begin{bmatrix} \tilde{A} - \lambda \tilde{E} & \tilde{B} \\ \tilde{C} & \tilde{D} \end{bmatrix}, \tag{4.13}$$

is a compressed matrix pencil.

Algorithm: Compressed Matrix Pencil

input(E, A, B, C, D),

output$(\tilde{E}, \tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$,

comment: compress rows and columns of matrix E using orthogonal transformation matrices U, V and perform system equivalence.

$$\left[\begin{array}{c|c} E_{11} & O \\ \hline O & O \end{array}\right] = U^T E V,$$
$$\left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right] = U^T A V,$$
$$\left[\begin{array}{c} B_1 \\ \hline B_2 \end{array}\right] = U^T B,$$
$$\left[\begin{array}{c|c} C_1 & C_2 \end{array}\right] = C V,$$

comment: define matrices.

$$\tilde{E} \triangleq E_{11},\ \tilde{A} \triangleq A_{11},\ \tilde{B} \triangleq \begin{bmatrix} A_{12} & B_1 \end{bmatrix},$$
$$\tilde{C} \triangleq \begin{bmatrix} A_{21} \\ C_1 \end{bmatrix},\ \tilde{D} \triangleq \begin{bmatrix} A_{22} & B_2 \\ C_2 & D \end{bmatrix},$$

comment: update the dimensions of the matrices.

$\tilde{n} = r$, $\tilde{m} = n - r + m$ and $\tilde{p} = n - r + p$.

S-Reduce

This is a recursive algorithm implemented on the compressed matrix pencil. For notational convenience, define $(E, A, B, C, D) := (\tilde{E}, \tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$, $n := r$, $m := n - r + m$ and $p := n - r + p$. Each iteration can be divided into the following three steps:

Step 1: Compress rows of the matrix D using orthogonal transformation matrix $W^{(i)}$, such that $W^{T(i)}D$ is a row compressed matrix and the matrix $W^{T(i)}C$ is partitioned into C_1 and C_2 as

$$W^{T(i)}D = \begin{bmatrix} D_1 \\ O \end{bmatrix}, \quad D_1 \in \mathbb{R}^{(p-\tau_i)\times m}, \quad O \in \mathbb{R}^{\tau_i \times m}, \tag{4.14}$$

$$W^{T(i)}C = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}, \quad C_1 \in \mathbb{R}^{(p-\tau_i)\times n}, \quad C_2 \in \mathbb{R}^{\tau_i \times n}. \tag{4.15}$$

Step 2: Compress columns of the matrix C_2 using orthogonal transformation matrix $V^{(i)}$, such that $C_2V^{(i)}$ is a column compressed matrix

$$C_2V^{(i)} = \begin{bmatrix} O & C_{22} \end{bmatrix}, \quad O \in \mathbb{R}^{\tau_i \times (n-\mu_i)}, \quad C_{22} \in \mathbb{R}^{\tau_i \times \mu_i}, \tag{4.16}$$

Step 3: Column compression on matrix C_2 destroys the diagonal or triangular structure of the matrix E. To maintain the upper triangular structure of matrix E simultaneously, an orthogonal transformation $U^{(i)}$ is determined using the QR decomposition of matrix E, such that

$$U^{T(i)}EV^{(i)} = \left[\begin{array}{c|c} E_{11} & E_{12} \\ \hline O & E_{22} \end{array}\right], \quad E_{11} \in \mathbb{R}^{(n-\mu_i)\times(n-\mu_i)}, \quad E_{22} \in \mathbb{R}^{\mu_i \times \mu_i}. \tag{4.17}$$

As a result of above mentioned steps and system equivalence on the matrix pencil, the matrix pencil is partitioned as

$$\begin{aligned} &\left[\begin{array}{c|c} U^{T(i)} & O \\ \hline O & W^{T(i)} \end{array}\right] \left[\begin{array}{c|c} A - \lambda E & B \\ \hline C & D \end{array}\right] \left[\begin{array}{c|c} V^{(i)} & O \\ \hline O & I \end{array}\right] \\ &\qquad = \left[\begin{array}{c:c|c} A_{11} - \lambda E_{11} & A_{12} - \lambda E_{12} & B_1 \\ \hdashline A_{21} & A_{22} - \lambda E_{22} & B_2 \\ \hline C_{11} & C_{21} & D_1 \\ \hdashline O & C_{22} & O \end{array}\right]. \end{aligned} \tag{4.18}$$

Here, the matrix C_{22} has full column rank. Based on the rank of the matrices D_1 and C_{22}, the matrices of the matrix pencil are redefine as

$$\left[\begin{array}{c|c} A-\lambda E & B \\ \hline C & D \end{array}\right] := \left[\begin{array}{c|c} A_{11}-\lambda E_{11} & B_1 \\ \hline A_{21} & B_2 \\ C_{11} & D_1 \end{array}\right], \tag{4.19}$$

where $A,\ E \in \mathbb{R}^{(n-\mu_i)\times(n-\mu_i)}$, $B \in \mathbb{R}^{(n-\mu_i)\times m}$, $C \in \mathbb{R}^{(p-\tau_i+\mu_i)\times(n-\mu_i)}$, and $D \in \mathbb{R}^{(p-\tau_i+\mu_i)\times m}$.

These steps, row compression on the matrix D, column compression on the matrix C, and triangularization on the matrix E are carried out on the reduced matrix pencil obtained at the end of each iteration, until a full row rank matrix D or a zero rank matrix C is encountered. At the end of this recursive algorithm the transformed matrix pencil has the following structure

$$\left[\begin{array}{c|c} U^T & O \\ \hline O & W^T \end{array}\right] \left[\begin{array}{c|c} A-\lambda E & B \\ \hline C & D \end{array}\right] \left[\begin{array}{c|c} V & O \\ \hline O & I \end{array}\right] = \left[\begin{array}{c|c} A_1-\lambda E_1 & * \\ \hline O & A_2-\lambda E_2 \end{array}\right], \tag{4.20}$$

where the matrix pencil $(A_1 - \lambda E_1)$ contains information regarding the finite null values and the Kronecker column indices of a matrix pencil and has the following form

$$A_1-\lambda E_1 = \left[\begin{array}{c|c} A_r-\lambda E_r & B_r \\ \hline C_r & D_r \end{array}\right]. \tag{4.21}$$

The matrix pencil $(A_2 - \lambda E_2)$ contains information regarding the infinite elementary divisors and the Kronecker row indices of the matrix pencil. The matrix pencil $(A_2 - \lambda E_2)$ has the structure described in (3.22).

$$\begin{bmatrix} A^c_{l,l} & A^c_{l,l-1}-\lambda E^c_{l,l-1} & \dots & A^c_{l,2}-\lambda E^c_{l,2} & A^c_{l,1}-\lambda E^c_{l,1} \\ 0 & A^c_{l-1,l-1} & \dots & A^c_{l-1,2}-\lambda E^c_{l-1,2} & A^c_{l-1,1}-\lambda E^c_{l-1,1} \\ 0 & 0 & \dots & A^c_{l-2,2}-\lambda E^c_{l-2,2} & A^c_{l-2,1}-\lambda E^c_{l-2,1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & & A^c_{2,2} & A^c_{2,1}-\lambda E^c_{2,1} \\ 0 & 0 & \dots & 0 & A^c_{1,1} \end{bmatrix}. \tag{4.22}$$

Information about the infinite elementary divisors and the Kronecker row indices, as described in [30] states,

- there are $d_i = \mu_i - \tau_{i+1}$ infinite elementary divisors of degree i, $(i = 1, \ldots, l)$;
- there are $r_i = \tau_i - \mu_i$ Kronecker row indices of size $(i-1)$, $(i = 1, \ldots, l)$.

Algorithm: S-Reduce

input$(E, A, B, C, D, n, m, p, \mu, \tau)$,

output$(E_r, A_r, B_r, C_r, D_r, nr, mr, pr, rr, dr, \mu_r, \tau_r)$,

step i

comment: compress rows of the matrix D

$$\left[\begin{array}{c|c} I & O \\ \hline 0 & W^{T(i)} \end{array}\right] \left[\begin{array}{c|c} A - \lambda E & B \\ \hline C & D \end{array}\right] = \left[\begin{array}{c|c} A - \lambda E & B \\ \hline C_1 & D_1 \\ \hdashline C_2 & O \end{array}\right],$$

$$D_1 \in \mathbb{R}^{(p-\tau_i)\times m}, \quad C_2 \in \mathbb{R}^{\tau_i \times n},$$

if $\tau_i = 0$, stop and jump to exit1,

comment: Using TRIANGULARIZATION algorithm discussed next, compress the columns of the matrix C_2 using $V^{(i)}$ and maintain the upper triangular form of E using $U^{(i)}$. Partition the matrices as

$$\left[\begin{array}{c|c} U^{T(i)} & O \\ \hline O & W^{T(i)} \end{array}\right] \left[\begin{array}{c|c} A - \lambda E & B \\ \hline C & D \end{array}\right] \left[\begin{array}{c|c} V^{(i)} & O \\ \hline O & I \end{array}\right]$$

$$= \left[\begin{array}{c:c|c} A_{11} - \lambda E_{11} & A_{12} - \lambda E_{12} & B_1 \\ \hdashline A_{21} & A_{22} - \lambda E_{22} & B_2 \\ \hline C_{11} & C_{21} & D_1 \\ \hdashline O & C_{22} & O \end{array}\right],$$

$$\left[\begin{array}{c|c} A - \lambda E & B \\ \hline C & D \end{array}\right] := \left[\begin{array}{c|c} A_{11} - \lambda E_{11} & B_1 \\ \hline A_{21} & B_2 \\ C_{11} & D_1 \end{array}\right],$$

if $\mu_i = 0$, then $p := p - \tau_i$, stop and jump to exit2.

comment: update dimensions. $n := n - \mu_i, \quad p := p - \tau_i + \mu_i$,

$i = i + 1$, jump to step i;

exit1:

$(E_r, A_r, B_r, C_r, D_r) := (E, A, B, C, D);$

$nr := n, mr := m, pr := p,$

$r_k := \tau_k - \mu_k, d_k := \mu_k - \tau_{k+1},$

$k = 1, ..., (i-1);$

exit2:

$(E_r, A_r, B_r, C_r, D_r) := (E, A, B, C, D);$

$nr := n, mr := m, pr := p,$

$r_k := \tau_k - \mu_k, d_k := \mu_k - \tau_{k+1},$

$k = 1, ..., (i-1), r_i = \tau_i;$

Algorithm: Triangularization

comment: compress columns of the matrix C_2.

$$\left[\begin{array}{c|c} A - \lambda E & B \\ \hline C_1 & D_1 \\ \hdashline C_2 & O \end{array}\right] \left[\begin{array}{c|c} V & O \\ \hline O & I \end{array}\right] = \left[\begin{array}{c:c|c} A_{11} - \lambda E_{11} & A_{12} - \lambda E_{12} & B_1 \\ \hdashline A_{21} - \lambda E_{21} & A_{22} - \lambda E_{22} & B_2 \\ \hline C_{11} & C_{21} & D_1 \\ \hdashline O & C_{22} & O \end{array}\right],$$

comment: maintain the upper triangular form of the matrix E.

$$\left[\begin{array}{c|c} U^T & O \\ \hline O & I \end{array}\right] \left[\begin{array}{c:c|c} A_{11} - \lambda E_{11} & A_{12} - \lambda E_{12} & B_1 \\ \hdashline A_{21} - \lambda E_{21} & A_{22} - \lambda E_{22} & B_2 \\ \hline C_{11} & C_{21} & D_1 \\ \hdashline O & C_{22} & O \end{array}\right]$$

$$= \left[\begin{array}{c:c|c} A_{11} - \lambda E_{11} & A_{12} - \lambda E_{12} & B_1 \\ \hdashline A_{22} & A_{22} - \lambda E_{22} & B_2 \\ \hline C_{11} & C_{21} & D_1 \\ \hdashline O & C_{22} & O \end{array}\right].$$

To compute finite null values of the matrix pencil, **S-Reduce** algorithm is applied to the **block transpose** of the matrix pencil (4.21), which yields invertible matrices E_{rc} and D_{rc}. The resulting matrix pencil has the following structure

$$\left[\begin{array}{c|c} U & O \\ \hline O & W \end{array}\right] \left[\begin{array}{c|c} A_r^T - \lambda E_r^T & C_r^T \\ \hline B_r^T & D_r^T \end{array}\right] \left[\begin{array}{c|c} V & O \\ \hline O & I \end{array}\right] = \left[\begin{array}{c|c} A_3 - \lambda E_3 & \times \\ \hline O & A_4 - \lambda E_4 \end{array}\right]. \qquad (4.23)$$

Here, the matrix pencil $(A_3 - \lambda E_3)$ contains just the information about finite null values and has the following form

$$A_3 - \lambda E_3 = \left[\begin{array}{c|c} A_{rc} - \lambda E_{rc} & B_{rc} \\ \hline C_{rc} & D_{rc} \end{array}\right]. \tag{4.24}$$

The matrix pencil $(A_4 - \lambda E_4)$ contains information regarding the Kronecker column indices of system matrix and has the structure described in (3.23).

$$\begin{bmatrix} A^r_{1,1} & A^r_{1,2} - \lambda E^r_{1,2} & \dots & A^r_{1,k-1} - \lambda E^r_{1,k-1} & A^r_{1,k} - \lambda E^r_{1,k} \\ O & A^r_{2,2} & \dots & A^r_{2,k-1} - \lambda E^r_{2,k-1} & A^r_{2,k} - \lambda E^r_{2,k} \\ O & O & \dots & A^r_{k-2,k-2} - \lambda E^r_{k-2,k-2} & A^r_{k-2,k} - \lambda E^r_{k-2,k} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ O & O & & A^r_{k-1,k-1} & A^r_{k-1,k-1} - \lambda E^r_{k-1,k-1} \\ O & O & \dots & O & A^r_{k,k} \end{bmatrix}. \tag{4.25}$$

The matrix D_{rc} of the matrix pencil (4.24) is invertible. To compute the finite null values of the matrix pencil, columns of matrix $\begin{bmatrix} C_{rc} & D_{rc} \end{bmatrix}$ are compressed using orthogonal transformation W as

$$\left[\begin{array}{c|c} A_{rc} & B_{rc} \\ \hline C_{rc} & D_{rc} \end{array}\right] W = \left[\begin{array}{c|c} A_f & * \\ \hline O & D_f \end{array}\right], \tag{4.26}$$

$$\left[\begin{array}{c|c} E_{rc} & O \\ \hline O & O \end{array}\right] W = \left[\begin{array}{c|c} E_f & O \\ \hline O & O \end{array}\right]. \tag{4.27}$$

Here, the matrix D_f is invertible and since W is orthogonal transformation matrix

$$\begin{aligned} \operatorname{rank}\left(\left[\begin{array}{c|c} A_{rc} - \lambda E_{rc} & B_{rc} \\ \hline C_{rc} & D_{rc} \end{array}\right]\right) &= \operatorname{rank}\left(\left[\begin{array}{c|c} A_f - \lambda E_f & * \\ \hline O & D_f \end{array}\right]\right) \\ &= \operatorname{rank}(D_f) + \operatorname{rank}(A_f - \lambda E_f). \end{aligned} \tag{4.28}$$

Thus, the values of λ for which the rank of pencil $(A_f - \lambda E_f)$ drops, are the finite null values of the system. Null values can be computed by applying the QZ algorithm on the matrix pencil $(A_f - \lambda E_f)$. Using the algorithms **Compressed matrix pencil**, **S-Reduce**, and **Triangularization**; **Complete algorithm** is outlined for computation of null values.

Complete Algorithm

comment: compress rows and columns of the matrix E, such that matrix pencil is in its compressed matrix pencil form.

$(\tilde{E}, \tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}) =$ compressed matrix pencil(E, A, B, C, D);

comment: reduce $(\tilde{E}, \tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ to $(E_r, A_r, B_r, C_r, D_r)$ such that D_r has full row rank and compute infinite elementary divisor and the Kronecker row indices.

$(E_r, A_r, B_r, C_r, D_r, nr, mr, pr, \mu_r, \tau_r) =$ S-Reduce$(\tilde{E}, \tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}, n, m, p, \mu, \tau)$;

comment: reduce $(E_r, A_r, B_r, C_r, D_r)$ to $(E_{rc}, A_{rc}, B_{rc}, C_{rc}, D_{rc})$ such that D_{rc} is invertible and compute the Kronecker column indices.

$(E_{rc}, A_{rc}, C_{rc}, B_{rc}, D_{rc}, n_{rc}, p_{rc}, m_{rc}, \mu_{rc}, \tau_{rc}) =$

S-Reduce$(E_r^T, A_r^T, C_r^T, B_r^T, D_r^T, n_r, p_r, m_r, \mu_r, \tau_r)$;

comment: compress columns of $[C_{rc} \;\; D_{rc}]$ to $[0 \;\; D_f]$ and perform system equivalence.

$$\left[\begin{array}{c|c} A_f & * \\ \hline O & D_f \end{array}\right] = \left[\begin{array}{c|c} A_{rc} & B_{rc} \\ \hline C_{rc} & D_{rc} \end{array}\right] W;$$

$$\left[\begin{array}{c|c} E_f & * \\ \hline O & O \end{array}\right] = \left[\begin{array}{c|c} E_{rc} & O \\ \hline O & O \end{array}\right] W;$$

comment: apply the QZ algorithm to get triangular matrices A_t, E_t. Determine finite null values of the system. $[A_t, E_t] = QZ(A_f, E_f)$;

$\{\lambda_1, ..., \lambda_{n_f}\} = a_{ii}/e_{ii}, \;\; i = 1, ..., n_f$.

4.4 Computation of Null Vectors

As defined in the definition 4.1, null values are the finite values of λ, which satisfy

$$\text{rank}(F - \lambda G) < min(s, t), \tag{4.29}$$

where F, $G \in \mathbb{R}^{s \times t}$, and $\lambda \in \mathbb{C}$. This indicates, that the matrix pencil is singular at $\lambda =$ (null value). Thus, it is possible to compute non-zero vector which satisfy

$$(F - \lambda G)\mathbf{v} = \mathbf{0}, \quad \mathbf{v} \in \mathbb{R}^{t \times 1}, \tag{4.30}$$

as well as

$$\mathbf{w}(F - \lambda G) = \mathbf{0}, \quad \mathbf{w} \in \mathbb{R}^{1\times s}, \tag{4.31}$$

$\mathbf{w}$ and $\mathbf{v}$ are respectively the left and the right null vectors of the matrix pencil corresponding to the null value λ.

To compute $\mathbf{w}$ and $\mathbf{v}$, the matrix pencil $(F - \lambda G)$ is first reduced to its row echelon form (3.18). Let λ_0 be a null value of the matrix pencil and the matrix $J \in \mathbb{R}^{s\times t}$ be a constant matrix defined as

$$J \triangleq F - \lambda_0 G, \tag{4.32}$$

and its row echelon form be denoted by J_e. Since the matrices F, $G \in \mathbb{R}^{s\times t}$ are non-square, the row echelon form J_e is a non-square matrix. Moreover, matrix J is singular because λ_0 is a null value of the matrix pencil $(F - \lambda G)$. If $\operatorname{rank}(J) = r$ for the null value λ_0, then the rank deficiency indicates the multiplicity of that null value. Hence, it is possible to compute null vectors of the matrix pencil corresponding to each null value. Since the matrix J_e is a non-square matrix it is possible to permute the columns of J_e using the permutation matrix P such that J_eP has the following form

$$J_eP = \begin{bmatrix} J_{11} & J_{12} \\ O & O \end{bmatrix}. \tag{4.33}$$

Since, $\operatorname{rank}(J) = r < \min(s,t)$, then $J_{11} \in \mathbb{R}^{r\times r}$ is an upper triangular matrix, $J_{12} \in \mathbb{R}^{r\times(t-r)}$. The null vector corresponding to λ_0 can be computed as follows

$$\begin{bmatrix} J_{11} & J_{12} \\ O & O \end{bmatrix} \begin{bmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \end{bmatrix} = \mathbf{0}, \tag{4.34}$$

here, $\mathbf{v}_1 \in \mathbb{R}^{r\times 1}$ and $\mathbf{v}_2 \in \mathbb{R}^{(t-r)\times 1}$. Values for the vector $\mathbf{v}_2 \in \mathbb{R}^{(t-r)\times 1}$ can be chosen arbitrarily, provided that $\mathbf{v}_2 \neq \mathbf{0}$, then the matrix equation can be rewritten as

$$J_{11}\mathbf{v}_1 = \mathbf{0} - J_{12}\mathbf{v}_2. \tag{4.35}$$

Since, J_{11} is an upper triangular matrix, $\mathbf{v}_1$ can be computed using the back substitution algorithm as detailed in chapter 3.

Similarly, left null vector corresponding to λ_0 can be computed by computing the right null vector of the transposed matrix pencil $(F^T - \lambda G^T)$.

Example: This example illustrates the computation of a null vector corresponding to a null value of a matrix pencil $(F - \lambda G)$. The matrices of the matrix pencil are:

$$\begin{aligned} F &= \begin{bmatrix} 38.533 & 49.200 & 22.533 & 0.53333 & 0.066667 & 0.26667 \\ -64.667 & -82.000 & -38.667 & -0.66667 & 0.66667 & -0.33333 \\ 71.520 & 95.360 & 35.760 & 5.9600 & 11.920 & 2.9800 \end{bmatrix}, \\ G &= \begin{bmatrix} 10.067 & 12.333 & 6.6667 & 0 & 0 & 0 \\ -16.333 & -19.667 & -11.333 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \end{aligned} \tag{4.36}$$

Using the **Complete algorithm** the null values of the matrix pencil are found to be $\lambda_1 = 1.6813$ and $\lambda_2 = 2$. Selecting $\lambda = 2$, the matrix $J = (F - \lambda G)$ is

$$\begin{aligned} J &= \begin{bmatrix} 38.533 & 49.200 & 22.533 & 0.53333 & 0.066667 & 0.26667 \\ -64.667 & -82.000 & -38.667 & -0.66667 & 0.66667 & -0.33333 \\ 71.520 & 95.360 & 35.760 & 5.9600 & 11.920 & 2.9800 \end{bmatrix} \\ &\quad - 2 \begin{bmatrix} 10.067 & 12.333 & 6.6667 & 0 & 0 & 0 \\ -16.333 & -19.667 & -11.333 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 18.400 & 24.533 & 9.2000 & 0.53333 & 0.066667 & 0.26667 \\ -32.000 & -42.667 & -16.000 & -0.66667 & 0.66667 & -0.33333 \\ 71.520 & 95.360 & 35.760 & 5.9600 & 11.920 & 2.9800 \end{bmatrix}. \end{aligned} \tag{4.37}$$

Transforming the matrix J to its row echelon form

$$J_e = \begin{bmatrix} 18.400 & 24.533 & 9.2000 & 0.53333 & 0.066667 & 0.26667 \\ 0 & 0 & 0 & -0.26087 & -0.78261 & -0.13043 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{4.38}$$

To compute the null vector using back substitution algorithm it is necessary to rearrange the columns of J_e as described in (4.33). Using a permutation matrix P

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \tag{4.39}$$

matrix J can be brought the following form

$$JP = \left[\begin{array}{rr|rrrr} 18.400 & 0.53333 & 9.2000 & 24.533 & 0.066667 & 0.26667 \\ 0 & -0.26087 & 0 & 0 & -0.78261 & -0.13043 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]. \tag{4.40}$$

The column permutation also affects the vector $\mathbf{v}$. The null vector can be computed by solving

$$\begin{aligned} \left[\begin{array}{rr|rrrr} 18.400 & 0.53333 & 9.2000 & 24.533 & 0.066667 & 0.26667 \\ 0 & -0.26087 & 0 & 0 & -0.78261 & -0.13043 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \left[\begin{array}{c} v_1 \\ v_4 \\ \hline v_3 \\ v_2 \\ v_5 \\ v_6 \end{array}\right] \\ = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \end{aligned} \tag{4.41}$$

Setting

$$\begin{aligned} J_{11} &= \begin{bmatrix} 18.400 & 0.53333 \\ 0 & -0.26087 \end{bmatrix}, \\ J_{12} &= \begin{bmatrix} 9.2000 & 24.533 & 0.066667 & -0.26667 \\ 0 & 0 & -0.78261 & -0.13043 \end{bmatrix}, \end{aligned} \tag{4.42}$$

then,

$$\begin{aligned} \begin{bmatrix} 18.400 & 0.53333 \\ 0 & -0.26087 \end{bmatrix} \begin{bmatrix} v_1 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ - \begin{bmatrix} 9.2000 & 24.533 & 0.066667 & -0.26667 \\ 0 & 0 & -0.78261 & -0.13043 \end{bmatrix} \begin{bmatrix} v_3 \\ v_2 \\ v_5 \\ v_6 \end{bmatrix}, \end{aligned} \tag{4.43}$$

assuming $v_2 = -3$, $v_3 = 3$, $v_5 = -5$, and $v_6 = 5$ we have

$$\begin{bmatrix} 18.400 & 0.53333 \\ 0 & -0.26087 \end{bmatrix} \begin{bmatrix} v_1 \\ v_4 \end{bmatrix} = \begin{bmatrix} -44.999 \\ 3.2609 \end{bmatrix}, \tag{4.44}$$

using back substitution values of v_1 and v_4 are found to be $v_1 = 2.0833$ and $v_4 = 12.5$. Thus, null vector of the matrix pencil is

$$\mathbf{v} = \begin{bmatrix} 2.0833 \\ -3.0000 \\ 3.0000 \\ 12.5000 \\ -5.0000 \\ 5.0000 \end{bmatrix}. \tag{4.45}$$

5 Applications

5.1 Circuit with input derivatives

To understand the computation of zeros of a circuit having input derivative term, consider a circuit as shown in fig. 5.1, [26]

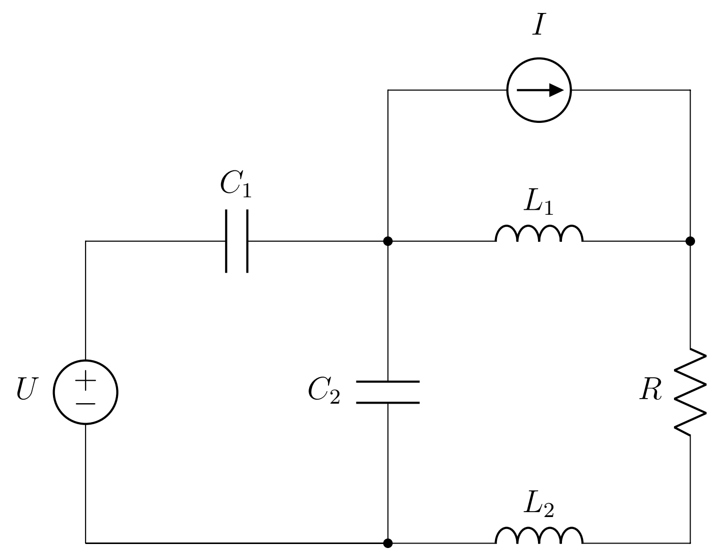

Figure 5.1: Circuit with input derivative.

The differential equations describing the circuit are

$$\frac{du_{C_1}}{dt} = \frac{1}{C_1 + C_2} i_{L_1} + \frac{C_2}{C_1 + C_2} \frac{dU}{dt}, \tag{5.1}$$

$$\frac{di_{L_1}}{dt} = -\frac{1}{L_1 + L_2} U_{C_1} - \frac{R}{L_1 + L_2} + \frac{1}{L_1 + L_2} U - L_2 L_1 + L_2 \frac{dI}{dt}. \tag{5.2}$$

It is possible to represent these differential equations in a generalized state space form. Let the states of the generalized state space model be the capacitor voltage u_{C_1} and the inductor current i_{L_1}. The inputs being the voltage source U and the current source I, the state vector and input vector are

$$x = \begin{bmatrix} u_{C_1} \\ i_{L_1} \end{bmatrix}, \quad u = \begin{bmatrix} U \\ I \end{bmatrix}. \tag{5.3}$$

Based on the chosen states, the state equation of the circuit is written as

$$\begin{aligned} \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_{C_1} \\ i_{L_1} \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{C_1+C_2} \\ \frac{-1}{L_1+L_2} & \frac{-R}{L_1+L_2} \end{bmatrix} \begin{bmatrix} u_{C_1} \\ i_{L_1} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ \frac{1}{L_1+L_2} & 0 \end{bmatrix} \begin{bmatrix} U \\ I \end{bmatrix} \\ + \lambda \begin{bmatrix} \frac{C_2}{C_1+C_2} & 0 \\ 0 & \frac{-L_2}{L_1+L_2} \end{bmatrix} \begin{bmatrix} U \\ I \end{bmatrix} \end{aligned} \tag{5.4}$$

where

$$E = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & \frac{1}{C_1+C_2} \\ \frac{-1}{L_1+L_2} & \frac{-R}{L_1+L_2} \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 0 & 0 \\ \frac{1}{L_1+L_2} & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} \frac{C_2}{C_1+C_2} & 0 \\ 0 & \frac{-L_2}{L_1+L_2} \end{bmatrix}. \tag{5.5}$$

To generalize the computation of null values (zeros) for a circuit with input derivatives, subsequent details are provided directly in terms of matrices (E, A, B_1, B_2, C, D). The generalized state space model of circuit in fig. 5.1 stated in (5.4), has the following general form

$$\begin{aligned} \lambda E &= Ax + B_1 u + \lambda B_2 u, \\ y &= Cx + Du. \end{aligned} \tag{5.6}$$

Then the generalized state space equation can be represented as

$$\left[\begin{array}{c|c} A - \lambda E & B_1 + \lambda B_2 \\ \hline C & D \end{array}\right] \begin{bmatrix} \mathbf{x} \\ \mathbf{u} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{y} \end{bmatrix}, \tag{5.7}$$

where $A,\ E \in \mathbb{R}^{n\times n}$, $B_1,\ B_2 \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$, and $D \in \mathbb{R}^{p\times m}$. Thus the system matrix of the circuit is

$$\left[\begin{array}{c|c} A - \lambda E & B_1 + \lambda B_2 \\ \hline C & D \end{array}\right]. \tag{5.8}$$

Now, to facilitate the computation of null values using the proposed algorithm, the matrix pencil is first reduced to standard matrix pencil form $(F - \lambda G)$. This can be accomplished be defining a new matrix pencil which is equivalent to the original system matrix of the circuit. Define

$$\hat{E} \triangleq \begin{bmatrix} E & -B_2 \end{bmatrix}, \quad \hat{A} \triangleq \begin{bmatrix} A & B_1 \end{bmatrix}, \quad \hat{C} \triangleq \begin{bmatrix} C & D \end{bmatrix}, \tag{5.9}$$

where $\hat{A}$, $\hat{E} \in \mathbb{R}^{(n\times(n+m))}$, and $\hat{C} \in \mathbb{R}^{(p\times(n+m))}$. Then null values are the finite values of λ for which

$$\operatorname{rank}\left[\begin{array}{c} \hat{A} - \lambda\hat{E} \\ \hline \hat{C} \end{array}\right] = \operatorname{rank}\left[\begin{array}{c|c} A - \lambda E & B_1 + \lambda B_2 \\ \hline C & D \end{array}\right] < n + \min(m, p). \tag{5.10}$$

Having defined new matrices for the matrix pencil, matrix $\hat{E}$ can be partitioned using row and column compression using orthogonal transformation matrices $Q \in \mathbb{R}^{(n\times(n+m))\times(n\times(n+m))}$ and $Z \in \mathbb{R}^{(n\times(n+m))\times(n\times(n+m))}$ such that

$$Q^T\hat{E}Z = \left[\begin{array}{c|c} \hat{E}_1 & O \end{array}\right], \tag{5.11}$$

$$\left[\begin{array}{c|c} Q^T & O \\ \hline O & I \end{array}\right]\left[\begin{array}{c} \hat{A} - \lambda\hat{E} \\ \hline \hat{C} \end{array}\right] Z = \left[\begin{array}{c|c} \hat{A}_1 - \lambda\hat{E}_1 & \hat{A}_2 \\ \hline \hat{C}_1 & \hat{C}_2 \end{array}\right]. \tag{5.12}$$

This matrix pencil now has a similar form as the matrix pencil in (4.3). Thus null values (zeros) of circuit can be computed easily using the **Complete algorithm**.

Let C_1, $C_2 = 1F$, L_1, $L_2 = 1H$, $R = 1\Omega$ be the values of components for the circuit shown in fig. 5.1. then the resulting state space system is

$$\begin{aligned} \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_{C_1} \\ i_{L_1} \end{bmatrix} = \begin{bmatrix} 0 & 0.5 \\ -0.5 & -0.5 \end{bmatrix} \begin{bmatrix} u_{C_1} \\ i_{L_1} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0.5 & 0 \end{bmatrix} \begin{bmatrix} U \\ I \end{bmatrix} \\ + \lambda \begin{bmatrix} 0.5 & 0 \\ 0 & -0.5 \end{bmatrix} \begin{bmatrix} U \\ I \end{bmatrix}. \end{aligned} \tag{5.13}$$

If the output equation of the generalized state space is

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_{C_1} \\ i_{L_1} \end{bmatrix}. \tag{5.14}$$

Then, based on the redefinition of matrix pencil as defined in (5.9) the matrices of matrix pencil are

$$\begin{gathered} \hat{E} = \begin{bmatrix} 1 & 0 & -0.5 & 0 \\ 0 & 1 & 0 & -0.5 \end{bmatrix}, \quad \hat{A} = \begin{bmatrix} 0 & 0.5 & 0 & 0 \\ -0.5 & -0.5 & 0.5 & 0 \end{bmatrix}, \\ \hat{C} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}. \end{gathered} \tag{5.15}$$

Using the **Complete algorithm**, null values (zeros) of the circuit are found to be $\lambda_1 = 0$ and $\lambda_2 = 0$.

5.2 Zero directions of linear system

Given, a generalized state space system

$$\begin{bmatrix} A - \lambda E & B \\ C & D \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{u} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{y} \end{bmatrix}, \tag{5.16}$$

transfer function of the generalized state space system is determined by

$$G(\lambda) = C(\lambda E - A)^{-1} B + D. \tag{5.17}$$

Assuming that the system is controllable as well as observable, the finite values of λ for which the rank of the system matrix drops are the null values of the system or equivalently the transmission zeros of the generalized state space system. If $\lambda = \lambda_0$ is a transmission zero, then there exists a non-zero vector $\begin{bmatrix} \mathbf{x}_0 \\ \mathbf{u}_0 \end{bmatrix}$ which satisfies

$$\begin{bmatrix} A - \lambda_0 E & B \\ C & D \end{bmatrix} \begin{bmatrix} \mathbf{x}_0 \\ \mathbf{u}_0 \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \end{bmatrix}, \quad \begin{bmatrix} \mathbf{x}_0 \\ \mathbf{u}_0 \end{bmatrix} \neq \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \end{bmatrix}. \tag{5.18}$$

Here, if the system has n states, m inputs, and p outputs then

- the column vector $\mathbf{x}_0 \in \mathbb{R}^{n\times 1}$ is termed as the **state zero direction**,
- the column vector $\mathbf{u}_0 \in \mathbb{R}^{m\times 1}$ is termed as the **input zero direction**,
- $\begin{bmatrix} \mathbf{x}_0 \\ \mathbf{u}_0 \end{bmatrix}$ is called the **right zero direction** of the system.

To verify that $\mathbf{u}_0$ is the input zero direction, the system matrix can be transformed as

$$\begin{bmatrix} (A-\lambda E)^{-1} & O \\ -C(A-\lambda E)^{-1} & I \end{bmatrix} \begin{bmatrix} A-\lambda E & B \\ C & D \end{bmatrix} = \begin{bmatrix} I & (A-\lambda E)^{-1}B \\ O & C(\lambda E - A)^{-1}B + D \end{bmatrix}. \tag{5.19}$$

Hence,

$$\begin{bmatrix} I & (A-\lambda E)^{-1}B \\ O & G(\lambda) \end{bmatrix} \begin{bmatrix} \mathbf{x}_0 \\ \mathbf{u}_0 \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \end{bmatrix}, \tag{5.20}$$

and the rank of the system matrix drops only when the rank of $G(\lambda)$ drops. Since, λ_0 is a transmission zero, $(A - \lambda_0 E)$ is invertible and $G(\lambda_0)$ is singular. Hence, $\mathbf{u}_0 \neq \mathbf{0}$ satisfies

$$G(\lambda)\mathbf{u}_0 = \mathbf{0}. \tag{5.21}$$

This shows that $\mathbf{u}_0$ is a input zero direction, which results in zero output.

This also means that, if the system is subjected to the input $\mathbf{u}(t) = \mathbf{u}_0 e^{\lambda_0 t}$ and have an initial state vector $\mathbf{x}_0$, resulting to the state response $\mathbf{x}(t) = \mathbf{x}_0 e^{\lambda_0 t}$. Then for such input, the output of the system $\mathbf{y}(t) = \mathbf{0}$, this is shown using numerical example of a 5^{th} order system.

Similarly, there also exist **left zero direction** corresponding to λ_0 of the system such that

$$\begin{bmatrix} \mathbf{z}_0 & \mathbf{f}_0 \end{bmatrix} \begin{bmatrix} A-\lambda_0 E & B \\ C & D \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{0} \end{bmatrix}, \quad \begin{bmatrix} \mathbf{z}_0 & \mathbf{f}_0 \end{bmatrix} \neq \begin{bmatrix} \mathbf{0} & \mathbf{0} \end{bmatrix}. \tag{5.22}$$

Here,

- the row vector $\mathbf{z}_0 \in \mathbb{R}^{1\times n}$ corresponds to the states of the system,
- the row vector $\mathbf{f}_0 \in \mathbb{R}^{1\times p}$ corresponds to the input of the system,
- $\begin{bmatrix} \mathbf{z}_0 & \mathbf{f}_0 \end{bmatrix}$ is called the **left zero direction** of the system.

Similar to (5.21) for right zero direction, the left zero direction satisfies

$$\mathbf{f}_0 G(\lambda) = \mathbf{0}. \tag{5.23}$$

We illustrate above application by means of the following example, to demonstrate that the computation of the left and the right null vectors can be used to determine the left and the right zero directions of a generalized state space system. Given a 5^{th} order system, with two inputs and three output [2] with the following matrices describing its state space representation:

$$E = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, A = \begin{bmatrix} -2 & -6 & 3 & -7 & 6 \\ 0 & -5 & 4 & -4 & 8 \\ 0 & 2 & 0 & 2 & -2 \\ 0 & 6 & -3 & 5 & -6 \\ 0 & -2 & 2 & -2 & 5 \end{bmatrix}, B = \begin{bmatrix} -2 & 7 \\ -8 & -5 \\ -3 & 0 \\ 1 & 5 \\ -8 & 0 \end{bmatrix},$$

$$C = \begin{bmatrix} 0 & -1 & 2 & -1 & -1 \\ 1 & 1 & 1 & 0 & -1 \\ 0 & 3 & -2 & 3 & -1 \end{bmatrix}, D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}. \tag{5.24}$$

The transfer function matrix for this system is

$$G(\lambda) = \begin{bmatrix} \frac{(\lambda+3)(\lambda-2.111)}{(\lambda-3)(\lambda-2)(\lambda-1)} & 0 \\ \frac{(\lambda+3)(\lambda+2.6)}{(\lambda-1)(\lambda+2)(\lambda+1)} & \frac{-2(\lambda-4)}{(\lambda+2)(\lambda+1)} \\ \frac{(\lambda+3)(\lambda-1.8571)}{(\lambda-3)(\lambda-2)(\lambda-1)} & 0 \end{bmatrix}. \tag{5.25}$$

Using the **Complete algorithm**, null values of the system are found to be $\lambda = -3$ and $\lambda = 4$. The right null vector computed corresponding to null values -3 and 4 are, respectivley

$$\mathbf{v}_1 = \begin{bmatrix} -3.1581e^{-14} \\ -1.0330e^{-14} \\ 5.0000e^{+00} \\ 5.0000e^{+00} \\ 5.0000e^{+00} \\ 5.0000e^{+00} \\ -5.7919e^{-16} \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -4.0000e^{+00} \\ 4.0000e^{+00} \\ 2.0938e^{-17} \\ -4.0000e^{+00} \\ -4.0982e^{-15} \\ -1.2353e^{-16} \\ -4.0000e^{+00} \end{bmatrix}. \tag{5.26}$$

For the given system, there are five states and two inputs, thus the right null vector consists of a zero state direction vector $\mathbf{x}_0 \in \mathbb{R}^{5\times 1}$ and a zero input direction vector $\mathbf{u}_0 \in \mathbb{R}^{2\times 1}$. To verify the right null vector corresponding to $\lambda = 4$, is indeed the right zero direction of the system, part of right null vector corresponding to the input zero direction $\mathbf{u}_0$ should satisfy (5.21)

$$G(4)\mathbf{u}_0 = \begin{bmatrix} -19.8333 & 0.0000 \\ 2.5667 & 0.0000 \\ 17.5000 & 0.0000 \end{bmatrix} \begin{bmatrix} -1.2353e^{-16} \\ -4.0000e^{+00} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \tag{5.27}$$

This indicates that the computed null vector is the zero direction of the system. The same result is verified using the zero input and zero state response of the system. Select the transfer function relation corresponding to the second input and the second output

$$G_{22}(\lambda) = \frac{-2(\lambda - 4)}{(\lambda + 2)(\lambda + 1)}. \tag{5.28}$$

From (5.25), it is clear that the null value corresponding to this transfer function is $\lambda = 4$ because the rank of $G(\lambda)$ drops. Let the input to the system be $\mathbf{u}(t) = -4e^{4t}$, which is equivalent to $U_2(\lambda) = \frac{-4}{(\lambda-4)}$ in the frequency domain. The zero state response $Y_{22}^1(\lambda)$ of the system can be computed as

$$Y_{22}^1(\lambda) = G_{22}(\lambda)U_2(\lambda) = \frac{-2(\lambda-4)}{(\lambda+2)(\lambda+1)}\frac{-4}{(\lambda-4)} = \frac{8}{(\lambda+2)(\lambda+1)}. \tag{5.29}$$

Here, the response $Y_{22}^1(\lambda)$ does not contain a component of the input frequency $\lambda = 4$. Thus it can be said that the input has been blocked. Now, the zero input response $Y_{22}^2(\lambda)$ with initial state vector $\mathbf{x}_0 = \begin{bmatrix} -4 & 4 & 0 & -4 & 0 \end{bmatrix}^T$ can be computed as

$$Y_{22}^2(\lambda) = C_2(A-\lambda E)^{-1}\mathbf{x}_0 = \frac{-8}{(\lambda+2)(\lambda+1)}, \tag{5.30}$$

where C_2 is the row vector corresponding to second output of the system. Thus, the total response corresponding to the transfer function $G_{22}(\lambda)$ is

$$Y_{22}(\lambda) = Y_{22}^1(\lambda) + Y_{22}^2(\lambda) = \frac{8}{(\lambda+2)(\lambda+1)} + \frac{-8}{(\lambda+2)(\lambda+1)} = 0. \tag{5.31}$$

This indicates that, for the input containing frequency at null value, the total output $Y_{22}(\lambda)$ of the transfer function $G_{22}(\lambda)$ is 0. And thus, the computed right null vector is also the right zero direction of the system.

Similar results is shown for left null vector. The computed left null vector corresponding to each null value of the system (5.24) are

$$\mathbf{w}_1^T = \begin{bmatrix} 2.2055e^{-15} \\ -6.1500e^{+00} \\ 5.9500e^{+00} \\ -6.1500e^{+00} \\ 3.1500e^{+00} \\ -4.0000e^{+00} \\ -2.4494e^{-15} \\ 5.0000e^{+00} \end{bmatrix}, \mathbf{w}_2^T = \begin{bmatrix} 8.3333e^{-01} \\ 3.3922e^{+00} \\ 1.7549e^{+00} \\ 2.2255e^{+00} \\ -3.9804e^{+00} \\ 3.2941e^{+00} \\ 5.0000e^{+00} \\ 3.0000e^{+00} \end{bmatrix}. \tag{5.32}$$

To verify the left null vector corresponding to $\lambda = 4$, is indeed the left zero direction of the system, part of left null vector corresponding to the input zero direction $\mathbf{f}_0$ should satisfy (5.23)

$$\mathbf{f}_0 G(4) = \begin{bmatrix} 3.2941e^{+00} & 5.0000e^{+00} & 3.0000e^{+00} \end{bmatrix} \begin{bmatrix} -19.8333 & 0.0000 \\ 2.5667 & 0.0000 \\ 17.5000 & 0.0000 \end{bmatrix} \tag{5.33}$$
$$= \begin{bmatrix} 0 & 0 \end{bmatrix}.$$

This indicates that the computed left null vector is the left zero direction of the system. Next, the same result is verified using the zero input and zero state response of the system for the transfer function relation corresponding to the second input and the second output

$$G_{22}(\lambda) = \frac{-2(\lambda-4)}{(\lambda+2)(\lambda+1)}. \tag{5.34}$$

Let the input to the system be $\mathbf{u}(t) = 5e^{4t}$, which in frequency domain is $U_2(\lambda) = \frac{5}{(\lambda-4)}$, the zero state response $Y_{22}^1(\lambda)$ of the system thus will be

$$Y_{22}^1(\lambda) = G_{22}(\lambda)U_2(\lambda) = \frac{-2(\lambda-4)}{(\lambda+2)(\lambda+1)}\frac{5}{(\lambda-4)} = \frac{-10}{(\lambda+2)(\lambda+1)}. \tag{5.35}$$

Here the response $Y_{22}^1(\lambda)$ does not contain any component at input frequency, which confirms that the input is blocked. Now, the zero input response $Y_{22}^2(\lambda)$ with initial state vector is

$$\mathbf{x}_0 = \begin{bmatrix} 8.3333e^{-01} & 3.3922e^{+00} & 1.7549e^{+00} & 2.2255e^{+00} & -3.9804e^{+00} \end{bmatrix} \tag{5.36}$$

be

$$Y_{22}^2(\lambda) = \mathbf{x}_0(A-\lambda E)^{-1}B_2 = \frac{-10}{(\lambda+2)(\lambda+1)}, \tag{5.37}$$

where B_2 is the column vector corresponding to second input to the system.Thus the total response of the transfer function $G_{22}(\lambda)$ is

$$Y_{22}(\lambda) = Y_{22}^1(\lambda) + Y_{22}^2(\lambda) = \frac{10}{(\lambda+2)(\lambda+1)} + \frac{-10}{(\lambda+2)(\lambda+1)} = 0, \tag{5.38}$$

This indicates that, for the input containing frequency at null value, the total output $Y_{22}(\lambda)$ of the transfer function $G_{22}(\lambda)$ is 0. And thus, the computed left null vector is also the left zero direction of the system.

6 Concluding Remarks

Numerous applications lend themselves to a differential-algebraic relationship of the form

$$F\mathbf{v} = \lambda G\mathbf{v}, \tag{6.1}$$

where F, $G \in \mathbb{R}^{s\times t}$. An important problem is to determine finite values of $\lambda \in \mathbb{C}$ which satisfy the rank condition

$$\text{rank}(F - \lambda G) < \min(s, t), \tag{6.2}$$

and non-zero vector $\mathbf{v} \in \mathbb{R}^{t\times 1}$ and $\mathbf{w} \in \mathbb{R}^{1\times s}$ corresponding to λ, such that

$$\begin{aligned} (F - \lambda G)\mathbf{v} &= \mathbf{0}, \\ \mathbf{w}(F - \lambda G) &= \mathbf{0}. \end{aligned} \tag{6.3}$$

A theoretical basis was presented for computation of null values λ, null vectors $\mathbf{v}$ and $\mathbf{w}$ for a non-square matrix pencil $(F - \lambda G)$. The method discussed in this research is used for computation of zeros of a circuit with input derivative, and for computation of zero directions of a linear system.

Thus, the computational principle followed in this research is to reduce a non-square matrix pencil $(F - \lambda G)$ to a strict system equivalent matrix pencil using rank preserving orthogonal transformations

$$U^T(F - \lambda G)V = \begin{bmatrix} A - \lambda E & B \\ C & D \end{bmatrix} \tag{6.4}$$

where sub-matrix pencil $(A - \lambda E)$ is square and non-singular and contains the information about the finite null values. This reduction procedure is based on the algorithms developed in [29], which also provides information regarding null values at infinity and the singularities of the non-square matrix pencil. Moreover, this approach also finds its applicability for computing null values for a matrix pencil of the form

$$\left[\begin{array}{c|c} A - \lambda E & B_1 + \lambda B_2 \\ \hline C & D \end{array}\right], \tag{6.5}$$

which may result from applications like circuit with input derivative. The proposed approach is applicable to non-square matrix pencils, computes exact null values and null vectors when they exists and also serve a framework to solve non-standard matrix pencil structure as shown above. Hence, it overcomes the limitations of existing methods which may not be applicable to non-square matrix pencils, or are unable to compute exact null values.

A future research direction includes generalization of these results to systems with arbitrarily higher input derivatives. To support this claim, consider a differential-algebraic relation consisting of a second order derivative term

$$\lambda E\mathbf{v} = A\mathbf{v} + B_1\mathbf{u} + \lambda B_2\mathbf{u} + \lambda^2 B_3\mathbf{u}, \tag{6.6}$$

resulting to a matrix pencil of the form

$$\left[\begin{array}{c|c} A - \lambda E & B_1 + \lambda B_2 + \lambda^2 B_3 \end{array}\right]. \tag{6.7}$$

Determination of finite λ for such a matrix pencil is not directly possible because it includes higher order derivative term. However, using the principle implemented for computing zeros of a circuit with input derivative in section 5, finite λ of such a matrix pencil can be computed. By defining the matrices of the matrix pencil as

$$\hat{G} = \left[\begin{array}{ccc} E & -B_2 & -B_3 \end{array}\right], \quad \hat{F} = \left[\begin{array}{ccc} A & B_1 & O \end{array}\right], \tag{6.8}$$

the matrix pencil in equation (6.7) is transformed to a standard matrix pencil ($\hat{F} - \lambda\hat{G}$), for which finite λ is easily computed using the method discussed in this research.

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