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Hamiltonicity in Bidirected Signed Graphs and Ramsey Signed Numbers

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Hamiltonicity in Bidirected Signed Graphs and Ramsey Signed Numbers

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by

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I HEREBY RECOMMEND THAT THE THESIS PREPARED UNDER MY SUPERVISION BY Mohammed A. Mutar ENTITLED Hamiltonicity in Bidirected Signed Graphs and Ramsey Signed Numbers BE ACCEPTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF Master of Science.

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ABSTRACT

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Strong connectivity, 2-factors, and their relevance to Hamiltonicity, have been intensively studied on various classes of directed and 2-colored graphs. In chapter one, we define strong connectivity and bidirected 2-factors on bidirected graphs as a common generalization for both directed graphs and 2-colored graphs. We give necessary and sufficient conditions for the existence of bidirected Hamilton cycles in the following bidirected signed graphs: $\pm K_n$, $\pm K_{n,n}$, and $-K_{n,n}$.

The Ramsey number problem is considered an interesting problem in graph theory which asks for the minimum positive integer r that assures a 2-colored complete K_r has a monochramatic clique K_n or K_m. In chapter two, we define $r^*(n,m)$ to be the minimum positive integer that guarantees that any signing on K_r has, up to switching, $-K_n$ or $+K_m$. Also, the following results are obtained: $r^*(n,m) = r^*(m,n)$, $r^*(n,m) \leq r(n-1,m-1)+1$, $r^*(4,4) = 7$, $r^*(4,5) = 8$, and $10 \leq r^*(4,6) \leq 15$.

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Acknowledgment

I would like to take this opportunity to express my sincere gratitude to my advisor Prof. Daniel for his patience, guidance, support, and inspiration. I am privileged and even lucky to be supervised by an outstanding person and professor like him. Besides my advisor, I am extremely grateful to the kindest professor I have ever met Prof. Evans. Indeed, Joining his class in January 2016 was a wise decision because it was a good start of my academic journey at Wright State University. Also, many thanks must be recorded to those wonderful people by whom I have been sincerely taught.

Finally, I am very thankful to my family for their best wishes and prayers and my friends for their support.

Dedication

This humble work is dedicated to those whom I love and owe the most:

Mom, Dad, and Grandfather.

Hamiltonicity in Bidirected Signed Graphs

1.1 Literature Review

In the mathematical field of graph theory, *a graph* G is an orderd pair (V, E) where V represents the set of vertices and E is the set of edges in which each edge connects a pair of vertices. *A bipartite graph* is a graph whose vertices partitioned into two disjoint sets X and Y such that each edges connects a vertex from X and another one from Y. *A directed graph* is a graph in which each edge is given a one-way directional arrow. A tournament is a highly studied class of directed graphs, having attracted the attention of many mathematicians. *A tournament* is a directed complete graph K_n on n vertices. *A directed cycle* in a directed graph is a cycle whose edges are pointed in the same direction. If such a cycle includes all the vertices of a directed graph, then it is called *a directed Hamilton cycle*. In addition, *a directed path* is a path between two distinct vertices where all the edges are pointed in same direction.

A vertex x is said to be *strongly connected* to a vertex y if there are two directed paths, one from x to y and another from y to x. Moreover, a directed graph is *strongly connected* if any two distinct vertices are strongly connected. *A 2-factor* is a spanning 2-regular subgraph. That is, a 2-factor is a vertex-disjoint union of directed cycles covering all of the vertices of a graph G. *A directed 2-factor* is a 2-factor where the cycles in it are directed.

In 1959, Pual Camion [1] gave a necessary and sufficient condition for the existence of a Hamilton cycle in Tournaments.

Theorem 1 (Camion[1])**.** *A tournament D has a directed Hamilton cycle if and only if D is strongly connected.*

One year later, Foulkes [2] independently obtained the same result. In 1979, Grotschel and Harary [9] proved that strong connectivity on a directed graph G does not imply Hamiltonicity unless the underlying graph is a cycle C_n or a complete graph K_n. Therefore, the 2-factor in Theorem 2 cannot be removed. So, what other conditions along with strong connectivity yields Hamiltonicity?

A complete bipartite graph is a bipartite graph where every vertex in X is connected to all the vertices of Y. *A bipartite tournament* is a complete bipartite graph where each edge is given a one-way directional arrow. In 1987, Haggkvist and Manousskis [4] gave necessary and sufficient conditions for the existence of a Hamilton cycle in bipartite graphs.

Theorem 2 (Haggkvist and Manoussakis [4])**.** *A bipartite tournament B contains a Hamilton cycle if and only if B is strongly connected and has a directed 2-factor.*

A 2-colored graph is graph G such that each edge is colored by one of two possible colors, say red and blue. *An alternating cycle* is defined as a cycle in G where the edges are colored in an alternating fashion. Apparently, any alternating cycle in a 2-colored complete graph must have even length. A 2-factor in 2-colored graphs is called *alternating 2-factor* if each cycle in it is alternating. Saad [5] defined that any two different vertices x, y are *strongly connected* to each other if there are two alternating paths P_1 and P_2 from x to y such that the path P_1 starts with color 1 and the path P_2 starts with color 2, and one of the paths ends in edge colored by color 1 and the other path ends in edge colored by color 2. A 2-colored complete graph is defined to be *strongly connected* if there is strong connectivity between any two vertices. Furthermore, Saad established the sufficient and necessary conditions for existence of Hamilton cycle in 2-colored graphs as presented next.

Theorem 3 (Saad [5]). *A 2-colored complete graph is Hamiltonian if and only if the graph is strongly connected and contains an alternating 2-factor.*

Arthur Busch's example shows that the 2-factor in Theorem 3 cannot be removed as a necessary and sufficient condition, see Figure 1.1.

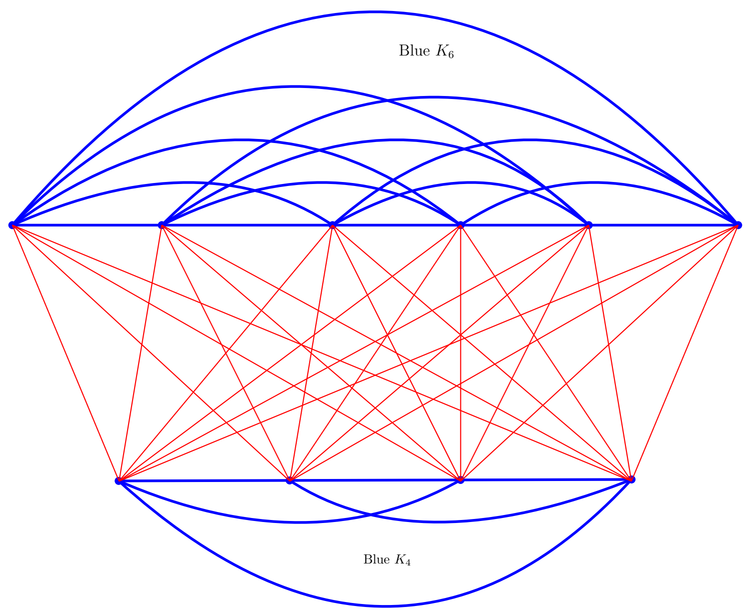


Figure 1.1: Busch's Example

1.2 A common generalization

A bidirected graph is an extraordinary graph introduced by Edmonds and Johnson in 1970 [6]. In a bidirected graph, each edge is given two arrows as in Figure 1.2.

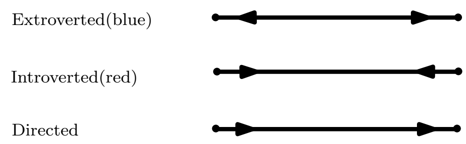


Figure 1.2: Types of bidirected graph's edges

Directed edges in bidirected graphs can be simply considered as directed edges of directed graphs. Similarly, we can think of introverted and extroverted as red and blue edges of 2-colored graphs, respectively. In accordance with the previous definition of a directed cycle in ordinary directed graphs and an alternating cycle in 2-colored graphs, *a bidirected cycle* is defined as a cycle in which no vertex on the cycle is solely a source or sink, see Figure 1.3. *A Hamilton cycle* in bidirected graph is a bidirected cycle including all the vertices of the graph.

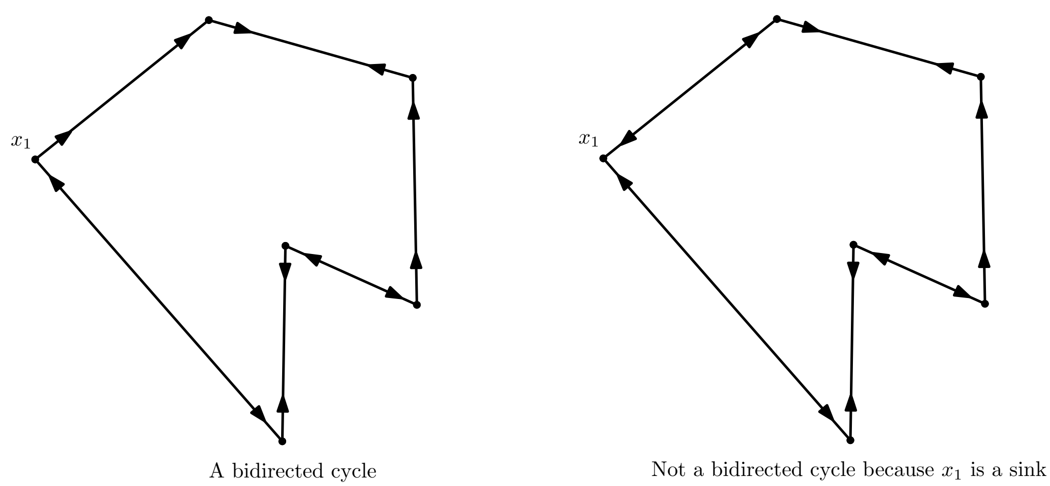


Figure 1.3: Bidirected and not bidirected cycles

A bidirected complete graph is defined as a complete graph K_n in which every edge is independently bidirected. An interesting problem of such graphs is finding necessary and sufficient conditions for the existence of a bidirected Hamilton cycle in a bidirected complete graph. This will generalize both ordinary tournaments and 2-colored complete graphs together into a common structure. This idea of generalization was noted by T. Zaslavsky [8](P.26).

In bidirected graphs we define that two vertices x, y in G are *strongly connected* when there are two bidirected $xy - paths$ whose arrows differ at x and y, see Figure 1.4. Therefore, G is *strongly connected* if there is strong connectivity between any two vertices.

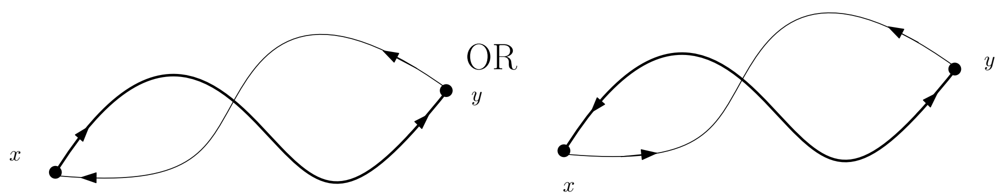


Figure 1.4: The two possible paths satisfying strong connectivity between x and y.

We thought that maybe a bidirected complete graph G is Hamiltonian if and only if G is strongly connected and contains a bidirected 2-factor. This statement, however, is false by the following example.

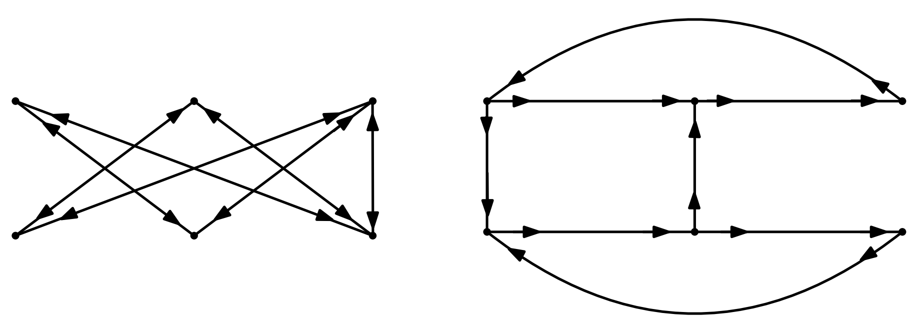


Figure 1.5: A bidirected complete graph that is strongly connected and has a bidirected 2-factor but fails to be Hamiltonian

Obviously, the bidirected edges on the right side of Figure 1.5 satisfy strong connectivity and existence of a bidirected 2-factor on 6 vertices. The remaining edges on the left side, all extroverted, prevent Hamiltonicity. However, a question arises whether or not there is a larger class of bidirected graphs in which strong connectivity along with bidirected 2-factors implies Hamiltonicity.

A signed graph is a triple (G, E, σ) where $\sigma : E(G) \longrightarrow \{+, -\}$. In other words, a signed graph is a graph such that each edge marked by a positive or negative sign. The emergence of signed graphs was in 1953 when Harary [9] published a mathematical paper defining such graphs and the notion of balance in signed graphs. In addition, $+G$ is the signed graph obtained from the graph G where every edge marked with a positive sign. Similarly, $-G$ is the signed graph in which every edge assigned a negative sign. $\pm G$ is a graph G in which every edge is replaced by a positive edge and negative edge.

Bidirection on signed graphs is an assignment of bidirected edges to each signed edge where the positive edge assigned a directed edge and the negative one assigned either an introverted or extroverted edge [10], see Figure 1.2.

Switching at a vertex v in signed graphs is defined as interchanging the sign of all incident edges of v with the opposite sign. *Switching* at a vertex v in directed graphs and bidirected graph is defined as interchanging the direction of arrow of all incident edges of v with opposite arrows, see Figure 1.6.

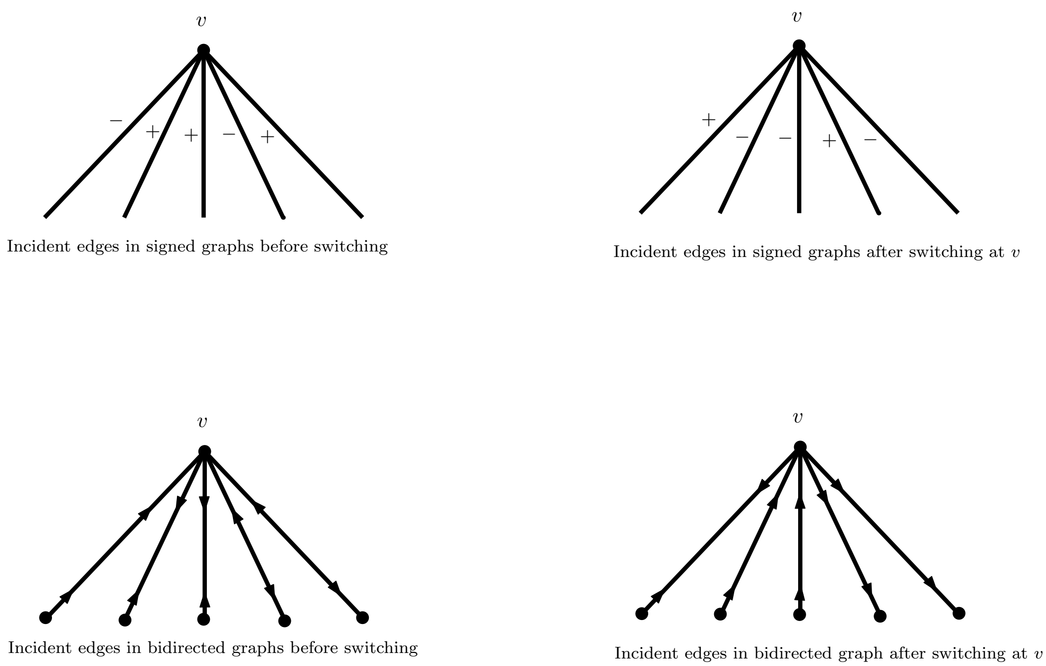


Figure 1.6: Switching in signed and (bi)-directed graphs

We will be looking for bidirected signed graphs which satisfy the following Predicates.

Predicate 1. *Let B be a bidirected signed graph, then B contains a bidirected Hamilton cycle if and only if B is strongly connected.*

Predicate 2. *Let B be a bidirected signed graph, then B contains a bidirected Hamilton cycle if and only if B is strongly connected and has a bidirected 2-factor.*

We have already seen that Predicate 1 holds for $B = +K_n$ and Predicate 2 holds for $B = -K_n$ and $B = +K_{n,n}$. We will prove that Predicate 1 holds for $B = \pm K_n$ and Predicate 2 holds for $B = \pm K_{n,n}$ and $B = -K_{n,n}$.

1.3 An important comment on bidirected 2-factors

It might seem that having a 2-factor is a heavy condition; however this is not the case. Finding a 2-factor in bidirected graph is equivalent to finding a complete matching which can be done in polynomial time $O(n^{2.5})$ [11]. The process of finding a 2-factor in directed is basically demonstrated as follows:

- Split each vertex into two vertices and label one of them as in-vertex and the other one as out-vertex with in-edges at the in-vertex and out-edges at the out-vertex. Call this new graph $\widetilde{B}$.
- Find a perfect matching in $\widetilde{B}$ and re-combine each in-vertex with its out-vertex to get a bidirected 2-factor.

1.4 Hamiltonicity in $\pm K_n$

Predicate 1 holds for $\pm K_n$.

Theorem 4. *Let B a bidirected $\pm K_n$. Then B is Hamiltonian if and only if B is strongly connected.*

Proof. Let B be a bidirected $\pm K_n$ on n vertices and assume x, y are two vertices in B. Since B is strongly connected, then there are two bidirected $xy - paths$ whose arrows differ at x and y. We claim there is a bidirected cycle containing x and y.

If P_1 and P_2 do not overlap at any vertex except x and y, then P_1 and P_2 together create a bidirected cycle C. Otherwise, as a result, neither P_1 or P_2 are of length one. Now, switch if necessary to make P_1 as a directed path from x to y, see Figure 1.7.

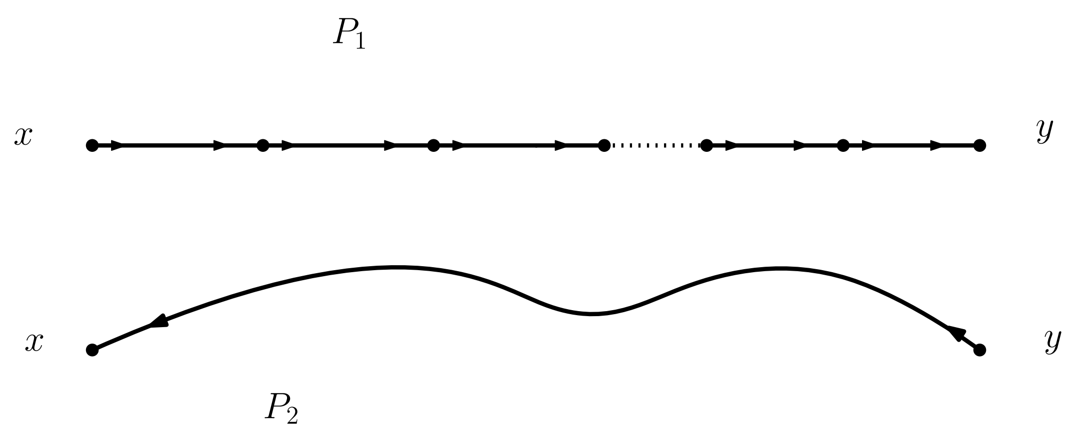


Figure 1.7: Two bidirected paths P_1, P_2 are made directed by switching

Obviously, $(x, y)^+$ guarantees existence of a bidirected cycle C. Assume C is the biggest possible bidirected cycle of length k given by $C = \{x_1, x_2, ..., x_k\}$. By switching we may assume that C is a directed cycle.

If $k = n$, then there is nothing left to prove, i.e. C is the required Hamilton cycle. Otherwise, there is an vertex v_1 in B is not included in C. Let us try to extend the cycle C by including the vertex v_1.

Assume without loss of generality $(v_1, x_i)^+$ is a directed edge from v_1 to x_i, then $(v_1, x_{i-1})^+$ is either directed from x_{i-1} to v_1, which in this case C can be extended by adding v_1 as illustrated in Figure 1.8 or $(v_1, x_{i-1})^+$ is directed from v_1 to x_{i-1}.

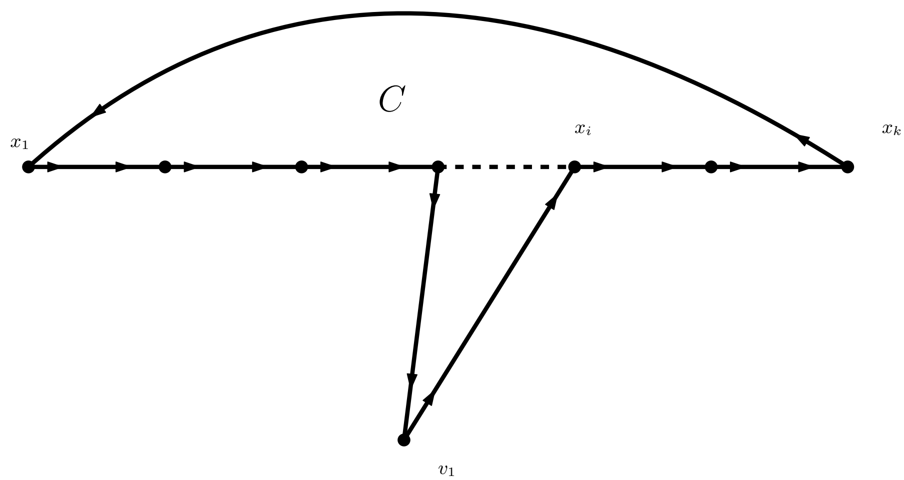


Figure 1.8: The extension of the cycle C by adding the vertex v_1

This leaves us with only two outcomes to consider if C cannot be extended by adding v_1. The two cases are either all the positive edges $\{(v_1, x_1)^+, (v_1, x_2)^+, ..., (v_1, x_k)^+\}$ are directed from v_1 to C, or equivalently directed from C to v_1. Also, by switching, all the negative edges $\{(v_1, x_1)^-, (v_1, x_2)^-, ..., (v_1, x_k)^-\}$ are introverted or all extroverted. As a result, there are only four possible edge-patterns as shown in Figure 1.9.

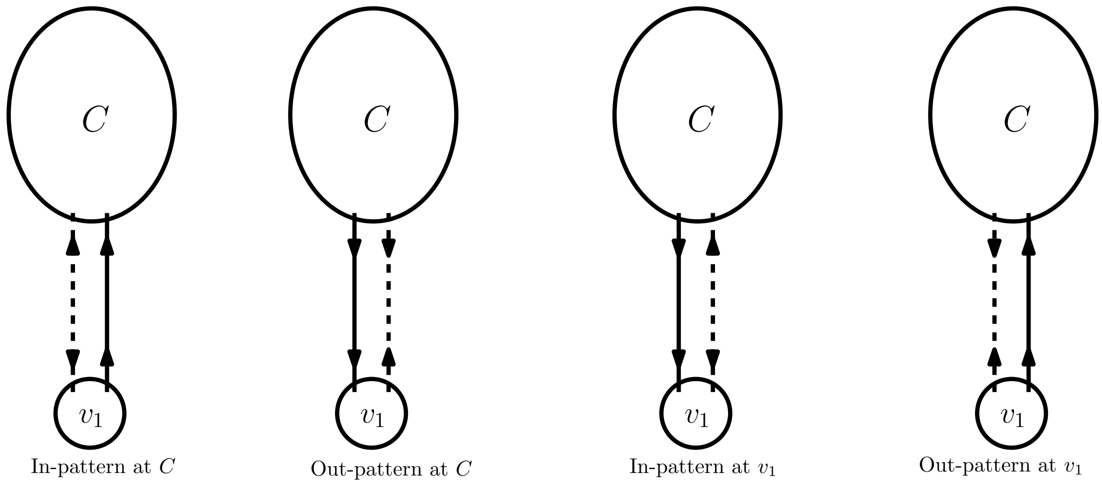


Figure 1.9: Four possible edge-patters for a vertex failing to be added to C

An introverted edge inside C can form an introverted-path of length $k-1$. Similarly, an extroverted edge inside C can form an extroverted-path of length $k-1$, see Figure 1.10.

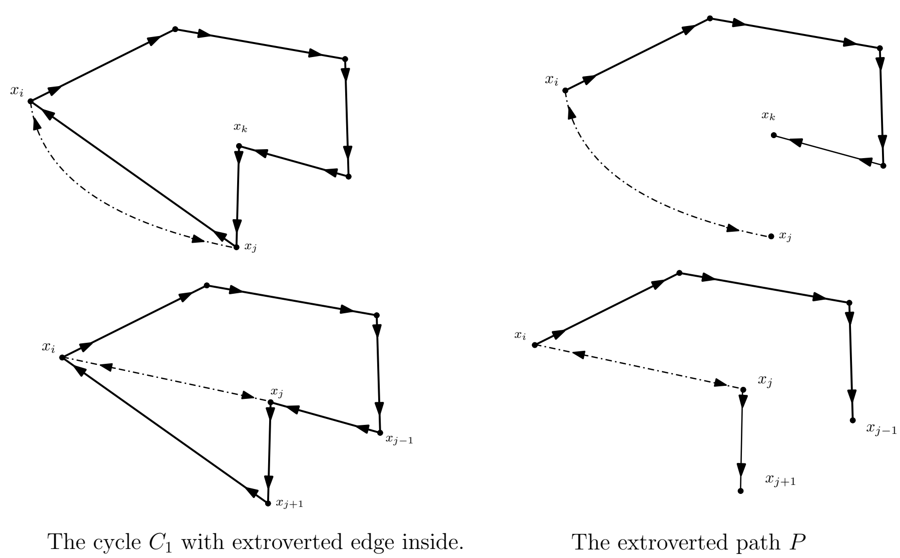
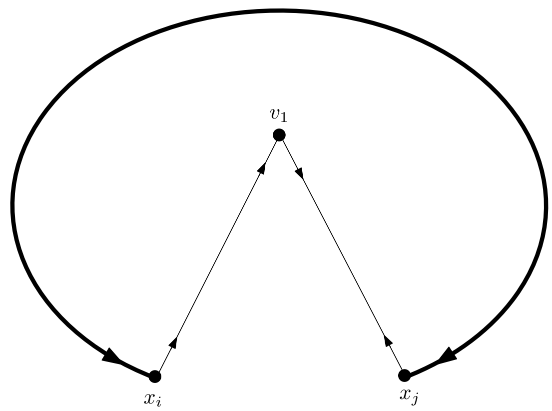


Figure 1.10: Extroverted paths generated by extroverted edges inside C.

At this stage, two cases must be considered.

(A) The connecting edges, all the negative and positive edges between C and v_1, form out-pattern at C, then the cycle C either can be extended by using an extroverted-path of length $k-1$ along with one negative connecting edge and another positive one as illustrated in Figure 1.11.



The extroverted path P is drawn in bold line.

Figure 1.11: Using an extroverted path for adding a vertex into C.

Or, the negative edges inside C are all introverted which means there is no strong connectivity between any vertex of C and v_1 because C behaves like a source preventing any path from entering C and coming back to v_1. Also, if the connecting edges form in-pattern, then either the cycle C can be extended following the same process above or the negative edges inside C are all extroverted which again ruins strong connectivity because C behaves like a sink.

(B) If the connecting edges between C and v_1 form in(out)-pattern at v_1, then there is no strong connectivity between any vertex of C and v_1 because v_1 is a sink(source).

We will proceed by contradiction. Assume C cannot be extended by adding any proper subset of $V = \{v_1, v_2, ..., v_{n-k}\}$. Then we observe the following

I. C cannot have only out-patterns or only in-patterns at its vertices because that ruins strong connectivity.

II. C cannot have both out-pattern and in-patterns simultaneously because that allows us to extend C into a cycle of length $k+1$ as explained before by using either an extroverted path or introverted one. Without loss of generality, assume C has no in-patterns at all.

Now, let $x \in C$ and $v \in V$, since x is strongly connected in B, there is a bidirected cycle S containing x and v.

If S intersects with C in only the vertex x, then using switching we may assume S is a directed cycle, see Figure 1.12.

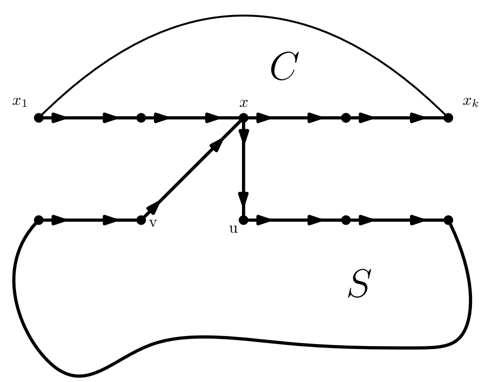


Figure 1.12: Two directed cycles overlapping in one vertex.

Using the directed edges $(x_1, v)^+$ and $(x_k, u)^+$ will assemble S and C into one directed cycle, contradiction.

If S intersects with C in more than one vertex, then that would create strands. *A strand* is an introverted, extroverted, or directed path generated on some vertices of V such that both ends of this path go into C. However, since the edges inside C are either introverted or directed edges, then there are only introverted and directed paths. Consequently, there is no extroverted strand.

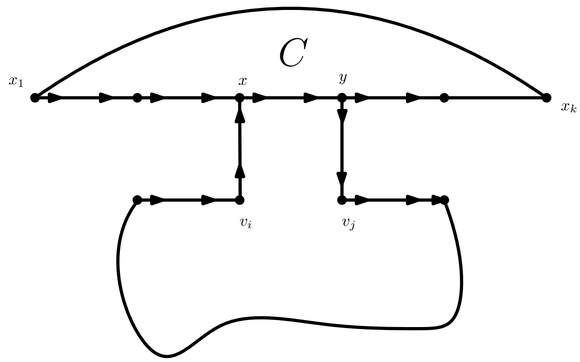


Figure 1.13: A directed strand created by the overlapping of C and S

Similarly, using the directed edges $(x_1, v_i)^+$ and $(x_k, v_j)^+$ will extend C by some vertices of V, contradiction.

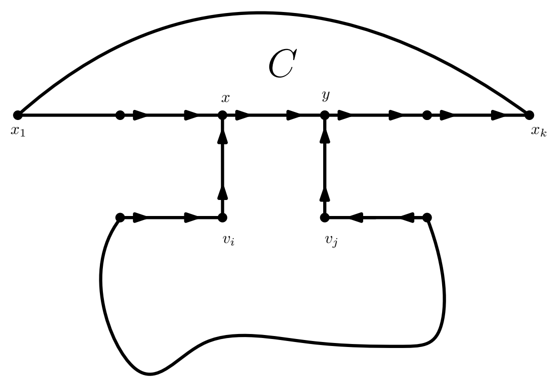


Figure 1.14: A extroverted strand created by the overlapping of C and S

Using the directed edge $(x_1, v_i)^+$ and the introverted $(x_k, v_j)^+$ will extend C by some vertices of V, contradiction.

QED

1.5 Hamiltonicity in $-K_{n,n}$

Predicate 2 holds for $-K_{n,n}$.

Theorem 5. *Let B be a bidirected $-K_{n,n}$. Then B is Hamiltonian if and only if B is strongly connected and contains a bidirected 2-factor.*

Proof. Assume that the vertices of $B = -K_{n,n}$ are distributed into two disjoint sets U and V. By switching at the vertices of U, all the negative edges will become positive edges. The introverted and extroverted edges will be interchanged into directed edges. That is, the graph $-K_{n,n}$ becomes $+K_{n,n}$. Moreover, bidirected 2-factors and strong connectivity are preserved under switching. Now, we can invoke Theorem 2 to get a directed Hamilton cycle C of $2n$ length. Since C covers all the vertices of the graph and half of them are the vertices of U, switch again at the vertices of U to get alternating Hamilton cycle.

QED

Similarly, Predicate 1 does not hold for $-K_{n,n}$ by switching and Grotschel/Harary's Theorem [3].

1.6 Hamiltonicity in $\pm K_{n,n}$

Predicate 2 holds for $\pm K_{n,n}$.

Theorem 6. *Let B be a bidirected $\pm K_{n,n}$. Then B contains a bidirected Hamilton cycle if and only if B is strongly connected and contains a bidirected 2-factor.*

Proof. The vertices of B are distributed into two disjoint sets U and V each of n vertices. Necessity is clear, we only need to show sufficiency. Assume B contains a 2-factor, then the vertices of B form vertex-disjoint bidirected cycles of even lengths where each cycle contains some vertices of U and equal number of vertices from V. Apply switching so that the bidirected cycles in the 2-factor are directed cycles.

I. Firstly, we will try to connect two bidirected cycles of any lengths. Let C_1 and C_2 be two bidirected cycles of length k and m (both of even lengths) where $k \leq m$. Let C_1={x_1, x_2,..., x_k} and C_2={y_1, y_2,..., y_m} such as

$$\{x_1, x_3, ..., x_{k-1}\} \cup \{y_2, y_4, ..., y_m\} \subseteq U$$

$$\{x_2, x_4, ..., x_k\} \cup \{y_1, y_3, ..., y_{m-1}\} \subseteq V$$

Assume that C_1 and C_2 are in the directions $\{x_1, x_2, x_3, ...\}$ and $\{y_1, y_2, y_3, ...\}$. Since the graph B is complete bipartite, the negative edges between C_1 and C_2 partitioned into precisely $m/2$ matchings. Same for the positive edges. These matchings are $M_0^\epsilon, M_2^\epsilon, ..., M_{m-2}^\epsilon$ where $M_i^\epsilon = \{(x_j, y_{j+i})^\epsilon : 1 \leq j \leq k\}$ and $\epsilon \in \{+, -\}$. A quadrangle Q is singular when each vertex in Q is a sink or source.

i. Suppose there is a singular quadrangle Q of length 4 involving 2 adjacent vertices of C_1 and another two adjacent vertices of C_2 where two vertices in Q are sinks and the others are sources. If a singular quadrangle exists on x_3, x_4, y_3, y_4, then the two bidirected cycles C_1 and C_2 can be easily assembled into a bidirected cycle of length $k + m$ as shown in Figure 1.15.

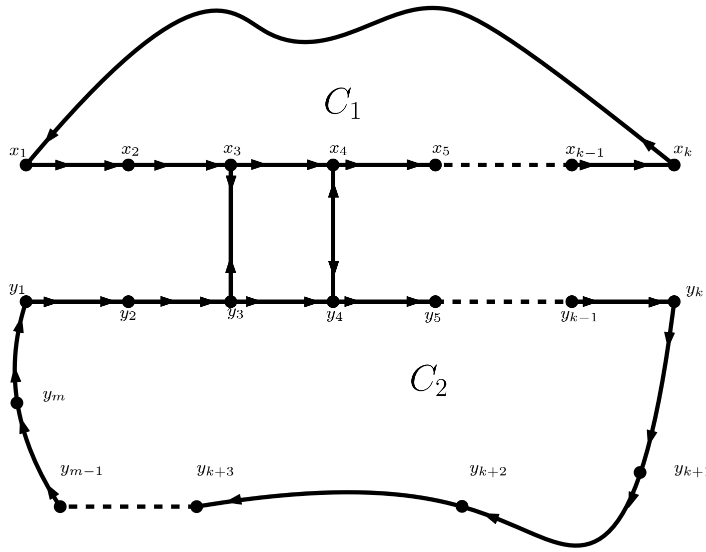


Figure 1.15: A singular quadrangle Q

ii. Suppose there is no singular quadrangle Q joining C_1 with C_2. Then, each matching M_i^- where $i = 0, 2, ..., m-2$ is identically bidirected.

Without loss of generality, assume the negative matching M_0^- is introverted and the negative matching M_2^- is extroverted as shown in Figure 1.16.

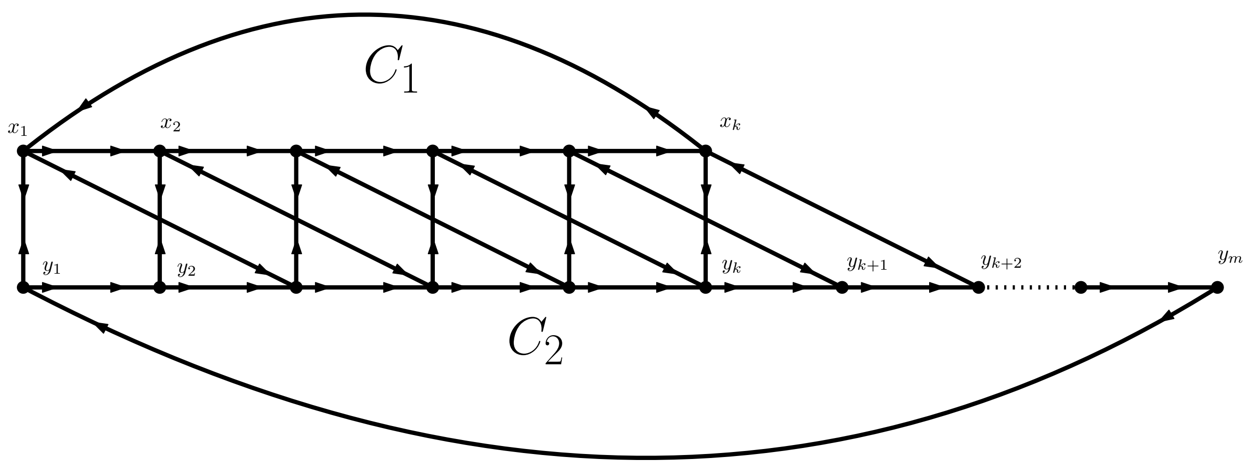


Figure 1.16: An example of two various negative matchings

This case directly yields a bigger bidirected cycle of length $m + k$ by assembling C_1 and C_2 using this technique in Figure 1.17.

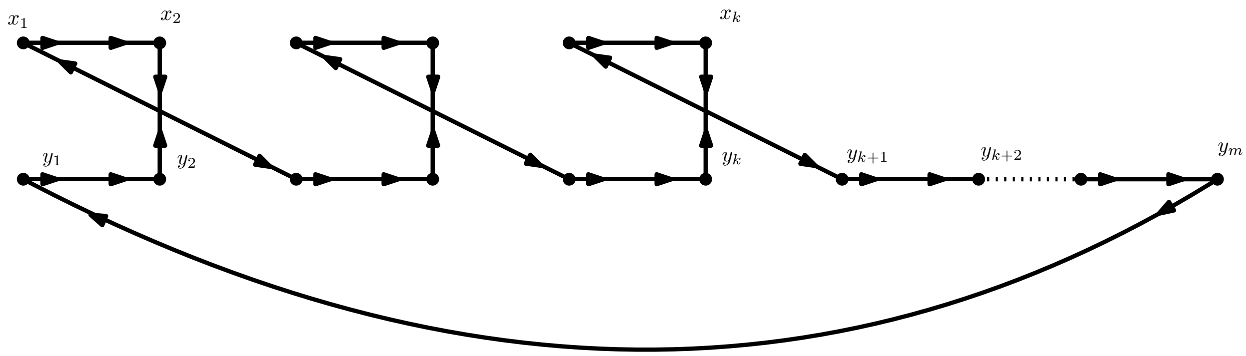


Figure 1.17: Assembling C_1 and C_2 by using various negative matchings.

In conclusion, If C_1 and C_2 cannot be assembled into a bigger bidirected cycle of length $m+k$, then the negative machings are all identically extroverted or introverted. Simultaneously, switching at the vertices of C_1 or C_2 yields the positive machings are identically directed. Accordingly, for any two bidirected cycles that fail to be joined together, there are four possible patterns of connecting edges with respect to C_1 and C_2 as illustrated in Figure 1.18.

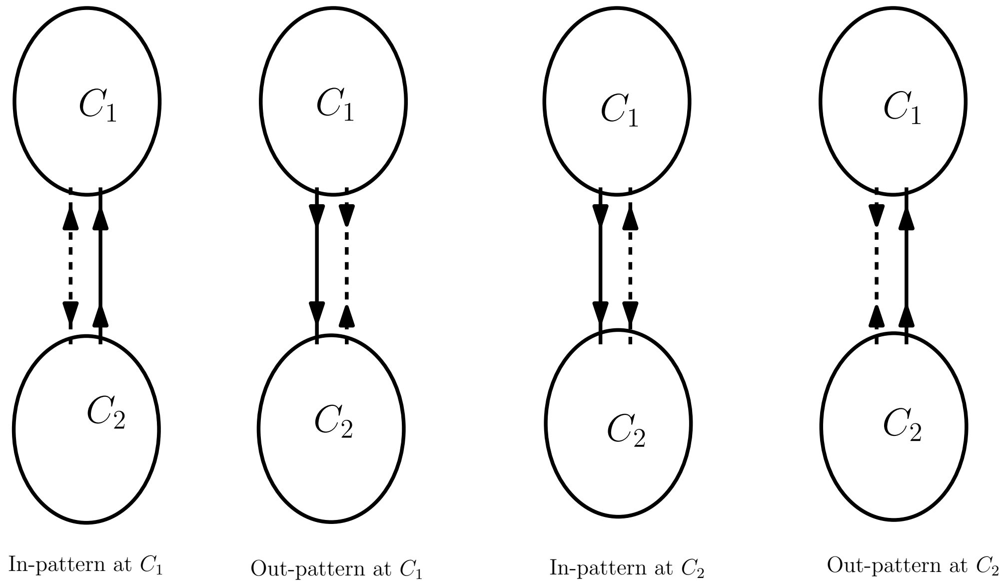


Figure 1.18: Possible edge-patterns between C_1 and C_2.

The negative edges inside each cycle have been deliberately ignored determining to the final steps. Let assume without loss of generality that the connecting edges between C_1 and C_2 create an out-pattern at the vertices of C_1, then the negative edges inside C_1 are all introverted because otherwise we obtain a bigger bidirected cycle of length $k + m$. To prove this assume that the connecting edges create an out-pattern at C_1 and there is an extroverted edge between x_i and x_j. Then, as illustrated before in Figure 1.10, there is an extroverted-path P of length $k - 1$ containing all vertices of C_1. The path P along with one negative connecting edge and another positive one give us a bigger bidirected cycle of length $m + k$.

At this stage, we find that if C_1 and C_2 cannot be assembled into a bigger bidirected cycle, then there is no strong connectivity between C_1 and C_2. This leads to hypothesize that the bidirected 2-factor B contains more than two bidirected cycles.

Let $C = \{C_1, C_2, ..., C_N\}$ be collection of vertex-disjoint directed cycles, covering all the the vertices of B, where this collection cannot be reduced into a smaller collection by assembling its cycles together. Strong connectivity implies that any bidirected cycle cannot have only in-patterns or only out-patterns at its vertices. Moreover, no cycle can have both in-patterns and out-patterns because the negative or positive edges inside that cycle will form a bigger bidirected cycle which contradicts our assumption of irreducible collection.

Let's consider the cycles of $C = \{C_1, C_2, ..., C_N\}$ as vertices and $\widetilde{B}$ denoted to the graph formed by these vertices. Also, consider the negative and positive edges between any two cycles in C as only one negative edge and only one positive edge, respectively. Evidently, $\widetilde{B}$ represents a bidirected $\pm K_N$. If $\widetilde{B}$ contains a bidirected cycle $C_{i_1}, C_{i_2}, ..., C_{i_k}$, then the cycles $C_{i_1}, C_{i_2}, ..., C_{i_k}$, can be assembled together into one bidirected cycle of length $|C_{i_1}| + |C_{i_2}| + ... + |C_{i_k}|$ as shown in Figure 1.19, where we assume by switching that the bidirected cycle in $\widetilde{B}$ is a directed cycle.

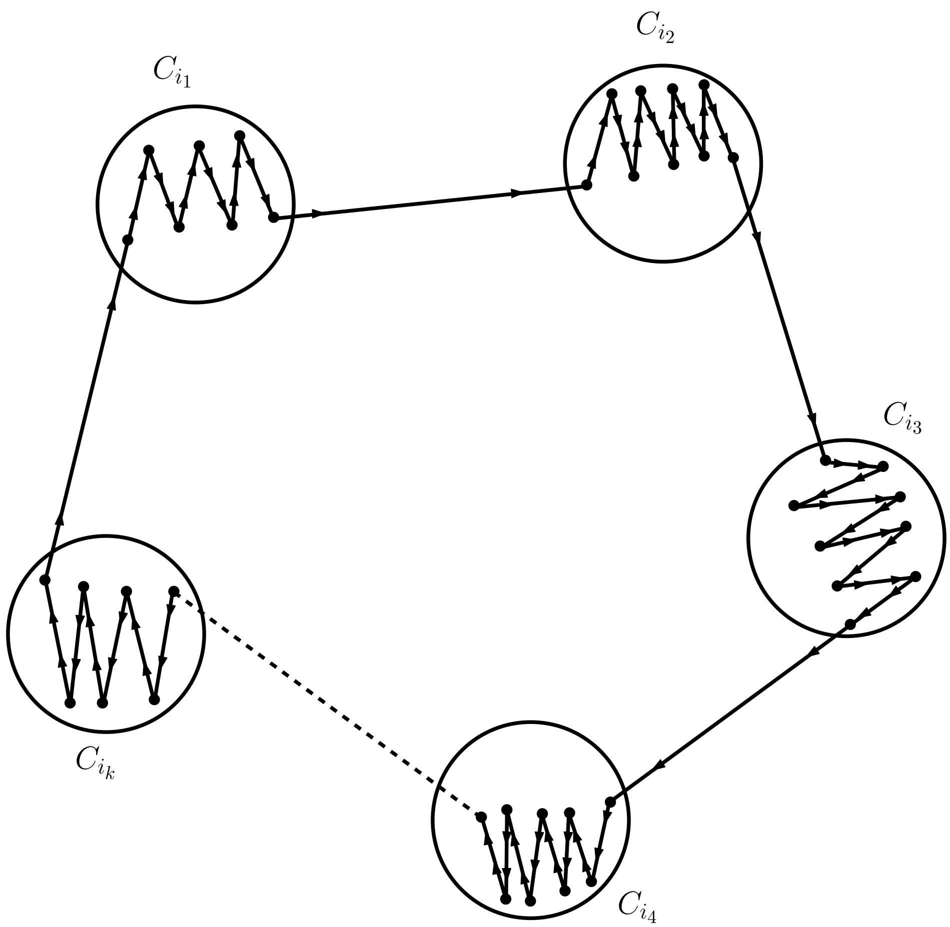


Figure 1.19: Assembling k bidirected cycles into one bidirected cycle.

Let us denote the connecting edges at the vertices of any cycle as a various pattern when these connecting edges do not form an in-pattern or out-pattern at that cycle. Therefore, because of strong connectivity, every cycle C_i has a various pattern at its vertices with edges connecting to some C_j. Let P the longest bidirected path created successively by using a various pattern at some vertices in $\widetilde{B}$. Switch if necessary to make P a directed path as shown in Figure 1.20.

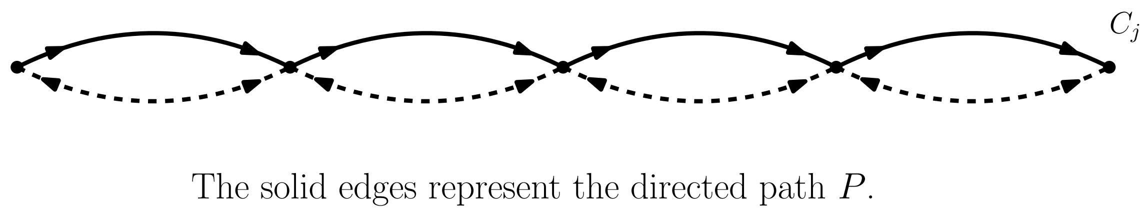


Figure 1.20: The longest bidirected path in the graph $\widetilde{B}$ after switching.

Since there is a various pattern at C_j and P is the longest bidirected path, then either there is a directed edge from C_j to C_i where $i \leq j-2$ and that would form a directed cycle. Or the introverted edge $(C_i, C_j)^-$ along with the extroverted $(C_i, C_{i+1})^-$ a bidirected cycle.

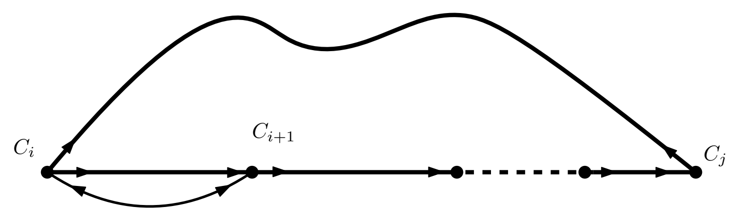


Figure 1.21: A bidirected cycle generated by the longest directed path in $\widetilde{B}$.

This contradicts our assumption of irreducible collection of bididirected cycles.

QED

Ramsey's Numbers on Signed Graphs

2.1 Literature Review

The Ramsey number is the solution to a well-known problem called *Party problem*. *The Ramsey number* is denoted by $r(n, m)$ referring to the minimum number of invited people in a party assuring that n of them are acquainted or m are unacquainted. In graph theory, the Ramsey number $r(n, m)$ represents the minimum sufficient number r of vertices to obtain red K_n or blue K_m in any 2-colored complete graph K_r. In 1930, Ramsey proved that the number $r(n, m)$ always exists and $r(n, m) = r(m, n)$ [12]. In 1955, Greenwood and Gleason [13] proved the following results: $r(3, 3) = 6$, $r(3, 4) = 9$, $r(3, 5) = 14$, and $r(4, 4) = 18$.

2.2 Signed Ramsey numbers

We define new type of Ramsey number $r^*(n, m)$ in signed graphs as the minimum number of vertices of a signed complete graph that guarantees obtaining, up to switching, $-K_n$ or $+K_m$ as a subgraph. It is worth pointing that there are exactly $[n/2] + 1$ different dashed degree sequences of $-K_n$ and equivalently the same number of $+K_n$.

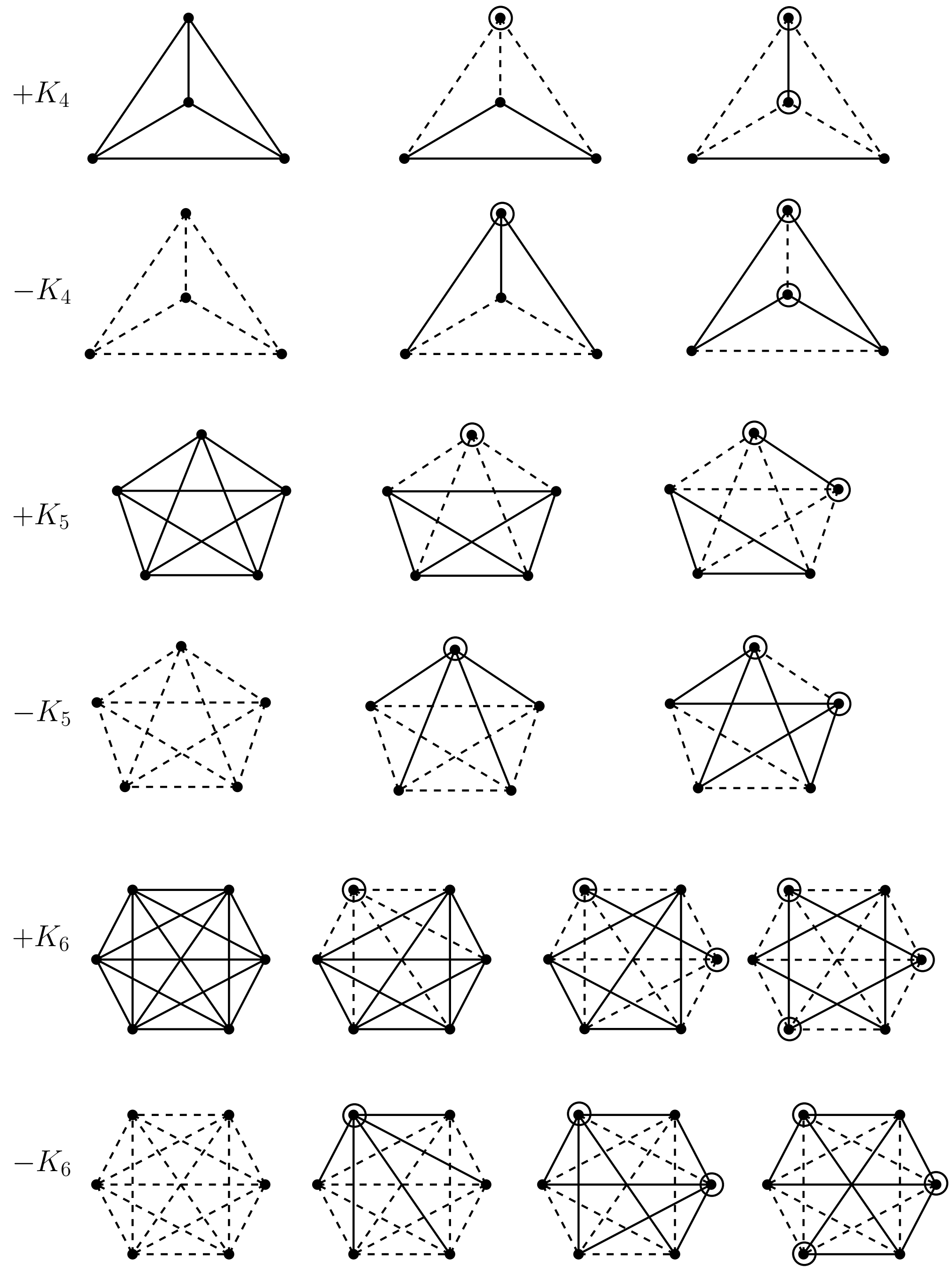


Figure 2.1: Up to switching $-K_n$ and $+K_n$ for $n = 4, 5, 6$.

2.3 Upper Bounds and Symmetry

In particular Proposion 1 shows that $r^*(n, m)$ exists because $r(n, m)$ exists.

Proposition 1. $r^*(n, m) \leq r(n-1, m-1) + 1$

Proof. Let σ be a signing on K_{r+1} where $r(n-1, m-1) = r$. Pick a vertex v and by switching make all the incident edges of v are positive. Since $r(n, m) = r$, the residual K_r has either all-positive K_{m-1} or all-negative K_{n-1}. Finally, Adding v yields either $+K_m$ or $-K_n$ up to switching. QED

Proposition 2. $r^*(n, m) = r^*(m, n)$

Proof. Assume $r^*(n, m) > r^*(m, n) = t$. Let $\sigma : E(G) \longrightarrow \{+, -\}$ be a signing on K_t such that there is no $-K_n$ and $+K_m$. This yields the signing $-\sigma$ on K_t contains no $-K_m$ and $+K_n$, contradiction.

QED

2.4 Some Small Signed Ramsey Numbers

Theorem 7. $r^*(4, 4) = 7$

Proof. By Proposition 1, $r^*(4, 4) \leq r(3, 3) + 1 = 7$. The example below shows $r^*(4, 4)$ is strictly greater than 6.

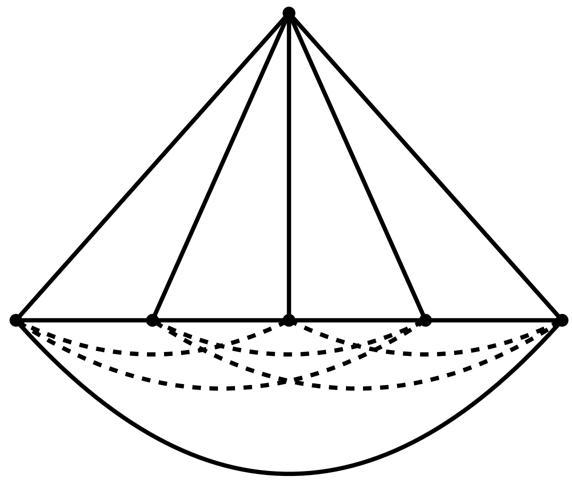


Figure 2.2: An example shows $r^*(4,4)$ is strictly greater than 6.

QED

Theorem 8. $r^*(4,5)=8$

Proof. Assume that $\sigma : E(K_8) \longrightarrow \{+,-\}$ is a given signing on K_8. Make the incident edges at a vertex v all positive by switching. Assume the residual K_7-graph has no all-positive K_4 or all-negative K_3 because the existence of such subgraphs along with the vertex v would directly form $+K_5$ or $-K_4$. We conclude that the dashed degree at each vertex in the residual K_7 must be $1 \leq dd \leq 3$ because if $dd_x = 0$ then the six incident edges of x in K_7 are all positive and these edges with the six vertices except x in K_7 would directly form $-K_4$ or all positive K_4 since $r(3,3)=6$. Also, the dashed degree cannot be strictly greater than 3 because if $dd_x \geq 4$ then the four vertices which are connected with x by the four incident negative edges would form all-positive K_4 on the purpose of avoiding all-negative K_3. Thus, the number of negative edges should be between 4 and 10.

If the number of the negative edges are in total 10, then the dashed degree sequence is 3333332. Up to isomorphism, there are only three graphs with this degree sequence, see Figure 2.3.

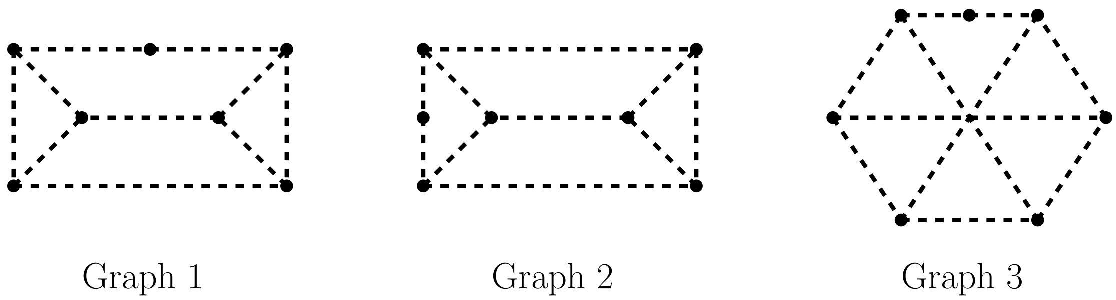


Figure 2.3: Subdivision of cubic graphs on six vertices.

The first two graphs have all-negative K_3. The third graph has all-negative $-K_{2,3}$, as an induced subgraph, which is along with the positive edges we get $+K_5$ (see row 3 of Figure 2.1).

Let us define that any two negative edges $(x, y)^-$, $(w, z)^-$ in a signed graph are *negatively isolated* if and only if the rest of the edges of $K_4 - subgraph$ on x, y, w, z are all positive. Note that if the subgraph on the dashed is disconnected, then there is a pair of negatively isolated edges.

Now, we claim that replacing a negative edge with a positive one in a signing σ on the residual K_7 containing at most 10 negative edges will make two edges negatively isolated.

In other words, generally, a connected graph on 7 vertices with at most 9 either has a triangle or there is a pair of negatively isolated edges.

Suppose there exists $(x, y)^- \in (K_7, \sigma)$ such that any e_1, e_2 in $(K_7, \sigma) - \{(x, y)^-\}$ are not negatively isolated. We will proceed by contradiction on the dashed degree of x and y.

If $dd_x = dd_y = 3$, then the only possible negative signing on 7 vertices such that the negative signing contains no triangle would be

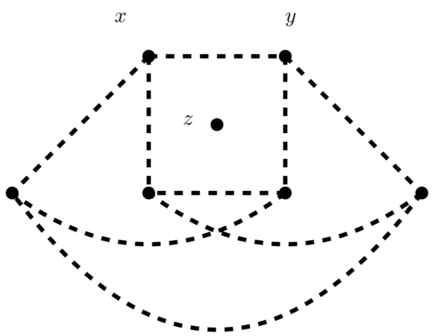


Figure 2.4: $dd_x = dd_y = 3$ and $(x, y)^-$ is not an edge of all negative K_3.

The vertex z is of zero dashed degree and that is not allowed because $1 \leq dd \leq 3$.

If $(x, y)^-$ is an edge of a triangle, then the potential signings would be

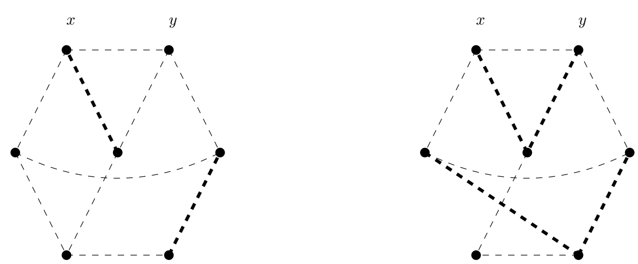


Figure 2.5: $dd_x = dd_y = 3$ and $(x, y)^-$ is an edge of all negative K_3.

Or their possible connected subgraphs after removing the edges generating negatively isolated edges, see Figure 2.6

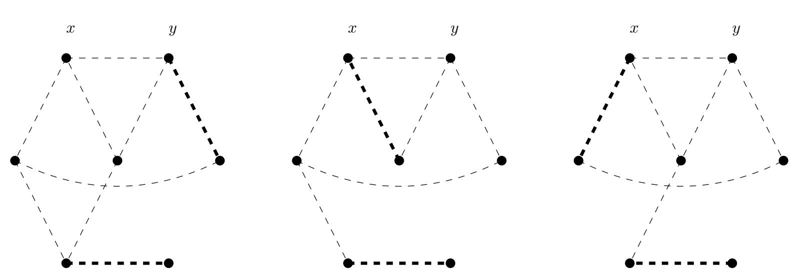


Figure 2.6: $dd_x = dd_y = 3$ and $(x, y)^-$ is an edge of all negative K_3 after removing some edges.

Obviously, the bold edges in each signing are negatively isolated.

Similarly, if $dd_x = 3, d_y = 2$:

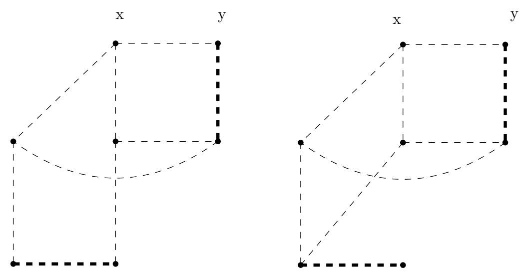


Figure 2.7: $dd_x = 3, dd_y = 2$ and $(x, y)^-$ is not an edge of all negative K_3.

Or their connected subgraph

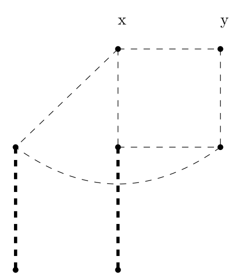


Figure 2.8: $dd_x = 3, dd_y = 2$ and $(x, y)^-$ is not an edge of all negative K_3 after removing some edges.

Obviously, the bold edges in each signing are negatively isolated.

If $(x, y)^-$ is an edge of a triangle, then the potential signing would be

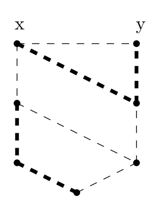


Figure 2.9: $dd_x = 3, dd_y = 2$ and $(x, y)^-$ is an edge of all negative K_3.

Similarly, If $dd_x = dd_y = 2$, then

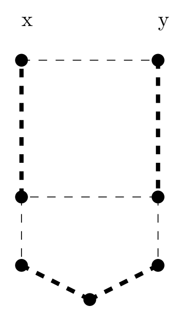


Figure 2.10: $dd_x = dd_y = 2$ and $(x, y)^-$ is not an edge of all negative K_3.

If $(x, y)^-$ is an edge of a triangle, then

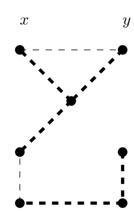


Figure 2.11: $dd_x = dd_y = 2$ and $(x, y)^-$ is an edge of all negative K_3.

The bold edges in each signing are negatively isolated.

Hence, we have just proved that any signing using 9 negative edges or less on 7 vertices such that the dashed degree at each vertex is $1 \leq dd \leq 3$ and the negative signing contains no triangle implies that there are two edges are negatively isolated. Any two negatively isolated edges forms $-K_4$(see row 2 in Figure 2.1). So, $r^*(4, 5) \leq 8$. The example below shows the $r^*(4, 5)$ strictly greater than 7.

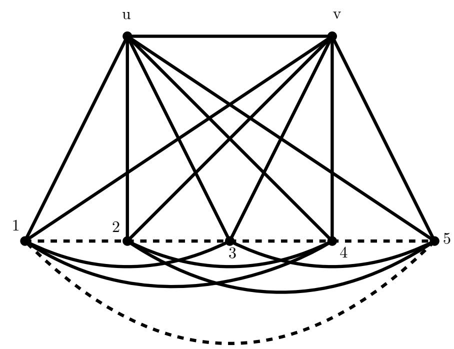


Figure 2.12: An example shows the $r^*(4,5)$ strictly greater than 7.

The subgraph K_5 on the vertices $1,2,3,4,5$ has no two negatively isolated edges or all negative triangle, so there is no $-K_4$. Also, It has no all-positive K_3, no all negative $K_{2,3}$ or $K_{1,4}$ as induced subgraphs, so there is no $+K_5$

QED

Theorem 9. $10 \leq r^*(4,6) \leq 15$

Proof. By Proposition 1, $r^*(4,6) \leq r(3,5)+1 = 15$. The example below shows $r^*(4,6)$ is strictly greater than 9.

Let the incident edges at a vertex v are all positive. Assume the residual K_8-graph has the following negative signing.

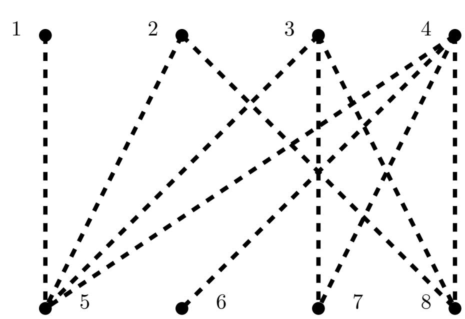


Figure 2.13: An example shows the $r^*(4,6)$ strictly greater than 9.

Clearly, in the residual K_8, there are no two negatively isolated edges or all-negative triangle, so there is no $-K_4$. Also, there is no all-positive K_5, no vertex of dashed degree 5, and no all-negative $K_{4,2}$ or $K_{3,3}$. Therefore, there is no $+K_6$ of any type.

QED

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