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John-Stromberg Inequality for Certain Anisotropic BMO Spaces

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John-Strömberg Inequality for Certain Anisotropic BMO Spaces

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ABSTRACT

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In this paper, we establish the John-Strömberg inequality in certain anisotropic BMO spaces and apply the inequality to anisotropic Hölder continuous function spaces. To achieve this goal, we define the median value, BMO space and sharp maximal function first. Secondly we constructed a family of continuously expanding rectangles and give a Calderón–Zygmund type decomposition. By using these tools, we establish the John-Strömberg inequality. Finally, an application to anisotropic Hölder continuous function spaces is presented.

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The John-Strömberg Inequality for Certain Anisotropic BMO Spaces

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§1 **Introduction**

The spcace $BMO(\mathbb{R}^n)$ consisting of all real-valued functions of bounded mean oscillation was introduced in [4]. Subsequently,$BMO(\mathbb{R}^n)$ space plays an important role in the study of harmonic analysis, function spaces, and partial differential equations. By [4], a function $f \in L^1_{loc}(\mathbb{R}^n)$ is in $BMO(\mathbb{R}^n)$ if

$$\sup_Q \fint_Q |f - f_Q| dx < \infty$$

where the supremum is taken over all Euclidean cubes with edges parallel to coordinate axes and $f_Q = \fint_Q f dx$ denotes the integral average of f over Q.

A deep characterization for measurable functions to be in $BMO(\mathbb{R}^n)$ using median values was given in [3] and [5]. Specifically, if there exist $0 < s < \frac{1}{2}$ and $M > 0$ such that for any cube Q

$$|\{x \in Q : |f - m_f(Q)| > M\}| \leqslant s|Q|$$

then $f \in BMO(\mathbb{R}^n)$. Here $|Q|$ denotes the Lebesgue measure of Q and $m_f(Q)$ is the median value of f over Q (see Def 2.2). This result was extended in [6] from the setting of the Lebesgue measure and Euclidean cubes to general setting of doubling measures and metric balls but the result in [6] requires s to be sufficiently small. This John-Strömberg inequality was recently used in the study of the regularity of solutions of fully nonlinear elliptic equations (see [1]).

Our main purpose here is to extend the results in [3],[5] to the setting of the Lebesgue measure and certain families of special rectangles with $0 < s <$

$\frac{1}{2}$. The advantage of our result (Theorem 3.7 below) is that s needs not to be sufficiently small. Instead, it could be any positive number less than $\frac{1}{2}$. Some applications to anisotropic Hölder continuous function spaces are also presented in §4.

§2 Preliminaries

Let $0 < s < \frac{1}{2}$ and $k = (k_1, k_2, ..., k_n) \in \mathbb{N}^n$. Denote $|k| = \sum_1^n k_i$.

Throughout this paper, all functions considered are Lebesgue measurable, real-valued, and finite almost everywhere.

***Definition* 2.1.** For $r > 0$, $x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$, the k-rectangle with center x and edgelengths $r^{k_1}, r^{k_2}, ..., r^{k_n}$ is given by

$$R_r(x) = \{y \in \mathbb{R}^n : |y_i - x_i| \leqslant \frac{r^{k_i}}{2}, 1 \leqslant i \leqslant n\}.$$

The collection

$$\mathcal{R} = \{R_r(x) : x \in \mathbb{R}^n, r > 0\}$$

will represent the family of all such k-rectangles.

***Definition* 2.2.** Let $f(x)$ be a measurable function in a bounded domain D. A median value $m_f(D)$ of f over D is defined by:

$$|\{x \in D : f(x) > m_f(D)\}| \leqslant \frac{1}{2}|D|$$

and

$$|\{x \in D : f(x) < m_f(D)\}| \leqslant \frac{1}{2}|D|.$$

The following two propositions can be easily verified by definition of median value.

***Proposition* 2.3.** If f is a measurable function and finite almost everywhere in D, then $m_f(D)$ always exists.

We point out that median value is not unique generally. But any median value serves our purpose in the paper. For the sake of specificity, one can choose

$$\inf\{N : |\{x \in D : f(x) > N\}| \leqslant \frac{1}{2}|D|\}$$

as the median value.

***Proposition* 2.4.** For any real number C, $m_{f-C}(D) = m_f(D) - C$.

***Lemma* 2.5.** For $f \in L^p(D)$ $(p > 0)$

$$\inf_{C\in\mathbb{R}} \int_D |f - C|^p dx \approx \int_D |f - m_f(D)|^p dx,$$

where $A \approx B$ means A/B is bounded between two positive constants.

Proof. Without loss of generality, assume that $m_f(D) = 0$. If $C > 0$, then $|f - C| = C - f \geqslant C$ on $\{f \leqslant 0\}$. Thus

$$\int_D |f - C|^p dx \geqslant \int_{\{f\leqslant 0\}} |f - C|^p dx \geqslant C^p |\{f \leqslant 0\}|.$$

Noting $m_f(D) = 0$, one obtains

$$|\{f \leqslant 0\}| = |D| - |\{f > 0\}| \geqslant |D| - \frac{1}{2}|D| = \frac{1}{2}|D|.$$

We concludes that

$$\int_D |f - C|^p dx \geqslant \ C^p |\{f \leqslant 0\}| \geqslant C^p \frac{1}{2}|D|.$$

Thus

$$2\int_D |f - C|^p dx \geqslant C^p |D|.$$

Similarly, when $C \leqslant 0$, one can show

$$2\int_D |f - C|^p dx \geqslant |C|^p |D|.$$

Therefore

$$\int_D |f|^p dx \leqslant C_p \int_D (|f - C|^p + |C|^p) dx \leqslant C_p \int_D |f - C|^p dx$$

where C_p depends only on p. $\square$

***Proposition* 2.6.** Let $R \in \mathcal{R}$ and $C \in \mathbb{R}$. Then

$$\inf\{\alpha : \frac{|\{x \in R : |f - C| > \alpha\}|}{|R|} \leqslant s\} \geqslant \frac{1}{2}\inf\{\alpha : \frac{|\{x \in R : |f - m_f(R)| > \alpha\}|}{|R|} \leqslant s\}.$$

Proof. One can assume $m_f(R) = 0$. Let $\alpha > 0$ be such that

$$|\{x \in R : |f(x) - C| > \alpha\}| \leqslant s|R|. \tag{2.1}$$

First show

$$0 = m_f(R) \in [C - \alpha, C + \alpha].$$

If $m_f(R) > C + \alpha$, by definition of $m_f(R)$

$$|\{x \in R : f < m_f(R)\}| \leqslant \frac{1}{2}|R|.$$

On the other hand

$$\begin{aligned} |\{x \in R : f < m_f(R)\}| &\geqslant |\{x \in R : f \leqslant C + \alpha\}| \\ &= |R| - |\{x \in R : f > C + \alpha\}| \\ &\geqslant (1 - s)|R| > \frac{1}{2}|R|. \end{aligned}$$

That is a contradiction. Similarly, we can show $m_f(R) \geqslant C - \alpha$. Therefore

$$C - \alpha \leqslant 0 \leqslant C + \alpha \quad \textit{and} \quad |C| \leqslant \alpha.$$

This results in

$$\{x \in R : |f(x)| > 2\alpha\} \subset \{x \in R : |f(x) - C| > \alpha\}. \tag{2.2}$$

By (2.1) and (2.2), one concludes

$$\frac{|\{x \in R : |f - m_f(R)| > 2\alpha\}|}{|R|} \leqslant s.$$

Therefore

$$\alpha = \frac{1}{2}(2\alpha) \geqslant \frac{1}{2}\inf\{\beta : \frac{|\{x \in R : |f - m_f(R)| > \beta\}|}{|R|} \leqslant s\}. \qquad \square$$

The following proposition follows from the Chebyshev inequality.

***Proposition* 2.7.** If $M^p \geqslant \fint_R |f - C|^p dx$, then for any $\lambda > Ms^{-\frac{1}{p}}$

$$|\{x \in R : |f - C| > \lambda\}| \leqslant s|R|.$$

We now give the definition of $BMO_{s,\mathcal{R}}(R_0)$ (anisotropic BMO space) and the sharp maximal function.

***Definition* 2.8.** Let $R_0 \in \mathcal{R}$ *and* $0 < s < \frac{1}{2}$, a measurable function f is in $BMO_{s,\mathcal{R}}(R_0)$ if there exists $M \geqslant 0$ such that

$$|\{x \in R : |f - m_f(R)| > M\}| \leqslant s|R|$$

for any $R = R_r(x_0) \subset R_0$.

***Definition* 2.9.** For $f \in BMO_{s,\mathcal{R}}(R_0)$, define

$$M^{\#}_{s,\mathcal{R},R_0} f(x) = \sup_{R_0 \supset R \ni x} \inf\{\lambda : \frac{|\{y \in R : |f - m_f(R)| > \lambda\}|}{|R|} \leqslant s\},$$

where the supremum is taken over all k-rectangles $R \subset R_0$ containing x.

By definition, we have

$$|\{x \in R : |f(x) - m_f(R)| > \inf_{y \in R} M^{\#}_{s,\mathcal{R},R_0} f(y)\}| \leqslant s|R|, \quad for\ R \subset R_0. \quad (2.3)$$

§3 The John-Strömberg inequality

In this section, our goal is to establish the John-Strömberg inequality (Theorem 3.7) for $BMO_{s,\mathcal{R}}(R_0)$. To achieve, we need to construct a family of continuously expanding special k-rectangles, use $M^{\#}_{s,\mathcal{R},R_0}f$ to estimate median values, and give a Calderón–Zygmund type decomposition in terms of median values and $M^{\#}_{s,\mathcal{R},R_0}f$. Then we use these tools to prove Theoem 3.7.

***Lemma* 3.1.** Let $R = R_r(x_0) \subset R_0$ and $R_\epsilon \subset R_0$ be k-subrectangles such that

$$R_{(1-\epsilon)r}(x_0) \subset R_\epsilon \subset R_{(1+\epsilon)r}(x_0).$$

Then there exists ϵ_0 dependent only on $s, |k|$ such that if $\epsilon \leqslant \epsilon_0$, then

$$|m_f(R) - m_f(R_\epsilon)| \leqslant \inf_{y\in R} M^{\#}_{s,\mathcal{R},R_0}f(y).$$

Proof. By (2.3) and $s < \frac{1}{2}$, we have for $\epsilon \leqslant \epsilon_0$

$$\begin{aligned}
&|\{x \in R_\epsilon : |f(x) - m_f(R)| > \inf_{y\in R} M^{\#}_{s,\mathcal{R},R_0}f(y)\}| \\
&\quad\leqslant |\{x \in R_\epsilon \cap R : |f(x) - m_f(R)| > \inf_{y\in R} M^{\#}_{s,\mathcal{R},R_0}f(y)\}| + |R_\epsilon - R| \\
&\quad\leqslant s|R| + [(1+\epsilon)^{|k|} - 1]|R| \\
&\quad\leqslant \frac{s + (1+\epsilon)^{|k|} - 1}{(1-\epsilon)^{|k|}}|R_\epsilon| < \frac{1}{2}|R_\epsilon|.
\end{aligned}$$

Similar to the proof of Proposition 2.6, by the definition of $m_f(R_\epsilon)$, we obtain

$$|m_f(R_\epsilon) - m_f(R)| \leqslant \inf_{y\in R} M^{\#}_{s,\mathcal{R},R_0}f(y).$$

We have completed the proof. $\square$

***Lemma* 3.2.** Let

$$\begin{aligned}
R_1 &= \{x : 0 \leqslant x_i \leqslant \sigma^{k_i}, 1 \leqslant i \leqslant n\}, \\
R_0 &= \{x : a_i \leqslant x_i \leqslant a_i + r^{k_i}, 1 \leqslant i \leqslant n\}
\end{aligned}$$

be two k-rectangles such that $R_0 \subset R_1$. Then there exists a family of expanding k-rectangles $\{R_\theta\}_{0\leqslant\theta\leqslant 1}$ such that $R_{\theta_1} \subset R_{\theta_2}$ if $\theta_1 < \theta_2$ and $\lim_{\theta\to\theta_0} |R_\theta| = |R_{\theta_0}|$.

Proof. Let $r_\theta = r + \theta(\sigma - r)$, Consider the k-rectangle

$$R_\theta = \{x : a_i - a_i\frac{r_\theta^{k_i} - r^{k_i}}{\sigma^{k_i} - r^{k_i}} \leqslant x_i \leqslant (a_i + r^{k_i}) + (\sigma^{k_i} - (a_i + r^{k_i}))\frac{r_\theta^{k_i} - r^{k_i}}{\sigma^{k_i} - r^{k_i}}, 1 \leqslant i \leqslant n\}.$$

Obviously R_θ has edgelengths $r_\theta^{k_1}, \cdots, r_\theta^{k_n}$ and the Lebesgue measure

$$|R_\theta| = r_\theta^{|k|}.$$

Since $a_i \geqslant 0$, it is easy to see $R_{\theta_1} \subset R_{\theta_2}$ for $0 \leqslant \theta_1 < \theta_2 \leqslant 1$. So $\{R_\theta\}$ continuously expands from R_0 to R_1. □

***Lemma* 3.3.** Let R_1 and R_2 be in $\mathcal{R}$ such that $R_1 \subset R_2 \subset R_0$ and $|R_2| \leqslant 2^j |R_1|$. Then

$$|m_f(R_1) - m_f(R_2)| \leqslant 6j \inf_{y \in R_1} M^{\#}_{s,\mathcal{R},R_0} f(y).$$

Proof. It suffice to prove the lemma for $j = 1$. Let

$$A = \inf_{y \in R_1} M^{\#}_{s,\mathcal{R},R_0} f(y).$$

Proceed with a contradiction. Assume

$$|m_f(R_1) - m_f(R_2)| > 6A. \tag{3.1}$$

By Lemma 3.2, let $\{R_t\}_{t \in [1,2]}$ be a family of k-rectangles continuously expanding from R_1 to R_2. Set

$$t^* = \sup\{t : 1 \leqslant t \leqslant 2, |m_f(R_t) - m_f(R_1)| \leqslant 3A\}.$$

By Lemma 3.1 and (3.1), $t^* \in (1, 2)$. We now claim

$$2A < |m_f(R_{t^*}) - m_f(R_1)| \leqslant 4A. \tag{3.2}$$

Let $t_l \to (t^*)^-$ such that $|m_f(R_{t_l}) - m_f(R_1)| \leqslant 3A$. On one hand, for large l,

$$\begin{aligned} |m_f(R_{t^*}) - m_f(R_1)| &\leqslant |m_f(R_{t^*}) - m_f(R_{t_l})| + |m_f(R_{t_l}) - m_f(R_1)| \\ &\leqslant A + 3A = 4A. \end{aligned}$$

On the other hand, if $|m_f(R_{t^*}) - m_f(R_1)| \leqslant 2A$, then

$$\begin{aligned} |m_f(R_{t^*+\epsilon}) - m_f(R_1)| &\leqslant |m_f(R_{t^*+\epsilon}) - m_f(R_{t^*})| + |m_f(R_{t^*}) - m_f(R_1)| \\ &\leqslant A + 2A = 3A, \end{aligned}$$

which contradicts with the definition of t^*. By assumption (3.1) and claim (3.2),

$$|m_f(R_{t^*}) - m_f(R_1)| > 2A$$

and

$$\begin{aligned}|m_f(R_{t^*}) - m_f(R_2)| &\geqslant |m_f(R_1) - m_f(R_2)| - |m_f(R_{t^*}) - m_f(R_1)| \\ &> 6A - 4A = 2A.\end{aligned}$$

We conclude that

$$\begin{aligned}&\{y \in R_1 : |f(y) - m_f(R_1)| \leqslant A\}, \\ &\{y \in R_2 : |f(y) - m_f(R_2)| \leqslant A\}, \\ &\{y \in R_{t^*} : |f(y) - m_f(R_{t^*})| \leqslant A\}\end{aligned}$$

are pairwise disjoint. Therefore

$$\begin{aligned}&|\{y \in R_2 : |f(y) - m_f(R_2)| > A\}| \\ &\geqslant |\{y \in R_1 : |f(y) - m_f(R_1)| \leqslant A\}| + |\{y \in R_{t^*} : |f(y) - m_f(R_{t^*})| \leqslant A\}|.\end{aligned}$$

It implies from (2.3)

$$\begin{aligned}s|R_2| &\geqslant (1-s)|R_1| + (1-s)|R_{t^*}| \\ &\geqslant 2(1-s)|R_1|,\end{aligned}$$

which contradicts with $s < \frac{1}{2}$. □

***Lemma* 3.4.** (Calderón–Zygmund decomposition) Let $R \subset R_0$ be a k-rectangle, $\xi \geqslant 0$, and $\beta > 0$ such that $|m_f(R)| \leqslant \xi$, and $\{y \in R : M^{\#}_{s,\mathcal{R},R_0} f(y) \leqslant \beta\} \neq \emptyset$. Then there exists a collection $\{R_i\}$ of nonoverlapping k-subrectangles of R such that

$$R_i \not\subset \{y \in R : M^{\#}_{s,\mathcal{R},R_0} f(y) > \beta\}$$

and

$$\xi < |m_f(R_i)| \leqslant \xi + 6|k|\beta.$$

In addition,

$$|f(x)| \leqslant \xi, \quad a.e. \;\; x \in R \setminus \Big[\{y \in R : M^{\#}_{s,\mathcal{R},R_0} f(y) > \beta\} \cup \bigcup_i R_i\Big].$$

Proof. Divide R into $2^{|k|}$ k-subrectangles $\{R_{i_1}\}$ each with edgelengths $(\frac{r}{2})^{k_1}, ..., (\frac{r}{2})^{k_n}$. Each R_{i_1} must satisfy one of the following

(i)$R_{i_1} \subset \{y \in R : M^{\#}_{s,\mathcal{R},R_0} f(y) > \beta\}$.

(ii)$R_{i_1} \not\subset \{y \in R : M^{\#}_{s,\mathcal{R},R_0} f(y) > \beta\}$ *and* $|m_f(R_{i_1})| > \xi$.

(iii)$R_{i_1} \not\subset \{y \in R : M^{\#}_{s,\mathcal{R},R_0} f(y) > \beta\}$ *and* $|m_f(R_{i_1})| \leqslant \xi$.

For case (i), we do nothing further with R_{i_1}. For case (ii), by Lemma 3.3,

$$\xi < |m_f(R_{i_1})| \leqslant |m_f(R)| + |m_f(R) - m_f(R_{i_1})| \leqslant \xi + 6|k|\beta.$$

R_{i_1} is selected into the collection.

For case (iii), we divide R_{i_1} and repeat this process again. Therefore, we obtain a collection $\{R_i\}$. Finally, by the Lemma 3.5 below, $|f(x)| \leqslant \xi$ for a.e. $x \in R \setminus \Big[\{y \in R : M^{\#}_{s,\mathcal{R},R_0} f(y) > \beta\} \cup \bigcup_i R_i\Big]$. □

***Lemma* 3.5.** Let E be a measurable set. Assume that for any $x \in E$, there exist k-rectangles $\{R_{r_j} = R_{r_j}(x_j)\}$ such that $x \in R_{r_j}, r_j \to 0$ and $|m_f(R_{r_j})| \leqslant \xi$. Then $|f(x)| \leqslant \xi$ a.e. in E.

Proof. It suffices to prove

$$|E_\epsilon| = |\{x \in E : f(x) > \xi + \epsilon\}| = 0, \quad \forall \epsilon > 0. \tag{3.3}$$

Assume (3.3) does not hold. By the Lebesgue differentiation theorem (see [2])

$$\frac{|R_{r_j} \cap \{y \in E : f(y) > \xi + \epsilon\}|}{|R_{r_j}|} \longrightarrow 1, \quad x \in E_\epsilon - \mathcal{N},$$

where $\mathcal{N}$ is a null set. Therefore, for any $x \in E_\epsilon - \mathcal{N}$, and associated $R_{r_j} \ni x$ with $r_j \to 0$ and $|m_f(R_{r_j})| \leqslant \xi$, we have

$$|\{y \in R_{r_j} : f(y) > \xi + \epsilon\}| > \frac{|R_{r_j}|}{2}, \quad as\ j \longrightarrow \infty$$

It implies

$$m_f(R_{r_j}) > \xi + \epsilon.$$

It contradicts with $|m_f(R_{r_j})| \leqslant \xi$. Hence $E_\epsilon - \mathcal{N} = \emptyset$ and $|E_\epsilon| = 0$. □

We next give distribution function estimates.

***Lemma* 3.6.** Let $R \subset R_0$ be a k-rectangle. Then there exists constants $C_1, C_2 > 0$ dependent only on s and $|k|$ such that for $\alpha, \gamma > 0$,

$$|\{x \in R : |f(x) - m_f(R)| > \alpha\}| \leqslant C_1 e^{-C_2 \frac{\alpha}{\gamma}} |R| + |\{x \in R : M^{\#}_{s,\mathcal{R},R_0} f(x) > \gamma\}|.$$

Proof. One can assume $\{x \in R : M^{\#}_{s,\mathcal{R},R_0} f(x) \leqslant \gamma\} \neq \emptyset$. Applying Lemma 3.4 to $f - m_f(R)$ and R with $\beta = \gamma$, and $\xi = 2\gamma$. We obtain nonoverlapping k-subrectangles $\{R_j\}$ such that

$$R_j \not\subset \{y \in R : M^{\#}_{s,\mathcal{R},R_0} f(x) > \gamma\},$$

$$2\gamma < |m_f(R_j) - m_f(R)| \leqslant (2 + 6|k|)\gamma,$$

$$|f(x) - m_f(R)| \leqslant 2\gamma, \quad a.e\ x \in R \setminus \Big[\bigcup_j R_j \cup \{y \in R : M^{\#}_{s,\mathcal{R},R_0} f(y) > \gamma\}\Big].$$

It implies for $\alpha > (2 + 6|k|)\gamma$ except a null set

$$\begin{aligned}
&\{y \in R : |f(y) - m_f(R)| > \alpha\} \\
&\quad \subset \cup_j \{y \in R_j : |f(y) - m_f(R)| > \alpha\} \cup \{y \in R : M^{\#}_{s,\mathcal{R},R_0} f(y) > \gamma\} \\
&\quad \subset \cup_j \{y \in R_j : |f(y) - m_f(R_j)| > \alpha - (2 + 6|k|)\gamma\} \\
&\qquad\qquad \cup \{y \in R : M^{\#}_{s,\mathcal{R},R_0} f(y) > \gamma\}.
\end{aligned}$$

We now estimate $\sum_j |R_j|$. Since $|m_f(R_j) - m_f(R)| > 2\gamma$,

$$\{x \in R : |f(x) - m_f(R)| \leqslant \gamma\} \cap \{x \in R_j : |f(x) - m_f(R_j)| \leqslant \gamma\} = \emptyset.$$

Therefore

$$\bigcup_j \{x \in R_j : |f(x) - m_f(R_j)| \leqslant \gamma\} \subset \{x \in R : |f(x) - m_f(R)| > \gamma\}.$$

By (2.3), it results in

$$\sum_j (1 - s)|R_j| \leqslant s|R|$$

and

$$\sum_j |R_j| \leqslant \frac{s}{1 - s}|R|.$$

For $\alpha > 2(2+6|k|)\gamma$, relabel R_j *as* R_{j_1} and repeat this procedure with R_{j_1}. Then we can obtain nonoverlapping k-rectangles $\{R_{j_1 j_2}\}$ such that

$$\begin{aligned}
&\{x \in R_{j_1} : |f(x) - m_f(R_{j_1})| > \alpha - (2+6|k|)\gamma\} \\
&\quad \subset \bigcup_{j_2} \{x \in R_{j_1 j_2} : |f(x) - m_f(R_{j_1 j_2})| > \alpha - 2(2+6|k|)\gamma\} \\
&\qquad\qquad \bigcup \{y \in R : M^{\#}_{s,\mathcal{R},R_0} f(y) > \gamma\}
\end{aligned}$$

and

$$\sum_{j_2} |R_{j_1 j_2}| \leqslant \frac{s}{1-s} |R_{j_1}|.$$

By repeating this procedure l times, we proceed to obtain for $\alpha > l(2+6|k|)\gamma$

$$\begin{aligned}
&\{x \in R : |f(x) - m_f(R)| > \alpha\} \\
&\quad \subset \bigcup \{x \in R_{j_1 \ldots j_l} : |f(x) - m_f(R_{j_1 \ldots j_l})| > \alpha - l(2+6|k|)\gamma\} \\
&\quad\quad \bigcup \{x \in R : M^{\#}_{s,\mathcal{R},R_0} f(y) > \gamma\}
\end{aligned}$$

and

$$\sum_{j_1 \ldots j_l} |R_{j_1 \ldots j_l}| \leqslant (\frac{s}{1-s})^l |R|. \tag{3.4}$$

If $\alpha > (2+6|k|)\gamma$, there exists $l \geqslant 1$ such that $l(2+6|k|)\gamma < \alpha \leqslant (l+1)(2+6|k|)\gamma$. Therefore by (3.4),

$$\begin{aligned}
&|\{x \in R : |f(x) - m_f(R)| > \alpha\}| \\
&\quad \leqslant (\frac{s}{1-s})^l |R| + |\{x \in R : M^{\#}_{s,\mathcal{R},R_0} f(y) > \gamma\}| \\
&\quad \leqslant C_1 e^{-C_2 \frac{\alpha}{\gamma}} |R| + |\{x \in R : M^{\#}_{s,\mathcal{R},R_0} f(x) > \gamma\}|.
\end{aligned}$$

Lemma 3.6 is proved. □

The following John-Strömberg inequality is the main result in this paper.

***Theorem* 3.7.** If $f \in BMO_{s,\mathcal{R}}(R_0)$ with $M^{\#}_{s,\mathcal{R},R_0} f \leqslant M$, then there exist constants C_3 and C_4 dependent only on s and $|k|$ such that

$$⨍_{R_0} e^{C_3 \frac{|f - m_f(R_0)|}{M}} dx \leqslant C_4.$$

Proof. Chosse $C_3 < C_2$ where C_2 is the constant in Lemma 3.6. By the Fubini theorem and Lemma 3.6 with $\gamma = M$, we have

$$\begin{aligned}
&\fint_{R_0} e^{C_3 \frac{|f-m_f(R_0)|}{M}} dx - 1 \\
&\quad = \fint_{R_0} dx \int_0^{\frac{|f-m_f(R_0)}{M}} C_3 e^{C_3\lambda} d\lambda \\
&\quad = C_3 \int_0^\infty e^{C_3\lambda} \fint_{R_0} \chi\{\frac{|f-m_f(R_0)|}{M} > \lambda\} dx \\
&\quad = \frac{C_3}{|R_0|} \int_0^\infty e^{C_3\lambda} |\{x \in R_0 : \frac{|f-m_f(R_0)|}{M} > \lambda\}| dx \\
&\quad \leqslant \frac{C_3}{|R_0|} \int_0^\infty e^{C_3\lambda} C_1 e^{-C_2\lambda} |R_0| d\lambda = C_4,
\end{aligned}$$

where χ_E denotes the characteristic function of E. The proof is completed. □

§4 Applications

We apply Theorem 3.7 to give some characterizations of anisotropic Hölder continuous functions.

***Definition* 4.1.** for $x = (x_1, ...x_n), y = (y_1, ...y_n) \in \mathbb{R}^n$, define

$$d(x, y) = \max_{1 \leqslant i \leqslant n} (2|x_i - y_i|)^{\frac{1}{k_i}}.$$

Note $(a + b)^\delta \leqslant a^\delta + b^\delta$ for $a, b \geqslant 0,\ 0 < \delta \leqslant 1$. It is easy to verify $d(x, y)$ is a distance in $\mathbb{R}^n$. Moreover, $R_r(x_0) = \overline{B}_d(x_0, r)$ is the close ball centered at x_0 with radius r with respect to the distance d.

***Theorem* 4.2.** Let f be a measurable function on R_0. Assume there exist $M > 0$ and $0 < \alpha \leqslant 1$ such that for any k-rectangle $R_r = R_r(z) \subset R_0$

$$|\{x \in R_r : |f - m_f(R_r)| > Mr^\alpha\}| \leqslant s|R_r|.$$

Then $f \in C_d^\alpha(R_0)$. More precisely

$$|f(x) - f(y)| \leqslant CMd^\alpha(x, y), \quad \forall x, y \in R_0.$$

Proof. Let $R_{r_1} = R_{r_1}(z_1) \subset R_0$ be a k-rectangle. To apply the John-Strömberg inequality to R_{r_1}, we note that

$$M^{\#}_{s,\mathcal{R},R_{r_1}} f(x) = \sup_{x\in R\subset R_{r_1}} \inf \left\{\lambda : |\{y \in R : |f - m_f(R)| > \lambda\}| \leqslant s|R|\right\} \leqslant Mr_1^{\alpha}.$$

Therefore by Theorem 3.7

$$\fint_{R_{r_1}} e^{c_3 \frac{|f-m_f(R_{r_1})|}{Mr_1^{\alpha}}} dx \leqslant C_4.$$

Since $x \leqslant e^x$ for all $x > 0$, it yields

$$\fint_{R_{r_1}} |f - f_{R_1}|dx \leqslant C_5 M r_1^{\alpha}.$$

Next for $R_r \subset R_0$ and $R_{\frac{r}{2}}$ (a dyadic sub k-rectangle of R_r),

$$|f_{R_r} - f_{R_{\frac{r}{2}}}| \leqslant \fint_{R_{\frac{r}{2}}} |f - f_{R_r}|dx \leqslant \frac{|R_r|}{|R_{\frac{r}{2}}|} \fint_{R_r} |f - f_{R_r}|dx \leqslant 2^{|k|} C_5 M r^{\alpha}.$$

Similarly

$$|f_{R_r(x_0)} - f_{R_{\frac{r}{2}}(x_0)}| \leqslant 2^{|k|} C_5 M r^{\alpha} \quad for\ R_r(x_0) \subset R_0.$$

Let $R' \subset\subset R_0$. For suitable small r and $l > m \geqslant 1$,

$$|f_{R_{r2^{-m}}(x_0)} - f_{R_{r2^{-l}}(x_0)}| \leqslant \sum_{j=m}^{l-1} 2^{|k|} C_5 M (r2^{-j})^{\alpha} \leqslant CM(r2^{-m})^{\alpha}. \tag{4.1}$$

So $\left\{f_{R_{r2^{-m}}(x_0)}\right\}$ is uniformly convergent. By the Lebesgue differentiation theorem, $f(x)$ is continuous in R_0.
For $x \in R_r$, consider dyadic k-subrectangles $\{R_{r2^{-i}}\}$ such that $x \in R_{r2^{-i}}$ for any i. Similar to (4.1), we have

$$|f_{R_{r2^{-i}}} - f_{R_{r2^{-l}}}| \leqslant CM(r2^{-i})^{\alpha}\ for\ l > i \geqslant 0.$$

Choosing $i = 0$ and letting $l \to \infty$, we obtain

$$|f_{R_r} - f(x)| \leqslant CMr^{\alpha}, \quad \forall x \in R_r.$$

It implies that

$$|f(x) - f(y)| \leqslant 2CMr^{\alpha} \quad for\ x, y \in R_r. \tag{4.2}$$

Finally, given $x, y \in R_0$, set $r = \max_{1\leqslant i\leqslant n} |y_i - x_i|^{\frac{1}{k_i}}$. Construct a k-rectangle R_r satisfying $x, y \in R_r \subset R_0$ as follows. Since $|x_i - y_i| \leqslant r^{k_i}$ by the definition of r, one can find $[a_i, b_i]$ such that $x_i, y_i \in [a_i, b_i]$ and $b_i - a_i = r^{k_i}$. Let $R_r = \prod_{i=1}^{n}[a_i, b_i]$. Obviously, $x, y \in R_r \subset R_0$. Therefore, from (4.2), $|f(x) - f(y)| \leqslant 2CMr^{\alpha} \leqslant 2CMd^{\alpha}(x, y)$.

$\square$

As an application of Theorem 4.2, we are ready to give the following characterization of parabolic Hölder continuous functions.

Consider the parabolic cylinders

$$P_0 = \{(x, t) \in R^{n+1} : |t - t_0| \leqslant \frac{r_0^2}{2}, |x_i - x_{0i}| \leqslant \frac{1}{2}r_0\ , 1 \leqslant i \leqslant n\}$$

and

$$P_r(x_1, t_1) = Q_{\frac{r}{2}}(x_1) \times [t_1 - \frac{r^2}{2}, t_1 + \frac{r^2}{2}],$$

where $Q_{\frac{r}{2}}(x_1)$ is the Euclidean cube centered at x_1 with edgelength r.

We have the following.

***Corollary* 4.3.** Let f be a measurable function. Assume there exist $M > 0$ and $0 < \alpha \leqslant 1$ such that

$$|\{x \in P_r : |f - m_f(P_r)| > Mr^{\alpha}\}| \leqslant s|P_r|$$

for all $P_r = P_r(x_1, t_1) \subset P_0$. Then $f \in C^{\alpha, \frac{\alpha}{2}}(P_0)$ and

$$|f(x, t) - f(y, \tau)| \leqslant CM(|x - y| + |t - \tau|^{\frac{1}{2}})^{\alpha}, \quad for\ (x, t), (y, \tau) \in P_0.$$

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