2020

Intersections of Deleted Digits Cantor Sets with Gaussian Integer Bases

Vincent T. Shaw
Wright State University

INTERSECTIONS OF DELETED DIGITS CANTOR SETS WITH GAUSSIAN INTEGER BASES

A thesis submitted in partial fulfillment

of the requirements for the degree of

Master of Science

By

VINCENT T. SHAW

B.S., Wright State University, 2017

2020

Wright State University

WRIGHT STATE UNIVERSITY

SCHOOL OF GRADUATE STUDIES

May 1, 2020

I HEREBY RECOMMEND THAT THE THESIS PREPARED UNDER MY SUPERVISION BY Vincent T. Shaw ENTITLED Intersections of Deleted Digits Cantor Sets with Gaussian Integer Bases BE ACCEPTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF Master of Science.

____

Steen Pedersen, Ph.D.

Thesis Director

____

Ayse Sahin, Ph.D.

Department Chair

Committee on
Final Examination

____
Steen Pedersen, Ph.D.

____
Qingbo Huang, Ph.D.

____
Anthony Evans, Ph.D.

____
Barry Milligan, Ph.D.
Interim Dean, School of Graduate Studies

Abstract

Shaw, Vincent T. M.S., Department of Mathematics and Statistics, Wright State University, 2020. Intersections of Deleted Digits Cantor Sets with Gaussian Integer Bases.

In this paper, the intersections of deleted digits Cantor sets and their fractal dimensions were analyzed. Previously, it had been shown that for any dimension between 0 and the dimension of the given deleted digits Cantor set of the real number line, a translate of the set could be constructed such that the intersection of the set with the translate would have this dimension. Here, we consider deleted digits Cantor sets of the complex plane with Gaussian integer bases and show that the result still holds.

Contents

List of Figures

List of Tables

Acknowledgements

First and foremost, I would like to thank my research advisor Dr. Steen Pedersen for providing me the opportunity to study mathematics under his guidance and leadership. The past two years working with him have given me a new found appreciation for those that contribute to field of mathematics. It was only through his encouragement, and profound patience that this thesis was possible. Also, special thanks is in order for Jason D. Phillips for his coauthorship to the predecessor of this paper that laid the foundation for our research. Thank you to both Dr. Qingbo Huang and Dr. Anthony Evans for reviewing my thesis and providing feedback.

I would also like to take a moment to thank the following individuals that had a positive impact on my personal development as a scholar: Dr. Ayse Sahin, Dr. Ann Farrell and Peggy Kelly in helping me succeed as a Graduate Teaching Assistant. Thank you to both Dr. Christopher Barton and Dr. Mateen Rizki for taking me on as a student in their MatLab course that proved to be an invaluable resource in conducting my research. Finally, I would like to thank my friend and colleague Joe Sjoberg for rekindling my interest in pursuing a degree in mathematics, as well as my fellow GTA's for making the last two years an enjoyable and worthwhile expreience.

Chapter 1

Introduction

1.1 Why study intersections of Cantor sets?

Cantor sets may appear to be rather special. However, they occur in mathematical models of many naturally occuring objects, play a role in number theory, in signal processing, in ergodic theory, and in limit-theorems from probability. We study the "size" of the intersection of two Cantor sets. The significance of this problem was noted by Furstenberg [Fur70] and Palis [Pal87]. Papers dealing with versions of our problem and related applications include: [Wil91], [Kra92], [PS98], [Kra00], [LN04], [DT07], [DHW08], [DT08a], [ZLL08], [DLT09], [KLD10], [LYZ11a], [Mor11], [ZG11], [LTZ18], [SRM$^+$20]. The literature in the subject, neighboring areas, and applications is vast. In the list above, we limit ourselves to a small sample of the literature closely related to our problem.

1.2 Prior work on this problem

Most of the prior work related to our problem has been done in the one dimensional case, so we will begin with a summary of that case.

Let $n \geq 3$ be an integer. Any real number $t \in [0, 1]$ has at least one *n-ary representation*

$$t = 0._n t_1 t_2 \cdots = \sum_{k=1}^{\infty} \frac{t_k}{n^k}$$

where each t_k is one of the digits $0, 1, \ldots, n-1$. Deleting some element from the full digit set $\{0, 1, \ldots n-1\}$ we get a set of *digits* $D := \{d_k \mid k = 1, 2, \ldots, m\}$ with $m < n$ digits $d_k < d_{k+1}$ and a corresponding *deleted digits Cantor set*

$$C = C_{n,D} := \left\{ \sum_{k=1}^{\infty} \frac{x_k}{n^k} \mid x_k \in D \text{ for all } k \in \mathbb{N} \right\}. \tag{1.1}$$

We are interested in the dimension of the sets $C \cap (C + t)$, where $C + t := \{x + t \mid x \in C\}$. Since the problem is invariant under translation we will assume $d_1 = 0$.

We say that D is *uniform,* if $d_{k+1} - d_k$, $k = 1, 2, \ldots, m-1$ is constant and ≥ 2. We say D is *regular,* if D is a subset of a uniform digit set. Finally, we say that D is *sparse,* if $|\delta - \delta'| \geq 2$ for all $\delta \neq \delta'$ in

$$\Delta := D - D = \{d_j - d_k \mid d_j, d_k \in D\}.$$

Clearly, a uniform set is regular and a regular set is sparse. The set $D = \{0, 5, 7\}$ is sparse and not regular. We abuse the terminology and say $C_{n,D}$ is uniform, regular, or sparse provided D has the corresponding property.

Previous studies of the sets $C \cap (C + t)$ include:

- When $C = C_{3,\{0,2\}}$ is the middle thirds Cantor set, a formula for the Hausdorff dimension $\dim (C \cap (C + t))$ of $C \cap (C + t)$ can be found in [DH95] and in [NL02]. Such a formula can also be found in [DT08b] if C is uniform and $d_m = n - 1$, and in [KP91] if C is regular. For sparse C, a formula can be found in [PP13].

- Let F^+ be the set of all $t \geq 0$ such that $C \cap (C + t)$ is non-empty. For $0 \leq \beta \leq 1$, let $F_\beta := \{t \in F^+ \mid \dim (C \cap (C + t)) = \beta \log_n(m)\}$. If C is the middle thirds Cantor set then $F^+ = [0, 1]$ and it is shown in [Haw75, DH95, NL02] that F_β is dense in F^+ for all $0 \leq \beta \leq 1$. This is extended to regular sets and to sets $C_{n,D}$ such that D satisfies $d_{k+1} - d_k \geq 2$ and $d_m < n - 1$ in [PP12]. It is also shown in [PP12] that F_β is not dense in F^+ for all $0 \leq \beta \leq 1$ for all deleted digits Cantor sets $C_{n,D}$.

- It is shown in [Haw75, Igu03] that, if C is the middle thirds Cantor set, then the Hausdorff dimension of $C \cap (C + t)$ is $\frac{1}{3} \log_3(2)$ for Lebesgue almost all t in the closed interval $[0, 1]$. This is extended to all deleted digits sets in [KP91].

- If C is the middle thirds Cantor set, then $C \cap (C + t)$ is self-similar if and only if the sequence $\{1 - |y_k|\}$ is "strong periodic", where $t = \sum_{k=1}^{\infty} \frac{2y_k}{3^k}$ and $y_k \in \{-1, 0, 1\}$ for all k by [LYZ11b]. This is extended to sparse C in [PP14]. In particular, $C \cap (C + t)$ is not, in general, a self-similar set.

Some of the cited papers only consider rational t and some consider Minkowski dimension in place of Hausdorff dimension. It is known, see e.g., [PP12] for an elementary proof, that the (lower) Minkowski dimensions of $C \cap (C + t)$ equals its Hausdorff dimension.

Palis [Pal87] conjectured that for dynamically defined Cantor sets typically the corresponding set F^+ either has Lebesgue measure zero or contains an interval. The papers [DS08], [MSS09] investigate this problem for random deleted digits sets and solve it in the affirmative in the deterministic case.

In two dimensions, where C are certain Sierpinski carpets, the dimension of $C \cap (C + t)$ is investigated in [LTZ18] and the dimension is calculated for certain translation vectors t. In [DT07] the authors consider the intersection of a Sierpinski carpet with its rational translates. They use the methods from and obtain results similar to the ones in [NL02]. Both in [LTZ18] and [NL02] the authors study the case where the base is a real number and the "digits" are vectors parallel to the coordinate axes.

In this work, we study the case where the base is a complex number of the form $-n \pm i$ for an integer $n > 1$ and the set of digits is a proper subset of the set $\{0, 1, \ldots, n^2\}$.

In Chapter 2 we discuss the one dimensional case, in particular, we show that the case where we use a negative integer as a base does not lead to new results.

In Chapter 3 we show that if we want to use a Gaussian integer as the base and have a digit set of the form $\{0,1,2,\ldots,N\}$, then we must have $b=-n\pm i$ for some positive integer and $N=n^2$.

In Chapter 4 we begin to investigate the set $C\cap(C+t)$ and its dimension, in particular we identify appropriate conditions on the subset of $\{0,1,2,\ldots,N\}$ that can serve as a good set of digits.

In Chapter 5 we show that for any $0\leq\alpha\leq 1$, there is a complex number t, such that $C\cap(C+t)$ has dimension $\alpha\dim(C)$.

Chapter 2

Negative Base Representations

2.1 Integer Representations

Suppose b is an integer such that $|b| \geq 2$. Let $D = \{0, 1, 2, \ldots, |b| - 1\}$. For $d_k \in D$, we say

$$n = \sum_{k=0}^{M} d_k b^k$$

is a *representation of n in base b with digit set D*. When $b = 10, 3, 2$ this gives the usual decimal, ternary, and binary representations. Here we are also interested in the case where $b \leq -2$. The case of negative bases is well known, see e.g., [Knu98] for the $n = 2$ case, the proof below is essentially taken from [GG79].

Proposition 1. *If $b \leq -2$, then every integer has a unique representation in base b with digit set D.*

Proof. Existence follows from the usual division algorithm. For a positive base, it is clear that the sequence of quotients is decreasing, hence the division algorithm terminated. Here termination of the division algorithm requires an argument. Suppose n is an integer, positive or negative. Let $q_0 = n$ and inductively $q_k = q_{k+1}b + d_k$, where $0 \leq d_k < |b|$.

(i) If $q_k > 0$ and $q_k \geq d_k$, then $|q_{k+1}| = \left|\frac{q_k - d_k}{b}\right| \leq \left|\frac{q_k}{b}\right| < |q_k|$.

(ii) If $q_k > 0$ and $q_k < d_k$, then $|q_{k+1}| = \left|\frac{q_k - d_k}{b}\right| \leq \left|\frac{d_k}{b}\right| < 1 \leq |q_k|$, since $d_k < |b|$.

(iii) If $q_k < -1$, then $|q_{k+1}| = \left|\frac{q_k - d_k}{b}\right| \leq \left|\frac{q_k}{b}\right| + \left|\frac{d_k}{b}\right| < \left|\frac{q_k}{b}\right| + 1 \leq |q_k|$, the last $\leq$ used $q_k \leq -2$.

(iv) If $q_k = -1$, then $q_{k+1}b = -1 - d_k$, so $1 + d_k$ is divisible by b, since $0 < 1 + d_k \leq -b$, we have $-1 - d_k = b$, hence $q_{k+1} = 1$ and therefore $q_{k+2} = 0$, by (i).

It follows from (i)-(iv) that eventually the quotient q_k must equal zero. If $q_{M+1} = 0$, then

$$\begin{aligned} n &= d_0 + q_1 b \\ &= d_0 + (d_1 + q_2 b)\, b \\ &= d_0 + d_1 b + (d_2 + q_3 b)\, b^2 \\ &= \cdots \\ &= d_0 + d_1 b + d_2 b^2 + \cdots + (d_M + q_{M+1} b)\, b^M \\ &= \sum_{k=0}^{M} d_k b^k. \end{aligned}$$

Thus every interger has a representation in base b.

Suppose $d_k \in D$, $e_k \in D$, and $\sum_{k=0}^{M} d_k b^k = \sum_{k=0}^{M} e_k b^k$. Then

$$b \sum_{k=1}^{M} (d_k - e_k)\, b^{k-1} = e_0 - d_0$$

so the left hand side is an integer divisible by $|b|$ and the right hand side is an integer $< |b|$. Hence, the right hand side $e_0 - d_0 = 0$. Therefore $d_0 = e_0$.

Using $d_0 = e_0$ it follows that $\sum_{k=1}^{M} d_k b^{k-1} = \sum_{k=1}^{M} e_k b^{k-1}$. Repeating the previous argument shows $d_1 = e_1$ and $\sum_{k=2}^{M} d_k b^{k-2} = \sum_{k=2}^{M} e_k b^{k-2}$.

Continuing in this manner we see that $d_k = e_k$ for all $k = 0, 1, \ldots, M$. $\square$

2.2 Fractals Obtained by Deleting Digits

Fix an integer $|b| > 1$ and let $D = \{0, 1, \ldots, |b| - 1\}$ as above. Let $f_j(x) = \frac{1}{b}(x + j)$ for $j \in D$.

Suppose $b > 1$. Let $\alpha = 0$ and $\beta = 1$, then $\bigcup_{j \in D} f_j[\alpha, \beta] = [\alpha, \beta]$. In fact, $f_j[0, 1] = \left[\frac{j}{b}, \frac{j+1}{b}\right]$, so $\bigcup_{j \in D} f_j[\alpha, \beta] = \bigcup_{j=0}^{b-1}\left[\frac{j}{b}, \frac{j+1}{b}\right] = \left[\frac{0}{b}, \frac{b-1+1}{b}\right]$. If D^* is a subset of D, then the compact non-empty set $C_{b,D^*} = C_{D^*}$ satisfying $\bigcup_{j \in D*} f_j(C_{D^*}) = C_{D^*}$ is the *deleted digits Cantor set* $C_{D^*} = \left\{\sum_{k=1}^{\infty} d_k b^{-k} : d_k \in D^*\right\}$. In [PP12] the authors studied the fractal dimensions of the intersections of C_{D^*} with its translates.

Suppose $b < -1$ and let $D = \{0, 1, \ldots, -b - 1\}$ as above. Let $\alpha = \frac{b}{1-b}$ and $\beta = \frac{1}{1-b}$, then $\alpha < 0$, $\beta > 0$, and $\bigcup_{j \in D} f_j[\alpha, \beta] = [\alpha, \beta]$. In fact, $\beta - \alpha = 1$, $f_0[\alpha, \beta]$ is an interval of length $\frac{1}{-b}$, and $f_j[\alpha, \beta]$ is obtained from $f_{j-1}[\alpha, \beta]$ by a translation by $\frac{1}{b}$. Hence the intervals $f_j[\alpha, \beta]$ and $f_{j-1}[\alpha, \beta]$ intersect at endpoints, and we just need to check that the right hand endpoint $f_0(\alpha)$ of $f_0[\alpha, \beta]$ is β and the left hand endpoint $f_{-b-1}(\beta)$ of $f_{-b-1}[\alpha, \beta]$ is α. Now

$$f_0(\alpha) = f_0\left(\frac{b}{1-b}\right) = \frac{1}{b} \cdot \frac{b}{1-b} = \frac{1}{1-b} = \beta$$

and

$$f_{-b-1}(\beta) = f_{-b-1}\left(\frac{1}{1-b}\right) = \frac{1}{b} \cdot \frac{1}{1-b} + \frac{-b-1}{b} = \frac{b}{1-b} = \alpha$$

Again, if D^* is a subset of D, then the compact non-empty set $C_{b,D^*} = C_{D^*}$ satisfying $\bigcup_{j \in D*} f_j(C_{D^*}) = C_{D^*}$ is the Cantor set

$$C_{D^*} = \left\{\sum_{k=1}^{\infty} d_k b^{-k} : d_k \in D^*\right\}.$$

In this case, the reflection caused by b being negative means we cannot simply copy iterated function systems arguments from [Gau32] when studying fractal dimensions of the intersection of C_{D^*} with its translates. However, it is easy to see that as a set $C_{-|b|,D^*}$ can be obtained by translating the set $C_{|b|,D^*}$ by $-\frac{|b|}{1+|b|}$. For any integer b, such that $|b| > 1$, the union of translated $\bigcup_{k\in\mathbb{Z}}(C_D+k)$ is a tiling of the real line.

Chapter 3

Gaussian Integers and Fractals

3.1 Gaussian Integer Representations

The set of Gaussian integers is the set $\mathbb{Z}[i] = \{x + iy : x, y \in \mathbb{Z}\}$. To extend the study of the dimension of the intersection of a deleted digits Cantor set with its translates we will study the case where the base b is a Gaussian integer and our Cantor set is a subset of the complex plane determined by a set of digits as above. We consider a digit set $D = \{0, 1, 2, \ldots, N\}$ and a Gaussian integer $b = m + in$. We say a Gaussian integer z has a representation in base b, with digit set D, if $z = \sum_{k=0}^{M} d_k b^k$, with $d_k \in D$. In analogy with Proposition 1, the first question is: Can we choose m, n, N such that every Gaussian integer has a unique representation in base $b = m + in$ with digit set $D = \{0, 1, 2, \ldots, N\}$?

We begin by showing that a necessary condition for every Gaussian integer to have a representation in base $b = m + in$, is that the imaginary part of b has modulus one.

Lemma 2. *If every Gaussian integer has a representation in base $b = m + in$, with digits set $D = \{0, 1, 2, \ldots, N\}$, then $n = \pm 1$.*

Proof. Since $(u + iv)(x + iy) = ux - vy + i(uv + vx)$, it follows by induction that for any integer k, n is a factor of y, when $b^k = x + iy$ where n divides y in $\mathbb{Z}$. Consequently, if $z = \sum_{k=0}^{M} d_k b^k$, then n divides the imaginary part of z in $\mathbb{Z}$. By assumption $i = \sum_{k=0}^{M} d_k b^k$, for some $d_k \in D$, so n divides 1 in $\mathbb{Z}$. Thus $n = \pm 1$. $\square$

Our next result establishes a connection between representations in base b and congruence classes modulo b.

Lemma 3. *If every Gaussian integer has a unique representation in base b, then every Gaussian integer is congruent to exactly one element of D, that is D is a complete set of representatives for $\mathbb{Z}[i] / (b)$.*

Proof. Let z be a Gaussian integer. By assumption $z = d_0 + b\sum_{k=1}^{M} d_k b^{k-1}$, hence $z \equiv d_0 \pmod{b}$. So, if every Gaussian integer has a representation in base b, then every Gaussian integer is congruent to some element of D.

Uniqueness of representation means, if $\sum_{k=0}^{M} d_k b^k = \sum_{k=0}^{M'} d'_k b^k$, then $d_k = d'_k$ for all k. Where if $M < M'$ we set $d_k = 0$ for all $M < k \leq M'$. And similarly, if $M > M'$, we set $d'_k = 0$ for all $M' < k \leq M$.

The second claim is that no Gaussian integer is congruent to more than one element of D. So suppose z is a Gaussian integer and z is congruent to $d \in D$ and to $e \in D$. Then $z = d + bu$ and $z = e + bv$ for some Gaussian integers u, v. Now $v - u$ is a Gaussian integer, so $v - u = \sum_{k=0}^{M} d_k b^k$ for some $d_k \in D$. Hence,

$$d = e + b(v - u) = e + \sum_{k=0}^{M} d_k b^{k+1}.$$

By uniqueness, $d = e$ and $d_k = 0$ for all k. $\square$

Due to Lemma 3 we ask: For a Gaussian integer $b = m + in$, does there exist an N, such that $D = \{0, 1, 2, \ldots, N\}$ is a complete set of representatives for the congruence classes $\mathbb{Z}[i] / (m + in)$. Gauss showed that if m, n are relatively prime the answer is affirmative. We reproduce Gauss' argument below.

Lemma 4 ([Gau32, Theorem 40])**.** *If $b = m + in$ is a Gaussian integer and m, n are relatively prime, then $D = \left\{0, 1, 2, \ldots, m^2 + n^2 - 1\right\}$ is a complete set of representatives for the congruence classes $\mathbb{Z}[i] / (b)$.*

Proof. Since m, n are relatively prime, there are integers α, β, such that $\alpha m + \beta n = 1$. Note

$$\begin{aligned}\alpha n - \beta m + b(\beta + i\alpha) &= \alpha n - \beta m + (m + in)(\beta + i\alpha) \\ &= \alpha n - \beta m + m\beta + (m\alpha + n\beta) i - n\alpha \\ &= i.\end{aligned}$$

Hence, if $x + iy$ is any Gaussian integer, then

$$x + iy = x + (\alpha n - \beta m) y + b(\beta y + i\alpha y). \tag{3.1}$$

Let h, k be integers such that $x + (\alpha n - \beta m) y = h + k\left(m^2 + n^2\right)$ and $0 \leq h < m^2 + n^2$. Then

$$x + (\alpha n - \beta m) y = h + bk(m - in), \tag{3.2}$$

since $b(m - in) = m^2 + n^2$. Combining Eqns. (3.1) and (3.2) we see

$$x + iy = h + b(\beta y + km + i(\alpha y - kn)).$$

Thus $x + iy \equiv h \pmod{b}$.

To complete the proof we need to show that no Gaussian integer is congruent to more than one element of D. Suppose $x+iy$ is a Gaussian integer, $h, h' \in D = \left\{0, 1, 2, \ldots, m^2 + n^2 - 1\right\}$, $x + iy \equiv h \pmod{b}$, and $x + iy \equiv h' \pmod{b}$, then $h - h' = b(u + iv)$ for some Gaussian integer $u + iv$. Hence,

$$(h - h')(m - in) = \left(m^2 + n^2\right)(u + iv).$$

So

$$(h - h') m\alpha = u\alpha\left(m^2 + n^2\right) \text{ and } (h - h') n\beta = -v\beta\left(m^2 + n^2\right).$$

Adding these equations and using $\alpha m + \beta n = 1$ it follows that

$$h - h' = (u\alpha - v\beta)\left(m^2 + n^2\right).$$

Since $h, h' \in \left\{0, 1, 2, \ldots, m^2 + n^2 - 1\right\}$ and $u\alpha - v\beta$ is an integer, $h = h'$. □

Recall, we are interested in when every Gaussian integer has a unique representation in base $b = m + in$ using digit set $D = \{0, 1, 2, \ldots, N\}$. By Lemma 2, we may assume $b = m \pm i$. and by Lemma 3 and Lemma 4, we may assume $D = \left\{0, 1, 2, \ldots, m^2\right\}$. To complete this line of reasoning we need to go back to the original question of representing every Gaussian integer uniquely in base $b = m + i$ with digits in $D = \left\{0, 1, 2, \ldots, m^2\right\}$.

Lemma 5. *Suppose $b = m + i$, $D = \left\{0, 1, 2, \ldots, m^2\right\}$, and every Gaussian integer has a unique representation in base b with digit set D, then $m < 0$.*

Proof due to [KS75]. The proof is by contradiction. If $m = 0$, then $D = \{0\}$ and we can only represent 0 in base b with digit set D.

If $m > 0$. Suppose $1 - m + i = \sum_{k=0}^{m^2} d_k b^k$. Multiplying by $1 - b = 1 - m - i$ we get

$$(1 - m)^2 + 1 = d_0 + \sum_{k=1}^{m^2} (d_k - d_{k-1})\, b^k - d_{m^2} b^{m^2+1}. \tag{3.3}$$

The left hand side $m^2 - 2m + 2$ is in D and the right hand side is $\equiv d_0 \pmod{b}$, so using Lemma 3 and Lemma 4 we conclude $m^2 - 2m = d_0$. So, by Eqn. (3.3),

$$\sum_{k=1}^{m^2} (d_k - d_{k-1})\, b^{k-1} - d_{m^2} b^{m^2} = 0.$$

Either $d_1 - d_0$ or $d_0 - d_1$ is in D, so we can repeat the argument to get $d_1 = d_0$. Continuing in this manner we get

$$d_k = d_{k-1}, \text{ for } k = 1, 2, 3, \ldots, m^2, \text{ and } d_{m^2} = 0.$$

Consequently, $d_0 = d_1 = d_2 = \cdots = d_{m^2-1} = d_{m^2} = 0$. Thus, we get the contradiction

$(1-m)^2+1=0$, from Eqn. (3.3). $\square$

We have shown, that if m, n, N are integers, $b=m+in$, and every Gaussian integer has a unique representation in base b with digit set $D=\{0,1,2,\ldots,N\}$, then $m<0$, $n=\pm 1$, and $N=m^2$. Conversely, it is known [KS75], that if $n>0$ is an integer, then every Gaussian integer has has unique representation in base $b=-n\pm i$ with digits form $D=\left\{0,1,2,\ldots,n^2\right\}$.

3.2 Gaussian Fractals Obtained by Deleting Digits

Fix an integer $n>0$ and let $D=\left\{0,1,\ldots,n^2\right\}$. Then every Gaussian integer has a unique representation in base $b=-n+i$ with digits set D. Every complex number is within $\frac{1}{\sqrt{2}}$ of some Gaussian integer $\mathbb{Z}[i]$. Hence every complex number is within $\frac{1}{\sqrt{2}}$ of some complex number of the form $\sum_{k=0}^{m} d_k b^k$, $d_k \in D$. Now, let z be a complex number and pick a Gaussian integer $\sum_{k=0}^{m} d_k b^k, d_k \in D$ within $\frac{1}{\sqrt{2}}$ of bz. Then $\frac{1}{b}\sum_{k=0}^{m} d_k b^k = \sum_{j=-1}^{m-1} d_{j-1} b^j$ is within $\frac{1}{|b|\sqrt{2}}$ of z. Similarly, given any complex number z, we can find $\sum_{k=-l}^{m} d_k b^k$, $d_k \in D$, within $\frac{1}{|b|^l\sqrt{2}}$ of z. It follows that every complex number has at least one representation of the form

$$z=\sum_{k=-\infty}^{m} d_k b^k = \sum_{k=0}^{m} d_k b^k + \sum_{k=1}^{\infty} d_{-k} b^{-k}, d_k \in D.$$

We say that z has radix expansion (or equivalently radix representation) $z=\sum_{k=-\infty}^{m} = d_m \ldots d_1 d_0 . d_{-1} d_{-2} \ldots$. When base $b=-n+i$, the radix expansion of z does not share the same properties when $b \in \mathbb{Z}$. For example, if $b=10$, it is well known that the geometric series $0.999\cdots = \sum_{k=-\infty}^{-1} 9(10)^k = 1$. But, if $b=-2+i$ with digits set $D=\{0,1,2,3,4\}$, then $0.444\cdots = \sum_{k=-\infty}^{-1} 4(-2+i)^k = \frac{2}{5}(-3-i) \neq 1$. Nevertheless, we will use radix representation with Gaussian integer base b where convenient. We are particularly interested in the set of complex numbers whose radix expansions have integer part 0. So, the define the set

$$T_0=\left\{\sum_{k=1}^{\infty} d_{-k} b^{-k} : d_k \in D\right\}.$$

This takes the place of the unit interval. See Figure 6.1 of T_0 when $n=2$.

We define some terms that will be useful in proving upcoming lemmas that will be necessary in proving the main theorem. First, let $D^* = \{d_1, d_2, ..., d_t\} \subset D$ denote a set of at least two integers such that $0 = d_1 < d_2 < ... < d_t < n^2 + 1, t < n^2$.

Definition 6. We say that D^* satisfies the *separation condition* if $\mid d_i - d_j \mid \geq 2,$ for all $i \neq j$ if $b = -2 + i$, or $|d_i - d_j| \geq 2n,$ for all $i \neq j$ if $b = -n + i,$ for all $n \geq 3$.

We then define our analagous Cantor set in the complex plane.

Definition 7. The set $T = T_{b,D^*} = \{z \in T_0 \mid d_i \in D^*\}$ is a *deleted digits Cantor set with Gaussian integer base b*. Here, T is obtained from T_0 by restricting attention to those representations that only contain digits from the set D^*.

We wish to refine the set T_0 to construct the deleted digits Cantor set T in the same way [PP12] constructed their deleted digits Cantor set. For that we have the following definition.

Definition 8. For all $k \geq 0$, the set $T_k = \{0.d_1 d_2 ... \mid d_i \in D^*, \forall i \leq k\}$ is called a *refinement of T_0 at the kth stage*. Here, T_k is obtained from T_0 by restricting attention to those representations whose first k digits past the radix point are from the set D^*.

See figures 6.2 and 6.3 for refinements T_1 and T_2 when $n = 2$. The fractal set T_0 is self-similar, meaning that T_0 has the same shape as one or more of its parts. [Fal85] refers to these parts as similitudes. In [PP12], similitudes of the unit interval were called subintervals. Hence, it is natural to introduce the following term.

Definition 9. Let $b^{-k}T_0 = \left\{b^{-k}x \mid x \in T_0\right\}$, and $b^{-k}T_0 + x = \left\{z + x \mid z \in b^{-k}T_0\right\}, x \in \mathbb{C}$. Then a *subtile of* T_k refers to a subset of T_k of the form $b^{-k}T_0 + \sum_{i=1}^{k} d_{-i}b^{-i}$ for fixed $d_{-i} \in D^*$.

In order to prove the theorem, we must first turn our attention to the entropy dimension of a subset of $\mathbb{R}^n$. To calculate the entropy dimension of intersections, we need to extend the definition of possible covering sets.

Chapter 4

Preliminaries

4.1 Entropy Dimension of the Intersections

In this chapter, we set up some necessary foundation that will aid in the proof of the theorem. Due to the structure of T, we study its *entropy dimension* (also called *box-counting dimension*, *Kolmogorov dimension*, or *Minkowski dimension*.)

Definition 10. Let $E \subset \mathbb{R}^n$ such that E is nonempty. Then the *entropy dimension of* E is defined as

$$\dim E = \lim_{\delta \to \infty} \frac{\log N_\delta(E)}{-\log \delta}$$

where $N_\delta(E)$ denotes the smallest number of sets each of diameter at most δ needed to cover E.

The *lower* and *upper entropy dimensions* are defined respectively as follows:

$$\underline{\dim} E = \underline{\lim}_{\delta \to \infty} \frac{\log N_\delta(E)}{-\log \delta}$$

$$\overline{\dim} E = \overline{\lim}_{\delta \to \infty} \frac{\log N_\delta(E)}{-\log \delta}.$$

If the limit does not exist, then we can talk about the upper and lower entropy dimensions obtained by replacing the limit by the limit superior and limit inferior, respectively. [Fal85] considers various definitions of $N_\delta(E)$, namely

- the smallest amount of closed balls of radius δ that cover E;
- the least amount of cubes of side length δ that cover E;
- the number of $\delta-$mesh cubes that intersect E;
- the smallest number of sets of diameter at most δ that cover E;
- the largest number of disjoint balls of radius δ with centers in E.

We extend the list to include the smallest number of subtiles of T_0; sets of the form $b^{-j}T_0 + 0.d_1d_2...d_j$ for some integer j, that cover the set E.

Lemma 11. *If $E \subset \mathbb{R}^n$ such that E is nonempty, then the entropy dimension of E is $\lim_{\delta\to\infty} \dfrac{\log N_\delta(E)}{-\log\delta}$ where $N_\delta(E)$ is the smallest number of subtiles of diameter at most δ needed to cover E.*

Proof. Denote the diameter of T_0 as $\text{diam}(T_0)$. Let $N_{\delta_j}(E)$ denote the smallest number of subtiles of T_0 each of diameter δ_j from a decreasing sequence $\{\delta_j\}_{j=0}^{\infty}$, where $\delta_j = \sqrt{n^2+1}^{-j}\text{diam}(T_0)$. Let $L_{\delta_j}(E)$ be the smallest number of closed balls of radius δ_j needed to cover E. Obviously, $L_{\delta_j}(E) \le N_{\delta_j}(E)$. Let $M_{\sqrt{2}\delta}(E)$ be the smallest number of $\delta-$mesh cubes that intersect E. Now, $M_{\sqrt{2}\delta}(E) \le N_{\delta_j}(E)$. If $\sqrt{2}\delta_j < 1$, then $\dfrac{\log M_{\sqrt{2}\delta}(E)}{-\log(\sqrt{2}\delta)} \le \dfrac{\log N_{\delta_j}(E)}{-\log(\sqrt{2})-\log(\delta)}$. As $\delta_j \to 0$ when $j\to\infty$, we get the following,

$$\underline{\text{dim}}E = \underline{\lim}_{\delta_j\to 0}\frac{\log N_{\delta_j}(E)}{-\log\delta_j}$$

and

$$\overline{\text{dim}}E \le \overline{\lim}_{\delta_j\to 0}\frac{\log N_{\delta_j}(E)}{-\log\delta_j}.$$

Any set of diameters at most δ_j is contained in $3^2 = 9$ δ_j-mesh cubes. Therefore, we get $N_{\delta_j}(E) \le 9M_{\delta_j}(E)$ and taking logarithms leads to the opposite inequalities of the equations above. As a result, we can calculate the entropy dimension using the subtiles of T_k. As vector spaces, $\mathbb{R}^2$ is isomorphic to $\mathbb{C}$. So if $T \subset \mathbb{C}$ is defined as above and $N_\delta(T)$ is the minimum number of subtiles in $T_k = \{z \in T_0 \mid d_i \in D^*, \forall i \leq k\}$ needed to cover T, then, by [Fal85], the entropy dimension of T is equal to $\log_{\sqrt{n^2+1}} \mid D^* \mid$. $\square$

Let

$$F = \{x \in \mathbb{C} \mid T \cap (T + x) \neq \emptyset\}$$

be the set of all $x \in \mathbb{C}$ such that the intersection of T and its translate by x, $T + x = \{z + x \mid z \in T\}$, is non-empty. We consider all x in F such that $T \cap (T + x)$ has entropy dimension $\alpha \log_{\sqrt{n^2+1}} \mid D^* \mid$ where $0 \leq \alpha \leq 1$. Under appropriate assumptions, we show that for any $\alpha \in [0, 1]$, there exists $x \in F$ such that $\dim(T \cap (T + x)) = \alpha \dim T$.

4.2 Construction of Deleted Digits Cantor Sets

In this section, we consider how the self-similarity construction of T can be used to study $T \cap (T + x)$.

Let $b = -n + i \in \mathbb{Z}[i]$ and $n \in \mathbb{N} - \{1\}$. Let $D = \left\{0, 1, ..., n^2\right\}$ be the digit set. Let $D^* = \{d_1, d_2, ..., d_t\} \subset D$ be a set of at least two distinct integers satisfying the separation condition. Let T_0 and T_k be defined as above. The set T_{k+1} is obtained by *refining* the set T_k; that is, by removing complex numbers in T_k with digits in the $k{+}1th$ position to the right of the radix point that are not in D^*. If we take $b^{-k}T_0 = \{0.0...0d_{-k-1}d_{-k-2}... \mid d_i \in D\}$ and consider its translates $b^{-k}T_0 + x = \{0.0...0d_{-k-1}d_{-k-2}... + x \mid d_i \in D\}$, $x \in \mathbb{C}$, then we can write $T_k = \bigcup\limits_{d_i \in D^*} \left(b^{-k}T_0 + 0.d_{-1}...d_{-k}\right)$. What this means, is that for each k, the set T_k consists of $|D^*|^k$ subtiles each $\dfrac{1}{(n^2+1)^k}$ the area of T_0. It should be clear that $T_k \subset T_{k+1}$ for all k, and that $T = \bigcap_{k=0}^{\infty} T_k$.

In order to use subtiles in calculating the entropy dimension of T, we need to show that the subtiles of T_k are pairwise disjoint. To prove this, we require the following lemma.

Lemma 12. *Let T_0 be given. It follows that* *i) if $n = 2$ and D^* satisfies the condition $|d_i - d_j| \geq 2$ for all $i \neq j$, then $T_0 \cap (T_0 + d) = \emptyset$ for $d = 3, 4$;*

ii) if $n \geq 3$ and D^* satisfies the condition $|d_i - d_j| \geq 2n$ for all $i \neq j$, then $T_0 \cap (T_0 + d) = \emptyset$ for all $d \in D^* - \{0\}$.

Proof. Let $z = \lceil \frac{n^2}{2} \rceil \in D$. Denote $c_s = 0.s\overline{z}$ where $s \in D$ is fixed. Then c_s represents the "center" of the subtile $b^{-1}(T_0 + s) \subset T_0$ as

$$|x - c_s| = \left| \sum_{j=2}^{\infty} (x_j - \lceil \frac{n^2}{2} \rceil) b^{-j} \right| \leq \sum_{j=2}^{\infty} \lceil \frac{n^2}{2} \rceil \mid b \mid^{-j} = \frac{\lceil \frac{n^2}{2} \rceil}{n^2 + 1 - \sqrt{n^2+1}} = \epsilon$$

for all $x \in b^{-1}(T_0 + s)$. Denote $B_\epsilon(c_s)$ as the ball centered at c_s with radius ϵ. As $b^{-1}(T_0 + s) \subset B_\epsilon(c_s)$, we get $T_0 \subset \bigcup_{s=0}^{n^2} B_\epsilon(c_s)$. Similarly, $b^{-1}(T_0 + t) + d \subset B_\epsilon(c_t + d)$ and $T_0 + d \subset \bigcup_{t=0}^{n^2} B_\epsilon(c_t + d)$ where $t \in D$ is fixed. For any $t, s = 0, 1, ..., n^2$, $\mid c_s - (c_t + d) \mid = \mid 0.s\overline{z} - d.t\overline{z} \mid = \mid d.(s-t) \mid$ is the distance between the centers of the balls $B_\epsilon(c_s)$ and $B_\epsilon(c_t + d)$. Therefore, it is sufficient to show that the ball centered at c_s with radius ϵ is disjoint from any ball of radius ϵ centered at $c_t + d$; that is, $B_\epsilon(c_s) \cap B_\epsilon(c_t + d) = \emptyset$. Figure 6.4 provides a visual representation when $b = -2 + i$, and $D^* = \{0, 3\}$.

The proof reduces to show that $\mid d.(s-t) \mid > 2\epsilon$. Algebraically, we get

$$\begin{aligned}
&\mid d.(s-t) \mid = \mid d + 0.(s-t) \mid = \mid d + (s-t)b^{-1} \mid \\
&= \left| d + \frac{(s-t)}{-n+i} \right| = \left| d + \frac{(s-t)(-n-i)}{n^2+1} \right| = \left| \left(d + \frac{-(s-t)n}{n^2+1} \right) + \left(\frac{-(s-t)}{n^2+1} \right) i \right| \\
&= \left| d + \frac{(s-t)(-n-i)}{n^2+1} \right| = \left| \left(d + \frac{-(s-t)n}{n^2+1} \right) + \left(\frac{-(s-t)}{n^2+1} \right) i \right| \\
&= \left(\left(d + \frac{-(s-t)n}{n^2+1} \right)^2 + \left(\frac{-(s-t)}{n^2+1} \right)^2 \right)^{1/2} = \sqrt{d^2 - \frac{2dn}{n^2+1}(s-t) + \frac{1}{n^2+1}(s-t)^2}.
\end{aligned}$$

Also, $2\epsilon = \dfrac{2\lceil \frac{n^2}{2} \rceil}{n^2 + 1 - \sqrt{n^2+1}} \leq \dfrac{n^2+1}{n^2 + 1 - \sqrt{n^2+1}}$. We will take a closer look at small values of n.

If $n = 2$, then $D^* = \{0, 3\}$, or $\{0, 4\}$ and we have $(2\epsilon)^2 = 4\epsilon^2 \leq \dfrac{5}{5 - \sqrt{5}} < 9 - \frac{12}{5}(s -$

$t) + \frac{1}{5}(s-t)^2 = |d.(s-t)|^2$. Table 6.1 compares 2ϵ with $|3.(s-t)|$ for all possible s and t values. Note that $2\epsilon < |3.(s-t)| \leq |4.(s-t)|$. Taking the square root yields the desired result.

If $3 \leq n < 5$, then $D^* = \{d_1, d_2\}$ where $d_i = \begin{cases} d \in D & \text{if } d \geq 2n \\ 0 & \text{if } i = 1 \end{cases}$. The reason $d_i = d \geq 2n$ is because $\sqrt{d^2 - 2\frac{dn}{n^2+1}(s-t) + \frac{1}{n^2+1}(s-t)^2} \approx \sqrt{d^2 - 2dn + 0 + 1} \geq \frac{1}{2}$. So, $d^2 - 2dn - \frac{3}{4} > 0$ implies $d = \dfrac{2n + \sqrt{4n^2+3}}{2}$ which leads to $d \geq 2n$. Clearly,

$$(2\epsilon)^2 = 4\epsilon^2 \leq \frac{n^2+1}{n^2+1-\sqrt{n^2+1}} < d^2 - 2\frac{dn}{n^2+1}(s-t) + \frac{1}{n^2+1}(s-t)^2 = |d.(s-t)|^2$$

for all sufficient d, and taking the square root gives us the desired result. Table 6.2 compares 2ϵ and $|6.(s-t)|$ when $n = 3$. Again, note that $2\epsilon < |6.(s-t)| \leq |d.(s-t)|$ for all digits $d > 6$. Table 6.3 compares 2ϵ and $|8.(s-t)|$ when $n = 4$. Again, note that $2\epsilon < |8.(s-t)| \leq |d.(s-t)|$ for all digits $d > 8$.

If $n \geq 5$, and D^* satisfies the separation condition, then we have,

$$(2\epsilon)^2 \leq \frac{n^2+1}{n^2+1-\sqrt{n^2+1}} < d^2 - 2\frac{dn^3}{n^2+1} < d^2 - 2\frac{dn(s-t)}{n^2+1} + \frac{1}{n^2+1}(s-t)^2 = |d.(s-t)|^2$$

for all $s, t \in D$. This is the first case where D^* can have more than two digits. Thus, for all cases, $2\epsilon < |d.(s-t)|$ implying $B_\epsilon(c_s) \cap B_\epsilon(c_t + d) = \emptyset$, and the desired result. $\square$

Now, for an important corollary.

Corollary 13. *If $b^{-k}T_0 + 0.d_{-1}...d_{-k}$ and $b^{-k}T_0 + 0.d'_{-1}...d'_{-k}$ are subtiles of T_k and $d_{-i} \neq d'_{-i}$ for at least one i, then the subtiles are disjoint.*

Proof. Assume $b^{-k}T_0 + 0.d_{-1}...d_{-k}$ and $b^{-k}T_0 + 0.d'_{-1}...d'_{-k}$ are subtiles of T_k and $d_{-i} \neq d'_{-i}$ for at least one i. Then $b^i\left(b^{-k}T_0 + 0.d_{-1}...d_{-k}\right) = b^{i-k}T_0 + d_{-1}...d_{-i}.d_{-i-1}...d_{-k}$ and $b^i\left(b^{-k}T_0 + 0.d'_{-1}...d'_{-k}\right) = b^{i-k}T_0 + d'_{-1}d'_{-2}...d'_{-i}.d'_{-i-1}...d'_{-k}$. Note that $d_{-j} = d'_{-j}$ for all $j < i$. Without loss of generality, assume $d_i > d'_i$. By translating, we get $b^{i-k}T_0 \subset T_0$ and

$b^{i-k}T_0 + \sum_{j=i}^{k}(d_{-j} - d'_{-j})b^{j-k} \subset T_0 + (d_{-j} - d'_{-j})$. By the result of lemma 12, the intersections of these two subtiles are disjoint. □

4.3 Investigating $T \cap (T + x)$

Assume D^* satisfies the separation condition. Consider the translation $T + x = \{z + x \mid z \in T, x \in F\}$. Realize that

$$T \cap (T + x) = \bigcap_{k=0}^{\infty} (T_k \cap (T_k + x)).$$

For $x = 0.d_1 d_2 d_3 ...$ let $\lfloor x \rfloor_k$ denote the truncation of x to the first k places; i.e.,

$$\lfloor x \rfloor_k = 0.d_1 d_2 d_3 ... d_k.$$

We will investigate $T_k \cap (T_k + \lfloor x \rfloor_k)$, since both T_k and $T_k + \lfloor x \rfloor_k$ consists of subtiles.

Remember a subtile in T_k is short for a similitude in T_k; that is, one of the tiles obtained from the refinement process. We apply similar terminology to a *subtile in* $T_k + x$.

Consider how $T_{k+1} \cap (T_{k+1} + \lfloor x \rfloor_{k+1})$ is beget from $T_k \cap (T_k + \lfloor x \rfloor_k)$ for $k \geq 0$. T_{k+1} is obtained from T_k by refining each subtile in T_k. Consequently, $T_{k+1} + \lfloor x \rfloor_{k+1}$ is obtained from $T_k + \lfloor x \rfloor_k$ by refining each subtile in $T_k + \lfloor x \rfloor_k$ and then translating the resulting subtiles in the direction of d_{k+1}/b^{k+1}. Figures 6.5, 6.6, and 6.7 provide a visual demonstration when $b = -2 + i$, and $D^* = \{0, 3\}$ for the first two possible refinements. The intersections are circled.

We study the subtiles of T_k and $T_k + \lfloor x \rfloor_k$ as we *transition* from $T_k \cap (T_k + \lfloor x \rfloor_k)$ to $T_{k+1} \cap (T_{k+1} + \lfloor x \rfloor_{k+1})$. The subtiles in $T_k + \lfloor x \rfloor_k$ either coincide with the subtiles in T_k, share boundaries in common with subtiles in T_k, or are disjoint from the subtiles of T_k. Provided that the digits of x are chosen from the set D^*, the subtiles of T_k and $T_k + \lfloor x \rfloor_k$ either coincide, or are disjoint. Notice that once the subtiles are disjoint, they stay disjoint upon further transition as a result from corollary 13.

Chapter 5

Proof of Theorem

And now for the pièce de rèsistance.

Theorem 14. *If D^* satisfies the separation condition $\mid d_i - d_j \mid \geq 2$ for all $i \neq j, b = -2 + i$ or $\mid d_i - d_j \mid \geq 2n$ for all $i \neq j, b = -n + i, n \geq 2$, then given any $0 \leq \alpha \leq 1$, there exists $x \in F$ such that $T \cap (T + x)$ has entropy dimension equal to $\alpha \log_{\sqrt{n^2+1}} \mid D^* \mid$. This x may be chosen to admit a terminating radix expansion.*

Proof. Let $0 \leq \alpha \leq 1$ be given. We use the transition process to construct an x such that $T \cap (T + x)$ has entropy dimension $\alpha \log_{\sqrt{n^2+1}} \mid D^* \mid$. We begin the refinement process starting with the set $T_0 \cap (T_0 + 0)$. The idea is if $x_{j+1} = 0$, then transitioning from $T_j \cap (T_j + \lfloor x \rfloor_j)$ to $T_{j+1} \cap (T_{j+1} + \lfloor x \rfloor_{j+1})$ multiplies the number of subtiles by $\mid D^* \mid$, and if $x_{j+1} = d_t$ the transition multiplies the number of subtiles by one. Using the procedure outlined in [PP12], let $h_j := [j\alpha]$. Then h_j is a positive integer such that $h_j \leq j\alpha < 1 + h_j$, and consequently, $h_j / j \to \alpha$ as $j \to \infty$. Since $0 \leq \alpha \leq 1$ we have $h_j \leq h_{j+1} \leq 1 + h_j$. Suppose $0 < \alpha < 1$. For $j \geq 1$ set

$$x_j = \begin{cases} d_t & \text{if } h_j = h_{j-1} \\ 0 & \text{if } h_j = 1 + h_{j-1} \end{cases}.$$

Then the number of subtiles in $T_j \cap (T_j + \lfloor x \rfloor_j)$ is $\mid D^* \mid^{h_j}$. Since $T \cap (T + x)$ is a subset of

$T_j \cap (T_j + \lfloor x \rfloor_j)$ this provides an upper bound for the number of subtiles of scale $\dfrac{1}{(n^2+1)^j}$ of T_0 needed to cover $T \cap (T + x)$:

$$N_{\delta_j}(T \cap (T + x)) \leq \mid D^* \mid^{h_j} .$$

To calculate the entropy dimension of $T \cap (T + x)$ it remains to check that any subtile in $T_j \cap (T_j + \lfloor x \rfloor_j)$ leads to points in $T \cap (T + x)$ so that the upper bound for $N_{\delta_j}(T \cap (T + x))$ is also a lower bound. Note that the diameter of the copies are given by $\delta_j = \sqrt{n^2+1}^{-j} \mathrm{diam}(T_0)$ for each j. But by the refinement process each subtile in $T_j \cap (T_j + \lfloor x \rfloor_j)$ transitions to one or $\mid D^* \mid$ subtiles in $T_{j+1} \cap (T_{j+1} + \lfloor x \rfloor_{j+1})$. Hence, it follows from Cantor's Intersection Theorem that each subtile in $T_j \cap (T_j + \lfloor x \rfloor_j)$ has infinitely many points in common with $T \cap (T + x)$. Using $h_j/j \to \alpha$, we conclude

$$\lim_{\delta_j \to 0} \frac{\log N_{\delta_j}(T \cap (T + x))}{-\log \delta_j} = \lim_{j \to \infty} \frac{\log \mid D^* \mid^{h_j}}{-\log \left(\sqrt{n^2+1}^j \mathrm{diam}(T_0) \right)} \to \alpha \, \mathrm{dim}(T).$$

□

We can change some of the digits $x_j = 0$ to $x_j = d_m$ or visa versa, as long as the limit remains unchanged. That is, we can make changes of this nature on a sparse set of j's. For $\alpha = 0$ or $\alpha = 1$, we can choose digits d_j such that the subtiles are disjoint at some jth refinement, or let $x = 0$, respectively. We leave the details to the reader.

Chapter 6

Figures and Tables

Figure 6.1: T_0, $b = -2 + i$

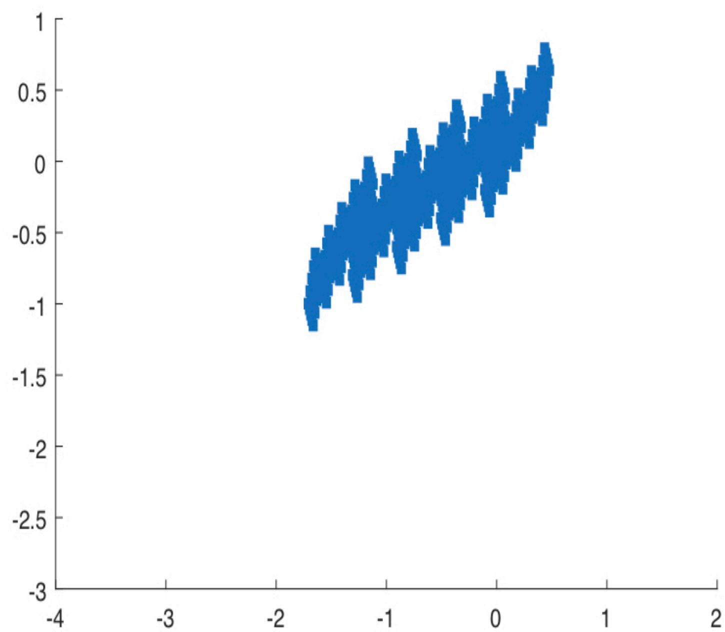

Figure 6.2: T_1, $b = -2 + i$, $D^* = \{0, 3\}$

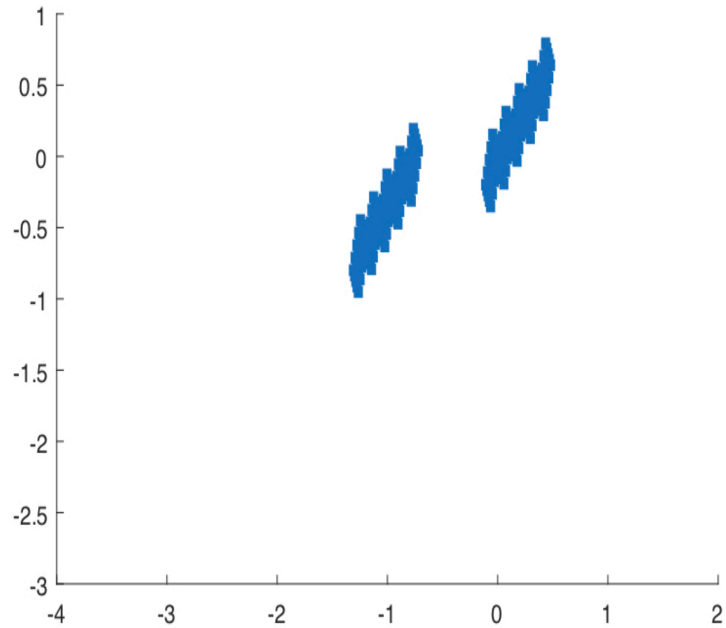

Figure 6.3: T_2, $b = -2 + i$, $D^* = \{0, 3\}$

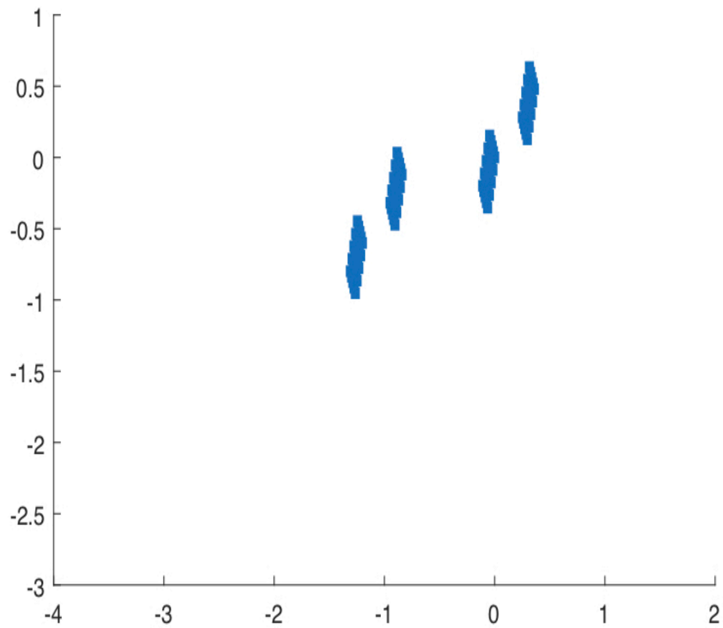

Figure 6.4: $T_0 \cap (T_0 + 3) = \emptyset$, $b = -2 + i$, $D^* = \{0, 3\}$

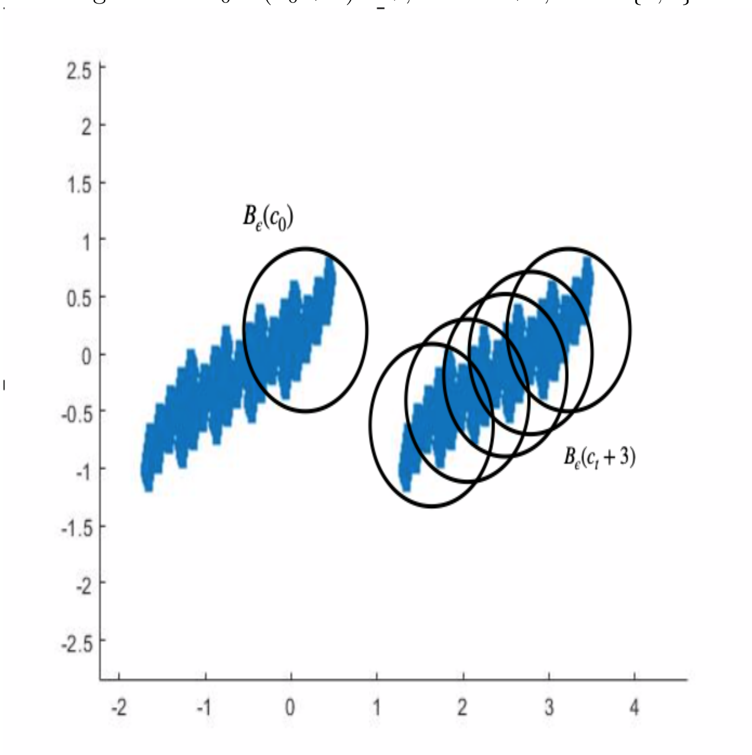

Figure 6.5: $T_1 \cap (T_1 + 0.3)$

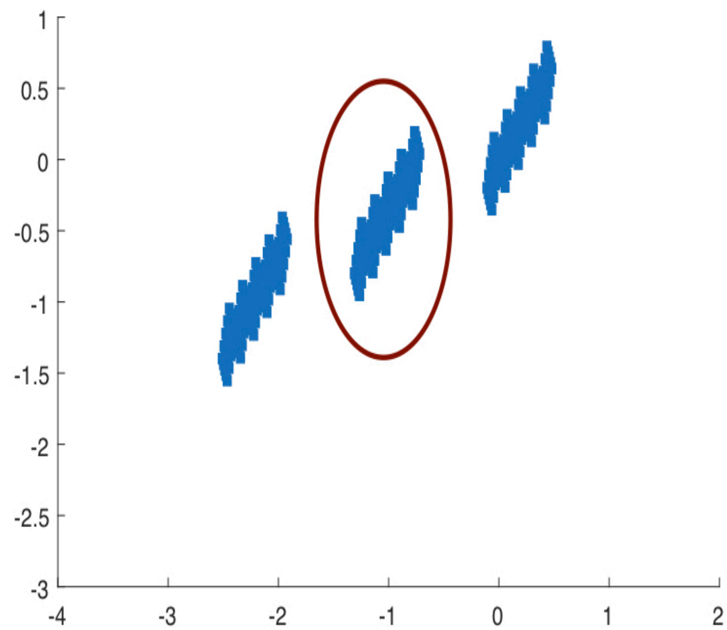

Figure 6.6: $T_2 \cap (T_2 + 0.30)$

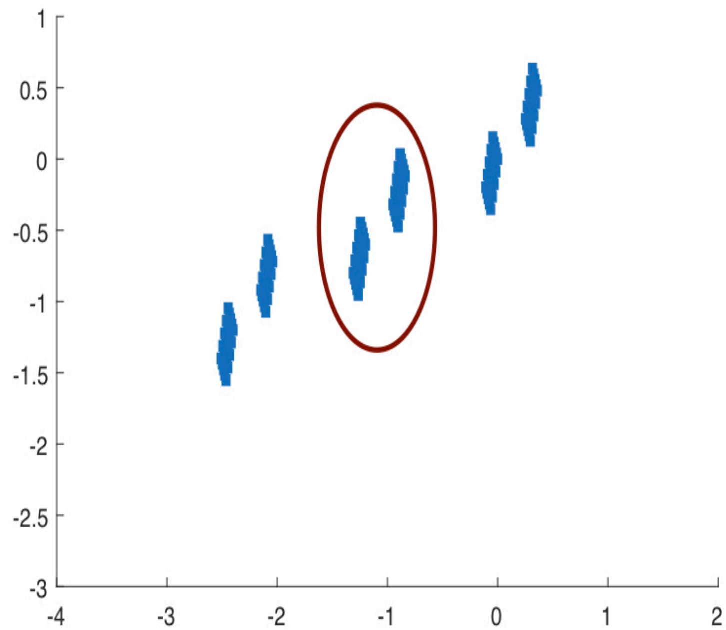

Figure 6.7: $T_2 \cap (T_2 + 0.33)$

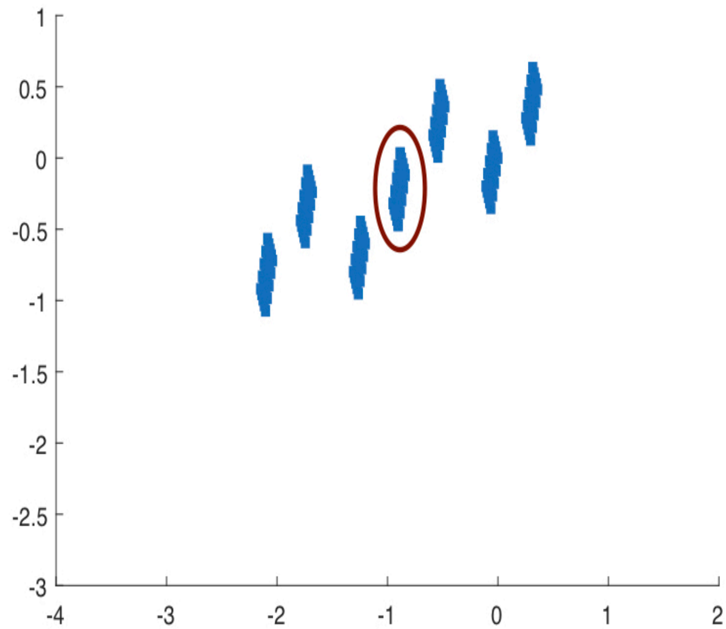

Table 6.1: $b = -2 + i$, $D^* = \{0, 3\}$

$s - t$	2ϵ	$\|3.(s - t)\|$
-4	1.80901699437495	4.6690470119715
-3	1.80901699437495	4.24264068711928
-2	1.80901699437495	3.82099463490856
-1	1.80901699437495	3.40587727318528
0	1.80901699437495	3
1	1.80901699437495	2.60768096208106
2	1.80901699437495	2.23606797749979
3	1.80901699437495	1.89736659610103
4	1.80901699437495	1.61245154965971

Table 6.2: $b = -3 + i$, $D^* = \{0, 6\}$

$s - t$	2ϵ	$\|6.(s - t)\|$
-9	1.46247529557426	8.74642784226795
-8	1.46247529557426	8.43800924389159
-7	1.46247529557426	8.13019064967114
-6	1.46247529557426	7.82304288624318
-5	1.46247529557426	7.51664818918645
-4	1.46247529557426	7.21110255092798
-3	1.46247529557426	6.9065186599328
-2	1.46247529557426	6.60302960768767
-1	1.46247529557426	6.30079360080935
0	1.46247529557426	6
1	1.46247529557426	5.70087712549569
2	1.46247529557426	5.40370243444252
3	1.46247529557426	5.10881590977792
4	1.46247529557426	4.81663783151692
5	1.46247529557426	4.52769256906871
6	1.46247529557426	4.24264068711928
7	1.46247529557426	3.96232255123179
8	1.46247529557426	3.68781778291716
9	1.46247529557426	3.42052627529741

Table 6.3: $b = -4 + i$, $D^* = \{0, 8\}$

$s-t$	2ϵ	$\lvert 8.(s-t) \rvert$
-16	1.3201941016011	11.8022928978677
-15	1.3201941016011	11.563125976696
-14	1.3201941016011	11.3241023537253
-13	1.3201941016011	11.085231298497
-12	1.3201941016011	10.8465228909328
-11	1.3201941016011	10.6079881110749
-10	1.3201941016011	10.3696389409037
-9	1.3201941016011	10.1314884801558
-8	1.3201941016011	9.89355107841348
-7	1.3201941016011	9.65584248616593
-6	1.3201941016011	9.41838002805903
-5	1.3201941016011	9.18118280218071
-4	1.3201941016011	8.94427190999916
-3	1.3201941016011	8.70767072251614
-2	1.3201941016011	8.47140518936221
-1	1.3201941016011	8.2355041990011
0	1.3201941016011	8
1	1.3201941016011	7.76492869555534
2	1.3201941016011	7.5303308262613
3	1.3201941016011	7.29625205962945
4	1.3201941016011	7.06274400931936
5	1.3201941016011	6.8298652126912
6	1.3201941016011	6.5976823024988
7	1.3201941016011	6.36627141776926
8	1.3201941016011	6.13571991077896
9	1.3201941016011	5.90612842234035
10	1.3201941016011	5.67761341741819
11	1.3201941016011	5.45031029877577
12	1.3201941016011	5.22437724968812
13	1.3201941016011	5
14	1.3201941016011	4.77739776570517
15	1.3201941016011	4.55683068396807
16	1.3201941016011	4.33860915637312

Chapter 7

Conclusion

All complex numbers z can be expressed as a radix expansion $z = d_M \ldots d_0.d_{-1}d_{-2}... = \sum_{k=-\infty}^{M} d_k b^k$ such that $b = -n + i$ is a Gaussian integer, $n \geq 1$ and d_k is a digit in the digits set $D = \{0, 1, ..., n^2\}$ for all k. The set $T_0 = \{0.d_1d_2d_3... \mid d_i \in D\}$ forms a tile in the complex plane. The subset $T_k = \{z \in T_0 \mid d_i \in D^*, \forall i \leq k\}$ is a refinement of T_0 by restricting d_i, $1 \leq i \leq k$, to the digits in $D^* = \{d_1, ..., d_t\} \subset D$ such that $0 = d_1 < d_2 < ... < d_t < n^2$ and D^* satsifies the separation condition. As $T_{k+1} \subset T_k$ for all k, we define $T = \bigcap_{k=0}^{\infty} T_k = \{z \in T_0 \mid d_i \in D^*\}$ to be the deleted digits Cantor set with Gaussian integer base b. The entropy dimension of T is $\dim T = \lim_{\delta \to \infty} \dfrac{\log N_\delta(T)}{-\log \delta}$, or equivalently, $\lim_{\delta_j \to 0} \dfrac{\log N_{\delta_j}(T)}{-\log \delta_j}$, where $N_{\delta_j}(T)$ denotes the smallest number of subtiles of T_j, each of diameter δ_j from a sequence $\{\delta_j\}_{j=0}^{\infty}$, where $\delta_j = \sqrt{n^2+1}^{-j} \operatorname{diam}(T_0)$. This is calculated to be $\dim T = \log_{\sqrt{n^2+1}} |D^*|$. We define a translate of T by $T + x = \{z + x \mid z \in T\}$. By theorem 14, if $\alpha \in [0, 1]$, then there exists $x = 0.x_1x_2x_3...$ for every $x_j \in \{0, d_t\} \subset D^*$, such that $T \cap (T + x)$ has entropy dimension $\alpha \dim T$.

Bibliography

[DH95] G. J. Davis and T-Y Hu, *On the structure of the intersection of two middle thirds Cantor sets*, Publ. Math. **39** (1995), 43–60.

[DHW08] Guo-Tai Deng, Xing-Gang He, and Zhi-Xiong Wen, *Self-similar structure on intersections of triadic Cantor sets*, J. Math. Anal. Appl. **337** (2008), no. 1, 617–631.

[DLT09] Shu-Juan Duan, Dan Liu, and Tai-Man Tang, *A planar integral self-affine tile with Cantor set intersections with its neighbors*, Integers **9** (2009), A21, 227–237.

[DS08] Michel Dekking and Karoly Simon, *On the size of the algebraic difference of two random Cantor sets*, Random Structures Algorithms **32** (2008), no. 2, 205–222.

[DT07] Meifeng Dai and Lixin Tian, *Intersection of the sierpinski carpet with its rational translate*, Chaos Solitons Fractals **31** (2007), no. 1, 179–187.

[DT08a] Meifeng Dai and Lixin Tian, *On the intersection of an m-part uniform Cantor set with its rational translation*, Chaos Solitons Fractals **38** (2008), no. 4, 962–969.

[DT08b] Meifeng Dai and Lixin Tian, *On the intersection of an m-part uniform Cantor set with its rational translations*, Chaos Solitons Fractals **38** (2008), 962–969.

[Fal85] Kenneth. J. Falconer, *The geometry of fractal sets*, Cambridge University Press, Cambridge, 1985.

[Fur70] Harry Furstenberg, *Intersections of Cantor sets and transversality of semigroups*, Problems in analysis (Sympos. Salomon Bochner, Princeton Univ., Princeton, N.J., 1969), Princeton Univ. Press, Princeton, N.J., 1970, pp. 41–59.

[Gau32] Carl F. Gauss, *Theoria residuorum biquadraticorum, commentatio secunda*, Comm. Soc. Reg. Sci. Göttingen **7** (1832), 1–34.

[GG79] W.J. Gilbert and R. James Green, *Negative based number systems*, Math. Mag. **52** (1979), 240–244.

[Haw75] John Hawkes, *Some algebraic properties of small sets*, Q. J. Math. Oxf. **26** (1975), 713–747.

[Igu03] K. Igudesman, *Lacunary self-similar fractal sets and intersection of Cantor sets*, Lobachevskii J. Math **12** (2003), 41–50.

[KLD10] Derong Kong, Wenxia Li, and F. Michel Dekking, *Intersections of homogeneous Cantor sets and beta-expansions*, Nonlinearity **23** (2010), no. 11, 2815–2834.

[Knu98] Donald E. Knuth, *The art of computer programming*, third ed., vol. 2 Seminumerical algorithms, Addison-Wesley, Reading, MA, 1998.

[KP91] Richard Kenyon and Yuval Peres, *Intersecting random translates of invariant Cantor sets*, Invent. Math. **104** (1991), 601–629.

[Kra92] Roger L. Kraft, *Intersections of thick Cantor sets*, Mem. Amer. Math. Soc. **97** (1992), no. 468, vi+119.

[Kra00] ———, *Random intersections of thick Cantor sets*, Trans. Amer. Math. Soc. **352** (2000), no. 3, 1315–1328.

[KS75] Imre Kàtai and János Szabo, *Canonical number systems for complex integers*, Acta Sci. Math. (Szeged) **37** (1975), 255–260.

[LN04] Jun Li and Fahima Nekka, *Intersection of triadic Cantor sets with their translates. II. Hausdorff measure spectrum function and its introduction for the clas-*

sification of Cantor sets, Chaos Solitons Fractals **19** (2004), no. 1, 35–46, Dedicated to our teacher, mentor and friend, Nobel laureate, Ilya Prigogine.

[LTZ18] Jian Lu, Bo Tan, and Yuru Zou, *Intersections of translation of a class of sierpinski carpets*, Fractals **26** (2018), no. 3, 10.

[LYZ11a] Wenxia Li, Yuanyuan Yao, and Yunxiu Zhang, *Self-similar structure on intersection of homogeneous symmetric Cantor sets*, Math. Nachr. **284** (2011), no. 2-3, 298–316.

[LYZ11b] Wenxia Li, Yuanyuan Yao, and Yunxiu Zhang, *Self-similar structure on intersection of homogeneous symmetric Cantor sets*, Math. Nachr. **284** (2011), no. 2–3, 298 – 316.

[Mor11] Carlos Gustavo Moreira, *There are no C^1-stable intersections of regular Cantor sets*, Acta Math. **206** (2011), no. 2, 311–323.

[MSS09] Peter Mora, Karoly Simon, and Boris Solomyak, *The Lebesgue measure of the algebraic difference of two random Cantor sets*, Indagationes Mathematicae **20** (2009), 131–149.

[NL02] Fahima Nekka and Jun Li, *Intersections of triadic Cantor sets with their rational translates - I*, Chaos Solitons Fractals **13** (2002), 1807–1817.

[Pal87] Jacob Palis, *Homoclinic orbits, hyperbolic dynamics and dimension of Cantor sets*, The Lefschetz centennial conference, Part III (Mexico City,1984) (Providence, RI), Contemp. Math., vol. 58, Amer. Math. Soc., 1987, pp. 203–216.

[PP12] Steen Pedersen and Jason D. Phillips, *Intersections of certain deleted digits sets*, Fractals **20** (2012), 105–115.

[PP13] ______, *On intersections of Cantor sets: Hausdorff measure*, Opuscula Math. **33** (2013), no. 3, 575–598.

[PP14] ______, *On intersections of Cantor sets: Self-similarity*, Commun. Math. Anal. **16** (2014), no. 1, 1–30.

[PS98] Yuval Peres and Boris Solomyak, *Self-similar measures and intersections of Cantor sets*, Trans. Amer. Math. Soc. **350** (1998), no. 10, 4065–4087.

[SRM+20] Julian Smith, Conor Rowland, Saba Moslehi, Richard Taylor, Anastasija Lesjak, Martin Lesjak, Sabrina Stadlober, Luis Lee, Jackie Dettmar, Mark Page, and Jeanette Himes, *Relaxing floors: Fractal fluency in the built environment*, Nonlinear Dynamics, Psychology, and Life Sciences **24** (2020), no. 1, 127–141.

[Wil91] R. F. Williams, *How big is the intersection of two thick Cantor sets?*, Continuum theory and dynamical systems (Arcata, CA, 1989), Contemp. Math., vol. 117, Amer. Math. Soc., Providence, RI, 1991, pp. 163–175.

[ZG11] Yun Xiu Zhang and Hui Gu, *Intersection of a homogeneous symmetric Cantor set with its translations*, Acta Math. Sinica (Chin. Ser.) **54** (2011), no. 6, 1043–1048.

[ZLL08] Yuru Zou, Jian Lu, and Wenxia Li, *Self-similar structure on the intersection of middle-*$(1-2\beta)$ *Cantor sets with* $\beta \in (1/3, 1/2)$, Nonlinearity **21** (2008), no. 12, 2899–2910.