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The Effect of Winding Curvature and Core Permeability on the Power Losses and Leakage Inductance of High-Frequency Transformers

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THE EFFECT OF WINDING CURVATURE AND CORE PERMEABILITY ON THE POWER LOSSES AND LEAKAGE INDUCTANCE OF HIGH-FREQUENCY TRANSFORMERS

A dissertation submitted in partial fulfillment of the requirements for the degree of
Doctor of Philosophy

by

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ABSTRACT

Whitman, Daniel J. Ph.D., Department of Electrical Engineering, Wright State University, 2021. The Effect of Winding Curvature and Core Permeability on the Power Losses and Leakage Inductance of High-Frequency Transformers.

At the frequencies used in switching dc-dc converters, the skin and proximity effects have a significant effect on both the losses and leakage inductance of the transformers used in these circuits. Analytical expressions that have been derived to calculate ac resistance and leakage inductance have primarily used a 1-D approximation. They also have used Cartesian coordinates or approximations that are equivalent to Cartesian coordinates, as well as usually assuming an ideal core of infinite permeability.

The classical result in the case of the resistance/losses is Dowell’s equation, and there are analogous results for leakage inductance. This dissertation derives new equations that take the effect of winding curvature into account by using cylindrical coordinates. These equations are also more general in that they permit any interleaving pattern as well as variable layer thicknesses and gaps between layers. Additionally, new equations for the magnetic field coefficients are derived that take the core permeability into account, which requires a full 2-D model and the method of images applied in two dimensions. These coefficients then allow calculations of resistance/losses and leakage inductance that also take the core permeability and winding width into account.

The accuracy of all of these equations is assessed by comparing their results with those of finite-element analysis (FEA) simulations. Due to the large number of parameters involved with the fully general equations, a statistical approach is used in which a large number of randomly generated devices are simulated. Finally, for a special class of more specific transformers, the effects of a reduced number of independent parameters on the resistance/losses and leakage inductance is determined empirically. The relative sensitivity of these quantities on these parameters is also determined.
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The following list is cumulative in that each chapter only lists new symbols introduced in that chapter. For example, a symbol introduced in Chapter 2 that is reused in Chapter 3 is not listed again under the Chapter 3 symbols. However, if a symbol is redefined in a later chapter for use there, then it is listed. If a chapter or appendix is not listed at all, it means that no new symbols are defined within.

Chapter 2

- \( E, \bar{E} \) Electric field in the positive \( \phi \) or \( y \) direction (see Figure 2.1), which is normalized to the dc electric field in one primary turn as \( \bar{E} = E/E_0 \).
- \( E_0 \) Normalization electric field, defined as the electric field at dc inside a primary turn \( E_0 = \rho J_0 \).
- \( F_n \) Cylindrical ac power factor for the \( n \)th layer, setting \( F_0 = 0 \) for convenience.
- \( g_n, \bar{g}_n \) Gap just outside the \( n \)th layer, i.e. the gap between layer \( n \) and layer \( n + 1 \), with \( g_0 \) being the gap between the core and the innermost layer. These are normalized to the thickness of the first layer as \( \bar{g}_n = g_n/h_1 \).
- \( G_n \) Cartesian ac power factor for the \( n \)th layer, setting \( G_0 = 0 \) for convenience.
- \( h_n, \bar{h}_n \) Thickness of the \( n \)th layer, which is normalized to the thickness of the first layer as \( \bar{h}_n = h_n/h_1 \), noting that of course \( \bar{h}_1 = 1 \) by definition. We also set \( h_0 = \bar{h}_0 = 0 \) for convenience.
- \( H, \bar{H} \) Magnetic field intensity in the positive \( z \) direction (see Figure 2.1), which is normalized to the magnetic field intensity due to the primary current as \( \bar{H} = H/H_0 \).
- \( H_0 \) Normalization magnetic field intensity due to the primary winding current \( H_0 = I_1/w \).
- \( H_{n_0} \) Constant magnetic field inside the \( g_n \) gap region, which is \( H_{n_0} = \alpha_{n_0} H_0 \).
- \( I_\alpha \) Modified Bessel function of the first kind and order \( \alpha \). In the case of \( \alpha \in \{1, 2\} \), this can be easily distinguished from the primary and secondary winding currents by the context.
- \( I_l, \bar{I}_l \) Magnitude of the primary \((l = 1)\) or secondary \((l = 2)\) current, which is normalized to the primary current as \( \bar{I}_l = I_l/I_1 \). These can be easily distinguished from the modified Bessel function of the first kind by the context.
- \( J_0 \) Normalization current density, defined as the current density at dc in one turn of a primary layer \( J_0 = I_1/wh_1 \).
- \( J_{n_0} \) Current density at dc in the \( n \)th layer, reasoned to be \( J_{n_0} = N_n I_1/wh_n \).
- \( k \) Complex Bessel equation factor \( k = (1 + j)/\delta_w \).
- \( K_\alpha \) Modified Bessel function of the second kind and order \( \alpha \).
- \( l_n, \bar{l}_n \) Turn length for the \( n \)th layer, which is normalized to the thickness of the first layer as \( \bar{l}_n = l_n/h_1 \).
- \( L_0 \) Convenient normalization inductance with no meaningful physical interpretation \( L_0 = 2W_0 I_1^2 \).
- \( L_{lk} \) Leakage inductance at ac referred to the primary winding \( L_{lk} = (W + W_g)/I_1^2 \).
- \( L_{lk0} \) Normalization leakage inductance referred to the primary winding \( L_{lk0} = 2W_0/I_1^2 \).
$L_{lk(cart)}$ Normalized ac leakage inductance in Cartesian coordinates $L_{lk(cart)} = L_{lk}/L_{lk0}$.

$\overline{L}_{lk(cyl)}$ Normalized ac leakage inductance in cylindrical coordinates $\overline{L}_{lk(cyl)} = L_{lk}/L_{lk0}$.

$M_l$ The set of $p_l$ integer layer numbers for the primary ($l = 1$) or secondary ($l = 2$) winding.

$n$ Denotes the layer number where the innermost layer is 1 and the outermost layer is $p$, regardless of the interleaving pattern.

$N_n$ Number of turns in the $n$th layer, which is used effectively only as a current multiplier since the layers are treated as foils.

$p$ Total number of all layers $p = p_1 + p_2$.

$p_{d\text{i}n0}$ Power density at dc in the $n$th layer, defined as $p_{d\text{i}n0} = \rho J_n^2$.

$p_{\text{l}}$ Number of primary ($l = 1$) or secondary ($l = 2$) layers.

$P_0$ Total dc power loss within all of the primary layers, defined as $P_0 = \sum_{n \in M_1} P_{n0}$.

$P_{\text{ac}}$ Total ac power dissipation in the primary winding $P_{\text{ac}} = \frac{1}{2} \sum_{n \in M_1} \Re\{P_n\}$.

$P_{\text{lo}}$ Convenient normalization power with no meaningful physical interpretation, defined as $P_{\text{lo}} = wh_1^2 \rho J_n^2$.

$P_n$ Complex power of the $n$th layer, reasoned to be the double integral of the Poynting vector over the surface of the layer conductor.

$P_{n0}$ Power loss at dc in the $n$th layer, reasoned to be the triple integral of the dc power density $p_{d\text{i}n0}$ over the layer conductor volume.

$r_{n\text{a}}, \tau_{n\text{a}}$ Center radius of the $n$th layer, which is normalized to the first layer thickness as $\tau_{n\text{a}} = r_{n\text{a}}/h_1$.

$r_{n\text{i}}, \overline{r}_{n\text{i}}$ Inner radius of the $n$th layer, which is normalized to the first layer thickness as $\overline{r}_{n\text{i}} = r_{n\text{i}}/h_1$.

$r_{n\text{o}}, \tau_{n\text{o}}$ Outer radius of the $n$th layer, which is normalized to the first layer thickness as $\tau_{n\text{o}} = r_{n\text{o}}/h_1$.

$\overline{R}_{\text{cart}}$ Normalized ac resistance and losses in Cartesian coordinates, defined as $\overline{R}_{\text{cart}} = 2P_{\text{ac}}/P_0$.

$\overline{R}_{\text{cyl}}$ Normalized ac resistance and losses in cylindrical coordinates, defined as $\overline{R}_{\text{cyl}} = 2P_{\text{ac}}/P_0$.

$U$ Normalization energy density in the entire winding window $U = \frac{1}{2} \mu_0 H_0^2$.

$U_{n\text{g}}$ Energy density within magnetic field in the $g_n$ gap region, defined to be $U_{n\text{g}} = \frac{1}{2} \mu_0 |H_{n\text{g}}|^2$.

$w, \overline{w}$ Winding breadth/width as shown in Figure 2.1, which is assumed to be the same for all layers. The width is normalized to the first layer thickness as $\overline{w} = w/h_1$.

$W$ Total ac energy inside all of the layers $W = \sum_{n=1}^{p} W_n$.

$W_0$ Total normalization energy of all layers and gap regions $W_0 = \sum_{n=0}^{p} W_{n0}$.

$W_{\text{g}}$ Total gap energy for all of the gap regions $W_{\text{g}} = \sum_{n=0}^{p} W_{n\text{g}}$.

$W_{\text{lo}}$ Normalization energy, defined to be the energy per unit length in the first layer at the normalization magnetic field $W_{\text{lo}} = \frac{1}{2} \mu_0 w h_1 H_0^2$.

$W_n$ Energy at ac in the magnetic field inside the $n$th layer $W_n = \Im\{P_n\}/2\omega$.

$W_{n0}$ Total normalization energy in the $n$th layer and the gap region outside of the $n$th layer, reasoned to be the triple integral of the normalization energy density $U$ over the layer conductor and gap region volume.
Chapter 3

\( W_{n_g} \) Total energy stored in the magnetic field in the \( g_n \) gap region, reasoned to be the triple integral of the gap energy density \( U_{n_g} \) over the gap region volume.

\( \alpha_{n_i} \) Generally complex magnetic field coefficient at the inner surface of the \( n \)th layer, normalized to the magnetic field intensity due to the primary current as \( \alpha_{n_i} = H(r_{n_i})/H_0 \).

\( \alpha_{n_o} \) Generally complex magnetic field coefficient at the outer surface of the \( n \)th layer, normalized to the magnetic field intensity due to the primary current as \( \alpha_{n_o} = H(r_{n_o})/H_0 \).

\( \delta_w \) Skin depth in the layer conductors \( \delta_w = \sqrt{\frac{2\rho}{\mu_0\omega}} \).

\( \Delta \) Penetration ratio of the first layer, defined as \( \Delta = \frac{h_1}{\delta_w} \).

\( \varepsilon_0 \) Permittivity of free space \( \varepsilon_0 \approx 8.854 \times 10^{-12} \text{ F/m} \).

\( \kappa \) Simple complex factor \( \kappa = 1 + j \).

\( \mu_0 \) Permeability of free space \( \mu_0 = 4\pi \times 10^{-7} \text{ H/m} \).

\( \rho \) Resistivity of copper \( \rho = 1/\sigma \approx 17.241 \text{ n}\Omega\text{m} \).

\( \sigma \) Conductivity of copper \( \sigma = 58 \text{ MS/m at 20° C} \).

\( \Psi_0 \) Kernel function of order zero, defined as \( \Psi_0(x,y) = I_0(x)K_0(y) - I_0(y)K_0(x) \) in cylindrical coordinates and \( \Psi_0(x,y) = \sinh(x-y) \) in Cartesian coordinates.

\( \Psi_1 \) Kernel function of order one, defined as \( \Psi_1(x,y) = I_0(x)K_1(y) + I_1(y)K_0(x) \) in cylindrical coordinates and \( \Psi_1(x,y) = \cosh(x-y) \) in Cartesian coordinates.

\( \omega \) Angular frequency of the driving current in the primary winding.

\( A_{k,m}, \overline{A}_{k,m} \) Magnetic field intensity contribution from image layer \( m \) (in the \( \hat{z} \) dimension) in image stack \( 2k \) (in the \( \hat{x} \) dimension) which is normalized to \( T_n H_0 \) as \( \overline{A}_{k,m} = A_{k,m}/T_n H_0 \).

\( b \) Overall winding window thickness, see Figure 3.6.

\( B_{k,m}, \overline{B}_{k,m} \) Magnetic field intensity contribution from image layer \( m \) (in the \( \hat{z} \) dimension) in image stack \( 2k + 1 \) (in the \( \hat{x} \) dimension) which is normalized to \( T_n H_0 \) as \( \overline{B}_{k,m} = B_{k,m}/T_n H_0 \).

\( H_i \) Magnetic field intensity to the left of an isolated layer in the 1-D model.

\( H_o \) Magnetic field intensity to the right of an isolated layer in the 1-D model.

\( H_c \) Magnetic field intensity at the innermost surface of the innermost layer in the 1-D model.

\( H_t \) Magnetic field intensity at the outermost surface of the outermost layer in the 1-D model.

\( H_z, \overline{H}_z \) The \( z \) component of the overall 2-D magnetic field intensity from all layers and their images, which is normalized to \( H_0 \) as \( H_z = \overline{H}_z/H_0 \).

\( \overline{H}_{zn}, \overline{H}_zn \) Base \( z \) component of the magnetic field intensity function for the 2-D solution of the rectangular \( n \)th layer, which is normalized to \( T_n H_0 \) as \( \overline{H}_{zn} = \overline{H}_zn = \overline{H}_{zn}/T_n H_0 \).

\( H_{zn}, \overline{H}_zn \) The \( z \) component of the 2-D magnetic field intensity due to the \( n \)th layer \textit{in situ} that includes the effects of every image layer which is normalized to \( T_n H_0 \) as \( \overline{H}_zn = H_{zn}/T_n H_0 \).
The \( H_{zn0}, \mathcal{P}_{zn0} \) \( z \) component of the 2-D magnetic field intensity outside of the isolated rectangular \( n \)th layer, which is normalized to \( \mathcal{T}_n H_0 \) as \( \mathcal{P}_{zn0} = H_{zn0}/\mathcal{T}_n H_0 \).

\( \hat{r}_{n_i} \) \( \hat{r}_{n_i} \) Inner \( x \) position of the \( n \)th layer with respect to the coordinate system of Figure 3.6.

\( s_l \) Simple sum of number of turns in the primary \( (l = 1) \) or secondary \( (l = 2) \) winding, defined as \( s_l = \sum_{n \in \mathcal{M}_l} N_n \).

\( \alpha_c \) Image current factor \( \alpha_c = (\mu_r - 1)/(\mu_r + 1) \).

\( \gamma_{c,l} \) Inner partial field at the inner surface of the innermost layer due to the primary \( (l = 1) \) or secondary \( (l = 2) \) winding.

\( \gamma_{t,l} \) Outer partial field at the outer surface of the outermost layer due to the primary \( (l = 1) \) or secondary \( (l = 2) \) winding.

\( \lambda_m \) Single-sided series factor defined as \( \lambda_m = 1/2 \) when \( m = 0 \) and \( \lambda_m = 1 \) when \( m > 0 \).

\( \mu_r \) Relative permeability of the magnetic core of the device.

\( \xi(k, m) \) Image current factor exponent for image layer \( (k, m) \) in the grid, defined as \( \xi(k, m) = \max(||k||, ||m||) \).

**Chapter 4**

\( r_c, \bar{r}_c \) Radius of the core, which is normalized to the first layer thickness as \( r_c = \bar{r}_c/h_1 \).

\( R^2 \) Denotes the coefficient of determination metric from statistics, see Appendix C.

\( \%RE^m \) Denotes the mean relative error metric, see Appendix C.

\( \%RE^a \) Denotes the specific relative error metric, see Appendix C.

**Chapter 5**

\( \bar{t}_i \) Insulation thickness for a basic transformer, which determines the layer gaps as \( \bar{g}_0 = \bar{t}_i \) and \( \bar{g}_p = 2\bar{t}_i \), and \( \bar{g}_n = 2\bar{t}_i \) for \( 1 \leq n < p \).

**Appendix B**

Here we list only symbols used throughout the entire appendix. Symbols used only within individual theorems and lemmas are not listed.

\( a \) Distance from the center of the layer to its edges in the \( x \) dimension, defined as \( a = h_n/2 \).

\( b \) Distance from the center of the layer to its edges in the \( z \) dimension, defined as \( b = w/2 \).

\( f, \tilde{f} \) Patched \( (f) \) and unpatched \( (\tilde{f}) \) functions representing the entire \( H_{zn0} \) function.

\( g, \tilde{g} \) Patched \( (g) \) and unpatched \( (\tilde{g}) \) functions representing the entire \( \tilde{H}_{zn} \) function.

\( g_1, \tilde{g}_1 \) Patched \( (g_1) \) and unpatched \( (\tilde{g}_1) \) functions representing the natural logarithm term in \( \tilde{H}_{zn} \).

\( g_2, \tilde{g}_2 \) Patched \( (g_2) \) and unpatched \( (\tilde{g}_2) \) functions representing a single arctangent term in \( \tilde{H}_{zn} \).
\( h_2 \)  
Patched function representing both of the arctangent terms in \( \tilde{H}_{zn} \).

\( T_{k,m} \)  
Terms of the iterated series whose convergence is being proven.

Appendix C

\( D \)  
Potentially multidimensional domain over which the physical function \( y \) and model function \( f \) are defined.

\( e_r \)  
Relative error chosen for the specific relative error metric.

\( \overline{e}_r \)  
Average relative error over the sample points, defined as \( \frac{1}{N} \sum_{i=1}^{n} e_{r_i} \).

\( e_{r_i} \)  
Absolute relative error of the model at each sample point, defined as \( e_{r_i} = \frac{|(f_i - y_i)/y_i|}{1/2} \) for \( i \in \{1, 2, \ldots, N\} \).

\( f \)  
Function that models the physical process function \( y \) on domain \( D \).

\( f_i \)  
Values of the model function defined at the sample points \( x_i \), defined as \( f_i = f(x_i) \) for \( i \in \{1, 2, \ldots, N\} \).

\( N \)  
Number of sample points where the physical function \( y \) is determined via experiment.

\( P \)  
Probability of the absolute error at any sample point being no greater than either \( \overline{e}_r \) for \( \%RE^m \) or \( e_r \) for \( \%RE^s \).

\( x_i \)  
Points on the domain \( D \) where function \( y \) is determined by experiment, defined for \( i \in \{1, 2, \ldots, N\} \).

\( y \)  
Function defined on domain \( D \) that represents the measured or actual values of some physical process.

\( \overline{y} \)  
Average of the all of the \( y_i \) values, defined as \( \frac{1}{N} \sum_{i=1}^{N} y_i \).

\( y_i \)  
Values of the physical function evaluated at the sample points \( x_i \), defined as \( y_i = y(x_i) \) for \( i \in \{1, 2, \ldots, N\} \) and determined by experiment.
Acknowledgment

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Dedicated to

My wife Jennifer and children Rebecca, Alexander, and Lucas.
Chapter 1

Introduction

The topic of determining the ac losses and leakage inductance in the windings of solenoidal transformers at high frequencies due to skin and proximity effects has been well-studied in the literature over the past decades. The primary motivation in accurately predicting these physical quantities is so that designers of magnetic devices can incorporate the skin and proximity effects into their design considerations. These effects become particularly important when designing components for use in switching dc-dc converters, which require these components to operate at frequencies at which these effects become prominent, both at the fundamental frequency and especially at the harmonic frequencies. This is largely why these topics are under the umbrella of power electronics despite effectively being an electromagnetics problem.

The resistance/losses problem and the leakage inductance problem seem to be treated separately in the literature, though they are quite similar. This chapter first contains brief literature reviews of both topics in separate sections. The objectives and structure of the dissertation are then discussed in the context of the existing literature.
1.1 Literature Review: Losses and Resistance

1.1.1 Foundational Works

The earliest works on this topic seem be those by the eminent physicist Stephen Butterworth in the 1920s in a series of papers [7, 9, 8]. In these papers he derives formulas for the eddy currents and resistances of various configurations of infinitely long cylindrical conductors in various alternating magnetic fields, which he then argues applies to coils with certain geometric assumptions. However, the seminal works on this topic seem to be those decades later by Bennet and Larson [5] and especially Dowell [21], in which Dowell’s equation was derived to calculate these losses. Both of these works treat the winding layers as foils with rectangular cross-sections and derive the losses in a Cartesian coordinate system. They also assume an ideal core with infinite permeability so that the magnetic field intensity in the core is zero. As practical inductors and transformers usually have wire windings and not foils, Dowell introduces the porosity factor to transform porous winding layers into a solid foil with equivalent dc resistance. It is worth noting here that it was later demonstrated in [54] that this porosity factor is theoretically unsound, though it is still used as a heuristic tool. This, along with some other simplifying assumptions, results in a fairly simple expression involving hyperbolic functions for the winding loss and resistance of the overall device.

1.1.2 Optimal Layer Thickness

One of the immediate applications of the early work on this problem is to determine an optimum foil layer thickness to minimize the ac losses, which was first explored back in [5]. In this early work, optimum layer thicknesses were only derived for each layer independently. In a later article by Perry [51], a single optimum layer thickness was also derived for device in which every layer has the same thickness, which is more useful for
practical inductors and transformers. Later papers derived optimum layer thickness under different conditions such as current waveforms with a modulated duty cycle [6] or a different winding structure tailored to specific applications [30]. Another interesting article is [73] by Wojda and Kazimierczuk, which renormalized Dowell’s equation to reveal other critical thicknesses for round wires.

1.1.3 Arbitrary Waveforms

One of the simplifying assumptions of Dowell’s equation is that only sinusoidal currents are assumed, which of course allows for a time-harmonic analysis. The understanding of course is that losses can be summed for arbitrary waveforms at each Fourier harmonic to determine the losses for arbitrary current waveforms. To this end Venkatraman [68] examined the losses for trapezoidal and rectangular waves. The intuitive result that rectangular waveforms have higher losses than trapezoidal waveforms is shown analytically, and losses are shown for various duty cycles and rise/fall times. These results are experimentally verified. A few years later, Vandelac and Ziogas in [67] predicted losses for transformers and current waveforms found in several specific converter topologies. Later papers such as [6] and [28] derived expressions for other waveforms as well as discovering a general formula for the optimized layer thickness that does not require summing a Fourier series, but instead by calculating the derivative of the current waveform.

1.1.4 Edge Effects

Though still widely used, Dowell’s equation is known to have a number of deficiencies. Perhaps the primary of these is its neglect of edge effects, which is where the magnetic field has non-negligible components that are orthogonal to the length of the solenoid near ends of the winding window. Another issue is orthogonal components due to the round-wire conductors used in most practical devices. This neglect of orthogonal field components is
understandable as it is one of the key assumptions that renders the problem analytically solvable. When computational power became sufficient that finite-element analysis (FEA) was feasible, the known deficiencies of Dowell’s equation could be quantitatively analyzed. This was done in [55], which concluded that the normalized resistance predicted by Dowell’s equation is too low at lower frequencies, and too high at higher frequencies, noting that the simulations were validated with experiment for a single geometry. The exact degree of the inaccuracy depends on the winding geometry and frequency but can be as much as 12%. Similar exercises were done in [61] and [12], though these works used FEA in a trial-and-error way to develop useful rules of thumb for designers to minimize the edge effects from various sources to reduce overall losses.

Prior to this, however, analytical methods of accounting for these effects had been developed. The seminal work on this was a couple of papers by Ferreira in the early 1990s [23, 24] in which the orthogonality of the skin and proximity effects were exploited to render an analytical solution for actual round-wire windings. This result was improved in [4] the following year to include the porosity factor and remove the assumption that the magnetic field is uniform across the conductor cross-section. In [36] and [34], Kutkut et al. replaced a foil winding with an equivalent ellipse to account for edge effects near the end of the winding window in an analytic way. Wang et al. also derived an analytical expression for the loss of a single foil that takes into account the edge effects at the end of the foil [69]. However, this is for foils only and only for a single layer and so does not take into account the proximity effect from neighboring layers.

1.1.5 Numerical Methods

With the advent of ever-increasing computational power available to researchers, there has been research on using this to attack the high-frequency winding loss problem. One of the issues with accurately simulating these magnetic devices using FEA is that it is very computationally expensive due to the fine meshes needed to capture the fields inside the
winding wire cross sections. One method developed to make this computationally cheaper is by replacing the winding area with material with a suitably chosen anisotropic complex permeability that accounts for the high-frequency effects, which allows for a much coarser mesh in this area. This was first developed in [43] then later refined in [44] and [45].

Other numerical or iterative methods have been developed as well that are not based on FEA. The square-field-derivative method was developed in [64] and claims to be much less computationally expensive than FEA methods. Other numerical methods are designed to take into account specific edge effects. For example, the method proposed in [52] is designed to take into account inter-turn and inter-layer edge effects, but does to take into account edge effects at the ends of the winding window. On the other hand, [62] proposes an iterative method to calculate the losses in a rod core inductor, taking into account the edge effects around the window.

1.1.6 Comparisons

A handful of papers have been published that compare the various analytical loss equations, typically comparing the analytical results against experimental and/or FEA results. The first of these seems to have been [55], which compares Dowell’s equation with FEA results as previously discussed in Section 1.1.4. Then, in [53] Reatti and Kazimierczuk compare the equations of Dowell, Perry, Ferreira, and their own result derived in [4] against experimental devices. Later, Dimitrakakis and Tatakos performed another comparison in [17] and the corresponding journal article [16], comparing the analytical equations with FEA results. The most recent of these comparisons was done by Kaymak et al. in [31], again using FEA simulation results as the comparison criteria.

These works agree in some of their conclusions but disagree in others. They all agree that Dowell’s equation has good accuracy but becomes less accurate as frequency increases, the number of layers increases, and especially when the porosity factor becomes much less than unity. They also reach the conclusion that Ferreira’s equation is actually considerably
less accurate than Dowell’s equation, despite taking some of the edge effects into account. However, [53] concludes that their improvement of Ferreira’s equation is generally the most accurate across the entire frequency range, whereas [16] and [31] seem to conclude that Dowell’s equation is actually the most accurate overall.

1.1.7 Semi-Empirical Models

A more recent trend in this area of research is to utilize FEA simulations to develop so-called “semi-empirical” models to determine losses. Typically these models are based on a modified version of one of the older analytical equations, with a number of extra parameters that are functions of various geometric parameters of the device. Expressions or lookup tables for these parameters are then determined by running a large number of FEA simulations with various combinations of geometric parameters. The earliest of these seems to be [56] by Robert et al., which is based on a modified Dowell’s equation with three additional parameters. This model was later used by the same authors in [57] to develop semi-quantitative design guidelines for the geometric parameters.

Another model was developed, also based on Dowell’s equation, by Nan and Sullivan in [46], which was later improved in the follow-up paper [47] to replace the lookup tables with mathematical expressions that can be calculated. Dimitrakakis et al. also developed another model and improved it in [19, 15, 18], which was also based on a modified Dowell’s equation. Holguin et al. joined the fray as well with their own model in [27], again based on Dowell, but this one also includes fringing field effects of gapped cores. Finally, the most recent model was developed by Yin and Li in [75], this time using the Ferreira equation as a basis, instead of Dowell’s.

These models have the advantage of being more accurate than the purely analytical equations by effectively correcting them over parameter ranges where they are known to be inaccurate. They also have an advantage over FEA simulations in that their expressions are typically only a little more complicated than the basic analytical equation and so do
not require time-consuming modeling and simulation. However, one disadvantage of these models is that they are only available for parameters within the design space in which they were developed, which is not a limitation of the stock analytical equations.

1.1.8 Other Topics

Other aspects of this problem have also been examined over the years, which is to say the investigation of other causes of losses. These include the effect of fringing fields due to air gaps in gapped core devices [33, 61, 35, 58, 27], which can be minimized by using distributed gaps. Also studied are the effects on losses of using Litz wire [23, 64, 52, 45, 74, 50] rather than single-strand windings. Of course high-frequency winding losses manifest as heat, and the increased temperature does have an effect on the losses. This thermal feedback aspect to the problem has been studied a bit as well, for example in [36] and [2]. Lastly, in [70] and [36], the effects of a non-ideal core have been investigated using the method of images, though this was done only for rod core devices.

1.2 Literature Review: Leakage Inductance

Overall, the leakage inductance problem has not received as much attention in the literature to date as the loss problem, though the approach is quite similar for both. Additionally, treatment of this problem is much more recent than treatments for the loss problem. The earliest calculations of leakage inductance seems to be those by Japanese researchers Ueda and Koshiha in [65] and Dauhajre and Middlebrook in [14]. These demonstrate how the leakage inductance can be determined by integrating the magnetic flux in the winding area. However, these early works assume uniform current density within the winding conductors and so do not take high-frequency effects into account. This neglect of high-frequency effects is also done in some much later papers like [39, 20, 37, 59]. Niemela et al. did a
very similar analysis in [48] a couple years after publication of the Ueda and Kishiba paper, taking the skin and proximity effects into account for infinitely long winding conductors in Cartesian coordinates. It is also worth noting that they introduced the technique of calculating the leakage inductance using the magnetic field energy within the winding area instead of the flux directly, which is a technique that would be used by several subsequent papers.

Of particular interest to this work is a 2013 article by Lambert et al. [37] in which the analytical leakage inductance for a pot core device of arbitrary core permeability is derived. This approach uses the solution of the magnetic vector potential for conductors with a rectangular cross section with a uniform current density along with a doubly-infinite set of image layers. The leakage inductance between each pair of primary and secondary layers is then calculated using the superposition of energies cause by the vector potentials from all of the image layers. However, this leakage inductance assumes a uniform current density and so does not take high-frequency effects into account.

Two excellent treatments were published in the same month of 2015, which are also relevant to the present work and both include skin and proximity effects. The first is [49] by Ouyang et al. in which an analysis was done similar to that in [48], though formulas were also derived for interleaved windings and were validated by both FEA and experiment. The second is [3] by Bahmani and Thiringer. This paper has a similar derivation in Cartesian coordinates but uses different turn lengths for each layer instead of a single average length as in [49] and [48]. Though this treatment only applies to non-interleaved windings, it also allows for differing insulation gaps between the primary and secondary windings instead of a constant gap size between all layers. A more recent paper by Das et al. [13] is similar to these derivations but allows for differing mean turn lengths for each layer, uses a porosity factor, and also treats several “fractional” interleaving configurations in which the magnetic field is not necessarily zero in between the primary and secondary sections.

Other treatments of this topic include [10] in which Cheng derives the leakage induc-
tance for stranded wire windings using the voltage per unit length of each of the strands. Other works focus on computationally efficient numerical methods to calculate the leakage inductance. Stadler and Albach develop in [63] an interesting numerical method to calculate leakage inductance and core losses in gapped-core transformers with different winding layouts. Another numerical method was developed by Duppalli and Sudhoff in [22] as an alternative to expensive FEA computations. However, it seems that neither of these methods actually take the skin and proximity effects into account within the windings.

1.3 Dissertation Objectives

One can see that this is a mature research area with numerous different sub-topics. A common aspect to all these different treatments of the problem, however, is that almost all of them model the windings as infinitely long, straight conductors so as to enable solving the problem in Cartesian coordinates, which is an acceptable approximation so long as the core radius is large relative to the layer thickness. A few of the previous treatments, such as [51] and the appendix of [28], do initially perform the derivation in cylindrical coordinates. However, at a certain point in the derivation, both take exponential approximations of the resulting Bessel functions, the result of which is Dowell’s equation when the derivation is carried to completion.

Additionally, most previous analytical treatments assume an ideal core, which results in zero magnetic field intensity in the core, and makes determining the fields in between each layer relatively simple. One paper [35] does derive the losses with a core of arbitrary permeability, but this is done only for rod core devices. A similar treatment was done for inductors by in the thesis of the author [70] in which it is pointed out that a similar analysis for transformers using the 1-D approach leads to a contradiction, though details about this are not discussed. There is also [37], which derived the leakage inductance for a core of arbitrary permeability, applicable to pot core devices as discussed above.
To the knowledge of the author, no previous work has been done to derive either the losses or the leakage inductance fully using cylindrical coordinates that take the winding curvature into account. Also, though an arbitrary core permeability for pot core devices was treated for the leakage inductance, this treatment does not take high-frequency effects into account, and this has not been applied to the loss problem. It is the objective of this dissertation to derive analytical equations for the losses and leakage inductance fully in cylindrical coordinates in a general and unified way that also takes high-frequency effects as well as the core permeability into account for both. The treatment here makes many of the same assumptions used in the previous major analytical works, which explicitly are the following:

1. Attention is restricted to pot core transformers in which core material surrounds the winding window.

2. The core is assumed to be lossless and isotropic, and it is assumed the frequencies of interest are far from the resonant frequency. We make this assumption regardless of the shape of the core, which can affect the resonance frequency as discussed in [25]. This results in a constant, real permeability.

3. Only foil windings are considered due to known methods to transform from round wires using an equivalent resistance, for example see Section 5.17 of [32].

4. The winding width $w$ is assumed to be much longer than the layer thickness $h$ so that the magnetic field throughout is entirely parallel to the layer surfaces. This means that edge effects are neglected.

5. The winding layers extend the entire width of the window, which also contributes to the magnetic field being parallel to the layer surfaces.

6. The current in the windings is purely sinusoidal.

These permit a 1-D, time-harmonic analysis that renders the analytical treatment feasible.
However, the derivation of Dowell’s equation and most previous leakage inductance analyses make additional assumptions that are not made in this treatment:

1. The radius of curvature of the winding layers is large relative to the layer thickness.

2. Each layer is assumed to have the same average turn length.

3. All layers have the same thickness.

4. The gaps between layers are the same for all layers.

5. There is no interleaving.

6. The core is ideal with infinite permeability.

Additionally, the accuracy of the new equations are assessed in their full generality by statistical comparison to a large number of FEA simulations. Lastly, for a specific class of transformers, the qualitative effects of the defining parameters on the losses and leakage inductance are determined empirically, and the sensitivity of these are quantitatively determined for each of the parameters.

1.4 Dissertation Organization

The remainder of this dissertation is organized as follows. First, Chapter 2 derives both the resistance/losses and the leakage inductance in both cylindrical coordinates and Cartesian coordinates for comparison. The resulting equations are much more general than previously presented results in that they allow for variable layer thicknesses, variable gaps between layers, and any interleaving pattern. In doing this, the notion of the magnetic field coefficients is introduced, which represent the normalized magnetic field intensity between each of the layers. These must be calculated first in order to calculate the resistance/losses
or leakage inductance. We note here that the results of the research on the cylindrical solution for the resistance/losses has been published in [71] in which these are quantitatively compared with Dowell’s equation.

The topic of Chapter 3 is how to determine these coefficients. First these are derived for an ideal core for later comparison, which permits a 1-D analysis. Then it is shown how a 1-D analysis for a non-ideal core leads to a fundamental contradiction, which is what was alluded to but not discussed in [70]. The remainder of the chapter (which constitutes most of it) is devoted to a 2-D analysis to calculate the coefficients in terms of the core permeability for a non-ideal core. As a kind of bonus, the effect of the winding width \( w \) is also taken into account in the resulting equations. Hence the topics of Chapter 2 and Chapter 3 are out of order in terms of how actual calculations must be done, but the losses and leakage inductance equations were derived first in order to better introduce, define, and provide context for the magnetic field coefficients.

Next, Chapter 4 assesses the accuracy of all of the previously derived equations by comparing them with the results of FEA simulations. In order to exercise the full generality of the resistance/losses and leakage inductance equations, in which there are a large number of independent parameters, a statistical approach is taken in which a large number of randomly generated devices are simulated. The various analytical methods (i.e. cylindrical or Cartesian coordinates with an ideal or non-ideal core) are all compared with the FEA results using three different metrics. The major contributors to inaccuracies are also identified. Finally, it is determined beyond what core permeability and core radius the ideal core and Cartesian approximations, respectively, become sufficiently accurate.

Chapter 5 looks at a specific class of transformers, dubbed “basic transformer”, which can be defined with a much smaller set of independent parameters. It is then determined how each of these parameters qualitatively affects the resistance/losses and leakage inductance in terms of whether increasing the parameter causes the resistance/losses and leakage inductance to increase monotonically, decrease monotonically, or whether there is a non-
monotonic relationship over a defined design space. The relative impact of these parameters on the resistance/losses and leakage inductance is also determined.

Lastly, the concluding chapter first summarizes the assumptions, parameters, and results of the previous chapters. Following this, the specific contributions of this dissertation to the body of knowledge in this topic area are discussed. Lastly, some ideas and recommendations for future research are presented and discussed.

Following the final chapter are several appendices that provide more detail on some of the topics introduced in the main chapters. In particular, Appendix A shows how the successive method of images is used to generate an infinite number of image currents for a current in between two semi-infinite regions of different permeability. This is relevant to Chapter 3 as the 2-D analysis for a non-ideal core involves applying this in both dimensions. Appendix B proves the convergence, in a mathematically rigorous way, of the iterated double series derived in Chapter 3 that is required to evaluate the magnetic field coefficients for a non-ideal core. Lastly, Appendix C discusses in more detail the three metrics used in Chapter 4 to compare the analytical equations with the FEA results.
Chapter 2

General Winding Losses and Leakage Inductance

The first order of business is to derive from first principles the field solutions of a multi-layer, pot core transformer. This is done in a more general way than what has been previously published, and that will allow for any winding configuration as well as to examine the effect of core permeability on the winding losses. Once the electromagnetic fields are determined, the ac complex power is then derived. From this the normalized losses and resistance can be determined as well as the leakage inductance. This is all done in cylindrical coordinates so as to take winding curvature into account, though we also derive the leakage inductance in Cartesian coordinates.

2.1 Electromagnetic Fields

To begin, shown in Figure 2.1 are single foil layers of a solenoidal magnetic device in cylindrical and Cartesian coordinates with the corresponding dimensions. Like many previous treatments, only foil windings are considered as there are well-established techniques for converting round, square, and rectangular windings into foil by introducing a porosity
factor as discussed in [70, 21, 32]. We assume the device has \( p_1 \) primary layers and \( p_2 \) secondary layers so that the total number of layers is \( p = p_1 + p_2 \). We number the layers from 1 starting from the innermost layer to \( p \) for the outermost layer, noting that we use this numbering convention regardless of the interleaving pattern. We also generally allow for each layer to have different thicknesses and numbers of turns so that \( h_n \) and \( N_n \) are the thickness and number of turns of the \( n \)th layer, respectively. However, in devices we consider in subsequent chapters (and indeed in most practical devices) these will all be the same. We will also normalize all physical lengths to the thickness of the first layer \( h_1 \) and denote these normalized lengths with bars over the quantities. Thus each normalized layer thickness becomes \( \bar{h}_n = h_n / h_1 \) and of course \( \bar{h}_1 = 1 \).

To determine the fields within the winding layers we begin with the time-harmonic Faraday and Ampère laws:

\[
\nabla \times \mathbf{E} = -\mu_0 j \omega \mathbf{H} \\
\nabla \times \mathbf{H} = \sigma \mathbf{E} + \varepsilon_0 j \omega \mathbf{E},
\]

where clearly \( \sigma \) is the homogeneous conductivity of the winding material (typically cop-
$\mu_0$ is the vacuum permeability (which assumes that the winding material is non-magnetic), $\varepsilon_0$ is the vacuum permittivity, and $\omega$ is the angular frequency of the sinusoidal driving current.

Now, we claim that a magnetoquasistatic approximation of these equations is quite sufficient for our purposes, in which of course the second (displacement current) term of (2.2) is neglected. The standard rule of thumb is that we are justified in this if the displacement current term is at most one hundred times smaller than the other term. That is

$$|\varepsilon_0 j \omega E| \leq \frac{|\sigma E|}{100}$$

$$\varepsilon_0 \omega |E| \leq \frac{\sigma |E|}{100}$$

$$\omega \leq \frac{\sigma}{100 \varepsilon_0}$$

$$2\pi f \leq \frac{\sigma}{100 \varepsilon_0}$$

$$f \leq \frac{\sigma}{200\pi \varepsilon_0} \quad (2.3)$$

where of course $f$ is the frequency of the driving current. If we use a copper conductivity of $\sigma = 58 \text{ MS/m}$ at $20^\circ \text{C}$ and the vacuum permittivity of $\varepsilon_0 = 8.854 \times 10^{-12} \text{ F/m}$, we get $f \leq 10426 \text{ THz}$. Certainly this maximum frequency is nowhere near those encountered in practice so that the approximation is justified.

Thus our foundational equations become:

$$\nabla \times E = -\mu_0 j \omega H \quad (2.4)$$

$$\nabla \times H = \sigma E \quad (2.5)$$

As in the solutions derived in [70], [21], and the appendix of [28], the magnetic field
intensity has only a \( z \) component and the electric field only a \( \phi \) component so that

\[
H_\phi = H_r = 0 \quad \quad E_z = E_r = 0 .
\]  \hspace{1cm} (2.6)

We can then simply refer to \( H(r) = H_z(r) \) and \( E(r) = E_{\phi}(r) \) without ambiguity, noting that these are a function of \( r \) only by symmetry.

Applying these to (2.5) results in

\[
-\frac{dH}{dr} = \sigma E \rightarrow E = -\rho \frac{dH}{dr} ,
\]  \hspace{1cm} (2.7)

where \( \rho = 1/\sigma \) is the resistivity of the winding material. Similarly, (2.4) becomes

\[
\frac{1}{r} \frac{d}{dr} (rE) = -j\omega \mu_0 H .
\]  \hspace{1cm} (2.8)

Substituting (2.7) into (2.8) then gives

\[
\frac{1}{r} \frac{d}{dr} \left( -r\rho \frac{dH}{dr} \right) = -\mu_0 j\omega H
\]
\[
-\frac{\rho}{r} \left( \frac{dH}{dr} + r \frac{d^2 H}{dr^2} \right) = -\mu_0 j\omega H
\]
\[
\frac{d^2 H}{dr^2} + \frac{1}{r} \frac{dH}{dr} = \frac{\mu_0 j\omega}{\rho} H
\]
\[
\frac{d^2 H}{dr^2} + \frac{1}{r} \frac{dH}{dr} = k^2 H ,
\]  \hspace{1cm} (2.9)

where we have let

\[
k = \sqrt{\frac{\mu_0 j\omega}{\rho}} = \frac{1 + j}{\delta_w} ,
\]  \hspace{1cm} (2.10)
and

\[ \delta_w = \sqrt{\frac{2\rho}{\mu_0 \omega}} \]  

is the skin depth of the fields in the winding. Now, (2.9) is the modified Bessel equation, whose well-known solution is

\[ H(r) = A_n I_0(kr) + B_n K_0(kr), \]  

where \( A_n \) and \( B_n \) are complex constants determined by the boundary conditions, and \( I_0 \) and \( K_0 \) are modified Bessel functions of the first and second kinds, respectively.

Consider next the \( n \)th layer, whether it be primary or secondary, and suppose that the magnetic field at the inside of the layer is \( \alpha_{n_i} H_0 \) and is \( \alpha_{n_o} H_0 \) at the outside for some generally complex constants \( \alpha_{n_i} \) and \( \alpha_{n_o} \) that depend on \( n \). We have normalized these boundary fields to the difference in magnetic field due to the primary current

\[ H_0 = \frac{I_1}{w}, \]  

where \( I_1 \) is the magnitude of the primary sinusoidal current, and \( w \) is the breadth/width of the windings as shown in Figure 2.1, noting that \( w \) is assumed to be the same for all layers. From (2.12) these boundary conditions then result in the equations

\[ H(r_{n_i}) = A_n I_0(kr_{n_i}) + B_n K_0(kr_{n_i}) = \alpha_{n_i} H_0 \]  

\[ H(r_{n_o}) = A_n I_0(kr_{n_o}) + B_n K_0(kr_{n_o}) = \alpha_{n_o} H_0, \]  

where \( r_{n_i} \) and \( r_{n_o} \) are the inner and outer radii of the layer, respectively. After some tedious
algebra that will be omitted here, the solution of this system of equations is found to be

\[
A_n = H_0 \frac{\alpha_{n_0} K_0(k r_{n_0}) - \alpha_{n_i} K_0(k r_{n_i})}{\Psi_0(k r_{n_0}, k r_{n_i})} \tag{2.16}
\]

\[
B_n = H_0 \frac{\alpha_{n_i} I_0(k r_{n_0}) - \alpha_{n_o} I_0(k r_{n_i})}{\Psi_0(k r_{n_0}, k r_{n_i})} \tag{2.17}
\]

where we introduce the notion of the kernel function of order zero for the cylindrical solution

\[
\Psi_0(x, y) = I_0(x) K_0(y) - I_0(y) K_0(x) \tag{2.18}
\]

Substituting (2.16) and (2.17) into (2.12) and rearranging results in the canonical form of the normalized magnetic field:

\[
\overline{H}(r) = \frac{H(r)}{H_0} = \frac{\alpha_{n_i} \Psi_0(k r_{n_o}, k r) + \alpha_{n_o} \Psi_0(k r, k r_{n_i})}{\Psi_0(k r_{n_0}, k r_{n_i})} \tag{2.19}
\]

We also have the electric field and current density, which are of course the same once the appropriate normalizations are done. From (2.7) and (2.12) we have

\[
E(r) = -\rho \frac{dH}{dr} = -\rho \frac{d}{dr} [A_n I_0(k r) + B_n K_0(k r)]
\]

\[
= -\rho k [A_n I_1(k r) - B_n K_1(k r)] \tag{2.20}
\]

Again we substitute in (2.16) and (2.17) and rearrange to get the canonical form of the normalized electric field and current density:

\[
\overline{E}(r) = \frac{E(r)}{E_0} = \frac{J(r)}{J_0} = \kappa \Delta \frac{\alpha_{n_i} \Psi_1(k r_{n_o}, k r) - \alpha_{n_o} \Psi_1(k r_{n_i}, k r)}{\Psi_0(k r_{n_0}, k r_{n_i})} \tag{2.21}
\]
where we have defined the kernel function of order one to be

\[
\Psi_1(x,y) = -\frac{\partial \Psi_0}{\partial y} = I_0(x)K_1(y) + I_1(y)K_0(x) .
\]

(2.22)

It was also reasoned that the uniform dc current density and electric fields in one turn of a primary layer are

\[
J_0 = \frac{I_1}{w h_1} = \frac{H_0}{h_1} ,
\]

(2.23)

\[
E_0 = \rho J_0 = \frac{\rho H_0}{h_1} ,
\]

(2.24)

which we use for normalization of all layers. Note that we have also let \( \kappa = 1 + j \) be a simple complex constant and \( \Delta = h_1/\delta_w \) be the penetration ratio, noting that this also encapsulates the frequency dependence of the fields (since the skin depth \( \delta_w \) is dependent on frequency).

In the Cartesian solution, the field equations (2.19) and (2.21) are still valid but the kernel functions become simply

\[
\Psi_0(x,y) = \sinh(x - y)
\]

(2.25)

\[
\Psi_1(x,y) = \cosh(x - y) .
\]

(2.26)

The first difference to notice is that, in the Cartesian case, clearly the kernel functions \( \Psi_0 \) and \( \Psi_1 \) exhibit odd and even symmetry, respectively, with respect to swapping \( x \) and \( y \). We also note that \( \Psi_0(x,x) = 0 \) while \( \Psi_1(x,x) = 1 \). However, while the cylindrical \( \Psi_0 \) does exhibit odd symmetry and \( \Psi_0(x,x) = 0 \), \( \Psi_1 \) does not exhibit any symmetry and \( \Psi_1(x,x) \) is not constant with respect to \( x \).

Another important difference between the cylindrical and Cartesian cases is that, in the latter, the field equations within the \( n \)th layer are dependent spatially only on \( r - r_{ni} \).
and $r_{n_o} - r$. That is the fields are dependent only on the position within the layer relative to the inner and outer surfaces, and the distance of the layer from the center of the core (the magnitude of $r_{n_i}$) is not relevant. This is not the case with the cylindrical solution and, if we have two identical layers (i.e. they have the same thickness and they both have layer number $n$) but one is closer to the core and the other farther away, then the fields within them will generally be different.

However, even in the cylindrical case we can perform some useful normalizations. As we would currently like to be able to handle arbitrary interleaving patterns, we shall put things in terms of the normalized inner radius $\bar{r}_{n_i} = r_{n_i}/h_1$. We then have the following:

$$kr_{n_i} = \frac{1 + j}{\delta_w} r_{n_i} = (1 + j) \left( \frac{h_1}{\delta_w} \right) \left( \frac{r_{n_i}}{h_1} \right) = \kappa \Delta r_{n_i}$$ (2.27)

and

$$kr_{n_o} = k (r_{n_i} + h_n) = \frac{1 + j}{\delta_w} (r_{n_i} + h_n)$$

$$= (1 + j) \left( \frac{h_1}{\delta_w} \right) \left( \frac{r_{n_i} + h_n}{h_1} \right) = \kappa \Delta \left( \bar{r}_{n_i} + \bar{h}_n \right).$$ (2.28)

The fields are also dependent on the radius $r_{n_i} \leq r \leq r_{n_o}$. For a given layer, though, we would like to express this dependence with a normalized layer position $\tau$ where within the layer we have $0 \leq \tau \leq 1$ so that $\tau = 0$ is at the inner surface of the layer while $\tau = 1$ is at the outer surface.

To accomplish this we set

$$\tau = \frac{r - r_{n_i}}{h_n}.$$(2.29)
so that

\[ r = r_n + h_n \Phi. \]  \hspace{1cm} (2.30)

We then have

\[
kr = k (r_n + h_n \Phi) = kh_1 (\rho_n + \hat{h}_n \Phi) \\
= 1 + \frac{j}{\delta_w} h_1 (\rho_n + \hat{h}_n \Phi) = \kappa \Delta (\rho_n + \hat{h}_n \Phi). \]  \hspace{1cm} (2.31)

### 2.2 Power Losses and Resistance

Previous treatments have derived the losses and resistance using Cartesian coordinates, which results in Dowell’s equation. However, Dowell’s equation is quite specific in that it assumes no winding interleaving, and also assumes an ideal core. Here we will first derive the losses and resistance in Cartesian coordinates, but keep things more general to allow for any interleaving pattern as well as a core of arbitrary permeability. After this, the corresponding equations will be derived again, this time entirely using cylindrical coordinates. As in previous work, we only consider the losses and resistance in the primary windings, though a very similar derivation could be done for the secondary winding as well.

#### 2.2.1 Cartesian Coordinates

Some previous Cartesian-coordinate treatments assume an average turn length that is the same for all layers [49, 48] while others use different turn lengths for each layer [3], which is more physically correct. To accommodate both of these, we assume the more general situation in which each layer has a different turn length \( l_n \) (see Figure 2.1b), where of
course \( n \) is the layer number as before. To begin, we first derive the losses and resistance at \( \text{dc} \).

The \( \text{dc} \) current density in the \( n \)th primary layer is reasoned to be

\[
J_{n0} = \frac{N_n I_1}{w h_n} = \frac{N_n}{h_n} \left( \frac{I_1}{w h_1} \right) = \frac{N_n}{h_n} J_0 \tag{2.32}
\]

so that the power density becomes

\[
p_{dn0} = \rho J_{n0}^2 = \rho \left( \frac{N_n}{h_n} \right)^2 J_0^2 . \tag{2.33}
\]

To find the total \( \text{dc} \) losses in the \( n \)th primary layer we integrate the \( \text{dc} \) power density within the straight layer shown in Figure 2.1b:

\[
P_{n0} = \int_0^w \int_{r_{n_i}}^{r_{n_o}} \int_0^{l_n} p_{dn0} dy \, dx \, dz
= w (r_{n_o} - r_{n_i}) l_n p_{dn0}
= w h_n l_n p_{dn0}
\tag{2.34}
\]

since \( r_{n_o} - r_{n_i} = h_n \). Applying normalization, this becomes

\[
P_{n0} = w h_n l_n p_{dn0}
= w \bar{h}_n l_1 l_n \bar{h}_1 \rho \left( \frac{N_n}{h_n} \right)^2 J_0^2
= \bar{h}_n l_n \left( \frac{N_n}{h_n} \right)^2 w h_1^2 \rho J_0^2
= P_{l0} \frac{\bar{h}_n N_n^2}{h_n}, \tag{2.35}
\]
where we have let \( l_n = l_n/h_1 \) be the normalized layer turn length. We have also defined

\[
P_{l_0} = wh_1^2 \rho J_0^2 = wh_1^2 \rho \left( \frac{H_0}{h_1} \right)^2 = w \rho H_0^2
\]

(2.36)
to be a convenient normalization power that has no useful physical interpretation. For any general interleaving pattern, let \( M_1 \) be the set of layer numbers belonging to the primary winding. Then, since \( p_1 \) is the number of primary layers, the set \( M_1 \) has \( p_1 \) positive integer elements. The total primary dc losses are then

\[
P_0 = \sum_{n \in M_1} P_{l_0} = \sum_{n \in M_1} P_{l_0} \frac{l_n N_n^2}{h_n}
= P_{l_0} \sum_{n \in M_1} \frac{l_n N_n^2}{h_n}.
\]

(2.37)

Regarding the ac losses, as discussed in [70] and [28], we can determine the power loss in each layer using the Poynting vector. To get the complex power, we are required to integrate the Poynting vector \( S = E \times H^* \) (where \( H^* \) is the complex conjugate of \( H \)) over the surface of the straight layer shown in Figure 2.1b. Recall that the Poynting theorem states that the complex power is

\[
P = - \oint_S S \cdot dA = - \oint_S (E \times H^*) \cdot dA.
\]

(2.38)

However, since our convention is that \( E \) is in the positive \( y \) direction and \( H \) is in the positive \( z \) direction, \( S \) is in the positive \( x \) direction. As a result, \( S \) is only normal to the surface at the inner and outer surfaces and is entirely tangential over the top and bottom surfaces so that these do not contribute anything. Therefore, we can integrate the Poynting vector around
the inner and outer surfaces to get the total complex power in the \( n \)th layer:

\[
P_n = - \left[ \int_0^w \int_0^{l_n} E(r_{no}) H_0 H(r_n)^* \, dy \, dz - \int_0^w \int_0^{l_n} E(r_n) H(r_n)^* \, dy \, dz \right]
= \int_0^w \int_0^{l_n} E(r_n) H(r_n)^* \, dy \, dz - \int_0^w \int_0^{l_n} E(r_n) H(r_n)^* \, dy \, dz
= w l_n \left[ E(r_n) H(r_n)^* - E(r_{no}) H(r_{no})^* \right]
= w l_n \left[ H_0 E_0 \alpha_n^* \kappa \Delta \alpha_m^* \Psi_1(kr_{no}, kr_n) - \alpha_{no} \Psi_1(kr_n, kr_{no}) \right]
\]

\[
= w l_n \left[ H_0 E_0 \alpha_n^* \kappa \Delta \alpha_m^* \Psi_1(kr_{no}, kr_n) - \alpha_{no} \Psi_1(kr_n, kr_{no}) \right]
\]

\[
= w l_n \left[ H_0 E_0 \alpha_n^* \kappa \Delta \alpha_m^* \Psi_1(kr_{no}, kr_n) - \alpha_{no} \Psi_1(kr_n, kr_{no}) \right]
\]

\[
= w l_n \left[ H_0 E_0 \alpha_n^* \kappa \Delta \alpha_m^* \Psi_1(kr_{no}, kr_n) - \alpha_{no} \Psi_1(kr_n, kr_{no}) \right]
\]

\[
= w l_n H_0 E_0 \Delta G_n = w l_n \left( \frac{\rho H_0}{h_1} \right) \Delta G_n
= \rho w H_0^2 \Delta \frac{l_n}{h_1} G_n = P_0 \Delta \frac{l_n}{h_1} G_n ,
\]

(2.39)

where in Cartesian coordinates the ac power factor becomes

\[
G_n = \frac{\kappa}{\Psi_0(kr_{no}, kr_n)} \left\{ \left| \alpha_{ni} \right|^2 \Psi_1(kr_{no}, kr_n) + \left| \alpha_{no} \right|^2 \Psi_1(kr_n, kr_{no}) \right\}
= \frac{\kappa}{\sinh(k(r_{no} - r_n))} \left\{ \left| \alpha_{ni} \right|^2 \cosh(k(r_{no} - r_n)) + \left| \alpha_{no} \right|^2 \cosh(k(r_n - r_{no})) \right\}
= \frac{\kappa}{\sinh(k r_n)} \left\{ \left| \alpha_{ni} \right|^2 \cosh(k r_n) + \left| \alpha_{no} \right|^2 \cosh(-k r_n) \right\}
= \frac{\kappa}{\sinh(k r_n)} \left\{ \left| \alpha_{ni} \right|^2 \cosh(k r_n) + \left| \alpha_{no} \right|^2 \cosh(-k r_n) \right\}
= \frac{\kappa}{\sinh(k r_n)} \left\{ \left( \left| \alpha_{ni} \right|^2 + \left| \alpha_{no} \right|^2 \right) \cosh(k r_n) - \alpha_{ni}^* \alpha_{no} - \alpha_{no}^* \alpha_{ni} \right\}
= \frac{\kappa}{\sinh(k r_n)} \left\{ \left( \left| \alpha_{ni} \right|^2 + \left| \alpha_{no} \right|^2 \right) \cosh(k r_n) - \alpha_{ni}^* \alpha_{no} - \alpha_{no}^* \alpha_{ni} \right\}
= \frac{\kappa}{\sinh(k r_n)} \left\{ \left( \left| \alpha_{ni} \right|^2 + \left| \alpha_{no} \right|^2 \right) \cosh(k r_n) - \alpha_{ni}^* \alpha_{no} - \alpha_{no}^* \alpha_{ni} \right\}
= \frac{\kappa}{\sinh(k r_n)} \left\{ \left( \left| \alpha_{ni} \right|^2 + \left| \alpha_{no} \right|^2 \right) \cosh(k r_n) - \alpha_{ni}^* \alpha_{no} - \alpha_{no}^* \alpha_{ni} \right\}
\]

(2.40)
and we note that

\[ kh_n = \frac{1 + j}{\delta_w} h_n = (1 + j) \frac{h_n}{\delta_w} = (1 + j) \frac{\overline{h_n} h_1}{\delta_w} = \kappa \Delta \overline{h_n}. \tag{2.41} \]

We then have that the total power dissipation in the primary winding of the device is

\[ P_{ac} = \frac{1}{2} \sum_{n \in M_1} \Re \{ P_n \} = \frac{1}{2} \Re \left\{ \sum_{n \in M_1} P_n \right\}, \tag{2.42} \]

where \( \Re \{ z \} \) denotes the real part of \( z \). Then the normalized resistance and losses are

\[ \overline{R}_{cart} = \frac{R_{ac}}{R_0} = \frac{2P_{ac}}{P_0} = \frac{\Re \left\{ \sum_{n \in M_1} P_n \right\}}{P_0} = \frac{\Re \left\{ \sum_{n \in M_1} P_0 \Delta I_n G_n \right\}}{P_0 \sum_{n \in M_1} I_n N_n^2 / \overline{h_n}} = \frac{\Delta \Re \left\{ \sum_{n \in M_1} I_n G_n \right\}}{\sum_{n \in M_1} I_n N_n^2 / \overline{h_n}}. \tag{2.43} \]

This equation is a more general analog to Dowell’s equation that still uses Cartesian coordinates.

### 2.2.2 Cylindrical Coordinates

While not the case in Cartesian coordinates, the cylindrical solutions generally depend on the inner radius \( r_{n_i} \) of each layer. Because of this, everything must be re-derived in cylindrical coordinates. In the cylindrical case, the dc current and power densities \( J_{n_0} \) and \( p_{d_{n_0}} \) are still the same as in Cartesian coordinates, given by (2.32) and (2.33), respectively.

To find the total dc losses in the \( n \)th primary layer we integrate the power density in the annular cylindrical volume of the layer shown in Figure 2.1a:

\[ P_{n_0} = \int_0^w \int_{r_{n_i}}^{r_{n_0}} \int_0^{2\pi} p_{d_{n_0}} r \, d\phi \, dr \, dz \]
\[
\begin{align*}
\frac{1}{\pi} \int_0^w \int_{r_{n_i}}^{r_{n_o}} 2\pi p_{\text{dno}} r \, dr \, dz &= \int_0^w 2\pi p_{\text{dno}} \frac{1}{2} \left[ r_{n_o}^2 - r_{n_i}^2 \right] \, dz \\
&= 2\pi w p_{\text{dno}} \frac{1}{2} \left[ r_{n_o}^2 - r_{n_i}^2 \right] \\
&= 2\pi w p_{\text{dno}} \frac{1}{2} \left( r_{n_o} - r_{n_i} \right) \left( r_{n_o} + r_{n_i} \right) \\
&= 2\pi w h_n p_{\text{dno}} \frac{r_{n_o} + r_{n_i}}{2}. \quad (2.44)
\end{align*}
\]

Here we note that \((r_{n_o} + r_{n_i})/2\) is the radius of the center of the layer, which could also be considered the average layer radius. We can again normalize this to the layer thickness so that we have

\[
\tau = \frac{r_{n_o} + r_{n_i}}{2h_1} = \frac{r_{n_o} + h_n + r_{n_i}}{2h_1} = \frac{2\tau_{n_i} + \overline{h_n}}{2} = \tau_{n_i} + \frac{\overline{h_n}}{2}. \quad (2.45)
\]

Then (2.44) becomes

\[
\begin{align*}
P_{n_o} &= 2\pi w h_n h_1 p_{\text{dno}} \tau = 2\pi w h_n h_1^2 \rho \left( \frac{N_n}{\overline{h_n}} \right)^2 J_0^2 \\
&= 2\pi P_0 \frac{N_n^2}{\overline{h_n}} \tau 
\end{align*}
\]

where again \(P_0\) is our convenient normalization power defined in (2.36). The total primary dc losses are then

\[
\begin{align*}
P_0 &= \sum_{n \in M_1} P_{n_o} = \sum_{n \in M_1} 2\pi P_0 \frac{N_n^2}{\overline{h_n}} \tau \\
&= 2\pi P_0 \sum_{n \in M_1} \frac{N_n^2}{\overline{h_n}} \tau = 2\pi P_0 \sum_{n \in M_1} \frac{N_n^2}{\overline{h_n}} \left( \tau + \frac{\overline{h_n}}{2} \right) \\
&= 2\pi P_0 \sum_{n \in M_1} N_n \left( \frac{\tau}{\overline{h_n}} + \frac{1}{2} \right). \quad (2.47)
\end{align*}
\]
Regarding the ac losses, we can again use the Poynting theorem to first determine the complex power in the \( n \)th layer, and once again only the inner and outer surfaces contribute to the integral of (2.38). Thus we integrate the Poynting vector around these surfaces to get the total complex power in the \( n \)th layer:

\[
P_n = - \left[ \int_0^w \int_0^{2\pi} E(r_{no})H(r_{no})^* r_{no} \, d\phi \, dz - \int_0^w \int_0^{2\pi} E(r_{ni})H(r_{ni})^* r_{ni} \, d\phi \, dz \right]
\]

\[
= \int_0^w \int_0^{2\pi} E(r_{ni})H(r_{ni})^* r_{ni} \, d\phi \, dz - \int_0^w \int_0^{2\pi} E(r_{no})H(r_{no})^* r_{no} \, d\phi \, dz
\]

\[
= 2\pi w \left[ r_{ni} E(r_{ni})H(r_{ni})^* - r_{no} E(r_{no})H(r_{no})^* \right]
\]

\[
= 2\pi w \left[ H_0 E_0 r_{ni} \alpha_{ni}^* \kappa \Delta \frac{\alpha_{ni}}{\Psi_0(kr_{ni}, kr_{ni})} \Psi_1(kr_{ni}, kr_{ni}) - \alpha_{no} \Psi_1(kr_{ni}, kr_{ni}) \Psi_0(kr_{no}, kr_{ni}) \right]
\]

\[
- H_0 E_0 r_{no} \alpha_{no}^* \kappa \Delta \frac{\alpha_{no}}{\Psi_0(kr_{no}, kr_{ni})} \Psi_1(kr_{no}, kr_{ni}) - \alpha_{no} \Psi_1(kr_{ni}, kr_{no}) \Psi_0(kr_{no}, kr_{ni}) \right]
\]

\[
= 2\pi w H_0 E_0 \Delta h_1 F_n = 2\pi w H_0 \left( \frac{\rho H_0}{h_1} \right) \Delta h_1 F_n = 2\pi w \rho H_0^2 \Delta F_n
\]

\[
= 2\pi P_{ac} \Delta F_n
\]

(2.48)

where we have let

\[
F_n = \frac{\kappa}{\Psi_0(k \Delta r_{no}, k \Delta r_{ni})} \left\{ \overline{r_{ni}} |\alpha_{ni}|^2 \Psi_1(k \Delta r_{ni}, k \Delta r_{ni}) + \overline{r_{no}} |\alpha_{no}|^2 \Psi_1(k \Delta r_{no}, k \Delta r_{no}) \right. \]

\[
- \overline{\tau_{ni}} \alpha_{ni}^* \overline{\alpha_{no}} \Psi_1(k \Delta r_{ni}, k \Delta r_{ni}) - \overline{\tau_{no}} \alpha_{no}^* \alpha_{ni} \Psi_1(k \Delta r_{no}, k \Delta r_{no}) \left. \right\}
\]

(2.49)

be the cylindrical complex ac power factor for the \( n \)th layer.

We then have that the total power dissipation in the primary winding of the device is

\[
P_{ac} = \frac{1}{2} \sum_{n \in M_1} \mathfrak{R} \left\{ P_n \right\} = \frac{1}{2} \mathfrak{R} \left\{ \sum_{n \in M_1} P_n \right\},
\]

(2.50)
where \(\Re\{z\}\) denotes the real part of \(z\). Thus the normalized resistance and losses are

\[
R_{\text{cyl}} = \frac{R_{\text{ac}}}{R_0} = \frac{2P_{\text{ac}}}{P_0} = \frac{\Re\{\sum_{n \in M_1} P_n\}}{P_0} = \frac{\Re\{\sum_{n \in M_1} 2\pi P_{l_0} \Delta F_n\}}{2\pi P_{l_0} \sum_{n \in M_1} N_n^2 \left(\frac{r_n}{h_n} + \frac{1}{2}\right)} = \frac{\Delta \Re\{\sum_{n \in M_1} F_n\}}{\sum_{n \in M_1} N_n^2 \left(\frac{r_n}{h_n} + \frac{1}{2}\right)}. \tag{2.51}
\]

Due to the general asymmetry of \(\Psi_1\) in the cylindrical case and the dependence of \(F_n\) on the radii, no further simplification can really be done. Equation (2.51) is the more general, cylindrical analog to Dowell’s equation. Though it looks simpler than Dowell’s equation, most the the complexity is wrapped up in the complex power factors \(F_n\).

### 2.3 Leakage Inductance

The leakage inductance is the inductance due to the portion of the magnetic field that does not link the windings. As discussed in [49], when the secondary coil is shorted, this consists of the inductance due the magnetic field stored in the entire winding area. The strategy employed there to determine the leakage inductance is to calculate the energy in the magnetic field \(W\) in the winding window area and then calculate the leakage inductance based on the well-known equation

\[
W = \frac{1}{2} L_{lk} I_1^2, \tag{2.52}
\]

where \(L_{lk}\) is the leakage inductance referred to the primary winding, which is why the primary current \(I_1\) is used.

The field energy in the winding area can be divided into energy within the winding layers and energy in the insulation/gap areas in between layers. The former of these is frequency-dependent since the fields within the layer depend on frequency due to the skin
and proximity effects. However, the latter is frequency-independent as well as having the advantage of the magnetic field being constant across the layer gap, which simplifies the calculation. As was done for the losses and resistance, we derive everything in both Cartesian and cylindrical coordinates in order to compare them later.

### 2.3.1 Cartesian Coordinates

First we derive the energy stored in the magnetic fields in the gaps between layers. As some transformers utilize larger isolation spaces between the primary and secondary layers (e.g. as in [3] and [13]), we shall assume the general case in which the gaps may be different between each layer. To this end, let $g_n$ be the gap between layer $n$ and $n + 1$. Then the gap region just outside of the $n$th layer has a constant magnetic field of $H_{ng} = \alpha_{no}H_0$, noting that the electric field is zero since the magnetic field is constant and the electric field is proportional to $dH/dr$. Though the gap energy derivation is the same for both dc and ac, we consider $H_{ng}$ to be generally complex so that the energy density within the gap is

$$U_{ng} = \frac{1}{2\mu_0} |H_{ng}|^2.$$  \hspace{1cm} (2.53)

Thus the gap energy outside the $n$th layer is then

$$W_{ng} = \int_0^w \int_{r_{no}}^{r_{no}+g_n} \int_{y_{no}} \int_{z_{no}} U_{ng} dx dy dz$$

$$= wl_n U_{ng} (r_{no} + g_n - r_{no}) = w l_n g_n U_{ng}$$

$$= \frac{1}{2} \mu_0 w l_n g_n |H_{ng}|^2 = \frac{1}{2} \mu_0 w l_n g_n |\alpha_{no}|^2 H_0^2$$

$$= \frac{1}{2} \mu_0 w h_1 H_0^2 l_n g_n |\alpha_{no}|^2$$

$$= W_{l_n} h_1 l_n g_n |\alpha_{no}|^2,$$  \hspace{1cm} (2.54)
where we have defined

$$W_{l_0} = \frac{1}{2} \mu_0 w h_1 H_0^2$$  (2.55)

to be the energy per unit length in the first layer at the normalization magnetic field $H_0$.

Now, we also allow for a gap between the the core and the first layer, which we denote with $g_0$ (and its normalized version $\tilde{g}_0 = g_0/h_1$). Completely out of convenience, we also introduce a “zeroth” layer that consists only of this gap region with no actual layer. Doing so means that (2.54) is valid for this $n = 0$ region, where we set

$$l_0 = l_1 \quad \quad \quad \quad \alpha_{0_o} = \alpha_{1_i}.$$  (2.56)

We then have that the total gap energy for all the layers (including the $g_0$ gap region) is

$$W_g = \sum_{n=0}^{p} W_{n_g} = \sum_{n=0}^{p} W_n h_1 l_n \tilde{g}_n |\alpha_{n_o}|^2 = W_n h_1 \sum_{n=0}^{p} l_n |\alpha_{n_o}|^2 \tilde{g}_n.$$  (2.57)

We note here that, since the $n$th gap energy is associated with the gap region just outside the $n$th layer, we allow for some gap $\tilde{g}_p$ between the outermost layer and the end of the winding window where the pot core material begins again. This situation is common as the winding layers do not always fill the winding window in a practical device and are more likely to hug the core than the outer window area. In the event where the window is entirely filled we could simply set $\tilde{g}_p = 0$ so that this term contributes nothing.

Regarding the ac leakage inductance, as previously mentioned, the gap energy remains unchanged, but the magnetic field energy within the layers becomes dependent on frequency. The approach here shall be to utilize the Poynting vector to calculate the complex power within the layers. As discussed in Appendix D of [70], the imaginary part of the
complex power is related to the energy stored within the electromagnetic fields inside the layers. Moreover, since the energy in the electric fields is dissipated as conduction, this is related to the real part of the complex power, leaving the imaginary part to correspond only to the energy in the magnetic field, which is exactly what we require to determine the leakage inductance. Now, we have already derived the complex power in Cartesian coordinates in Section 2.2.1, which is given by (2.39) and (2.40).

Here we digress for a moment to relate the normalization power to the normalization energy:

\[
\frac{P_n}{\Delta} = \frac{\rho w H_0^2}{\omega} = \frac{\rho w H_0^2 h_1}{\omega \delta_w} = \frac{\rho w H_0^2 h_1 \sqrt{\mu_0 \omega}}{\omega \sqrt{2\rho}} = \frac{w H_0^2 h_1 \sqrt{\mu_0 \rho}}{\sqrt{2\omega}}
\]

\[
= \frac{w H_0^2 h_1 \mu_0 \sqrt{2\rho}}{2\sqrt{\mu_0 \omega}} = \frac{1}{2} \mu_0 w h_1 H_0^2 \delta_w = \frac{\mu_0 w h_1 H_0^2 h_1 \delta_w}{2h_1} = \frac{\mu_0 w h_1 H_0^2 h_1}{2\Delta} = \frac{W_{lo} h_1}{\Delta} .
\]

(2.58)

Again, by (2.39) and what was shown in Appendix D of [70], we then have that the ac layer energy is

\[
W_n = \Im \{ P_n \} = \frac{\Im \{ P_n \Delta I_n G_n \}}{2\omega} = \frac{P_n \Delta}{2\omega} I_n \Im \{ G_n \}
\]

\[
= \frac{W_{lo} h_1}{2\Delta} I_n \Im \{ G_n \} ,
\]

(2.59)

where \( \Im \{ z \} \) denotes the imaginary part of \( z \). The total ac energy of all the winding layers is then

\[
W = \sum_{n=1}^{p} W_n = \sum_{n=1}^{p} \frac{W_{lo} h_1}{2\Delta} I_n \Im \{ G_n \} = \frac{W_{lo} h_1}{2\Delta} \sum_{n=1}^{p} I_n \Im \{ G_n \} .
\]

(2.60)
Then the total ac leakage inductance referred to the primary winding is

\[
L_{lk} = \frac{2}{I_1^2} (W + W_g)
\]

\[
= \frac{2}{I_1^2} \left( \frac{W_{10} h_1}{2 \Delta} \sum_{n=1}^{p} I_n \Im \{ G_n \} + W_{10} h_1 \sum_{n=0}^{p} I_n |\alpha_{n_o}|^2 \overline{g_n} \right)
\]

\[
= \frac{2}{I_1^2} \left( \frac{W_{10} h_1}{2 \Delta} \sum_{n=0}^{p} I_n \Im \{ G_n \} + W_{10} h_1 \sum_{n=0}^{p} I_n |\alpha_{n_o}|^2 \overline{g_n} \right)
\]

\[
= \frac{W_{10} h_1}{I_1^2} \sum_{n=0}^{p} I_n \left[ \Im \left\{ \frac{G_n}{\Delta} \right\} + 2 |\alpha_{n_o}|^2 \overline{g_n} \right]
\]

\[
= \frac{1}{2} L_0 \sum_{n=0}^{p} I_n \left[ \Im \left\{ \frac{G_n}{\Delta} \right\} + 2 |\alpha_{n_o}|^2 \overline{g_n} \right],
\]

where we have let

\[
L_0 = \frac{2W_{10} h_1}{I_1^2} = \frac{2h_1}{I_1^2} \left( \frac{1}{2} \mu_0 w h_1 H_0^2 \right) = \frac{\mu_0 w h_1^2 H_0^2}{I_1^2} = \frac{\mu_0 h_1^2}{w}
\]

be a convenient normalization inductance that has no useful physical interpretation. We have also simply set \( G_0 = 0 \) to keep the equations simple and allow for inclusion of the \( g_0 \) gap region.

As shall be discussed more in Chapter 4, it is not particularly useful to normalize the leakage inductance to that at dc. However, we would like to normalize it in order to avoid having to assign arbitrary values to every constant. As we have some freedom to choose our normalization inductance, for simplicity, we use the inductance that would result if the entire winding window were filled with the normalization magnetic field \( H_0 \). This results in the constant energy density

\[
U = \frac{1}{2} \mu_0 H_0^2
\]

across every layer and gap region. We then derive the magnetic field energy of the \( n \)th layer.
and the gap region outside of that layer together:

\[ W_{n0} = \int_0^w \int_0^l \int_{r_{n_i}}^{r_{n_o}+g_n} Udxdydz \]

\[ = w l_n U (r_{n_o} + g_n - r_{n_i}) = w l_n U (r_{n_i} + h_n + g_n - r_{n_i}) \]

\[ = w l_n U (h_n + g_n) = \frac{1}{2} \mu_0 h_1 H_0^2 h_1 \bar{I}_n (\bar{h}_n + \bar{g}_n) \]

\[ = W_{h_1} h_1 \bar{I}_n (\bar{h}_n + \bar{g}_n) \]  

(2.64)

so that the total magnetic field energy is

\[ W_0 = \sum_{n=0}^{p} W_{n0} = \sum_{n=0}^{p} W_{h_1} h_1 \bar{I}_n (\bar{h}_n + \bar{g}_n) = W_{h_1} h_1 \sum_{n=0}^{p} \bar{I}_n (\bar{h}_n + \bar{g}_n) , \]  

(2.65)

noting that we set \( \bar{h}_0 = 0 \) so that we can include the \( g_0 \) gap region and only the gap contributes. This results in a normalization leakage inductance of

\[ L_{lk0} = \frac{2}{I_1^2} W_0 = \frac{2W_{h_1} h_1}{I_1^2} \sum_{n=0}^{p} \bar{I}_n (\bar{h}_n + \bar{g}_n) = L_0 \sum_{n=0}^{p} \bar{I}_n (\bar{h}_n + \bar{g}_n) , \]  

(2.66)

and hence the fully normalized Cartesian leakage inductance is

\[ \bar{L}_{lk(cart)} = \frac{L_{lk}}{L_{lk0}} = \frac{\frac{1}{2} L_0 \sum_{n=0}^{p} \bar{I}_n \left[ 3 \left\{ \frac{G_{n}}{\Delta} \right\} + 2 |\alpha_{n_o}|^2 \bar{g}_n \right]}{L_0 \sum_{n=0}^{p} \bar{I}_n (\bar{h}_n + \bar{g}_n)} = \frac{\sum_{n=0}^{p} \bar{I}_n \left[ 3 \left\{ \frac{G_{n}}{\Delta} \right\} + 2 |\alpha_{n_o}|^2 \bar{g}_n \right]}{2 \sum_{n=0}^{p} \bar{I}_n (\bar{h}_n + \bar{g}_n)}. \]  

(2.67)
2.3.2 Cylindrical Coordinates

As before, we begin with the gap energy, which again is the same for both dc and ac. Similar to what was done before, the gap energy outside the \( n \)th layer is

\[
W_n = \int_0^w \int_0^{2\pi} \int_{r_{no}}^{r_{no}+g_n} U_{nk} r \, dr \, d\phi \, dz
\]

\[
= \pi w U_{nk} \left[ (r_{no} + g_n)^2 - r_{no}^2 \right] = \pi w U_{nk} g_n (2r_{no} + g_n)
\]

\[
= \frac{1}{2} \pi \mu_0 w |H_{gn}|^2 g_n (2r_{no} + g_n) = \frac{1}{2} \pi \mu_0 w |\alpha_{no}|^2 H_0^2 g_n (2r_{no} + g_n)
\]

\[
= \frac{1}{2} \pi \mu_0 w h_1^2 H_0^2 |\alpha_{no}|^2 \bar{g}_n (2\bar{r}_{no} + \bar{g}_n)
\]

\[
= \pi W_{n0} h_1 |\alpha_{no}|^2 \bar{g}_n (2\bar{r}_{no} + \bar{g}_n)
\]

\[
= \pi W_{n0} h_1 |\alpha_{no}|^2 \bar{g}_n (2\bar{r}_{ni} + 2\bar{h}_n + \bar{g}_n)
\]

(2.68)

We then have that the total gap energy for all the layers is

\[
W_g = \sum_{n=0}^{p} W_{ng} = \sum_{n=0}^{p} \pi W_{n0} h_1 |\alpha_{no}|^2 \bar{g}_n (2\bar{r}_{ni} + 2\bar{h}_n + \bar{g}_n)
\]

\[
= \pi W_{n0} h_1 \sum_{n=0}^{p} |\alpha_{no}|^2 \bar{g}_n (2\bar{r}_{ni} + 2\bar{h}_n + \bar{g}_n)
\]

(2.69)

again noting that we allow for a gap outside the outermost layer and the end of the winding window, and that we include the inside \( g_0 \) gap region by setting \( \alpha_{0o} = \alpha_1, \bar{r}_{0i} = \bar{r}_c, \) and \( \bar{h}_0 = 0. \)

For the ac leakage inductance we can directly utilize the complex power that was derived in Section 2.2.2, given by (2.48) and (2.49). Doing so, we have that the ac layer energy is

\[
W_n = \frac{\Im \{ P_n \}}{2\omega} = \frac{\Im \{ 2\pi P_{n0} \Delta F_n \}}{2\omega} = \frac{2\pi P_{n0} \Delta}{2\omega} \Im \{ F_n \}
\]

\[
= \pi W_{n0} h_1 \Delta \Im \{ F_n \}
\]

(2.70)
where we again refer to the relationship between power and energy derived in (2.58). We also again note that the relationship between the complex power and field energy was derived in Appendix D of [70]. The total ac energy of all the winding layers is then

$$ W = \sum_{n=1}^{P} W_n = \sum_{n=1}^{P} \frac{\pi W_{b1} h_1}{\Delta} \Im \{F_n\} = \frac{\pi W_{b1} h_1}{\Delta} \sum_{n=1}^{P} \Im \{F_n\} . \quad (2.71) $$

Thus the total ac leakage inductance is

$$ L_{lk} = \frac{2}{I_1^2} (W + W_g) $$

$$ = \frac{2}{I_1^2} \left( \frac{\pi W_{b1} h_1}{\Delta} \sum_{n=1}^{P} \Im \{F_n\} + \pi W_{b1} h_1 \sum_{n=0}^{P} |\alpha_{n_o}|^2 \overline{g}_n \left(2r_{n_i} + 2\overline{r}_{n} + \overline{g}_n \right) \right) $$

$$ = \frac{2}{I_1^2} \left( \frac{\pi W_{b1} h_1}{\Delta} \sum_{n=0}^{P} \Im \{F_n\} + \pi W_{b1} h_1 \sum_{n=0}^{P} |\alpha_{n_o}|^2 \overline{g}_n \left(2r_{n_i} + 2\overline{r}_{n} + \overline{g}_n \right) \right) $$

$$ = \frac{2 \pi W_{b1} h_1}{I_1^2} \sum_{n=0}^{P} \left[ \Im \left\{ \frac{F_n}{\Delta} \right\} + |\alpha_{n_o}|^2 \overline{g}_n \left(2r_{n_i} + 2\overline{r}_{n} + \overline{g}_n \right) \right] $$

$$ = \pi L_0 \sum_{n=0}^{P} \left[ \Im \left\{ \frac{F_n}{\Delta} \right\} + |\alpha_{n_o}|^2 \overline{g}_n \left(2r_{n_i} + 2\overline{r}_{n} + \overline{g}_n \right) \right] \quad (2.72) $$

in cylindrical coordinates, where we have again set $F_0 = 0$ to include the $g_0$ gap region in a single sum.

We normalize the cylindrical leakage inductance in the same way as the Cartesian case. This time we have that the normalization energy in the $n$th layer is

$$ W_{n0} = \int_{0}^{w} \int_{0}^{2\pi} \int_{r_{n_i}}^{r_{n_o} + g_n} Ur dr d\phi dz $$

$$ = \pi wU \left[ (r_{n_o} + g_n)^2 - r_{n_i}^2 \right] = \pi wU \left[ (r_{n_i} + h_n + g_n)^2 - r_{n_i}^2 \right] $$

$$ = \pi wU \left[ 2r_{n_i} (h_n + g_n) + (h_n + g_n)^2 \right] $$

$$ = \pi wU \left( h_n + g_n \right) \left( 2r_{n_i} + h_n + g_n \right) $$

$$ = \frac{1}{2} \mu_0 w h_1 H_0^2 h_1 \left( \overline{r}_{n} + \overline{g}_n \right) \left( 2\overline{r}_{n} + \overline{r}_{n} + \overline{g}_n \right) $$
\[ W_0 = \sum_{n=0}^{p} W_{n0} = \sum_{n=0}^{p} \pi W_{l0} h_1 (\bar{h}_n + \bar{g}_n) \left(2\tau_{n_i} + \bar{h}_n + \bar{g}_n\right) \]

so that the total energy is

\[ W_0 = \pi W_{l0} h_1 \left(\bar{h}_n + \bar{g}_n\right) \left(2\tau_{n_i} + \bar{h}_n + \bar{g}_n\right) \]

and the normalization leakage inductance is

\[ L_{lk0} = \frac{2}{I_1^2} W_0 = \frac{2\pi W_{l0} h_1}{I_1^2} \sum_{n=0}^{p} (\bar{h}_n + \bar{g}_n) \left(2\tau_{n_i} + \bar{h}_n + \bar{g}_n\right) \]

\[ = \pi L_0 \sum_{n=0}^{p} (\bar{h}_n + \bar{g}_n) \left(2\tau_{n_i} + \bar{h}_n + \bar{g}_n\right) . \]

It then follows that the cylindrical normalized leakage inductance is

\[ \mathcal{L}_{lk(cyl)} = \frac{L_{lk}}{L_{lk0}} = \frac{\pi L_0 \sum_{n=0}^{p} \left[3 \left\{ \frac{E_n}{A_n} \right\} + |\alpha_{n_o}|^2 \bar{g}_n \left(2\tau_{n_i} + 2\bar{h}_n + \bar{g}_n\right)\right]}{\pi L_0 \sum_{n=0}^{p} (\bar{h}_n + \bar{g}_n) \left(2\tau_{n_i} + \bar{h}_n + \bar{g}_n\right)} \]

\[ = \frac{\sum_{n=0}^{p} \left[3 \left\{ \frac{E_n}{A_n} \right\} + |\alpha_{n_o}|^2 \bar{g}_n \left(2\tau_{n_i} + 2\bar{h}_n + \bar{g}_n\right)\right]}{\sum_{n=0}^{p} (\bar{h}_n + \bar{g}_n) \left(2\tau_{n_i} + \bar{h}_n + \bar{g}_n\right)} . \]

\[ \text{(2.76)} \]

### 2.4 Conclusion

It is worth noting that, while fairly compact, the equations derived in this chapter for both the losses and the leakage inductance are quite general and are valid for a device with different layer thicknesses for each layer, a non-ideal core, any interleaving pattern, and potentially variable layer gaps. The interleaving pattern, core permeability, winding width \( w \), and numbers of layers all generally affect the field coefficients \( \alpha_{n_i} \) and \( \alpha_{n_o} \) for each layer while of course the layer thicknesses affect \( \bar{h}_n \) and the layer gaps affect \( \bar{g}_n \). The following
chapter will focus on ways in which the field coefficients can be analytically calculated to take the core permeability and winding width $w$ into account.
Chapter 3

Ideal and Non-Ideal Cores

Most previous treatments assume an ideal core in which the magnetic field intensity is taken to be zero at the inside and outside of the winding window. In [35] and [70] a non-ideal core with an arbitrary relative permeability $\mu_r$ was treated using the 1-D model in the same way, but only for rod core devices. In this chapter we attempt the same approach for transformers, which results in a fundamental contradiction. This will lead us to derive the needed results using a 2-D model instead.

With each approach we shall first calculate the shorted secondary current $I_2$, and then use this to determine the magnetic field constants $\alpha_n$ and $\alpha_{n_o}$ for every layer. Generally, these will all depend on the core permeability. From these the losses and leakage inductance can be calculated using the equations derived in Chapter 2. We also note that we must use the Cartesian coordinate approximation for all approaches to determine $I_2$ and the magnetic field constants, as analytical solutions for the cylindrical case could not be found despite research into the problem. As such, the complete analytical solution could be considered as using a hybrid of Cartesian and cylindrical coordinates.

As briefly discussed in Section 1.2, a similar approach was taken by Lambert et al. in [37] to treat a non-ideal core in service of calculating the leakage inductance. In that work, the magnetic vector potential was used, whereas here we shall work with the $z$-component
of the magnetic field solution directly. We note that, after some tedious calculus and algebra, it was confirmed that the solution for the magnetic vector potential produced by a conductor with rectangular cross section presented in [37] is consistent with the solution from [66] that is used herein. The direct magnetic field solution has many less terms than the magnetic vector potential and so is easier to work with, though we note that we neglect the $x$-component, which [37] does not do since the vector potential is used. The advantage to this, however, is that it allows us to decouple the magnetic field coefficients from the leakage equation so that the results can be easily applied to the losses as well. Additionally, this also allows for high-frequency effects to be taken into account, noting that [37] only calculates the leakage inductance without taking high-frequency effects into account.

### 3.1 Ideal Core

As with previous treatments, if an ideal core is assumed, then the magnetic fields in the core and outside the window are zero. This results in the magnetic field constants $\alpha_1 = \alpha_p = 0$, i.e. these constants are zero at the innermost surface of the first layer and the outer surface of the outermost layer $p$. Consider then the Ampèrian loop shown in Figure 3.1 that surrounds all of the layers, whatever their configuration. In the 1-D model the magnetic field has only a $z$ component so that the top and bottom of the loop contribute nothing.

If we take both the primary current $I_1$ and secondary current $I_2$ as going into the page, then the total current within the loop is clearly

$$I = \sum_{n \in M_1} N_n I_1 + \sum_{n \in M_2} N_n I_2 = I_1 \sum_{n \in M_1} N_n + I_2 \sum_{n \in M_2} N_n,$$

where again $M_1$ and $M_2$ are the sets of layer numbers for the primary and secondary layers, respectively. Since the field at the inside and outside of all layers is zero, the loop then
Figure 3.1: Ampèreian loop to determine $I_2$ for an ideal core.

results in

$$w\alpha_1 H_0 - w\alpha_p H_0 = 0 = I = I_1 \sum_{n \in M_1} N_n + I_2 \sum_{n \in M_2} N_n.$$  \hspace{1cm} (3.2)

Solving this for $I_2$ for fixed primary current $I_1$ results in a shorted secondary current of

$$I_2 = -\frac{\sum_{n \in M_1} N_n}{\sum_{n \in M_2} N_n} I_1,$$  \hspace{1cm} (3.3)

noting that if $N_n = 1$ for all layers then this becomes the familiar ideal transformer shorted current of

$$I_2 = -\frac{p_1}{p_2} I_1.$$  \hspace{1cm} (3.4)

For a given winding configuration, once the secondary current $I_2$ has been calculated using (3.3), all of the magnetic field constants can be calculated in the following way. Consider the Ampèreian loop of Figure 3.2, which is used to calculate the outer coefficient
\( \alpha_n \) of the \( n \)th layer, noting that in general \( \alpha_{(n+1)i} = \alpha_{no} \) so that it suffices to determine only the outer constants. Clearly this loop results in

\[
\begin{align*}
 w\alpha_1 H_0 - w\alpha_{no} H_0 &= \sum_{k=1}^{n} N_k I_k , \tag{3.5}
\end{align*}
\]

where we let \( I_n \) be the current of the \( n \)th layer, i.e. \( I_n = I_1 \) for \( n \in M_1 \) and \( I_n = I_2 \) for \( n \in M_2 \). Since, in the ideal case we are considering, \( \alpha_1 = 0 \), this becomes

\[
\begin{align*}
 0 - w\alpha_{no} H_0 &= \sum_{k=1}^{n} N_k I_k \\
 -w\alpha_{no} H_0 &= \sum_{k=1}^{n} N_k \tilde{I}_k I_1 \\
 -\alpha_{no} H_0 &= \sum_{k=1}^{n} N_k \tilde{I}_k I_1 = \sum_{k=1}^{n} N_k \tilde{I}_k H_0 \\
 \alpha_{no} &= -\sum_{k=1}^{n} N_k \tilde{I}_k , \tag{3.6}
\end{align*}
\]

where we have let \( \tilde{T}_n = I_n/I_1 \) be the normalized layer current so that \( \tilde{T}_n = 1 \) for \( n \in M_1 \).
and \( T_n = I_2/I_1 \) for \( n \in M_2 \). We note that equations (3.3) and (3.6) do not depend on the core permeability \( \mu_r \) as this is an ideal-core approximation.

### 3.2 Non-Ideal Core in 1-D

In [35] and [70], the losses for an inductor with a core of arbitrary permeability was determined using the method of images in the 1-D model. It was found that the losses depend not on \( \mu_r \) directly, but rather on the image current factor

\[
\alpha_c = \frac{\mu_r - 1}{\mu_r + 1},
\]

(3.7)

which should not be confused with the magnetic field constants \( \alpha_{n_1} \) and \( \alpha_{n_o} \). In these previous approaches, the outside of the inductor was not assumed to be core material so that only one set of image currents was needed.

However, in the case of a pot core transformer that we are considering, the magnetic material at both the inside of the window and the outside necessitates an infinite number of image currents. If we lump all the windings layers together as a single total current

\[
I = I_1 \sum_{n \in M_1} N_n + I_2 \sum_{n \in M_2} N_n,
\]

(3.8)

then the infinite number of image currents in the inner and outer core areas are as shown in Figure 3.3, noting that the magnetic field generated by this configuration is valid only in the winding window area (and not in either of the core regions), including at the interfaces themselves. See Appendix A for details on how this configuration was derived using successive images.

Now, in a 1-D analysis, the layers are effectively infinitely long in the \( z \) direction, and as a result the magnetic field generated by a layer or set of layers does not vary with the
Figure 3.3: Infinite number of image currents for the 1D non-ideal core analysis.

distance from the layer(s). Therefore the distances involved in the Figure 3.3 configuration are not relevant, which is why they are not shown. All that matters in determining the field is which side of the layer(s) we are on. An Ampèreian loop is shown in Figure 3.4 that allows us to determine the magnetic field on either side of a layer in isolation, image or otherwise. If the layer carries a current $I$, then clearly, by symmetry, the field on both sides are equal but of opposite sign (if we take all fields to be in the positive $z$ direction). This is to say that $H_o = -H_i$ in the figure. The loop then results in

$$wH_i - wH_o = I$$
$$wH_i + wH_i = I$$
$$2wH_i = I$$
$$H_i = \frac{I}{2w}.$$  

(3.9)
where then of course the outside field is

$$H_o = -H_i = -\frac{I}{2w}. \quad (3.10)$$

Returning then to our infinite image configuration of Figure 3.3, we abandon the approach using Ampèrian loops and instead determine the magnetic fields using a superposition of the fields created from every layer. Recalling again that $I$ is the lumped current of all winding layers, whatever that might be, we then have that the magnetic field at the left interface (i.e. the field in the core and the inner surface of the first layer) is

$$H_c = \sum_{n=1}^{\infty} -\alpha^n c I \frac{1}{2w} + \frac{I}{2w} + \sum_{n=1}^{\infty} \alpha^n c I \frac{1}{2w}, \quad (3.11)$$

noting that the first term is from the infinite number of image layers to the left of the interface and the remaining terms are from the layers to the right. However, this simplifies to

$$H_c = \sum_{n=1}^{\infty} -\alpha^n c I \frac{1}{2w} + \frac{I}{2w} + \sum_{n=1}^{\infty} \alpha^n c I \frac{1}{2w}$$
\[ I_2 w \sum_{n=1}^{\infty} \alpha_c^n + I_2 w \sum_{n=1}^{\infty} \alpha_c^n = -I_2 w \left( \frac{\alpha_c}{1 - \alpha_c} \right) + I_2 w \left( \frac{\alpha_c}{1 - \alpha_c} \right) = I_2 w \]  

(3.12)

since the geometric series converge because \(0 \leq \alpha_c < 1\), as noted at the end of Appendix A.

Similarly, the field at the right interface (i.e. at the outer surface of the outermost layer \(p\)) is

\[ H_t = \sum_{n=1}^{\infty} \frac{\alpha_c^n I}{2w} - \frac{I}{2w} \sum_{n=1}^{\infty} \frac{-\alpha_c^n I}{2w} \]

\[ = \frac{I}{2w} \sum_{n=1}^{\infty} \alpha_c^n - \frac{I}{2w} \sum_{n=1}^{\infty} \alpha_c^n \]

\[ = \frac{I}{2w} \left( \frac{\alpha_c}{1 - \alpha_c} \right) - \frac{I}{2w} \left( \frac{\alpha_c}{1 - \alpha_c} \right) = -\frac{I}{2w}. \]  

(3.13)

Notice that these are the same as as the fields produced by the original windings in free space with no core present at all! So clearly they have no dependence on the core permeability \(\mu_r\) as one would expect. This is the fundamental contradiction one reaches when attempting this analysis within the 1-D framework, which ultimately stems from the fact that the field does not decrease as we move farther from the current source in the 1-D model. We can remedy this by using moving to a 2-D analysis in which the field does depend on its distance from the source.
3.3 Non-Ideal Core in 2-D

In the 1-D analysis it was assumed that the magnetic field has only a \(z\) component. While the solutions in what follows do include the \(x\) component of the field (noting that we are using the Cartesian coordinates illustrated in Figure 2.1), we shall only concern ourselves again with the \(z\) component and neglect the \(x\) component. In [66], the magnetic field outside of an infinitely-long conductor with a rectangular cross-section carrying a uniform current is derived using the Biot-Savart law. Hence our analysis will be at dc with the understanding that the fields outside the layers do not change significantly with frequency since the total current in each layer is constant across all frequencies.

Consider the 2-D rectangular model for the \(n\)th layer in isolation in free space shown in Figure 3.5, noting that the point \((x_0, z_0)\) is affixed to the middle of the layer vertically and on the left represents the location of the layer in some Cartesian coordinate system. Supposing that the layer has a uniform current density of \(N_n I_n / wh_n\) into the page, then the \(z\) component of the magnetic field at a point \((x, z)\) outside the layer is \(H_{z0}(x-x_0, z-z_0)\),
where

\[ H_{zn0}(x, z) = \tilde{H}_{zn}(x, z + \frac{w}{2}) - \tilde{H}_{zn}(x, z - \frac{w}{2}) \]  

(3.14)

and

\[ \tilde{H}_{zn}(x, z) = \frac{N_n I_n}{4\pi wh_n} \left[ z \ln \left(1 + \frac{h_n^2 - 2xh_n}{x^2 + z^2}\right) + 2(x - h_n) \arctan \left( \frac{z}{x - h_n} \right) - 2x \arctan \left( \frac{z}{x} \right) \right]. \]  

(3.15)

as derived in [66]. Note that the arctangent functions in (3.15) refer to the principle branch whose the range is the open interval \((-\pi/2, \pi/2)\) regardless of the individual signs of the numerator and denominator of their arguments.

In our actual situation our rectangular layers are contained in a rectangular winding window surrounded by core material with some finite permeability \(\mu_r\) on all sides. We would like to determine the \(z\) component of the magnetic field at any location in the 2-D winding window, which we do by superposing the fields generated by every layer. So consider the \(n\)th layer in situ as shown in Figure 3.6. Note that we have affixed the origin of our Cartesian coordinate system in what follows to the inside of the winding window in the middle vertically. We denote positions with respect to this coordinate system (as opposed to that shown in Figure 2.1 centered at the center of the core) using variables with a hat. Hence the position of the left of the layer in this coordinate system is \(\hat{r}_{ni} = r_{ni} - r_c\). Also note that we have denoted the horizontal size of the winding window by \(b\).

Now, because the winding window is surrounded in both the \(x\) and \(z\) dimensions, the successive method of images must be applied in both dimensions, which results in an infinite grid of image currents all just for the \(n\)th layer. Similar to Figure 2 of [37], these are shown in Figure 3.7, which shows the current carried by each image according to the method of images. Note that an implicit assumption here is that the pot core shell is...
Figure 3.6: The $n$th layer in the winding window.

Figure 3.7: Infinite grid of images for the $n$th layer.
We identify each image layer in the grid with two integers $k$ and $m$, where $k$ numbers the images from left to right and $m$ from bottom to top, setting the original layer to indices $(k, m) = (0, 0)$ as illustrated in the figure. We note that each vertical stack of images (with the images forming each stack having the same $k$) forms an infinitely tall solid layer (though each image in the stack carries a different current) on account of the fact that we assume that every original layer fills the entire window vertically. As a result of this, we expect the magnetic field to decrease very slowly as we move away from an isolated stack, and to be nearly entirely in the $z$ direction.

Now, it can be observed from Figure 3.7 that there are two types of stacks with respect to their horizontal positions in their image cells. Namely, the left side of the stacks with even $k$ are positioned at a distance of $\hat{r}_{ni}$ from the left of their cells, whereas those with odd $k$ are positioned such that their right sides are a distance $\hat{r}_{ni}$ from the right side of their cells. With this in mind, the pertinent parameters of image layer $(k, m)$ were reasoned to be those shown in Table 3.1, noting that $x_0$ and $z_0$ are the position of the point on the image layer shown in Figure 3.5 and relative to the Cartesian origin shown in Figure 3.6 and Figure 3.7. For brevity in what follows, we have also defined

$$\xi(k, m) = \max(|k|, |m|).$$

(3.16)
With this in mind, the total field caused by all the image layers is then

\[
H_{zn}(\hat{x}, \hat{z}) = \sum_{k=-\infty}^{\infty} \left( \sum_{m=-\infty}^{\infty} \left[ \alpha_{\xi}^{(2k,m)} H_{zn0}(\hat{x} - 2kb - \hat{r}_n, \hat{z} - mw) + \alpha_{\xi}^{(2k+1,m)} H_{zn0}(\hat{x} - 2(k + 1)b + \hat{r}_n + \hat{h}_n, \hat{z} - mw) \right] \right),
\] (3.17)

where we have of course superposed (3.14) for each image, and \( \hat{x} \) and \( \hat{z} \) are with respect to the coordinate system of Figure 3.6. Here we have re-purposed the index \( k \) such that \( 2k \) represents the even stacks as discussed above and \( 2k + 1 \) represents the odd stacks, which clearly correspond to the first and second terms inside the sums, respectively, in (3.17). Hence each term of the double sum (a single term being what is added for a single combination of \( k \) and \( m \)) adds the fields caused by two stacks, an even one, and the odd one immediately to its right.

To digress for a moment, if \( \{a_m\} \) denotes a double-sided sequence of real numbers indexed by the integers, then the convergent double-sided series \( \sum_{m=-\infty}^{\infty} a_m \) of course really means

\[
\sum_{m=-\infty}^{\infty} a_m = \sum_{m=1}^{\infty} a_m + a_0 + \sum_{m=-1}^{-\infty} a_m = a_0 + \sum_{m=1}^{\infty} (a_m + a_{-m}) = \sum_{m=0}^{\infty} \lambda_m (a_m + a_{-m}),
\]

(3.18)

where we have introduced

\[
\lambda_m = \begin{cases} 
\frac{1}{2} & \text{if } m = 0 \\
1 & \text{if } m > 0
\end{cases}
\]

(3.19)

to account for the fact that \( a_0 \) is added twice in the equivalent single-sided series. Therefore,
if $a_{k,m}$ is a double sequence indexed by integers $k$ and $m$, then the iterated series means

$$
\sum_{k=-\infty}^{\infty} \left( \sum_{m=-\infty}^{\infty} a_{k,m} \right) = \sum_{k=-\infty}^{\infty} \left( \sum_{m=0}^{\infty} \lambda_m \left( a_{k,m} + a_{k,-m} \right) \right) \\
= \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \lambda_m \left( a_{k,m} + a_{k,-m} \right) + \sum_{m=0}^{\infty} \lambda_m \left( a_{-k,m} + a_{-k,-m} \right) \\
= \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \lambda_k \lambda_m \left( a_{k,m} + a_{k,-m} + a_{-k,m} + a_{-k,-m} \right), \quad (3.20)
$$

and so can be expressed as a standard iterated series. We here note that there is generally a distinction between the following three double series:

$$
\sum_{k=0}^{\infty} \left( \sum_{m=0}^{\infty} a_{k,m} \right) \\
\sum_{m=0}^{\infty} \left( \sum_{k=0}^{\infty} a_{k,m} \right) \\
\sum_{k,m=0}^{\infty} a_{k,m} \quad (3.21)
$$

for general double sequence $\{a_{k,m}\}$. The first two are called iterated series and the last simply a double series, and each has a different formal definition. An excellent and rigorous treatment of these can be found in [26], in which it is shown that generally these three series can converge to different values even if they all converge.

For brevity in what follows we denote

$$
A_{k,m} = H_{zn0}(\hat{x} - 2kb - \hat{r}_n, \hat{z} - mw) \quad (3.22)
$$

$$
B_{k,m} = H_{zn0}(\hat{x} - 2(k+1)b + \hat{r}_n + h_n, \hat{z} - mw), \quad (3.23)
$$

where we suppress the fact that these are functions of $\hat{x}$ and $\hat{z}$. In light of (3.20) it follows
that we can express (3.17) as a standard iterated series:

\[
H_{zn}(\hat{x}, \hat{z}) = \sum_{k=-\infty}^{\infty} \left( \sum_{m=-\infty}^{\infty} \left[ \alpha_c^{(2k,m)} A_{k,m} + \alpha_c^{(2k+1,m)} B_{k,m} \right] \right)
\]

\[
= \sum_{k=0}^{\infty} \left( \sum_{m=0}^{\infty} \lambda_k \lambda_m \left[ \alpha_c^{(2k,m)} A_{k,m} + \alpha_c^{(2k+1,m)} B_{k,m} \right] + \alpha_c^{(2k-m)} A_{k,-m} + \alpha_c^{(2k+1-m)} B_{k,-m} \right)
\]

\[
= \sum_{k=0}^{\infty} \left( \sum_{m=0}^{\infty} \lambda_k \lambda_m \left[ \alpha_c^{(2k,m)} (A_{k,m} + A_{k,-m} + A_{-k,m} + A_{-k,-m}) + \alpha_c^{(2k+1,m)} (B_{k,m} + B_{k,-m}) + \alpha_c^{(2k+1-m)} (B_{k,m} + B_{k,-m}) \right] \right)
\]

\[
(3.24)
\]

since of course

\[
\xi(-k, -m) = \max(|-k|, |-m|) = \max(|k|, |m|) = \xi(k, m).
\]

(3.25)

We note that these manipulations and rearrangements are valid only if the series are convergent. However, we prove in Appendix B that the iterated series of (3.24) converges to the same value regardless of the order of iteration or whether a proper double series is used, and does so mathematically at any field point \((\hat{x}, \hat{z})\), even though this solution is valid only in the window and outside the layers. Of course everything we have derived up to this point has been for a single layer, so to calculate the field at a point in the window due to all layers, we of course simply sum over the field equations for each layer:

\[
H_z(\hat{x}, \hat{z}) = \sum_{n=1}^{p} H_{zn}(\hat{x}, \hat{z}).
\]

(3.26)
Here we normalize all of the magnetic field equations we have derived. First, we note that
\[
\frac{N_n I_n}{4\pi w h_n} = \frac{N_n I_1}{4\pi w h_1} = \frac{N_n H_0}{4\pi h_1}.
\]

(3.27)

As done in Chapter 2, we normalize all lengths to the first layer thickness \(h_1\) and normalize the overall magnetic field to \(H_0\), denoting these normalized quantities with bars over them. These considerations allow us to re-normalize (3.15) as
\[
\bar{H}_{zn}(x, z) = \frac{\bar{H}_{zn}(xh_1, zh_1)}{\bar{T}_n H_0} = \frac{N_n}{4\pi h_n} \left[ \bar{z} \ln \left( 1 + \frac{\bar{r}_{n}^2 - 2\bar{r}_{n}h_n}{x^2 + z^2} \right) + 2 \frac{\bar{r}_{n}}{x - \bar{r}_{n}} \arctan \left( \frac{\bar{z}}{x - \bar{r}_{n}} \right) - 2\pi \arctan \left( \frac{\bar{z}}{\bar{x}} \right) \right],
\]

(3.28)

and (3.14) becomes
\[
\bar{H}_{zn0}(x, z) = \frac{H_{zn0}(xh_1, zh_1)}{\bar{T}_n H_0} = \bar{H}_{zn}(x, z + \frac{\bar{w}}{2}) - \bar{H}_{zn}(x, z - \frac{\bar{w}}{2}).
\]

(3.29)

The fields from the image grids (3.22) and (3.23) are normalized to
\[
\bar{A}_{k,m} = \frac{A_{k,m}}{\bar{T}_n H_0} = \bar{H}_{zn0}(\bar{x} - 2k\bar{b} - \bar{r}_{n}, \bar{z} - m\bar{w})
\]

(3.30)
\[
\bar{B}_{k,m} = \frac{B_{k,m}}{\bar{T}_n H_0} = \bar{H}_{zn0}(\bar{x} - 2(k+1)\bar{b} + \bar{r}_{n} + \bar{h}_{n}, \bar{z} - m\bar{w})
\]

(3.31)

where we note that of course \(\bar{r}_{n} = \hat{r}_{n}/h_1 = \bar{r}_{n} - \bar{r}_{c}\). Finally, (3.24) becomes
\[
\bar{H}_{zn}(x, z) = \frac{H_{zn}(xh_1, zh_1)}{\bar{T}_n H_0} = \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \lambda_k \lambda_m \left( \omega_c^{(2k,m)} (\bar{A}_{k,m} + \bar{A}_{k,-m} + \bar{A}_{-k,m} + \bar{A}_{-k,-m}) \right)
\]

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\[ + \alpha_c^{(2k+1,m)} \left( \overline{B}_{k,m} + \overline{B}_{k,-m} \right) + \alpha_c^{(2k-1,m)} \left( \overline{B}_{-k,m} + \overline{B}_{-k,-m} \right) \right) \ \right) \]

(3.32)

and (3.26) normalizes to

\[
\mathcal{H}_z (\hat{x}, \hat{z}) = \frac{H_z (\tilde{x}h_1, \tilde{z}h_1)}{H_0} = \sum_{n=1}^{p} \frac{H_{zn} (\tilde{x}h_1, \tilde{z}h_1)}{H_0} = \sum_{n=1}^{p} I_n \mathcal{H}_{zn} (\tilde{x}, \tilde{z}), \tag{3.33}
\]

where we have brought the \( I_n \) factor all the way out for reasons that will become apparent later.

Now, in the 1-D model in which we derived equations for the losses and leakage inductance, we assume that the magnetic field is constant in the gap regions between the layers. As we now have in hand equations to calculate the \( z \)-component of the magnetic field at any 2-D point in the window, we must choose points at which to evaluate these equations to obtain our \( \alpha_{ni} \) and \( \alpha_{no} \) values. As discussed in Section 3.1, it suffices to determine \( \alpha_1 \) and the outer coefficients \( \alpha_{no} \) since \( \alpha_{(n+1)i} = \alpha_{no} \). The natural choice for these points is in the very center of the gap regions:

\[
\alpha_1 = \frac{H_z (g_0/2, 0)}{H_0} = \mathcal{H}_z (\bar{g}_0/2, 0) \tag{3.34}
\]

\[
\alpha_{no} = \frac{H_z (\bar{r}_n + h_n + g_n/2, 0)}{H_0} = \mathcal{H}_z (\bar{r}_n + h_n + g_n/2, 0) \tag{3.35}
\]

Having to only ever evaluate \( \mathcal{H}_z (\tilde{x}, \tilde{z}) \) for \( \tilde{z} = 0 \) allows us to exploit some symmetry in the functions and simplify them. In particular, it is easy to verify by inspection of (3.28) that \( \overline{H}_{zn} (\tilde{x}, -\tilde{z}) = -\overline{H}_{zn} (\tilde{x}, \tilde{z}) \), which is to say that \( \overline{H}_{zn} (\tilde{x}, \tilde{z}) \) exhibits odd symmetry in \( \tilde{z} \). From this it follows that

\[
\overline{H}_{zn0} (\tilde{x}, \tilde{z}) + \overline{H}_{zn0} (\tilde{x}, -\tilde{z}) = \overline{H}_{zn} \left( \tilde{x}, \frac{\tilde{z} + \overline{w}}{2} \right) - \overline{H}_{zn} \left( \tilde{x}, \frac{\tilde{z} - \overline{w}}{2} \right)
\]
\begin{align*}
&= \overline{H}_{zn}(\overline{x}, \overline{z} - \overline{w}) - \overline{H}_{zn}(\overline{x}, \overline{z} + \overline{w}) \\
&\quad + \overline{H}_{zn}(\overline{x}, \overline{z} + \overline{w}) - \overline{H}_{zn}(\overline{x}, \overline{z} - \overline{w}) \\
&\quad = 2\overline{H}_{zn}(\overline{x}, \overline{z} - \overline{w}) - 2\overline{H}_{zn}(\overline{x}, \overline{z} + \overline{w})
\end{align*}

From this and (3.30) it follows that for \( \overline{z} = 0 \) we have

\[
\overline{A}_{k,m} + \overline{A}_{k,-m} = \overline{H}_{zn0}(\overline{x} - 2k\overline{b} - \overline{r}_{ni}, -m\overline{w}) + \overline{H}_{zn0}(\overline{x} - 2k\overline{b} - \overline{r}_{ni}, m\overline{w})
\]

\[
= 2\overline{H}_{zn0}(\overline{x} - 2k\overline{b} - \overline{r}_{ni}, -m\overline{w}) = 2\overline{A}_{k,m},
\]

and similarly \( \overline{B}_{k,m} + \overline{B}_{k,-m} = 2\overline{B}_{k,m} \) by (3.31). The geometric interpretation of this is that the corresponding layers above and below in a vertical stack of image layers contribute the same magnetic field along the vertical center of the window at \( \overline{z} = 0 \). Hence (3.32) becomes

\[
\overline{H}_{zn}(\overline{x}, 0) = \sum_{k=0}^{\infty} \left( \sum_{m=0}^{\infty} \lambda_k \lambda_m \left[ \alpha_c^{\xi(2k,m)} (\overline{A}_{k,m} + \overline{A}_{k,-m} + \overline{A}_{-k,m} + \overline{A}_{-k,-m}) 
\right. \right. \\
\left. \left. + \alpha_c^{\xi(2k+1,m)} (\overline{B}_{k,m} + \overline{B}_{k,-m}) + \alpha_c^{\xi(2k-1,m)} (\overline{B}_{-k,m} + \overline{B}_{-k,-m}) \right] \right)
\]

\[
= \sum_{k=0}^{\infty} \left( \sum_{m=0}^{\infty} \lambda_k \lambda_m \left[ \alpha_c^{\xi(2k,m)} (2\overline{A}_{k,m} + 2\overline{A}_{-k,m}) 
\right. \right. \\
\left. \left. + 2\alpha_c^{\xi(2k+1,m)} \overline{B}_{k,m} + 2\alpha_c^{\xi(2k-1,m)} \overline{B}_{-k,m} \right] \right)
\]
\[
\sum_{k=0}^{\infty} \left( \sum_{m=0}^{\infty} \lambda_k \lambda_m \left[ \alpha \xi^{(2k,m)} (A_{k,m} + A_{-k,m}) + \alpha \xi^{(2k+1,m)} (B_{k,m} + \alpha \xi^{(2k-1,m)} B_{-k,m}) \right] \right). \quad (3.38)
\]

Of course the coefficients \( \alpha_{n_i} \) and \( \alpha_{n_o} \) are still calculated using (3.34) and (3.35) with no further simplification there.

In order to calculate these field coefficients on either side of each layer, the normalized secondary current \( I_2 \) must be determined. For the non-ideal case we could simply use the ideal equation (3.3) to determine this. However, this can be done a little more precisely using the 2-D magnetic fields that we have derived. To do this, first define the inner and outer partial fields

\[
\gamma_{c,l} = \sum_{n \in M_l} H_{\text{zn}} (\bar{x}_c, 0), \quad (3.39)
\]

\[
\gamma_{t,l} = \sum_{n \in M_l} H_{\text{zn}} (\bar{x}_t, 0), \quad (3.40)
\]

where we let \( \bar{x}_c = g_0/2 \) and \( \bar{x}_t = \bar{r}_p + \bar{h}_p + g_p/2 \) so that these are related to the field inside all the layers or outside all the layers, respectively. Of course \( l \in \{1, 2\} \) so that these can be summed over either the primary or secondary layers. We note that

\[
\alpha_{1_i} = H_z (\bar{x}_c, 0) = \sum_{n=1}^{p} I_n H_{\text{zn}} (\bar{x}_c, 0) = \sum_{n \in M_1} I_1 H_{\text{zn}} (\bar{x}_c, 0) + \sum_{n \in M_2} I_2 H_{\text{zn}} (\bar{x}_c, 0) = \gamma_{c,1} + I_2 \gamma_{c,2} \quad (3.41)
\]
and similarly

\[ \alpha_{p_0} = \mathcal{H}_z (\bar{x}_t, 0) = \sum_{n=1}^{p} I_n \mathcal{H}_{zn} (\bar{x}_t, 0) \]

\[ = \sum_{n \in M_1} I_1 \mathcal{H}_{zn} (\bar{x}_t, 0) + \sum_{n \in M_2} I_2 \mathcal{H}_{zn} (\bar{x}_t, 0) \]

\[ = \sum_{n \in M_1} \mathcal{H}_{zn} (\bar{x}_t, 0) + I_2 \sum_{n \in M_2} \mathcal{H}_{zn} (\bar{x}_t, 0) \]

\[ = \gamma_{t,1} + I_2 \gamma_{t,2} \tag{3.42} \]

Lastly, for brevity, set

\[ s_l = \sum_{n \in M_l} N_n , \tag{3.43} \]

again for \( l \in \{1, 2\} \). The Ampèreian loop of Figure 3.1 then results in

\[ w_{\alpha_1} H_0 - w_{\alpha_{p_0}} H_0 = I_1 \sum_{n \in M_1} N_n + I_2 \sum_{n \in M_2} N_n \]

\[ H_0 \left( \gamma_{c,1} + \bar{T}_2 \gamma_{c,2} - (\gamma_{t,1} + \bar{T}_2 \gamma_{t,2}) \right) = \frac{I_1}{w} s_1 + \frac{I_2}{w} s_2 \]

\[ H_0 \left( \gamma_{c,1} + \bar{T}_2 \gamma_{c,2} - \gamma_{t,1} - \bar{T}_2 \gamma_{t,2} \right) = H_0 (s_1 + \bar{T}_2 s_2) \]

\[ \bar{T}_2 \gamma_{c,2} - \bar{T}_2 \gamma_{t,2} - \bar{T}_2 s_2 = \gamma_{t,1} - \gamma_{c,1} + s_1 \]

\[ \bar{T}_2 = -\frac{s_1 + \gamma_{t,1} - \gamma_{c,1}}{s_2 + \gamma_{t,2} - \gamma_{c,2}} , \tag{3.44} \]

which is a more general version of (3.3). Though derived in the reverse order, in practice, \( \bar{T}_2 \) must be calculated first using (3.44) in order for the field coefficients to be calculated using (3.34) and (3.35). Finally, we note that, as a kind of bonus to the 2-D approach, the equations derived here also incorporate the influence of the winding width \( \bar{w} \) as this appears in (3.29) and so influences all subsequent expressions that depend on this.
3.4 Conclusion

It was found in this chapter that a 2-D analysis is really needed for treatment of a core with arbitrary permeability. The bulk of the chapter was spent deriving these 2-D field equations in the winding window and then using them to determine the magnetic field coefficients $\alpha_{n_1}$ and $\alpha_{n_o}$. These equations are quite general, noting in particular that they are valid for any interleaving pattern. The material here was discussed after the losses and leakage inductance in Chapter 2 to better motivate the clear goal of determining the field coefficients. However, in practice the results of this chapter must be calculated before those of Chapter 2 can be applied.

In the following chapter, all of the equations derived up to this point are validated using freely available FEA tools. Issues related to computer calculation of the analytical equations derived here and in Chapter 2 will be discussed there, such as evaluating the iterated series of (3.38).
Chapter 4

Accuracy Assessment

In order to assess the accuracy of the equations derived in Chapters 2 and 3, a statistical approach is used in which many randomly generated (technically pseudo-randomly generated since a computer was used) transformers were modeled and simulated using FEA. We use FEA because, with physical transformers, there is great difficulty when taking measurements to separate the winding losses from the core losses, and we are interested only in the former. Additionally, to exercise the full generality of the equations, we wish to use transformers that generally have different numbers of turns and different thicknesses for each individual layer as well as different gap sizes between layers, which would be quite impractical to produce physically. This is also the reason why a statistical approach is used as there are too many independent variables to vary in a systematic or comprehensive way.

4.1 Random Transformers

A total of 1000 devices were simulated, each with certain randomly generated parameters. First, as it is unimportant and arbitrary, all devices use a first layer thickness of $h_1 = 1$ mm. Also, all layers for a given devices use the same normalized width $\overline{w} = w/h_1$ as this was an
Transformer parameters:

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Min</th>
<th>Max</th>
<th>Type</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p_1$ and $p_2$</td>
<td>1</td>
<td>5</td>
<td>Integer</td>
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<tr>
<td>$\bar{r}_c$</td>
<td>5</td>
<td>40</td>
<td>Real</td>
</tr>
<tr>
<td>$\bar{w}$</td>
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<td>80</td>
<td>Real</td>
</tr>
<tr>
<td>$\alpha_c$</td>
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<td>Real</td>
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Layer parameters:

<table>
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</tr>
</thead>
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<tr>
<td>$\bar{h}_n$</td>
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<td>2</td>
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</tr>
<tr>
<td>$\bar{g}_n$</td>
<td>0.1</td>
<td>2</td>
<td>Real</td>
</tr>
</tbody>
</table>

Table 4.1: Device parameter ranges and types for random devices.

assumption made in the derivation of the equations. However, each device has a randomly assigned $\bar{w}$, normalized core radius $\bar{r}_c$, initial gap $\bar{g}_0$, core relative permeability $\mu_r$, and number of primary and secondary layers $p_1$ and $p_2$. Additionally, the order of the primary and secondary layers were randomly shuffled to create an arbitrary interleaving pattern for each device. For each layer, a random normalized thickness $\bar{h}_n$ (with the first layer always having $\bar{h}_1 = 1$), normalized gap size $\bar{g}_n$, and number of turns $N_n$ were chosen. The normalized inner radii $\bar{r}_m$ for each layer were then calculated from $\bar{r}_c$, $\bar{g}_0$, and $\bar{h}_n$ and $\bar{g}_n$ for the previous layers, the equation of course being

$$\bar{r}_n = \bar{r}_c + \bar{g}_0 + \sum_{m=1}^{n-1} (\bar{h}_m + \bar{g}_m)$$

(4.1)

with the usual convention that $\sum_{m=a}^{b} = 0$ when $b < a$.

Table 4.1 shows the transformer and layer parameters that were randomized and the range and type of each. For real parameters, the values were selected from a uniform distribution over the closed interval, and for integer parameters each of the finite number of values in the range (including the endpoints) had an equal probability of selection. Since the non-ideal core equations do not depend on $\mu_r$ directly, but rather on the image current
Transformer parameters:

<table>
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<th>$h_1$ (mm)</th>
<th>$\bar{r}_c$</th>
<th>$\bar{w}$</th>
<th>$\mu_r$</th>
<th>$p_1$</th>
<th>$p_2$</th>
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<td>6.2153</td>
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Layers:

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<th>$\bar{g}_n$</th>
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<td>1</td>
<td>1.6878</td>
<td>1</td>
</tr>
<tr>
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<td>3</td>
<td>14.474</td>
<td>0.85588</td>
<td>1.159</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>10</td>
<td>16.489</td>
<td>0.90462</td>
<td>1.8726</td>
<td>1</td>
</tr>
<tr>
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<td>5</td>
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<td>1.9055</td>
<td>1.5527</td>
<td>1</td>
</tr>
<tr>
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<td>1</td>
<td>22.724</td>
<td>1.8952</td>
<td>1.3928</td>
<td>1</td>
</tr>
</tbody>
</table>

Table 4.2: Parameters for random Device A, chosen for a large minimum $\bar{g}_n$.

factor $\alpha_c$, it is this parameter that was chosen uniformly instead of $\mu_r$. The somewhat odd maximum value for $\alpha_c$ comes from having $\mu_r$ stay in the interval $[1, 200]$. Also, the initial gap $\bar{g}_0$ was chosen from the same distribution as the layer gaps $\bar{g}_n$ so is not listed.

It would be impractical to enumerate all 1000 devices, so instead we choose three to examine in detail based on certain criteria that illustrate different things. The parameters of Device A are shown in Table 4.2, and this was selected based on having a large minimum gap size, i.e. if we take the minimum $\bar{g}_n$ across the layers (including the initial gap $\bar{g}_0$) for each device, the minimum of Device A is among the largest. This is mainly to create clear figures to illustrate the geometry and meshing guidelines, which are shown below.

Devices B and C, whose parameters are shown in Tables 4.3 and 4.4, respectively, were chosen as random devices with a small core relative permeability $\mu_r$ and small normalized core radius $\bar{r}_c$, respectively. The reasons for these selections will become apparent when we examine their simulation results. In the device parameter tables, we note that each has a phony $n = 0$ layer for the sole purpose of showing the initial gap $\bar{g}_0$, and that the “Group” column indicates whether each layer is a primary or secondary layer.

As has been assumed in the analytical derivations in previous chapters, in the FEA models, the layers always fill the entire window in the $z$ dimension. Pot core devices are
**Transformer parameters:**

<table>
<thead>
<tr>
<th>$h_1$ (mm)</th>
<th>$\tau_c$</th>
<th>$\overline{w}$</th>
<th>$\mu_r$</th>
<th>$p_1$</th>
<th>$p_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>5.183</td>
<td>36.37</td>
<td>1.6632</td>
<td>2</td>
<td>3</td>
</tr>
</tbody>
</table>

**Layers:**

<table>
<thead>
<tr>
<th>$n$</th>
<th>$N_n$</th>
<th>$\overline{\tau}_{n_i}$</th>
<th>$\overline{h}_n$</th>
<th>$\overline{\eta}_n$</th>
<th>Group</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>5.183</td>
<td>0</td>
<td>1.9964</td>
<td>N/A</td>
</tr>
<tr>
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<td>6</td>
<td>7.1794</td>
<td>1</td>
<td>1.6907</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>8</td>
<td>9.8702</td>
<td>0.86209</td>
<td>0.53761</td>
<td>2</td>
</tr>
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<td>3</td>
<td>1</td>
<td>11.27</td>
<td>1.1692</td>
<td>0.95601</td>
<td>2</td>
</tr>
<tr>
<td>4</td>
<td>6</td>
<td>13.395</td>
<td>1.4044</td>
<td>0.49864</td>
<td>1</td>
</tr>
<tr>
<td>5</td>
<td>8</td>
<td>15.298</td>
<td>1.5149</td>
<td>1.46</td>
<td>1</td>
</tr>
</tbody>
</table>

Table 4.3: Parameters for random Device B, chosen for a small $\mu_r$.

**Transformer parameters:**

<table>
<thead>
<tr>
<th>$h_1$ (mm)</th>
<th>$\tau_c$</th>
<th>$\overline{w}$</th>
<th>$\mu_r$</th>
<th>$p_1$</th>
<th>$p_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>5.0792</td>
<td>78.303</td>
<td>5.989</td>
<td>5</td>
<td>3</td>
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</tbody>
</table>

**Layers:**

<table>
<thead>
<tr>
<th>$n$</th>
<th>$N_n$</th>
<th>$\overline{\tau}_{n_i}$</th>
<th>$\overline{h}_n$</th>
<th>$\overline{\eta}_n$</th>
<th>Group</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
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<td>5.0792</td>
<td>0</td>
<td>0.11502</td>
<td>N/A</td>
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<td>0.54155</td>
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<tr>
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<td>8</td>
<td>6.7358</td>
<td>1.0458</td>
<td>0.32101</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>9</td>
<td>8.1026</td>
<td>1.5223</td>
<td>1.3929</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
<td>5</td>
<td>11.018</td>
<td>1.4924</td>
<td>1.1065</td>
<td>1</td>
</tr>
<tr>
<td>5</td>
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<td>13.617</td>
<td>1.5959</td>
<td>1.4109</td>
<td>2</td>
</tr>
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<td>6</td>
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<td>16.624</td>
<td>1.7203</td>
<td>0.77696</td>
<td>2</td>
</tr>
<tr>
<td>7</td>
<td>2</td>
<td>19.121</td>
<td>1.9255</td>
<td>1.8607</td>
<td>1</td>
</tr>
<tr>
<td>8</td>
<td>1</td>
<td>22.907</td>
<td>1.9222</td>
<td>0.1859</td>
<td>1</td>
</tr>
</tbody>
</table>

Table 4.4: Parameters for random Device C, chosen for a small $\overline{\tau}_c$.  

63
simulated so that core material surrounds the winding window on all sides. For the core that surrounds the window at the top, bottom, and outside, a somewhat arbitrary thickness of \( t_c = 10h_1 = 10 \text{ mm} \) was used for all devices. Figure 4.1 shows the geometry for Device A along with its simulated magnetic field intensity, which is illustrative of the geometry for all devices. We note that, for this and Figure 4.2 below, Device A was simulated with a penetration ratio of \( \Delta = 5 \).

Axisymmetric, time-harmonic simulations for the random transformers were performed using the open-source xfemm software package [11], which is essentially a command-line version of the Finite Element Method Magnetics (FEMM) software [40], which is also open-source software but requires the use of a Graphical User Interface (GUI). We note here that capacitive effects are not included in the simulations as these require electric fields in the radial direction. The simulation geometry, on the other hand, assume electric fields only in the \( \phi \) direction since that is the direction of current flow. Semi-circular boundary was used on which asymptotic boundary conditions (ABC) were enforced with \( n = 1 \), as discussed in Appendix A.3 of [41].
In the simulations, adaptive mesh sizes were used such that the mesh size within a layer is never coarser than one third of the skin depth. An example of this meshing can be observed in Figure 4.2, which shows the upper-left corner of the window for Device A. Notice that the gaps and layers have mesh densities based on their thicknesses, with gaps generally having coarser meshing than the layers since the fields are known to be roughly constant there. Device A was also simulated using the original FEMM software in order to create Figure 4.1 and Figure 4.2, noting that xfemm and FEMM seem to produce identical results under identical conditions.

Though the mesh size is not constant over the geometry of the device as Figure 4.2 shows, the overall mesh size can be controlled with a single parameter, with the regional mesh sizes being various fractions of this. This parameter was tuned by setting up a simulation containing a single rectangular layer in the center of the window and using a core permeability of $\mu_r = 1$, which is to say that the layer is in free space although the core geometry is present in terms of the mesh. The analytical solution for the magnetic field of this situation is given by (3.14) at dc. The mesh size parameter of dc simulations was
decreased until the magnetic field along the horizontal line in the center of the window showed good agreement with the analytical equation, since this is where the field will be evaluated in practice by (3.34) and (3.35). In particular, if \( x_i \) are the \( N \) sample field values along the center line from the simulation and \( y_i \) are the values from the analytical equation then

\[
e = \sqrt{\frac{1}{N} \sum_{i=1}^{N} (x_i - y_i)^2}
\]  

(4.2)

is the \( L_2 \) norm of the residual error. The relative error is then \( e_r = e/\overline{y} \), where \( \overline{y} \) is the average of the analytic field sample points. The mesh size parameter was then decreased until \( e_r \) was less than 1\% as calculated over \( N = 101 \) sample points across the horizontal center line.

## 4.2 Analytical Calculations

As we would like to compare the simulated results with our analytical results, we shall here take a short digression to discuss how exactly the analytical results were calculated and some computational issues that were encountered. The actual simulation results for losses and leakage inductance will be compared with several different analytical methods that make various assumptions:

1. Cartesian coordinates assuming an ideal core (labeled “Cartesian IC”)
2. Cylindrical coordinates assuming an ideal core (labeled “Cylindrical IC”)
3. Cartesian coordinates using a non-ideal core (labeled “Cartesian NIC”)
4. Cylindrical coordinates using a non-ideal core (labeled “Cylindrical NIC”)
5. Cylindrical coordinates using magnetic field coefficients from the simulation (labeled “FEA Coefficients”)

We now briefly discuss how each of these is computed.

### 4.2.1 Cartesian IC

As an ideal core is assumed, the secondary current $I_2$ and magnetic field coefficients $\alpha_{ni}$ and $\alpha_{no}$ are calculated using the equations of Section 3.1. In particular $I_2$ is first calculated using (3.3). We then have that $\alpha_{1i} = 0$ due to the ideal core assumption, and the remainder of the field coefficients are calculated using (3.6) and the fact that $\alpha_{(n+1)i} = \alpha_{no}$. The losses are then calculated using the Cartesian equations derived in Section 2.2.1. Specifically, $\overline{R}_{\text{cart}}$ is calculated using (2.43), where the Cartesian ac power factors $G_n$ for each primary layer are given by (2.40). Lastly, the normalized leakage inductance $L_{\text{lk(cart)}}$ is calculated using (2.67) from Section 2.3.1 using the same ac power factors $G_n$.

### 4.2.2 Cylindrical IC

Since this also uses the ideal core assumption, $I_2$ and the magnetic field coefficients are calculated in exactly the same way as the preceding section. This time, however, the cylindrical normalized resistance equation (2.51) derived in Section 2.2.2 is used to calculate $\overline{R}_{\text{cyl}}$ with the cylindrical ac power factors $F_n$ given by (2.49). Analogously, the leakage inductance $L_{\text{lk(cyl)}}$ is calculated using (2.76) from Section 2.3.2, which uses the very same ac power factors $F_n$.

A numerical issue was found during the calculation of the cylindrical ac power factors $F_n$, which shall now be discussed. Computing $F_n$ as derived in (2.49) requires computing our cylindrical kernel functions $\Psi_0$ and $\Psi_1$, which are given by (2.18) and (2.22), respectively. These in turn depend on calculating the products of the modified Bessel functions $I_0(z)K_0(w)$, $I_1(z)K_0(w)$, and $I_0(z)K_1(w)$ for generally complex arguments $z$ and $w$. The
problem is that

\[
\lim_{|z| \to \infty} I_0(z) = \lim_{|z| \to \infty} I_1(z) = \infty
\]

(4.3)

and that these increase quickly enough that, for some of the floating point arguments that were encountered, these Bessel functions returned an \( \text{inf} \) value, which is a special type of floating point that represents positive infinity. Similarly,

\[
\lim_{|z| \to \infty} K_0(z) = \lim_{|z| \to \infty} K_1(z) = 0
\]

(4.4)

and these decrease quickly enough that pure \( 0 \) values were returned for some of the same arguments. The result of multiplying \( \text{inf} \) by \( 0 \) is the special \( \text{nan} \) value, which stands for “not a number” and represents a value that could not be determined.

Of course, in a pure mathematical sense, the returned values of \( I_0 \) and \( I_1 \) would merely be very large but finite, whereas those for \( K_0 \) and \( K_1 \) would be very small but not zero. Their product could then be a number that is neither excessively large nor excessively small. After some experimentation, these \( \text{inf} \) and \( 0 \) values were found to only ever occur in any of these Bessel functions when the absolute value of their arguments were greater than 695. The solution to avoid these \( \text{nan} \) values was then to utilize asymptotic expansions for these Bessel function products. As listed in [38], some of the simplest asymptotic expansions for the modified Bessel functions of any order are

\[
I_\nu(z) \approx \frac{e^z}{\sqrt{2\pi z}} \quad K_\nu(w) \approx \frac{e^{-w}}{\sqrt{2\pi w}},
\]

(4.5)

which are valid when \( |z| \) and \( |w| \) are large. Therefore the general product is approximately

\[
I_{\nu_1}(z)K_{\nu_2}(w) \approx P(z, w) = \left( \frac{e^z}{\sqrt{2\pi z}} \right) \left( \frac{e^{-w}}{\sqrt{2\pi w}} \right) = \frac{e^{z-w}}{2\pi \sqrt{zw}},
\]

(4.6)
noting that this does not depend on the Bessel function orders $\nu_1$ and $\nu_2$. Hence, for computational purposes, the Bessel function products in (2.18) and (2.22) are replaced with

\[
I_{\nu_1}(z)K_{\nu_2}(w) = \begin{cases} 
I_{\nu_1}(z)K_{\nu_2}(w) & \text{if } |z| \leq 695 \text{ and } |w| \leq 695 \\
P(z, w) & \text{otherwise}
\end{cases}
\] (4.7)

Note that some numerical testing was conducted, which showed that the transition from the actual products to $P(z, w)$ is a smooth one for all of the three products.

### 4.2.3 Cartesian NIC

This method takes both core permeability $\mu_r$ and winding width $w$ into account and uses the equations derived in Section 3.3. First, the normalized secondary current $\bar{T}_2$ is calculated from the geometry using (3.44), noting that the core permeability must be converted to the image current factor $\alpha_c$ using (3.7). With this in hand, the magnetic field coefficients $\alpha_{ni}$ and $\alpha_{no}$ are then calculated for each layer using (3.34) and (3.35) and the fact that $\alpha_{(n+1)i} = \alpha_{no}$. We note that these coefficients will always be real since the entire analysis of Section 3.3 was done at dc. Once these are known, the normalized losses $\bar{R}_{cyl}$ and leakage inductance $\bar{L}_{lk(cyl)}$ are calculated in the same way as in Section 4.2.1 for ideal Cartesian coordinates.

Now, in computing $\bar{T}_2$ and $\alpha_{ni}$ and $\alpha_{no}$ for each layer, we are required to evaluate the iterated series of (3.38). It is of course not possible to numerically calculate a series with an infinite number of terms on a computer, and so these series must be truncated since a closed-form solution is not known. Being an iterated series, we have to truncate (3.38) for both the $m$ and $k$ indices. For convenience, we truncate both series to the same number of terms $N$ so that both $m$ and $k$ run over the same integer values.

In order to determine where to safely truncate, we first note that that the normalized magnetic field from a single image layer $\bar{H}_{zn0}$, which was defined in (3.29), can be ex-
pressed as

\[
\tilde{H}_{zn0}(x, z) = \frac{H_{zn0}(\bar{x}h_1, \bar{z}h_1)}{H_0} = \frac{1}{H_0} \left[ \frac{N_n I_n}{4\pi w h_n} f(\bar{x} - \bar{a}, \bar{z}) \right]
\]

\[
= \frac{N_n I_n}{4\pi H_0 w h_n} h_1 f(x - \bar{a}, z)
\]

\[
= \frac{N_n I_1}{4\pi h_1 w h_n} f(x - \bar{a}, z) = \frac{N_n I_1}{4\pi I_1 w h_n} f(x - \bar{a}, z)
\]

\[
= \frac{N_n}{4\pi h_n} f(x - \bar{a}, z) \tag{4.8}
\]

where we have used (B.20) and (B.19) from Appendix B and defined \( \bar{a} = a/h_1 = h_n/2h_1 = T_n/2 \). We also note that it is trivial to show that

\[
f(\bar{x}h_1, \bar{z}h_1) = h_1 f(x, z). \tag{4.9}
\]

Therefore, we have

\[
\left| \tilde{H}_{zn0}(x, z) \right| = \left| \frac{N_n}{4\pi h_n} f(x - \bar{a}, z) \right| = \left| \frac{N_n}{4\pi h_n} \right| \left| f(x - \bar{a}, z) \right|
\]

\[
= \frac{N_n}{4\pi h_n} \left| f(x - \bar{a}, z) \right|
\]

\[
\leq \frac{N_n}{4\pi h_n} 4(\pi + 1) \bar{a} = \frac{4(\pi + 1)N_n \bar{a}}{4\pi h_n}
\]

\[
= \frac{(\pi + 1)N_n T_n}{2\pi h_n} = \frac{(\pi + 1)N_n}{2\pi} \tag{4.10}
\]

for all \((x, z) \in \mathbb{R}^2\), where we have used the boundedness of \( f \) proved in Theorem B.6 of Appendix B. Thus also clearly

\[
|\overline{A}_{k,m}| \leq M \quad \quad |\overline{B}_{k,m}| \leq M \tag{4.11}
\]

for any integers \( k \) and \( m \) by definitions (3.30) and (3.31), where we let \( M = (\pi + 1)N_n/2\pi \).
for brevity.

Now, the case of $\alpha_c = 0$ only occurs when $\mu_r = 1$, which physically means that the core is effectively not present with regard to its effect on the magnetic fields. In this case the iterated series of (3.38) totally collapses so that the only contributing term is the first when $(m, k) = (0, 0)$. This terms represents the original layer itself so that there are no image layers. As this is a degenerate case that requires only one term, in the following argument we assume that $0 < \alpha_c < 1$. As discussed in the proof of Theorem B.8 in Appendix B, a consequence of this is that

$$\sqrt{\alpha_c^{2k}} \leq \sqrt{\alpha_c^k} \leq \sqrt{\alpha_c^{2k+1}} \leq \sqrt{\alpha_c^k} \leq \sqrt{\alpha_c^{2k-1}} \leq \sqrt{\alpha_c^k} \quad (4.12)$$

for all $k \geq 0$. It is also shown in the same proof that

$$\alpha_c^{\xi(k,m)} \leq \sqrt{\alpha_c^{|k|}} \sqrt{\alpha_c^{|m|}} \quad (4.13)$$

for any integers $k$ and $m$.

Now, we define the terms of (3.38) as

$$T_{k,m} = 2\lambda_k \lambda_m \left[ \alpha_c^{\xi(2k,m)} \left( \bar{A}_{k,m} + \bar{\Lambda}_{-k,m} \right) + \alpha_c^{\xi(2k+1,m)} \bar{B}_{k,m} + \alpha_c^{\xi(2k-1,m)} \bar{B}_{-k,m} \right] \quad (4.14)$$

so that the iterated series becomes

$$\mathcal{H}_{zn} \left( \vec{x}, 0 \right) = \sum_{k=0}^{\infty} \left( \sum_{m=0}^{\infty} T_{k,m} \right) \quad (4.15)$$

We also define

$$C_{k,m} = 12M \sqrt{\alpha_c^m} \sqrt{\alpha_c^k} \quad (4.16)$$
for all integers \(k, m \geq 0\). Then, for any such \(k\) and \(m\), we have that

\[
|T_{k,m}| = |2\lambda_k \lambda_m [\alpha_c^{\xi(2k,m)} (\overline{A}_{k,m} + \overline{A}_{-k,m}) + \alpha_c^{\xi(2k+1,m)} B_{k,m} + \alpha_c^{\xi(2k-1,m)} B_{-k,m}]|
\]

\[
= 2\lambda_k \lambda_m [\alpha_c^{\xi(2k,m)} (\overline{A}_{k,m} + \overline{A}_{-k,m}) + \alpha_c^{\xi(2k+1,m)} B_{k,m} + \alpha_c^{\xi(2k-1,m)} B_{-k,m}]
\]

\[
\leq 2\lambda_k \lambda_m \alpha_c^m [\alpha_c^{\xi(2k,m)} (|\overline{A}_{k,m}| + |\overline{A}_{-k,m}|) + |\alpha_c^{\xi(2k+1,m)} B_{k,m}| + |\alpha_c^{\xi(2k-1,m)} B_{-k,m}|]
\]

\[
= 2\lambda_k \lambda_m \alpha_c^m [\alpha_c^{\xi(2k,m)} |\overline{A}_{k,m}| + \alpha_c^{\xi(2k+1,m)} |B_{k,m}| + \alpha_c^{\xi(2k-1,m)} |B_{-k,m}|]
\]

\[
\leq 2 \left[ \alpha_c^{\xi(2k,m)} (M + M) + \alpha_c^{\xi(2k+1,m)} M + \alpha_c^{\xi(2k-1,m)} M \right]
\]

\[
= 2 \left[ 2M \alpha_c^{\xi(2k,m)} + M \alpha_c^{\xi(2k+1,m)} + M \alpha_c^{\xi(2k-1,m)} \right]
\]

\[
\leq 2 \left[ 2M \alpha_c^{\xi(2k,m)} + 2M \alpha_c^{\xi(2k+1,m)} + 2M \alpha_c^{\xi(2k-1,m)} \right]
\]

\[
= 4M \left[ \alpha_c^{\xi(2k,m)} + \alpha_c^{\xi(2k+1,m)} + \alpha_c^{\xi(2k-1,m)} \right]
\]

\[
\leq 4M \left[ \sqrt{\alpha_c^{|m|}} \sqrt{\alpha_c^{|2k|}} + \sqrt{\alpha_c^{|m|}} \sqrt{\alpha_c^{|2k+1|}} + \sqrt{\alpha_c^{|m|}} \sqrt{\alpha_c^{|2k-1|}} \right]
\]

\[
= 4M \sqrt{\alpha_c^m} \left[ \sqrt{\alpha_c^{|2k|}} + \sqrt{\alpha_c^{|2k+1|}} + \sqrt{\alpha_c^{|2k-1|}} \right]
\]

\[
\leq 4M \sqrt{\alpha_c^m} \left[ \sqrt{\alpha_c^k} + \sqrt{\alpha_c^k} + \sqrt{\alpha_c^k} \right]
\]

\[
= 12M \sqrt{\alpha_c^m} \left[ \sqrt{\alpha_c^k} \right]
\]

\[
= C_{k,m} \quad (4.17)
\]

since \(\lambda_k, \lambda_m \leq 1\) by their definition (3.19) and \(M > 0\), where we have also of course repeatedly used the triangle inequality.

Next we evaluate the closed-form solutions of some iterated series, noting that these all involve geometric series since \(0 < \sqrt{\alpha_c} < 1\). First we note that, in general,

\[
\sum_{n=0}^{N-1} \sqrt{\alpha_c^n} = \frac{1 - \sqrt{\alpha_c^N}}{1 - \sqrt{\alpha_c}} \quad (4.18)
\]
and

\[ \sum_{n=N}^{\infty} \sqrt{\alpha_c^n} = \sum_{n=0}^{\infty} \sqrt{\alpha_c^n} - \sum_{n=0}^{N-1} \sqrt{\alpha_c^n} = \frac{1}{1 - \sqrt{\alpha_c}} - \frac{1 - \sqrt{\alpha_c}^N}{1 - \sqrt{\alpha_c}} = \frac{\sqrt{\alpha_c}^N}{1 - \sqrt{\alpha_c}}. \]  \hspace{1cm} (4.19)

Hence, in particular, we have

\[ \sum_{k=0}^{N-1} \left( \sum_{m=N}^{\infty} C_{k,m} \right) = \sum_{k=0}^{N-1} \left( \sum_{m=N}^{\infty} 12M \sqrt{\alpha_c^{-m}} \sqrt{\alpha_c^k} \right) = 12M \sum_{k=0}^{N-1} \sqrt{\alpha_c^k} \left( \sum_{m=N}^{\infty} \sqrt{\alpha_c^{-m}} \right) \]
\[ = 12M \sum_{k=0}^{N-1} \sqrt{\alpha_c^k} \frac{\sqrt{\alpha_c}^N}{1 - \sqrt{\alpha_c}} = 12M \frac{\sqrt{\alpha_c}^N}{1 - \sqrt{\alpha_c}} \sum_{k=0}^{N-1} \sqrt{\alpha_c^k} \]
\[ = 12M \left( \frac{\sqrt{\alpha_c}^N}{1 - \sqrt{\alpha_c}} \right) \left( \frac{1 - \sqrt{\alpha_c}^N}{1 - \sqrt{\alpha_c}} \right) \]
\[ = 12M \frac{\sqrt{\alpha_c}^N (1 - \sqrt{\alpha_c}^N)}{(1 - \sqrt{\alpha_c})^2}. \]  \hspace{1cm} (4.20)

We also have

\[ \sum_{k=N}^{\infty} \left( \sum_{m=0}^{N-1} C_{k,m} \right) = \sum_{k=N}^{\infty} \left( \sum_{m=0}^{N-1} 12M \sqrt{\alpha_c^{-m}} \sqrt{\alpha_c^k} \right) = 12M \sum_{k=N}^{\infty} \sqrt{\alpha_c^k} \left( \sum_{m=0}^{N-1} \sqrt{\alpha_c^{-m}} \right) \]
\[ = 12M \sum_{k=N}^{\infty} \sqrt{\alpha_c^k} \frac{1 - \sqrt{\alpha_c}^N}{1 - \sqrt{\alpha_c}} = 12M \frac{1 - \sqrt{\alpha_c}^N}{1 - \sqrt{\alpha_c}} \sum_{k=N}^{\infty} \sqrt{\alpha_c^k} \]
\[ = 12M \left( \frac{1 - \sqrt{\alpha_c}^N}{1 - \sqrt{\alpha_c}} \right) \left( \frac{\sqrt{\alpha_c}^N}{1 - \sqrt{\alpha_c}} \right) \]
\[ = 12M \frac{\sqrt{\alpha_c}^N (1 - \sqrt{\alpha_c}^N)}{(1 - \sqrt{\alpha_c})^2}. \]  \hspace{1cm} (4.21)

and

\[ \sum_{k=N}^{\infty} \left( \sum_{m=N}^{\infty} C_{k,m} \right) = \sum_{k=N}^{\infty} \left( \sum_{m=N}^{\infty} 12M \sqrt{\alpha_c^{-m}} \sqrt{\alpha_c^k} \right) = 12M \sum_{k=N}^{\infty} \sqrt{\alpha_c^k} \left( \sum_{m=N}^{\infty} \sqrt{\alpha_c^{-m}} \right) \]
\[ = 12M \sum_{k=N}^{\infty} \sqrt{\alpha_c^k} \frac{\sqrt{\alpha_c}^N}{1 - \sqrt{\alpha_c}} = 12M \frac{\sqrt{\alpha_c}^N}{1 - \sqrt{\alpha_c}} \sum_{k=N}^{\infty} \sqrt{\alpha_c^k} \]
\[
= 12M \left( \frac{\sqrt{\alpha} c^N}{1 - \sqrt{\alpha} c} \right) \left( \frac{\sqrt{\alpha} c^N}{1 - \sqrt{\alpha} c} \right) = 12M \frac{\sqrt{\alpha} c^{2N}}{(1 - \sqrt{\alpha} c)^2}
\]

(4.22)

Now, since it was shown in Appendix B that the original iterated series (3.24) converges absolutely, it follows that the normalized series (3.32) and (4.15) (which is just (3.38)) also converge absolutely. Therefore any rearrangements of these series also converge to the same value. Hence we clearly have

\[
\sum_{k=0}^{\infty} \left( \sum_{m=0}^{\infty} T_{k,m} \right) = \sum_{k=0}^{\infty} \left( \sum_{m=0}^{N-1} T_{k,m} + \sum_{m=N}^{\infty} T_{k,m} \right) = \sum_{k=0}^{N-1} \left( \sum_{m=0}^{N-1} T_{k,m} + \sum_{m=N}^{\infty} T_{k,m} \right) + \sum_{k=N}^{\infty} \left( \sum_{m=0}^{N-1} T_{k,m} + \sum_{m=N}^{\infty} T_{k,m} \right)
\]

(4.23)

where the first term of this is the iterated series truncated to \(N\) terms for both indices. Thus the magnitude of the error of the truncation is

\[
\left| \sum_{k=0}^{\infty} \left( \sum_{m=0}^{\infty} T_{k,m} \right) - \sum_{k=0}^{N-1} \left( \sum_{m=0}^{\infty} T_{k,m} \right) \right|
\]

\[
\leq \sum_{k=0}^{N-1} \left( \sum_{m=N}^{\infty} |T_{k,m}| \right) + \sum_{k=N}^{\infty} \left( \sum_{m=0}^{N-1} |T_{k,m}| \right) + \sum_{k=N}^{\infty} \left( \sum_{m=N}^{\infty} |T_{k,m}| \right)
\]

\[
\leq \sum_{k=0}^{N-1} \left( \sum_{m=N}^{\infty} C_{k,m} \right) + \sum_{k=N}^{\infty} \left( \sum_{m=0}^{N-1} C_{k,m} \right) + \sum_{k=N}^{\infty} \left( \sum_{m=N}^{\infty} C_{k,m} \right)
\]

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\[\begin{align*}
&\quad = 12M \sqrt{\alpha_c}^N \frac{(1 - \sqrt{\alpha_c}^N)}{(1 - \sqrt{\alpha_c})^2} + 12M \sqrt{\alpha_c}^N \frac{(1 - \sqrt{\alpha_c}^N)}{(1 - \sqrt{\alpha_c})^2} + 12M \frac{\alpha_c^N}{(1 - \sqrt{\alpha_c})^2} \\
&\quad = 12M \frac{2\sqrt{\alpha_c}^N (1 - \sqrt{\alpha_c}^N)}{(1 - \sqrt{\alpha_c})^2} + \sqrt{\alpha_c}^{2N} \\
&\quad = 12M \frac{\sqrt{\alpha_c}^N (2 - \sqrt{\alpha_c}^N)}{(1 - \sqrt{\alpha_c})^2} \\
&\quad = \frac{6(\pi + 1)N_n \sqrt{\alpha_c}^N (2 - \sqrt{\alpha_c}^N)}{\pi (1 - \sqrt{\alpha_c})^2} \\
&\quad = \frac{6(\pi + 1)N_n \sqrt{\alpha_c}^N (2 - \sqrt{\alpha_c}^N)}{\pi (1 - \sqrt{\alpha_c})^2} \\
&\quad = \frac{6(\pi + 1)N_n \sqrt{\alpha_c}^N (2 - \sqrt{\alpha_c}^N)}{\pi (1 - \sqrt{\alpha_c})^2} (4.24)
\end{align*}\]

where we have used the triangle inequality, (4.17), the closed-form series previously derived, and (4.10).

Thus (4.24) is an upper bound on the truncation error. The method to calculate the series (4.15) is then to calculate the truncated series

\[S_T = \sum_{k=0}^{N-1} \left( \sum_{m=0}^{N-1} T_{k,m} \right)\]

and the error bound

\[E = \frac{6(\pi + 1)N_n \sqrt{\alpha_c}^N (2 - \sqrt{\alpha_c}^N)}{\pi (1 - \sqrt{\alpha_c})^2} \]

for some \(N\), and then increase \(N\) and recalculate until the error is guaranteed to be much less than the magnitude of the truncated sum so that neglecting the remaining terms has a negligible effect. In particular, we stop increasing \(N\) and use \(S_T\) as the value for the infinite series when

\[\frac{E}{|S_T|} \leq \frac{1}{100}.\]

(4.27)

In practice, we begin with \(N = 50\) and then increase \(N\) by 50 at each step until the condition
(4.27) is met. It was noticed that convergence is much slower (i.e. larger values of \( N \) are required) when \( \mu_r \) is large, which makes sense since then \( \alpha_c \) and \( \sqrt{\alpha_c} \) are only slightly less than unity so that \( E \) generally becomes larger, primarily due to the \( (1 - \sqrt{\alpha_c})^2 \) factor in the denominator.

### 4.2.4 Cylindrical NIC

Here the magnetic field coefficients \( \alpha_{ni} \) and \( \alpha_{no} \) are again calculated as in the previous section, but the normalized losses \( \overline{R}_{cyl} \) and leakage inductance \( \overline{L}_{lk(cyl)} \) are of course then computed in the same way as in Section 4.2.2 for ideal cylindrical coordinates.

### 4.2.5 FEA Coefficients

In this method, the magnetic field coefficients \( \alpha_{ni} \) and \( \alpha_{no} \) are extracted from the simulation results at exactly the same points as in (3.34) and (3.35), i.e. in the centers of the gap regions. This is done at each simulated penetration ratio \( \Delta \), and these coefficients are allowed to be complex, since this will generally be the case for ac (i.e. when \( \Delta > 0 \)) simulations. It was found though that the imaginary parts of these complex values are generally very small relative to the real parts, indicating a nearly real value as we have in theory. Then, as in the previous two sections, the cylindrical equations (2.51) and (2.76) are used to calculate the normalized resistance \( \overline{R}_{cyl} \) and leakage inductance \( \overline{L}_{lk(cyl)} \), respectively.

### 4.3 Simulation Results

We now examine the results of the FEA simulations and compare them with our analytical solutions. As it is not feasible to present graphically the results of all 1000 devices that were simulated, we first examine the results of two of the devices described above, which illustrate certain things, using plots. Afterward, we then look at some statistics across all
of the devices that were simulated.

### 4.3.1 Specific Devices

Here we look at the results of Devices B and C as Device A was selected primarily to generate the geometry figures and there is nothing particularly interesting to observe in its results. The simulated results (labeled “FEA Actual”) along with the analytical solutions described above are shown in Figure 4.3 for Device B as a function of penetration ratio $\Delta$, noting that the vertical axis is logarithmic. First note that the simulation results as well as the FEA Coefficient analytical results (which depend on the simulations) are only at discrete values with a curve fitted in between points. This is of course because simulations were only run at 31 linearly spaced $\Delta$ values for each device due to computation time constraints. The other analytical solutions, on the other hand, could be calculated at many more points, making them appear continuous. Lastly, we note that the FEA Coefficients solutions are difficult to see in Figure 4.3 because they very nearly overlap the actual FEA results.

The reason Device B was chosen is to illustrate the difference between the ideal core approximations (i.e. the Cartesian IC and Cylindrical IC curves) and the non-ideal core solutions (i.e. the Cartesian NIC and Cylindrical NIC curves). As expected, this difference is most prominent when $\mu_r$ is low, as it is in Device B in which $\mu_r = 1.6632$ ($\alpha_c = 0.24902$), and we note that the non-ideal core solutions are closer to the actual simulation results for both the resistance and leakage inductance, though this easier to observe in the latter where things are more spaced out. Finally, we note that this device happens also to have a quite low $\bar{r}_c$ of 5.183 so that the Cartesian and cylindrical solutions are observably different as well.

The results for Device C are shown in Figure 4.4. This device was chosen for its very low $\bar{r}_c$ value of 5.0792, which illustrates more dramatically the difference between the Cartesian and cylindrical solutions, especially in the case of the leakage inductance.
Figure 4.3: Resistance and leakage results for Device B.
Figure 4.4: Resistance and leakage results for Device C.
Here it is difficult to distinguish between the ideal and non-ideal core curves for both the Cartesian and cylindrical solutions as these completely overlap. This is not a surprise when we consider that this device has a $\mu_r$ of 5.989 ($\alpha_c = 0.71384$). Although $\mu_r$ does not seem particularly large relative to Device B, notice how much larger $\alpha_c$ is. Notice that the Cartesian curves are significantly different from their cylindrical counterparts, and that the cylindrical solutions are closer to the simulated results. This difference is of course most prominent at low $\tau_c$ as the layer curvature is greater than for larger $\tau_c$.

One observation about these plots is that the different curves seem to have the same shape but are simply offset from one another by a constant amount. This is not the case so much with the resistance as with the leakage inductance. Since the plots are in logarithmic space on the $L_{lk}$ axis, this constant offset translates to the two curves having a constant ratio with respect to $\Delta$. Some analysis was conducted on this and, when comparing the Cylindrical IC to the Cylindrical NIC curves in Figure 4.3, the relative standard deviation (i.e. the standard deviation divided by the mean) of the ratio was only 0.29549% over 500 different samples of $\Delta$ over the range in the plots. Similarly, the relative standard deviation of the ratio of the Cartesian NIC curve to the Cylindrical NIC curve in Figure 4.4 is only 0.48565% over the same number of samples. Therefore the ratios are very nearly constant over $\Delta$ but not exactly since this would result in a relative standard deviation of zero.

This nearly constant ratio is curious, but the mathematical reasons are not immediately clear. In the case of the IC/NIC ratio, the same leakage inductance equation (2.76) is used, but the magnetic field coefficients are different. Interestingly these coefficients all appear only in the numerator of (2.76), and indeed, if every $\alpha_{n_o}$ and $\alpha_{n_i}$ were scaled by the same amount from their ideal-core versions, this should in fact result in $L_{lk}$ scaling constantly as well. However, examination showed that this is not the case for the coefficients of Figure 4.3, so the cause of the near-constant ratio must not be that straightforward. In the case of the Cartesian/Cylindrical ratio, on the other hand, the equations for $L_{lk}$ are entirely different for the two curves (the Cartesian equation is (2.67) while (2.76) is the cylindrical
equation), making it less clear why the ratio is nearly constant.

However, there is high confidence that this phenomenon is not caused by an implementation error in the equations. This is firstly because the ratio is not exactly constant (since the standard deviation is nonzero), which would likely be the case if the scaling were caused by a simple implementation error. Secondly, the same code produced both the plots of Figure 4.3 and Figure 4.4, and the Cartesian and cylindrical solutions have a completely different and much smaller ratio in the former plot (due to a larger $\bar{r}_c$), whereas the IC and NIC solutions in the latter plot are nearly overlapping (due to a larger $\mu_r$). In essence, the same code produces all of the plots, and the ratios really depend on the parameters ($\mu_r$ and $\bar{r}_c$), just as one would expect.

Now, looking at the FEA Actual leakage inductance curves in Figure 4.3 and Figure 4.4, there is an apparent anomaly happening at dc (i.e. for $\Delta = 0$) that warrants some explanation. First, it seems from these plots that only the single point $D = 0$ is anomalous, which is to say that curves are continuous at every point except at $\Delta = 0$. However, Figure 4.5 shows FEA simulation results for Device B for penetration ratios clustered around $\Delta = 0$, noting that the upper range of $\Delta$ in these plots is only 0.5. From this it is evident that the curve is in fact continuous at $\Delta = 0$ but quickly stabilizes from this anomalous value.

The explanation for this is that these simulations were run with a genuinely shorted secondary coil rather than simply setting the secondary current to that calculated from (3.3) or (3.44). At dc the flux generated by the primary winding current is of course constant so that no voltage is induced in the secondary winding, and hence there is no secondary current. From Figure 4.5, this anomaly seems to be observable in the leakage inductance only and not the resistance. This is because the resistance depends only on the fields immediately surrounding the primary layers, which are not strongly affected by the secondary current. The leakage inductance, on the other hand, depends on the magnetic field across the entire window, which is strongly influenced by the secondary current. This is the reason why the
Figure 4.5: Resistance and leakage results for Device B clustered at $\Delta = 0$. 
leakage inductance at dc is a not good choice for normalization purposes as alluded to in Section 2.2.1. Both the derivations of the ideal core and non-ideal core analytical equations assume coupling between the primary and secondary windings, which is why this anomaly is not present in those solutions.

Lastly, we note that both the Cartesian and cylindrical equations for the leakage inductance cannot be evaluated at dc, mathematically due to the $\Delta$ in the denominators of the $\Im \{G_n/\Delta\}$ term of (2.67) and the $\Im \{F_n/\Delta\}$ term of (2.76). This is most apparent in the FEA Coefficients curves of plots in Figure 4.3 and Figure 4.4, which simply lack a data point at $\Delta = 0$. This is also the case for the other analytical curves but this is impossible to tell due to the much higher density of evaluation points. As inductance is a fundamentally ac phenomenon, this does not pose a problem. The ac equations for the normalized resistance also cannot be calculated at dc. This time because of $\tanh(\kappa \Delta \overline{h}_n)$ and $\sinh(\kappa \Delta \overline{h}_n)$ in the denominators of the terms of $G_n$ needed to evaluate $R_{\text{cart}}$, and because $K_0(x)$ and $K_1(x)$ are undefined at $x = 0$ so that $\Psi_0$, $\Psi_1$, and $F_n$ cannot be evaluated for $R_{\text{cyl}}$. However we have specific dc versions for these equations (that are needed for normalization) and so the normalized resistance is just set to unity at $\Delta = 0$ in the above plots, which is of course the case by definition.

4.3.2 All Devices

Now we look at the results from all 1000 devices, which must be done in a statistical manner using metrics calculated over all devices. In particular, we are interested in comparing the various analytical solutions with the simulated results. First, as mentioned previously, the primary driver in differences between the ideal core and non-ideal core solutions is the core permeability $\mu_r$, while the primary driver of differences between Cartesian and cylindrical solutions is the core radius $\overline{r}_c$. As such, Figure 4.6 shows a scatter plot that shows the $\mu_r$ and $\overline{r}_c$ values of all 1000 devices. It can be observed that, while the $\overline{r}_c$ values are evenly distributed, the $\mu_r$ values are heavily skewed in favor of low $\mu_r$ values. This is again because
it was $\alpha_c$ that was selected from a uniform distribution rather than $\mu_r$, and this can be seen in Figure 4.7 in which $\alpha_c$ values are plotted in place of $\mu_r$.

Now, in order to get a handle on the accuracy of the analytical solutions across so many devices, we shall use three different metrics to get different perspectives. The first metric is the well-known coefficient of determination from statistics, which is commonly denoted as $R^2$. The other two are validation metrics based on relative error introduced in [29]: the mean relative error metric, denoted as $\%RE^m$, and the specific relative error metric, denoted as $\%RE^s$. Details of these three metrics are briefly discussed in Appendix C for completeness. As discussed there, there are certain precautions with the use of these metrics. First, regarding the potentially misleading $R^2$ metric that can occur when overfitting a regression model, this is not an issue with our data as our models are just as smooth as the simulated data as evidenced by the plots in Figure 4.3 and Figure 4.4. Regarding the care needed in preparing the data for the $\%RE^m$ and $\%RE^s$ metrics, this is also not an

Figure 4.6: Values of $\mu_r$ and $\bar{r}_c$ for all devices.
Figure 4.7: Values of $\alpha_c$ and $\tau_c$ for all devices.

issue for us as our simulated $R$ and $L_{\text{lk}}$ values are always nonzero. We do note though that dc points for the leakage inductance are not included in the metric calculations as the analytical solutions are undefined there as previously discussed.

The results of these metrics calculated over all 1000 devices are shown in Table 4.5. For each system response quantity (SRQ) the results are sorted by the $R^2$ metric in descending order (i.e. from the best fit to the worst fit). Also included as an SRQ are the analytical magnetic field coefficients $\alpha_{ni}$ and $\alpha_{no}$ over all devices relative to those from the simulation results, which is the top row. Note that the number of points compared for this SRQ are larger than the rest as there are of course numerous coefficients for each device at each $\Delta$. For the $\%\text{RE}^*$ metric a specific relative error of 10% was somewhat arbitrarily chosen.

First we compare the purely analytical solutions, that is the four Cylindrical/Cartesian NIC/IC solutions. As Table 4.5 shows, the NIC solutions are significantly more accurate
<table>
<thead>
<tr>
<th>SRQ</th>
<th>Method</th>
<th>Points</th>
<th>$R^2$</th>
<th>$%RE^m$</th>
<th>$%RE^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha_{ni}$ and $\alpha_{no}$</td>
<td>NIC Coefficients</td>
<td>214024</td>
<td>0.905695</td>
<td>$\leq 540.4%$ with $P = 95.983%$</td>
<td>$\leq 10%$ with $P = 45.214%$</td>
</tr>
<tr>
<td>$R_1$</td>
<td>FEA Coefficients</td>
<td>31000</td>
<td>0.999384</td>
<td>$\leq 2.6628%$ with $P = 68.119%$</td>
<td>$\leq 10%$ with $P = 96.058%$</td>
</tr>
<tr>
<td>$R_1$</td>
<td>Cartesian NIC</td>
<td>31000</td>
<td>0.915445</td>
<td>$\leq 11.531%$ with $P = 72.332%$</td>
<td>$\leq 10%$ with $P = 68.594%$</td>
</tr>
<tr>
<td>$R_1$</td>
<td>Cylindrical NIC</td>
<td>31000</td>
<td>0.873565</td>
<td>$\leq 12.099%$ with $P = 77.455%$</td>
<td>$\leq 10%$ with $P = 73.929%$</td>
</tr>
<tr>
<td>$R_1$</td>
<td>Cartesian IC</td>
<td>31000</td>
<td>0.763528</td>
<td>$\leq 20.412%$ with $P = 69.848%$</td>
<td>$\leq 10%$ with $P = 51.19%$</td>
</tr>
<tr>
<td>$R_1$</td>
<td>Cylindrical IC</td>
<td>31000</td>
<td>0.698814</td>
<td>$\leq 21.564%$ with $P = 72.79%$</td>
<td>$\leq 10%$ with $P = 53.326%$</td>
</tr>
<tr>
<td>$T_{lk}$</td>
<td>FEA Coefficients</td>
<td>30000</td>
<td>0.999931</td>
<td>$\leq 0.37575%$ with $P = 72.753%$</td>
<td>$\leq 10%$ with $P = 99.987%$</td>
</tr>
<tr>
<td>$T_{lk}$</td>
<td>Cylindrical NIC</td>
<td>30000</td>
<td>0.844793</td>
<td>$\leq 17.562%$ with $P = 76%$</td>
<td>$\leq 10%$ with $P = 59.747%$</td>
</tr>
<tr>
<td>$T_{lk}$</td>
<td>Cartesian NIC</td>
<td>30000</td>
<td>0.844423</td>
<td>$\leq 19.996%$ with $P = 73.753%$</td>
<td>$\leq 10%$ with $P = 51.717%$</td>
</tr>
<tr>
<td>$T_{lk}$</td>
<td>Cylindrical IC</td>
<td>30000</td>
<td>0.635905</td>
<td>$\leq 29.705%$ with $P = 72.257%$</td>
<td>$\leq 10%$ with $P = 58.03%$</td>
</tr>
<tr>
<td>$T_{lk}$</td>
<td>Cartesian IC</td>
<td>30000</td>
<td>0.633966</td>
<td>$\leq 31.633%$ with $P = 72.343%$</td>
<td>$\leq 10%$ with $P = 33.753%$</td>
</tr>
</tbody>
</table>

Table 4.5: Metrics for each of the analytical solutions relative to the axisymmetric simulation results.
than the corresponding IC solutions for both $\bar{R}$ and $\bar{L}_{ik}$ by every metric. Comparing the Cartesian and cylindrical solutions, we see that, for both the NIC and IC solutions for $\bar{R}$, the Cartesian solution is actually a better fit according to the $R^2$ metric, but the reverse is true by the $\%RE^s$ metric, at least in the sense that a random device has a higher probability having relative errors less than 10% in the cylindrical case. For $\bar{L}_{ik}$ the cylindrical solutions are the clear winners by every metric. This seems to suggest that the winding curvature has more of an effect on the leakage inductance than the resistance.

Now, for both $\bar{R}$ and $\bar{L}_{ik}$, the best fit is the FEA Coefficients solution, and this is significantly better than the NIC solutions. This is not terribly surprising since this uses information from the simulation results (i.e. $\alpha_{ni}$ and $\alpha_{no}$) in the analytical equations, whereas the Cartesian and Cylindrical NIC solutions are purely analytical and so can be calculated a priori. This along with the fact that the NIC Coefficients SRQ is only a moderately good fit, hints that the analytical magnetic field coefficients derived in Chapter 3 may not be exceptionally accurate. However, the fact that the FEA Coefficients solution is such a good fit shows that the resistance and loss equations derived in Chapter 2 are themselves still quite accurate.

There were two assumptions made in the derivation of the NIC magnetic field coefficients that are violated in the simulations. Namely they were derived only at dc, and were derived in Cartesian coordinates, both of these being out of necessity to make the problem solvable analytically. It was noted above that the simulated $\alpha_{ni}$ and $\alpha_{no}$ were found to be generally very nearly purely real, suggesting that perhaps the dc assumption may be a good one. To test the hypothesis that the primary culprit of the magnetic field coefficient accuracy is the Cartesian/cylindrical difference, everything was simulated again with the only difference being that the simulations were done in Cartesian coordinates instead of axisymmetric coordinates. The results of these are shown in Table 4.6.

Here the Cartesian solutions are more accurate than the cylindrical counterparts across the board by every metric, which makes sense. Also note that the NIC Coefficients are a
<table>
<thead>
<tr>
<th>SRQ</th>
<th>Method</th>
<th>Points</th>
<th>$R^2$</th>
<th>$% RE^m$</th>
<th>$% RE^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha_{n_i}$ and $\alpha_{n_o}$</td>
<td>NIC Coefficients</td>
<td>214024</td>
<td>0.962854</td>
<td>$\leq 99.097%$ with $P = 85.304%$</td>
<td>$\leq 10%$ with $P = 59.14%$</td>
</tr>
<tr>
<td>$\bar{R}$</td>
<td>Cartesian NIC</td>
<td>31000</td>
<td>0.987504</td>
<td>$\leq 3.51%$ with $P = 65.303%$</td>
<td>$\leq 10%$ with $P = 92.548%$</td>
</tr>
<tr>
<td>$\bar{R}$</td>
<td>FEA Coefficients</td>
<td>31000</td>
<td>0.975784</td>
<td>$\leq 5.4533%$ with $P = 63.555%$</td>
<td>$\leq 10%$ with $P = 84.203%$</td>
</tr>
<tr>
<td>$\bar{R}$</td>
<td>Cylindrical NIC</td>
<td>31000</td>
<td>0.964721</td>
<td>$\leq 6.194%$ with $P = 66.558%$</td>
<td>$\leq 10%$ with $P = 81.277%$</td>
</tr>
<tr>
<td>$\bar{R}$</td>
<td>Cartesian IC</td>
<td>31000</td>
<td>0.911461</td>
<td>$\leq 11.152%$ with $P = 66.129%$</td>
<td>$\leq 10%$ with $P = 61.932%$</td>
</tr>
<tr>
<td>$\bar{R}$</td>
<td>Cylindrical IC</td>
<td>31000</td>
<td>0.873011</td>
<td>$\leq 13.092%$ with $P = 66.313%$</td>
<td>$\leq 10%$ with $P = 58.548%$</td>
</tr>
<tr>
<td>$\bar{T}_{lk}$</td>
<td>FEA Coefficients</td>
<td>30000</td>
<td>0.992688</td>
<td>$\leq 4.8997%$ with $P = 61.407%$</td>
<td>$\leq 10%$ with $P = 88.54%$</td>
</tr>
<tr>
<td>$\bar{T}_{lk}$</td>
<td>Cartesian NIC</td>
<td>30000</td>
<td>0.969231</td>
<td>$\leq 6.6456%$ with $P = 62.517%$</td>
<td>$\leq 10%$ with $P = 79.95%$</td>
</tr>
<tr>
<td>$\bar{T}_{lk}$</td>
<td>Cylindrical NIC</td>
<td>30000</td>
<td>0.963016</td>
<td>$\leq 8.0624%$ with $P = 61.697%$</td>
<td>$\leq 10%$ with $P = 71.533%$</td>
</tr>
<tr>
<td>$\bar{T}_{lk}$</td>
<td>Cartesian IC</td>
<td>30000</td>
<td>0.855045</td>
<td>$\leq 16.591%$ with $P = 64.78%$</td>
<td>$\leq 10%$ with $P = 45.38%$</td>
</tr>
<tr>
<td>$\bar{T}_{lk}$</td>
<td>Cylindrical IC</td>
<td>30000</td>
<td>0.848572</td>
<td>$\leq 16.872%$ with $P = 64.73%$</td>
<td>$\leq 10%$ with $P = 46.463%$</td>
</tr>
</tbody>
</table>

Table 4.6: Metrics for each of the analytical solutions relative to Cartesian simulation results.
significantly better fit than in the cylindrical simulations. Though the FEA Coefficients is still the best first for $L_{lk}$, the NIC solutions for $L_{lk}$ are a much better fit than with the cylindrical simulations. For $R$ on the other hand, the Cartesian NIC solution is actually a better fit than the FEA Coefficients! This second batch of Cartesian simulations seems to confirm that the primary cause of inaccuracy in the NIC solutions is in fact the Cartesian assumption in the derivation of the magnetic field coefficients. This is a recognized shortcoming with the NIC solutions, though we again note that the NIC solutions are still generally significantly more accurate than the IC solutions in all cases.

With these data we can also answer the question of how large $\mu_r$ needs to be before the ideal core approximations become sufficiently accurate. To do this, we will look at the metrics for the Cylindrical IC solution relative to the axisymmetric simulation results only for devices that have $\mu_r \geq \mu_{rf}$, where $\mu_{rf}$ is a filter relative permeability. These are shown in Table 4.7 for increasing $\mu_{rf}$, noting that this shows the number of random devices for which $\mu_r \geq \mu_{rf}$ and the corresponding number of points over which the metrics were calculated. It can be observed from this that, even for $\mu_r$ as low as 4 (and correspondingly an $\alpha_c$ as low as 0.6), the Cylindrical IC solution is already more accurate than the Cylindrical NIC solution was over all devices by every metric for both $L_{lk}$ and $R$ (compare the $\mu_{rf} = 4$ rows of Table 4.7 with the Cylindrical NIC methods in Table 4.5). So, with the exception of high-frequency air-core transformers, the non-ideal core solutions may be of limited practical use as even devices with a gapped core will likely have an effective $\mu_r \geq 4$.

A similar analysis could be performed to investigate which normalized core radii $\overline{r}_c$ are large enough such that the simpler Cartesian solutions are sufficiently accurate. However, as Table 4.5 shows, even for axisymmetric simulations, the Cartesian solutions are only marginally worse than the cylindrical solutions for $L_{lk}$ and are actually more accurate for $R$, at least by the $R^2$ metric. If we focus on the $\%RE^s$ metric, recalling that the cylindrical solutions are more accurate with respect to this, then we can look at the metrics of the Cartesian NIC solution for devices in which $\overline{r}_c \geq \overline{r}_{cf}$ for various filter normalized core

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Table 4.7: Metrics for the Cylindrical IC solution for various $\mu_{\text{ref}}$.

<table>
<thead>
<tr>
<th>SRQ</th>
<th>$\mu_{\text{ref}}$</th>
<th>Devices</th>
<th>Points</th>
<th>$R^2$</th>
<th>$%\text{RE}^m$</th>
<th>$%\text{RE}^s$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\overline{R}$ 1</td>
<td>1000</td>
<td>31000</td>
<td>0.698814</td>
<td>$\leq 21.564%$</td>
<td>$\leq 10%$</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>with $P = 72.79%$</td>
<td>with $P = 53.326%$</td>
<td></td>
</tr>
<tr>
<td>$\overline{R}$ 2</td>
<td>663</td>
<td>20553</td>
<td>0.807240</td>
<td>$\leq 14.088%$</td>
<td>$\leq 10%$</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>with $P = 72.5%$</td>
<td>with $P = 63.699%$</td>
<td></td>
</tr>
<tr>
<td>$\overline{R}$ 3</td>
<td>493</td>
<td>15283</td>
<td>0.889457</td>
<td>$\leq 10.254%$</td>
<td>$\leq 10%$</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>with $P = 73.029%$</td>
<td>with $P = 72.244%$</td>
<td></td>
</tr>
<tr>
<td>$\overline{R}$ 4</td>
<td>387</td>
<td>11997</td>
<td>0.952731</td>
<td>$\leq 8.1664%$</td>
<td>$\leq 10%$</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>with $P = 71.785%$</td>
<td>with $P = 78.411%$</td>
<td></td>
</tr>
<tr>
<td>$\overline{R}$ 5</td>
<td>310</td>
<td>9610</td>
<td>0.977839</td>
<td>$\leq 6.1809%$</td>
<td>$\leq 10%$</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>with $P = 68.543%$</td>
<td>with $P = 83.923%$</td>
<td></td>
</tr>
<tr>
<td>$\overline{R}$ 10</td>
<td>173</td>
<td>5363</td>
<td>0.991082</td>
<td>$\leq 3.2429%$</td>
<td>$\leq 10%$</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>with $P = 66.362%$</td>
<td>with $P = 94.723%$</td>
<td></td>
</tr>
<tr>
<td>$\overline{R}$ 20</td>
<td>72</td>
<td>2232</td>
<td>0.986932</td>
<td>$\leq 2.2104%$</td>
<td>$\leq 10%$</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>with $P = 71.64%$</td>
<td>with $P = 96.685%$</td>
<td></td>
</tr>
<tr>
<td>$\overline{L}_{\text{lk}}$ 1</td>
<td>1000</td>
<td>30000</td>
<td>0.635905</td>
<td>$\leq 29.705%$</td>
<td>$\leq 10%$</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>with $P = 72.257%$</td>
<td>with $P = 38.03%$</td>
<td></td>
</tr>
<tr>
<td>$\overline{L}_{\text{lk}}$ 2</td>
<td>663</td>
<td>19890</td>
<td>0.801195</td>
<td>$\leq 18.539%$</td>
<td>$\leq 10%$</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>with $P = 71.493%$</td>
<td>with $P = 51.021%$</td>
<td></td>
</tr>
<tr>
<td>$\overline{L}_{\text{lk}}$ 3</td>
<td>493</td>
<td>14790</td>
<td>0.887088</td>
<td>$\leq 13.474%$</td>
<td>$\leq 10%$</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>with $P = 71.21%$</td>
<td>with $P = 60.92%$</td>
<td></td>
</tr>
<tr>
<td>$\overline{L}_{\text{lk}}$ 4</td>
<td>387</td>
<td>11610</td>
<td>0.932928</td>
<td>$\leq 10.759%$</td>
<td>$\leq 10%$</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>with $P = 71.483%$</td>
<td>with $P = 69.251%$</td>
<td></td>
</tr>
<tr>
<td>$\overline{L}_{\text{lk}}$ 5</td>
<td>310</td>
<td>9300</td>
<td>0.947725</td>
<td>$\leq 8.0316%$</td>
<td>$\leq 10%$</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>with $P = 68.914%$</td>
<td>with $P = 76.86%$</td>
<td></td>
</tr>
<tr>
<td>$\overline{L}_{\text{lk}}$ 10</td>
<td>173</td>
<td>5190</td>
<td>0.981804</td>
<td>$\leq 4.3826%$</td>
<td>$\leq 10%$</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>with $P = 62.659%$</td>
<td>with $P = 92.177%$</td>
<td></td>
</tr>
<tr>
<td>$\overline{L}_{\text{lk}}$ 20</td>
<td>72</td>
<td>2160</td>
<td>0.977017</td>
<td>$\leq 2.8941%$</td>
<td>$\leq 10%$</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>with $P = 73.565%$</td>
<td>with $P = 96.204%$</td>
<td></td>
</tr>
</tbody>
</table>
Table 4.8: Metrics for the Cartesian NIC solution for various \( r_{cf} \). These results are shown in Table 4.8, from which it can be observed that the Cartesian NIC solution surpasses the Cylindrical NIC solution over the whole data set for both \( R \) and \( L_{lk} \) when \( r_c \geq 25 \) by every metric (compare the \( r_{cf} = 25 \) lines of Table 4.8 with the Cylindrical NIC methods in Table 4.5).

### 4.4 Conclusion

Both axisymmetric and Cartesian simulations were run for 1000 randomly-generated transformers using a shorted secondary coil, which is important for the leakage inductance calculations. The accuracies of ideal core and non-ideal core solutions derived in Chapter 3 were assessed for both the Cartesian and cylindrical solutions derived in Chapter 2. The
analytical magnetic field coefficients were derived for the NIC solutions in Cartesian coordinates in order to make the problem solvable analytically, and this was found to limit the accuracy of the resistance and leakage inductance predictions for cylindrical devices of arbitrary core permeability. Finally, for devices with core relative permeabilities $\mu_r \geq 4$ ($\alpha_c \geq 0.6$), the ideal core approximations were found to be sufficiently accurate, limiting the usefulness of the NIC solutions for most practical devices. Similarly, for devices with a normalized core radius of $\bar{r}_c \geq 25$, the Cartesian solutions for the resistance and leakage inductance were found to be sufficiently accurate.
Chapter 5

Qualitative Effects

In this short chapter we study the qualitative effects of various parameters on the normalized resistance and leakage inductance. In particular, we determine whether the resistance or leakage inductance increases or decreases with a certain parameter or whether neither of these is the case and the relationship is not monotonic. Such information is useful for designers in that, in trying to minimize losses or leakage inductance, it is useful to know which parameters could also be minimized or maximized to achieve the desired effect. Unfortunately, showing these relationships rigorously using the equations of Chapter 2 is generally not feasible, so these are shown empirically using both the analytical equation results and using the results of FEA simulations.

5.1 Basic Transformers

The equations derived in previous chapters for losses/resistance and leakage inductance are very general and so involve a multitude of parameters. As just one example, they allow for each layer to have a different thickness. In order to make the number of parameters manageable, we restrict our attention to what we shall call basic transformers. We define a
basic transformer as a pot core transformer that we have been considering thus far but with the following additional restrictions:

1. There is no interleaving and, in particular, all of the primary layers are on the inside adjacent to the core, and all of the secondary layers on the outside.

2. There are an equal number of primary and secondary layers, i.e. \( p_1 = p_2 \).

3. All layers have the same thickness so that \( h_n = h_1 \), and hence \( \bar{h}_n = 1 \), for all layers.

4. The gaps between are defined in terms of an insulation thickness \( t_i \), which is the same for all layers, and that we normalize to the layer thicknesses as \( \bar{t}_i = t_i/h_1 \) as we do for all lengths. The idea is of course that each layer is wrapped in insulation of thickness \( t_i \) on both sides. The normalized gaps between the core and first layer and between the outermost layer and the pot core shell are then \( g_0 = \bar{t}_1 \) and \( g_p = \bar{t}_1 \), respectively, whereas the normalized gaps in between the layers are \( g_n = 2\bar{t}_1 \) for \( 1 \leq n < p \) since each of the two adjacent layers contributes one layer of insulation.

With these conditions we can examine the effects of the following six parameters: \( p_1 = p_2 \), \( \tau_c \), \( \bar{t}_1 \), \( \bar{w} \), \( \mu_r \), and \( \Delta \) on the normalized losses and leakage inductance. Finally, we note that basic transformers are exactly those for which Dowell’s equation is applicable.

### 5.2 FEA Simulations

In order to examine these effects for parameters, 6480 basic transformers were simulated using a shorted secondary coil as before. Each of the six parameters listed above were divided into a small number of linearly spaced samples over certain ranges, though for \( \mu_r \) it is the image current factor \( \alpha_c \) that is linearly spaced rather than \( \mu_r \) itself. This results in the discrete parameter values shown in Table 5.1, noting that all parameters are unitless and that \( \alpha_c \) is also shown. We then take the Cartesian product of the parameter values for a
Table 5.1: Parameter values used for basic transformer FEA simulations.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Values</th>
<th>Number</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p_1 = p_2$</td>
<td>1, 2, 3, 4, 5</td>
<td>5</td>
</tr>
<tr>
<td>$\bar{r}_c$</td>
<td>5, 12, 19, 26, 33, 40</td>
<td>6</td>
</tr>
<tr>
<td>$\bar{t}_i$</td>
<td>0.05, 0.24, 0.43, 0.62, 0.81, 1</td>
<td>6</td>
</tr>
<tr>
<td>$\bar{w}$</td>
<td>20, 32, 44, 56, 68, 80</td>
<td>6</td>
</tr>
<tr>
<td>$\mu_r$</td>
<td>1, 1.4938, 2.3114, 3.9265, 8.6172, 200</td>
<td>6</td>
</tr>
<tr>
<td>$\alpha_c$</td>
<td>0, 0.19801, 0.39602, 0.59403, 0.79204, 0.99005</td>
<td>6</td>
</tr>
<tr>
<td>$\Delta$</td>
<td>0, 2, 4, 6, 8, 10</td>
<td>6</td>
</tr>
</tbody>
</table>

total of 6480 devices that were each simulated at six different values of $\Delta$. This of course results in a total of 38880 distinct points in the design space. Note that the number of sample points for each parameter had to be kept to a fairly low number due to computation time limitations, though this number of points in each parameter is sufficient to determine the qualitative behavior. For the analytical results we use the Cylindrical NIC solution (see Section 4.2.4) as this is theoretically the most accurate. This solution is evaluated at exactly the same 38880 points in the design space as the FEA simulations.

5.3 Determining the Effects

As previously mentioned, for most parameters it is not feasible to show the effect analytically. Together, $\bar{r}_c$ and $\bar{t}_i$ determine the layer radii $\bar{r}_{ni}$ and $\bar{r}_{no}$ and these are deeply entangled inside the complex cylindrical power factor function $F_n$ defined by (2.49), as is $\Delta$. The parameters $\mu_r$ and $\bar{w}$, on the other hand, are deeply entangled in the terms of the iterated series $\bar{H}_{zn}$ given by (3.38). The remaining parameter $p_1 = p_2$ is the only one for which we can easily predict the effect. In particular, we expect increasing $p_1 = p_2$ to increase both the normalized losses/resistance and the leakage inductance. For the former, we have more conducting layers as $p_1 = p_2$ increases so that of course there will be more losses since these additional layers carry current. Regarding the leakage inductance, in-
creasing $p_1 = p_2$ with all else being the same will make the winding window larger so that there is more magnetic field to contribute to the leakage inductance, so we expect this to increase as well.

As such, we determine the effect of each parameter by exclusively using the FEA results and Cylindrical NIC analytical solution evaluated at the same points in the design space. For each parameter we determine the resulting behavior of both $\overline{R}$ and $\overline{L}_{lk}$, which we call the system response quantities (SRQs), throughout the design space as that parameter is varied. Specifically, we look at how the SRQs vary as the parameter is increased with the rest of the parameters held constant. For example, for $\mu_r$, there are 6480 combinations of the other parameters not including $\mu_r$. So we look at how each SRQ behaves as $\mu_r$ increases (where each SRQ is evaluated at the six different values of $\mu_r$) for each of these combinations.

For each combination, either the SRQ values are all equal (i.e. all of the differences between successive SRQ samples are zero), increase monotonically (all of the differences are $\geq 0$), decrease monotonically (all of the differences are $\leq 0$), or are not monotonic (neither of the previous conditions are met) as $\mu_r$ increases. We then only declare that the SRQ increases with $\mu_r$ when the SRQ is either increasing or equal for all of the different combinations, which is to say that $\mu_r$ increases the SRQ over the entire design space, and similarly for the decreasing case. If $\mu_r$ is found to increase for some combinations and decrease for others, or is non-monotonic for any of the combinations, then it is declared to be non-monotonic overall. Lastly, $\mu_r$ must cause the SRQ to increase/decrease for both the FEA results and the analytical results for the SRQ to be declared as increasing/decreasing with $\mu_r$ as a whole.

For each parameter we also quantitatively assess the sensitivity of each SRQ. To explain how this is done, suppose we are assessing a particular parameter $x$, and let $x_i$ be the $N$ samples of that parameter in increasing order. For example, if $x$ represents $\mu_r$ as before, then each $x_i$ has the values listed in Table 5.1 for $\mu_r$ where of course $1 \leq i \leq N = 6$.
For each combination of the other parameters suppose that \( y_i \) are the six SRQ values corresponding to each sample point \( x_i \). For all parameters, the parameter values \( x_i \) are normalized to independent values \( \bar{x}_i \) so that \( \bar{x}_1 = 0 \) and \( \bar{x}_N = 1 \), which is to say that

\[
\bar{x}_i = \frac{x_i - x_1}{x_N - x_1}
\] (5.1)

for \( 1 \leq i \leq N \). Since the different parameters can have wildly different ranges in our design space (e.g. the parameter \( \bar{t}_i \) ranges only from 0.05 to 1.0 but \( \mu_r \) ranges from 1 to 200), this ensures that the sensitivities can be directly compared.

We then perform a linear regression on the independent values \( \bar{x}_i \) and dependent values \( y_i \) and take the absolute value of the slope of the regression line as the sensitivity for that combination of other parameters so that the sensitivity is always non-negative. The sensitivities for each combination of other parameters are then averaged to get an overall sensitivity, and these averages are then averaged between the FEA results and the Cylindrical NIC analytical result to get the overall sensitivity for the SRQ with respect to the given parameter \( x \). Note that these sensitivities are unitless since the SRQs are normalized and therefore unitless themselves.

Lastly, we note that, for both the qualitative effect and the sensitivities, we do not include parameter combinations where \( \Delta = 0 \) for parameters of interest other than \( \Delta \), though we do include it when looking at \( \Delta \) itself. This is because \( \bar{R} = 1 \) by definition when \( \Delta = 0 \) so this contains no useful information for parameters other than \( \Delta \) since, for any combination where \( \Delta = 0 \), \( \bar{R} \) will be constant for all values of the parameter of interest. We also do not include \( \Delta = 0 \) when evaluating any of the leakage inductance effects and sensitivities, including when assessing the effect of \( \Delta \). This is because the leakage inductance is anomalous in the FEA results due to the truly shorted secondary coil, and we note that \( \overline{T_{lk}} \) should certainly be settled from this anomalous value by the next sample value of \( \Delta = 2 \) as indicated in the plots of Figure 4.3 through Figure 4.5.
Table 5.2: Effects of each parameter on $R$ and $L_{lk}$.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>SRQ</th>
<th>Trend</th>
<th>Sensitivity</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p_1 = p_2$</td>
<td>$\bar{R}$</td>
<td>Increasing</td>
<td>71.662</td>
</tr>
<tr>
<td></td>
<td>$\bar{L}_{lk}$</td>
<td>Increasing</td>
<td>3.0866</td>
</tr>
<tr>
<td>$\bar{r}_c$</td>
<td>$\bar{R}$</td>
<td>Non-monotonic</td>
<td>8.8672</td>
</tr>
<tr>
<td></td>
<td>$\bar{L}_{lk}$</td>
<td>Non-monotonic</td>
<td>0.32404</td>
</tr>
<tr>
<td>$\bar{t}_i$</td>
<td>$\bar{R}$</td>
<td>Non-monotonic</td>
<td>9.9216</td>
</tr>
<tr>
<td></td>
<td>$\bar{L}_{lk}$</td>
<td>Non-monotonic</td>
<td>0.84429</td>
</tr>
<tr>
<td>$\bar{w}$</td>
<td>$\bar{R}$</td>
<td>Non-monotonic</td>
<td>6.1284</td>
</tr>
<tr>
<td></td>
<td>$\bar{L}_{lk}$</td>
<td>Non-monotonic</td>
<td>0.31291</td>
</tr>
<tr>
<td>$\mu_r$</td>
<td>$\bar{R}$</td>
<td>Non-monotonic</td>
<td>16.397</td>
</tr>
<tr>
<td></td>
<td>$\bar{L}_{lk}$</td>
<td>Increasing</td>
<td>0.80444</td>
</tr>
<tr>
<td>$\Delta$</td>
<td>$\bar{R}$</td>
<td>Increasing</td>
<td>61.699</td>
</tr>
<tr>
<td></td>
<td>$\bar{L}_{lk}$</td>
<td>Decreasing</td>
<td>0.68317</td>
</tr>
</tbody>
</table>

Additionally, the leakage inductance cannot be calculated analytically at $\Delta = 0$. Refer to Section 4.3.1 for a discussion of these issues.

5.4 Effects Results

Table 5.2 shows the results of this process over the design space we have been considering. We observe that the only purely monotonic relationships are for parameters $p_1 = p_2$, $\mu_r$, and $\Delta$. As was argued before, increasing $p_1 = p_2$ increases both SRQs. Increasing $\mu_r$ also increases $\bar{L}_{lk}$, which was not entirely expected, though its effect on $\bar{R}$ is generally non-monotonic. Regarding $\Delta$, this increases $\bar{R}$, which is unsurprising given the plots of Figure 4.3 and Figure 4.4. It is also a well-known phenomenon that, as the frequency increases, the losses relative to dc get worse due to the skin and proximity effects becoming more severe. Increasing $\Delta$ was also found to monotonically decrease $\bar{L}_{lk}$, which is also a result that is well-reported in the literature and is also unsurprising given Figure 4.3 and Figure 4.4. Unfortunately, the remaining parameters $r_c$, $t_i$, and $w$ do not exhibit monotonic
Regarding the sensitivities, first note that the normalized leakage inductance generally does not vary as much as the normalized resistance in terms of magnitude. As a result, the sensitivities for $\bar{R}$ will generally be larger than for $\bar{L}_{lk}$, and they cannot really be directly compared. Referring again to Table 5.2, it can be observed that, for $\bar{R}$, $p_1 = p_2$ and $\Delta$ by far have the largest impacts and the remaining parameters $\tau_c$, $\bar{t}_i$, $\bar{w}$, and $\mu_r$ have much more modest impacts. This is not terribly surprising given the meanings of the parameters. In contrast, for $\bar{L}_{lk}$, $p_1 = p_2$ is again the largest driver by far, followed by $\bar{t}_i$ and $\mu_r$, and then the remaining parameters $\Delta$, $\tau_c$, and $\bar{w}$. It is particularly interesting to note that $\bar{L}_{lk}$ is more sensitive to changes in $\bar{t}_i$ and $\mu_r$ than to changes in $\Delta$.

### 5.5 Conclusion

In this chapter, the effect of the parameters $p_1 = p_2$, $\tau_c$, $\bar{t}_i$, $\bar{w}$, $\mu_r$, and $\Delta$ and their impacts on the normalized resistance $\bar{R}$ and normalized leakage inductance $\bar{L}_{lk}$ were assessed for basic transformers. This was determined using both FEA simulation results and the Cylindrical NIC analytical solution over 38880 distinct points in the design space. It was found that both $\bar{R}$ and $\bar{L}_{lk}$ increase monotonically with $p_1 = p_2$, whereas increasing $\Delta$ increases $\bar{R}$ but decreases $\bar{L}_{lk}$. Similarly, increasing $\mu_r$ always increases the leakage inductance but neither always increases nor decreases the resistance. The other parameters $\tau_c$, $\bar{t}_i$, and $\bar{w}$ neither increase or decrease $\bar{R}$ or $\bar{L}_{lk}$ over the entire design space. Both SRQs are most sensitive to changes in the number of layers $p_1 = p_2$, whereas $\bar{R}$ is also very sensitive to $\Delta$ but $\bar{L}_{lk}$ is more sensitive to changes in $\bar{t}_i$ and $\mu_r$ than to $\Delta$. 
Chapter 6

Conclusions

In this chapter we summarize what was done in the preceding chapters and present the major results therefrom. Refer to Figure 2.1 for the coordinate systems axes referenced in what follows. We then discuss the specific contributions of this dissertation as well as some recommendations for possible future research.

6.1 Summary

First, in Chapter 2, very general equations for the normalized resistance/losses and primary leakage inductance were derived in both Cartesian and cylindrical coordinates. These results along with the underlying assumptions for the analysis are presented below:

Assumptions/Notes:

1. Windings are treated as foils with rectangular cross sections. This was done for simplicity and there are well established methods in the literature for transforming non-foil windings into foil windings with an equivalent resistance.

2. The winding width $w$ is assumed to be much longer than its thickness $h$ so that the magnetic field throughout is entirely parallel to the layer surfaces, i.e. has
only a $z$ component. This means that edge effects are neglected.

3. A pot core is assumed and the winding fills up the entire window space in the $z$ direction. This also reinforces the magnetic field having only a $z$ component.

4. The magnetic field is assumed to be uniform in the gaps between layers.

5. The electric field is assumed to be strictly in the $\phi$ direction along with current. This means that capacitive effects are not modeled.

6. The primary and secondary currents are purely sinusoidal.

7. The above assumptions permit a 1-D and time-harmonic analysis.

**Parameters:**

\[
\begin{align*}
\bar{g}_0 & = \text{Normalized gap between the core and first layer} \\
\bar{g}_n & = \text{Normalized gap between the } n \text{th and } (n+1)\text{th layer} \\
\bar{h}_n & = \text{Normalized thickness of the } n \text{th layer} \\
\bar{\ell}_n & = \text{Normalized turn length for the } n \text{th layer in Cartesian coordinates} \\
M_1 & = \text{Set of integer layer numbers for the primary layers} \\
   & n = \text{Layer number with 1 being the innermost layer, counting outwards} \\
   & \quad \text{regardless of interleaving} \\
N_n & = \text{Number of turns in the } n \text{th layer} \\
p & = \text{Total number of primary and secondary layers} \\
\bar{r}_m, \bar{r}_o & = \text{Normalized inner and outer radii of the } n \text{th layer in cylindrical coordinates} \\
\alpha_{mi}, \alpha_{mo} & = \text{Inner and outer magnetic field coefficients of the } n \text{th layer} \\
\delta_w & = \text{Skin depth} \\
\Delta & = \text{Penetration ratio of the first layer } \Delta = h_1/\delta_w \\
\kappa & = \text{Simple complex factor } 1 + j
\end{align*}
\]

**Results:**

For the cylindrical solutions, the so-called kernel functions of order zero and one, respectively, are

\[
\begin{align*}
\Psi_0(x, y) &= I_0(x)K_0(y) - I_0(y)K_0(x) \\
\Psi_1(x, y) &= I_0(x)K_1(y) + I_1(y)K_0(x)
\end{align*}
\]

where $I_\alpha$ and $K_\alpha$ are the modified Bessel functions of order $\alpha$ and of the first and second kinds, respectively.
The Cartesian and cylindrical ac complex power factors are, respectively,

\[ G_n = \kappa \left[ \frac{|\alpha_n|^2 + |\alpha_{no}|^2}{\tanh(\kappa \Delta \bar{T}_n)} - \frac{\alpha_n^* \alpha_{no} + \alpha_n \alpha_{no}^*}{\sinh(\kappa \Delta \bar{T}_n)} \right] \]

\[ F_n = \frac{\kappa}{\Psi_0(\kappa \Delta \bar{T}_{no}, \kappa \Delta \bar{T}_{ni})} \left\{ \bar{T}_{ni} |\alpha_n|^2 \Psi_1(\kappa \Delta \bar{T}_{no}, \kappa \Delta \bar{T}_{ni}) + \bar{T}_{no} |\alpha_{no}|^2 \Psi_1(\kappa \Delta \bar{T}_{ni}, \kappa \Delta \bar{T}_{no}) \right. \]

\[ - \bar{T}_{ni} \alpha_n^* \alpha_{no} \Psi_1(\kappa \Delta \bar{T}_{ni}, \kappa \Delta \bar{T}_{ni}) - \bar{T}_{no} \alpha_n \alpha_{no}^* \Psi_1(\kappa \Delta \bar{T}_{ni}, \kappa \Delta \bar{T}_{no}) \} . \]

The normalized resistance/losses in Cartesian and cylindrical coordinates are

\[ R_{\text{cart}} = \frac{\Delta R \left\{ \sum_{n \in M1} \bar{I}_n G_n \right\}}{\sum_{n \in M1} \bar{I}_n N_n^2 / \bar{h}_n} \]

\[ R_{\text{cyl}} = \frac{\Delta R \left\{ \sum_{n \in M1} F_n \right\}}{\sum_{n \in M1} N_n^2 \left( \frac{r_n}{\bar{h}_n} + \frac{1}{2} \right)} . \]

The normalized primary leakage inductance in Cartesian and cylindrical coordinates are

\[ L_{\text{lk(cart)}} = \sum_{n=0}^{P} \bar{I}_n \left[ 3 \left\{ \frac{G_n}{\Delta} \right\} + 2 |\alpha_{no}|^2 \bar{g}_n \right] \]

\[ 2 \sum_{n=0}^{P} \bar{I}_n \left( \bar{h}_n + \bar{g}_n \right) \]

\[ L_{\text{lk(cyl)}} = \sum_{n=0}^{P} \left[ 3 \left\{ \frac{F_n}{\Delta} \right\} + |\alpha_{no}|^2 \bar{g}_n \left( 2 \bar{T}_{ni} + 2 \bar{h}_n + \bar{g}_n \right) \right] \]

\[ \sum_{n=0}^{P} \left( \bar{h}_n + \bar{g}_n \right) \left( 2 \bar{T}_{ni} + \bar{h}_n + \bar{g}_n \right) . \]

In Chapter 3 we determined the magnetic field coefficients for both an ideal core approximation as well as for a core of arbitrary permeability. While the former can be done with a 1-D model, the latter required the use of full 2-D model. Due to using the method of images in two dimensions, the result is an double iterated series. The results are again summarized below:

**Assumptions/Notes:**

1. Windings are treated as foils with rectangular cross sections.
2. A pot core is assumed and the winding fills up the entire window space in the \( z \) direction.
3. Both analyses assume Cartesian coordinates, which is required for analytical solutions.

4. The ideal core is a 1-D analysis but may be time-harmonic.

5. The non-ideal core analysis is 2-D and is done at dc, which is required so that the current density within the layers is uniform. It is understood that the magnetic field coefficients do not change significantly with frequency.

6. For the non-ideal core solutions, we neglect the $x$ component of the magnetic fields and take only the $z$ component to match the analyses of Chapter 2.

7. For the non-ideal core solutions, the magnetic fields are calculated in the very center of the layer gaps in both dimensions to determine the magnetic field coefficients.

---

**Parameters:**

- $\overline{g}_0$ = Normalized gap between the core and first layer
- $\overline{g}_n$ = Normalized gap between the $n$th and $(n+1)$th layer
- $\overline{r}_n$ = Normalized thickness of the $n$th layer
- $I_l, I_n$ = Primary or secondary current for $l \in \{1, 2\}$, or current in the $n$th layer
- $\overline{I}_l, \overline{I}_n$ = Current normalized to the primary current so that $\overline{I}_1 = 1$
- $M_1, M_2$ = Set of integer layer numbers for the primary or secondary winding, respectively
- $N_n$ = Number of turns in the $n$th layer
- $\overline{\hat{r}}_n$ = Normalized inner radius of the $n$th layer relative to the origin in Figure 3.6
- $\overline{w}$ = Normalized winding width in the $z$ dimension
- $\overline{x}, \overline{z}$ = Normalized position relative to $(x_0, z_0)$ in Figure 3.5
- $\alpha_c$ = Image current factor $\alpha_c = (\mu_t - 1)/(\mu_t + 1)$
- $\alpha_{n_i}, \alpha_{n_o}$ = Inner and outer magnetic field coefficients of the $n$th layer
- $\mu_t$ = Core relative permeability

---

**Results:**

For an ideal core approximation, the secondary current can be found as

$$I_2 = -\frac{\sum_{n \in M_1} N_n}{\sum_{n \in M_2} N_n} I_1$$
and the magnetic field coefficients are

\[ \alpha_{n_o} = - \sum_{k=1}^{n} N_k I_k \]

\[ \alpha_{n_i} = \begin{cases} 
0 & \text{if } n = 1 \\
\alpha_{(n-1)o} & \text{otherwise} 
\end{cases} \]

The non-ideal core solutions are substantially more complicated. First, we have the base normalized magnetic field from a single rectangular layer:

\[ \overline{H}_{zn0}(\bar{x}, \bar{z}) = \overline{H}_{zn}(\bar{x}, \bar{z} + \frac{w}{2}) - \overline{H}_{zn}(\bar{x}, \bar{z} - \frac{w}{2}) \]

where

\[ \overline{H}_{zn}(\bar{x}, \bar{z}) = \frac{N_n}{4\pi h_n} \left[ \bar{z} \ln \left( 1 + \frac{\bar{r}_n^2 - 2\pi h_n}{\bar{x}^2 + \bar{z}^2} \right) + 2(\bar{x} - \bar{r}_n) \arctan \left( \frac{\bar{z}}{\bar{x} - \bar{r}_n} \right) - 2\bar{x} \arctan \left( \frac{\bar{z}}{\bar{x}} \right) \right] . \]

Then there is the normalized magnetic field from a single layer and all of its images:

\[ \overline{H}_{zn}(\bar{x}, 0) = \sum_{k=0}^{\infty} \left( \sum_{m=0}^{\infty} T_{k,m} \right) \]

where the terms are

\[ T_{k,m} = 2\lambda_k \lambda_m \left[ \alpha_c^{\xi(2k,m)}(\overline{A}_{k,m} + \overline{A}_{-k,m}) + \alpha_c^{\xi(2k+1,m)}\overline{B}_{k,m} + \alpha_c^{\xi(2k-1,m)}\overline{B}_{-k,m} \right] \]

and

\[ \lambda_m = \begin{cases} 
\frac{1}{2} & \text{if } m = 0 \\
1 & \text{otherwise} 
\end{cases} \]

\[ \overline{A}_{k,m} = \overline{H}_{zn0} \left( \bar{x} - 2k\bar{b} - \bar{r}_n, \bar{z} - m\bar{w} \right) \]

\[ \overline{B}_{k,m} = \overline{H}_{zn0} \left( \bar{x} - 2(k+1)\bar{b} + \bar{r}_n + \bar{r}_n, \bar{z} - m\bar{w} \right) \]

for brevity. In Section 4.2.3 it was shown that this iterated series can be safely truncated for computational purposes to \( N \) terms for both sums as

\[ S_T = \sum_{k=0}^{N-1} \left( \sum_{m=0}^{N-1} T_{k,m} \right) . \]
Terms are added until
\[ \frac{E}{|S_T|} \leq \frac{1}{100}, \]
where the truncation error bound is
\[ E = \frac{6(\pi + 1)N_n \sqrt{\alpha_c}^N (2 - \sqrt{\alpha_c}^N)}{\pi (1 - \sqrt{\alpha_c})^2}. \]

In order to determine the secondary current, we first have the partial fields at the core and outside the device, respectively,
\[ \gamma_{c,l} = \sum_{n \in M_l} \gamma_n (g_0/2, 0) \]
\[ \gamma_{t,l} = \sum_{n \in M_l} \gamma_n (\hat{r} + \hat{h} + \hat{g}/2, 0), \]

We also have the sum of numbers layers
\[ s_l = \sum_{n \in M_l} N_n. \]

In the preceding three equations we let \( l \in \{1, 2\} \) so that we sum over the primary or secondary layers. The secondary current can then be calculated as
\[ I_2 = -\frac{s_1 + \gamma_{t,1} - \gamma_{c,1}}{s_2 + \gamma_{t,2} - \gamma_{c,2}}. \]

Lastly, we have the total normalized magnetic field from all layers and their images:
\[ H_z (\hat{x}, \hat{z}) = \sum_{n=1}^p I_n H_{zn} (\hat{x}, \hat{z}). \]

From this we can finally calculate the magnetic field coefficients as
\[ \alpha_{n_o} = H_z (\hat{r} + \hat{h} + \hat{g}/2, 0) \]
\[ \alpha_{n_i} = \begin{cases} H_z (g_0/2, 0) & \text{if } n = 1 \\ \alpha_{(n-1)o} & \text{otherwise}. \end{cases} \]

In Chapter 4 the accuracy of the purely analytical solutions is assessed by comparing their results with those of FEA. A total of 1000 devices were simulated, each with certain
randomly generated parameters. In particular, each device has a randomly assigned \( \mu_r \), \( \mu_\tau \), \( \mu_g \), \( \mu_p \), and \( \mu_\sigma \) (see the parameter list below). Additionally, each device features a random interleaving pattern. Moreover, each layer of each device was assigned a random \( \bar{h}_n \) (with the first layer always having \( \bar{h}_1 = 1 \)), \( \bar{g}_n \), and \( \bar{N}_n \) (again, see the parameter list). The normalized resistance/losses and leakage inductance were both extracted from the FEA results, and were compared with the corresponding SRQs from various analytical solutions. Namely the four combinations of Cartesian or cylindrical coordinates with an ideal or non-ideal core (labeled Cartesian/Cylindrical IC/NIC). Again, the results of this analysis are summarized below:

---

**Assumptions/Notes:**
All of the assumptions of the previous chapters apply to the analytical results. For FEA simulations we have the following assumptions:

1. The geometry was set up as a 2-D axisymmetric problem.

2. As it is not theoretically important, all simulated devices had a first-layer thickness of \( h_1 = 1 \) mm, which was chosen arbitrarily.

3. All devices simulated had a pot core, with a somewhat arbitrary shell thickness of \( t_c = 10h_1 = 10 \) mm on the top, bottom, and outside.

4. The solver assumes that magnetic fields are entirely in the 2-D \( r-z \) plane and the electric fields are entirely in the \( \phi \) direction along with the current. As a result, capacitive effects are not included in the simulations.

5. Simulations are time-harmonic so that quantities are generally complex phasors.

6. Finer meshes were used in areas where the quantities are expected to change significantly as a function of spatial position, such as the insides of the layers. Coarser meshes were used in areas where the quantities are expected to be relatively constant, such as the gaps between layers. This is nicely illustrated in Figure 4.2.

7. The mesh size in the layers was never larger than one third of the skin depth.

8. Overall mesh size was controlled with a single parameter, that was calibrated for a simplistic problem with a known analytical solution. See Section 4.1 for more details.
9. As required to correctly calculate leakage inductance, the simulations were run with a truly shorted secondary coil.

10. Three different metrics were used to compare the result across so many devices. See Appendix C for details on these.

**Parameters:**

- $\bar{g}_0$ = Normalized gap between the core and first layer
- $\bar{g}_n$ = Normalized gap between the $n$th and $(n + 1)$th layer
- $\bar{h}_n$ = Normalized thickness of the $n$th layer
- $N_n$ = Number of turns in the $n$th layer
- $p_1, p_2$ = Number of primary and secondary layers, respectively
- $r_c$ = Normalized core radius
- $\bar{w}$ = Normalized winding width in the $z$ dimension
- $\Delta$ = Penetration ratio of the first layer $\Delta = h_1/\delta_w$
- $\mu_r$ = Core relative permeability

**Results:**

In comparing the analytical results with those from the FEA we drew the following conclusions:

1. The analytical solutions always assume coupling between the primary and secondary coils. However, at dc ($\Delta = 0$) the primary current is constant and so the linking flux is also constant. As the flux is unchanging, no voltage or current is induced in the secondary coil. The simulations were run with a shorted secondary rather than simply setting the secondary current to that predicted by an ideal transformer. As a result of this, the simulation results for leakage inductance show an apparent anomaly at dc that deviates from the analytical results. It was found that this quickly stabilizes to closely match the analytical results somewhere when $0 < \Delta < 0.5$.

2. The NIC analytical solutions better matched the simulated results for both $\bar{R}$ and $\bar{L}_{lk}$ by every metric.

3. For $\bar{R}$ the Cartesian solutions better matched the simulation results for the $R^2$ metric but the cylindrical solutions were a better match by the $\%RE^a$ metric, noting that it is difficult to make direct comparisons with the $\%RE^m$ metric.

4. For $\bar{L}_{lk}$ the cylindrical solutions were a better match for all metrics. This suggests that the winding curvature has a stronger effect on the leakage inductance than on the resistance/losses.
5. Evidence suggests that the use of Cartesian coordinates in the derivation of the magnetic field coefficients for a non-ideal core is the primary cause of differences between the theoretical coefficients and those extracted from the simulations. The fact that dc was used in determining the magnetic field coefficients, on the other hand, does not play as much of a role.

6. Evidence suggests that the resistance/loss and leakage inductance equations are quite accurate when the magnetic field coefficients can be accurately determined.

7. For about $\mu_r \geq 4$, the IC solutions were found to be as good as the NIC solutions for both $\overline{R}$ and $\overline{L_{lk}}$. This suggests that the NIC solutions may be of limited practical utility except for air-core devices.

8. For about $\tau_e \geq 25$, the Cartesian solutions were found to be as good as the cylindrical solutions for both $\overline{R}$ and $\overline{L_{lk}}$.

Lastly, Chapter 5 examines the qualitative effect of parameters on the resistance/losses and leakage inductance, where attention is restricted to so-called basic transformers to keep the number of independent parameters manageable. In particular, for each parameter, we determine whether increasing it causes $\overline{R}$ and/or $\overline{L_{lk}}$ to increase or decrease monotonically for all combinations of other parameters, or whether the relationship is non-monotonic. We also determine which parameters have the most significant effects on $\overline{R}$ and $\overline{L_{lk}}$. The results are summarized below:

**Assumptions/Notes:**

1. Basic transformers have all the assumptions for the pot core devices that have been considered hitherto, with the following additional restrictions:
   
   (a) There is no interleaving and all of the primary layers are on the inside, and all of the secondary layers on the outside.
   
   (b) There are an equal number of primary and secondary layers, i.e. $p_1 = p_2$.
   
   (c) All layers have the same thickness so that $\overline{h}_n = 1$ for all layers.
   
   (d) The gaps between layers are defined in terms of a normalized insulation thickness $\overline{t}_i$, which is the same for all layers. The normalized layer gaps in terms of this are $\overline{g}_0 = \overline{t}_i$, $\overline{g}_p = \overline{t}_i$, and $\overline{g}_n = 2\overline{t}_i$ for $1 \leq n < p$. 
Hence a point in the design space is defined in terms of the following six parameters: \( p_1 = p_2, \tau_c, \bar{t}_i, w, \mu_r, \) and \( \Delta. \)

2. Due to the complexity of the equations involved, it is generally not feasible to determine the effects of the parameters analytically, which would involve taking derivatives with respect to each parameter.

3. Only the Cylindrical NIC equation is used for the analytical results as this is theoretically the most accurate.

4. The trends are shown empirically using both analytical equation results and simulation results, evaluated/simulated at 38880 points in the design space, taken as the Cartesian product of linearly spaced points for each parameter. For the values used for each parameter, refer to Table 5.1.

5. A parameter is declared as having a particular effect (increasing or decreasing) on an SRQ only if it does so across the entire design space for both the analytical and FEA results, otherwise it is declared non-monotonic.

6. The sensitivity of an SRQ to a parameter is assessed as an average over the design space and over the analytical and FEA results. See Section 5.3 for more detail on how the sensitivity was quantitatively determined.

7. For \( \bar{R} \), design space points at dc (\( \Delta = 0 \)) are excluded from the analyses for parameters other than \( \Delta \). This is because \( \bar{R} = 1 \) for \( \Delta = 0 \) by definition and so will always be constant for all values of the other parameters.

8. For \( \bar{T}_{lk} \), design space points at dc (\( \Delta = 0 \)) are excluded from the analyses for all parameters because of the anomalous values in the simulated results, and because the analytical result cannot be evaluated here.

---

**Parameters:**

\[
\begin{align*}
\bar{g}_0 &= \text{Normalized gap between the core and first layer} \\
\bar{g}_n &= \text{Normalized gap between the } n\text{th and } (n+1)\text{th layer} \\
\bar{h}_n &= \text{Normalized thickness of the } n\text{th layer} \\
p &= \text{Total number of primary and secondary layers } p = p_1 + p_2 \\
p_1, p_2 &= \text{Number of primary and secondary layers, respectively} \\
\tau_c &= \text{Normalized core radius} \\
\bar{t}_i &= \text{Normalized insulation thickness for all layers} \\
\bar{w} &= \text{Normalized winding width in the } z \text{ dimension} \\
\Delta &= \text{Penetration ratio of the first layer } \Delta = h_1/\delta_w \\
\mu_r &= \text{Core relative permeability}
\end{align*}
\]
Results:

1. Increasing $p_1 = p_2$ was found to monotonically increase both $\overline{R}$ and $\overline{L}_{lk}$.

2. Increasing $\mu_r$ was found to monotonically increase $\overline{L}_{lk}$ but neither always increase nor decrease $\overline{R}$.

3. Increasing $\Delta$ was found to monotonically increase $\overline{R}$ and monotonically decrease $\overline{L}_{lk}$.

4. The remaining basic transformer parameters $\overline{r}_c$, $\overline{t}_1$, and $\overline{w}$ were found not to have consistent monotonic relationships with either $\overline{R}$ or $\overline{L}_{lk}$ across the design space.

5. Both SRQs were found to be the most sensitive to changes in $p_1 = p_2$.

6. $\overline{R}$ was also found to be very sensitive to changes in $\Delta$, whereas it much more modestly sensitive to changes in $\overline{r}_c$, $\overline{t}_1$, $\overline{w}$, and $\mu_r$.

7. $\overline{L}_{lk}$ was found to be more sensitive to changes in $\overline{t}_1$ and $\mu_r$ than to changes in $\Delta$, and is even less sensitive to changes in $\overline{r}_c$ and $\overline{w}$.

6.2 Contributions

Refer to Section 1.1 and Section 1.2 for the literature review that discusses the current body of knowledge in this topic area. The specific contributions of this research to this body of knowledge given this context are the following:

1. New equations were derived and presented for both the winding resistance/losses and leakage inductance for pot core transformers operating at high frequencies typically found in switching dc-dc converters. Previously, equations had only been derived using Cartesian coordinates, but this research also produced equations derived in cylindrical coordinates to take winding curvature into account. In both cases, these equations are more general than what has been presented in the past as they allow for each layer to have a different thickness, and for variable gaps between layers. Additionally, they are agnostic to any interleaving pattern.
Aside from the frequency and geometric parameters, the primary input to these equations are the magnetic field coefficients, which effectively decouple these equations from the effects of core permeability, winding width, and the interleaving pattern. Moreover, these coefficients can be determined through any means such as FEA simulations, semi-empirical models, or analytical results. In addition to adding to the overall theoretical understanding of the problem, these allow designers to more accurately predict the resistance/losses and leakage inductance, especially for devices with small core radii, low core permeability, and hitherto unstudied interleaving patterns.

2. Equations were derived to calculate the magnetic field coefficients for a core of arbitrary permeability. This is in contrast with most previous works in which an ideal core with infinite permeability was assumed, adding to the overall theoretical understanding of the problem. As discussed in the introduction to Chapter 3, a similar approach was taken by Lambert et al. in [37]. It is worth discussing the specific contributions of this work relative to this:

(a) We formally proved the convergence of the iterated series in Appendix B.

(b) Whereas truncation of the series is discussed only empirically for a particular device in [37], we showed how it can be safely truncated in general in Section 4.2.3.

(c) The way in which we have decoupled the loss and leakage inductance equations from the core effects permits our results to be easily applied to the losses as well, and will also allow our final results to take the skin and proximity effects into account for both the losses and leakage inductance. In contrast, [37] only derives the leakage inductance, and does not take high-frequency effects into account.

3. In support of the previous contributions, the accuracy of the derived equations rel-
ative to FEA simulations was assessed. This demonstrated that the equations are correct, which provides confidence to future researchers or designers who wish to use them. Also assessed were the conditions under which the Cartesian and ideal core approximations show comparable accuracy to the full cylindrical and non-ideal core equations. These allow designers to potentially utilize much simpler equations (especially in the case of an ideal core) where applicable.

4. For a special class of transformers, so-called basic transformers, the effects of their defining parameters on the resistance/losses and leakage inductance was also determined. Additionally, the sensitivities to the parameters on these was also assessed. Though not all showed monotonic behavior, this information is useful for designers to be able to optimize their designs by knowing which parameters can be maximized or minimized (if requirements allow) to minimize the losses and/or leakage inductance.

In particular, the effects of the core radius $r_c$, insulation thickness $t_i$, winding width $w$, and core permeability $\mu_r$ were assessed, which had not previously been examined to the author’s knowledge.

### 6.3 Recommendations for Future Research

Throughout the course of this research, numerous ideas for additional research in this area were conceived. First, it was discussed in Section 1.1.7 that several different semi-empirical models have been developed to predict the losses, which enjoy the advantage of generally being more accurate than simpler purely analytical equations but without the computational expense of full FEA simulations. However, to the knowledge of the author, no direct comparison of these models has been conducted to determine which is the most accurate under various conditions. There is also a dearth of analogous models for the leakage inductance. Additionally, perhaps with the aid of the equations derived herein, these models
could be adapted to include additional parameters that we have taken into account in the analytical results here, viz. the core radius, core permeability, and/or winding width. A semi-empirical model could also potentially be developed for the magnetic field coefficients, taking these into account, rather than for the losses or leakage inductance directly.

In the derivation of the non-ideal core equations for the magnetic field coefficients, the first major assumption was that the system was at dc in order for the layer current density to be uniform and render the magnetic field solution feasible. Perhaps this assumption could be dispensed with and it could be attempted to solve for the fields caused by a layer current for any ac frequency. An anticipated challenge with this, however, would be that, at least with the present approach to analytical solutions, the current distributions inside the layers depend on the magnetic fields at the inner and outer surfaces. At the same time these fields depend on the current distribution at ac, and so this sort of feedback loop would have to be overcome somehow. Perhaps an iterative numerical method could be employed to tackle this.

The second assumption in this derivation was that we were forced to use Cartesian coordinates, noting that the work done in Chapter 4 suggests that this is the more impactful of the two assumptions. The problem with attempting the problem in cylindrical coordinates was that the analogous image current placements and magnitudes could not be found despite research into the problem. However, this does not mean that this problem has not been solved somewhere or that a solution could not be derived, noting that this falls firmly into the realm of pure physics.

Another topic that was left totally unexplored in this work is the use of a more sophisticated model for the core magnetic material and how this affects the winding losses. In the fully general case this would include lossy material, anisotropic materials, and situations in which the frequency is near resonance and the permeability can no longer be treated as constant with respect to frequency. The latter of these could be treated by either assuming a known resonant frequency and permeability response or by predicting this based on the
shape of the core. For a comprehensive treatment of these properties of magnetic materials, refer to [25].

One final idea regarding the calculation of the magnetic field coefficients would be to explore the effect of the pot core shell thickness on the accuracy of the magnetic field coefficients. In particular, it would be useful to know how thin the shell can be before the magnetic field coefficients generally become unacceptably inaccurate. This could potentially be confounded by other factors as well such as the winding width or window breadth, and it would be interesting to see what the critical quantity would be here. For example, it could be the shell thickness relative to the layer thicknesses, relative to the winding width, perhaps both, or something else entirely.

An unrelated and general avenue of potential research would be to bring machine learning techniques to bear on the problem of predicting the losses, leakage inductance, and/or the magnetic field coefficients, which has not been done to the author’s knowledge. One potential challenge with this include the potentially large number of independent parameters depending on how general the class of considered devices is. Another challenge is generating a sufficiently large number of FEA simulation results or experimental results in a reasonable amount of time in order to train a network. One approach might be to initially develop and train a network using computationally inexpensive analytical results as a sort of proof of concept, and then use results from FEA simulations or even experiments later.

Finally, in [3], it is mentioned that, for certain applications, it is desirable to tune the leakage inductance to a specific value. An interesting research problem, and one in which the research presented herein would likely be useful, is to determine how to optimize certain transformer parameters to achieve a desired leakage inductance. Given the complexity of the equations involved, an analytical solutions seems infeasible, at least ostensibly. However, codes could be developed to restrict certain parameters as required, and find an optimum set of values for the remaining parameters to achieve a leakage inductance as close to the desired value as possible. Obviously, design optimization techniques could
be used to solve this problem. Of particular interest could be fixing all other parameters and determining the optimum interleaving configuration as this does not affect the overall voltage and current transfer between the primary and secondary coils, at least not for an ideal transformer.
Appendix A

Derivation of Infinite Image Currents

Similar to what was done in [1], we can use the successive method of images to determine the infinite image current layout needed to calculate the magnetic field in the winding window of a pot core transformer. Forget for a moment about transformer windings and suppose we have an infinite current filament in free space carrying current $I$ and centered between two infinite regions of finite permeability $\mu_r$, as shown in Figure A.1. Note that the distance between the filament and the interfaces is $d$ and that the interfaces are labeled A and B. A current filament in free space has a well-known 2-D solution for the magnetic field intensity as a function of space, though this solution is not important for this discussion.

Now, recall that the purpose of the method of images is to ensure that the correct magnetic boundary conditions are satisfied at the interfaces between media, which then ensures that the field is correct in the region of interest by the uniqueness theorem. This

![Figure A.1: The problem for which we want to find the image currents.](image)

Figure A.1: The problem for which we want to find the image currents.
generally requires different sets of image currents depending on where in the space one is interested in finding the magnetic field. In our case we are only interested in the field in the free space region containing the original current, as this of course represents our winding window. Hence our task is to find the set of images such that the boundary conditions are satisfied at both interfaces, and we will successively add image currents until this is accomplished.

We first concern ourselves with interface A and ignore interface B for the moment. By the method of images, Figure A.2a shows the single image current needed such that the boundary conditions are satisfied at interface A, noting that this is valid only for the magnetic field to the right of interface A. Of course the image current is \( \alpha_c I \), where

\[
\alpha_c = \frac{\mu_r - 1}{\mu_r + 1} \tag{A.1}
\]

is the image current factor. However, in this configuration, the boundary conditions at interface B cannot be satisfied since we have not placed any image currents for that interface. So next we place such image currents for interface B as shown in Figure A.2b, which is of course done by reflecting the original currents over that boundary and multiplying by the \( \alpha_c \) factor, noting that \( \alpha_c \) is the same for both interfaces since \( \mu_r \) is the same. This once again invalidates the boundary conditions at interface A so that we need to again add image currents to the left of that boundary for the new currents that have been added on the right, as shown in Figure A.2c.

Continuing this process \textit{ad infinitum} clearly results in the infinite number of image currents on either side as shown in Figure A.2d. We again note that the magnetic field generated by this configuration is valid only in the region between the two interfaces. We also note the symmetry in the image currents on both sides and how this is to be expected given the symmetry in the original problem.
Figure A.2: Successive steps for producing the infinite image currents.
Lastly, we note that, since $\mu_r \geq 1$, we have

$$\mu_r \geq 1$$

$$\mu_r - 1 \geq 0$$

$$\alpha_c = \frac{\mu_r - 1}{\mu_r + 1} \geq 0.$$  \hspace{1cm} \text{(since $\mu_r + 1 > \mu_r \geq 1 > 0$)} \hspace{1cm} (A.2)

For any finite $\mu_r$ we also have that

$$0 < 2$$

$$\mu_r < \mu_r + 2$$

$$\mu_r - 1 < \mu_r + 1$$

$$\alpha_c = \frac{\mu_r - 1}{\mu_r + 1} < 1,$$ \hspace{1cm} (A.3)

again noting that $\mu_r + 1 > 0$. Therefore we always have $0 \leq \alpha_c < 1$, which also clearly means that $0 \leq \alpha_c^n < 1$ for any positive integer $n$. From this it also follow that the sequence $\{\alpha_c^n\}$ monotonically decreases as $n$ increases, and that this is strict when $\mu_r > 1$ so that $0 < \alpha_c < 1$. As a result, the contribution of the image currents decrease the further they get from the region of interest. Moreover, the geometric series

$$\sum_{n=1}^{\infty} \alpha_c^n$$ \hspace{1cm} (A.4)

converges to $\alpha_c/(1 - \alpha_c)$ since $0 \leq \alpha_c < 1$, which is important in using this method to actually calculate the magnetic field.
Appendix B

Convergence of Layer Image Fields

In Section 3.3 we derived the $z$ component of the magnetic field caused by a single winding layer in a device with a core of arbitrary permeability $\mu_r$ and with the associated image current factor $\alpha_c$. This required summing the field contributions from an infinite grid of image layers, and the resulting single-sided iterated series is

$$ H_{zn}(\hat{x}, \hat{z}) = \sum_{k=0}^{\infty} \left( \sum_{m=0}^{\infty} \lambda_k \lambda_m \left[ \alpha_c^{\xi(2k,m)} (A_{k,m} + A_{k,-m} + A_{-k,m} + A_{-k,-m}) + \alpha_c^{\xi(2k+1,m)} (B_{k,m} + B_{k,-m}) + \alpha_c^{\xi(2k-1,m)} (B_{-k,m} + B_{-k,-m}) \right] \right), $$

(B.1)

where, for brevity,

$$ \xi(k, m) = \max(|k|, |m|) $$

(B.2)

$$ A_{k,m} = H_{zn0}(\hat{x} - 2kb - \hat{r}_n, \hat{z} - mw) $$

(B.3)

$$ B_{k,m} = H_{zn0}(\hat{x} - 2(k+1)b + \hat{r}_n + h_n, \hat{z} - mw) $$

(B.4)
and

\[
H_{zn0}(x, z) = \tilde{H}_{zn} \left( x, z + \frac{w}{2} \right) - \tilde{H}_{zn} \left( x, z - \frac{w}{2} \right) \tag{B.5}
\]

\[
\tilde{H}_{zn}(x, z) = \frac{N_n I_n}{4 \pi w h_n} \left[ z \ln \left( 1 + \frac{h_n^2 - 2xh_n}{x^2 + z^2} \right) + 2(x - h_n) \arctan \left( \frac{z}{x - h_n} \right) \right.
\]

\[
- 2x \arctan \left( \frac{z}{x} \right) \right]. \tag{B.6}
\]

Again we note that the arctangent functions throughout refer to the principle branch whose range is the open interval \((-\pi/2, \pi/2)\). The primary goal of this chapter is to show that the iterated series (B.1) converges at any point \((\hat{x}, \hat{z})\), that is that

\[
H_{zn}(\hat{x}, \hat{z}) = \sum_{k=0}^{\infty} \left( \sum_{m=0}^{\infty} T_{k,m} \right) \tag{B.7}
\]

converges, where

\[
T_{k,m} = \lambda_k \lambda_m \left[ \alpha_c^{\xi(2k,m)} \left( A_{k,m} + A_{k,-m} + A_{-k,m} + A_{-k,-m} \right) \right.
\]

\[
+ \alpha_c^{\xi(2k+1,m)} \left( B_{k,m} + B_{k,-m} \right) + \alpha_c^{\xi(2k-1,m)} \left( B_{-k,m} + B_{-k,-m} \right) \right] \tag{B.8}
\]

and moreover that this series converges to the same results as the corresponding proper double series and the reverse iterated series, i.e.

\[
H_{zn}(\hat{x}, \hat{z}) = \sum_{k=0}^{\infty} \left( \sum_{m=0}^{\infty} T_{k,m} \right) = \sum_{k,m=0}^{\infty} T_{k,m} = \sum_{m=0}^{\infty} \left( \sum_{k=0}^{\infty} T_{k,m} \right) \tag{B.9}
\]

so that it does not matter in what order the terms are summed. See Section 3.3 and [26] for a discussion of these different forms of double series.
In order to do this, we first define the following functions:

\[
\bar{g}_1(x, z) = z \ln \left[ \frac{(x - a)^2 + z^2}{(x + a)^2 + z^2} \right] 
\]

(B.10)

\[
\bar{g}_2(x, z) = 2x \arctan \left( \frac{z}{x} \right) 
\]

(B.11)

with

\[
\bar{g}(x, z) = \bar{g}_1(x, z) + \bar{g}_2(x - a, z) - \bar{g}_2(x + a, z) 
\]

(B.12)

and

\[
\bar{f}(x, z) = \bar{g}(x, z + b) - \bar{g}(x, z - b) , 
\]

(B.13)

where we have let \( a = h_n/2 \) and \( b = w/2 \) for brevity. Note that \( \bar{f} \) is the same function as \( H_{zn0} \) defined in (B.5) except without the inconsequential (for our purposes here) scaling factor of \( N_n I_n/4\pi wh_n \), and with the origin shifted to the center of the layer instead of fixed at the left side centered vertically, as this will better let us exploit the symmetry in the functions. It is therefore trivial to show that

\[
H_{zn0}(x, z) = \frac{N_n I_n}{4\pi wh_n} \bar{f}(x - a, y) . 
\]

(B.14)

Now, regarding the function \( \bar{g}_1 \) defined in (B.10), we note that the argument of the logarithm is always non-negative since all terms are squared. However, there is a singularity at \( (x, z) = (-a, 0) \) where the denominator of the argument becomes zero. Likewise, there is another singularity at \( (x, z) = (a, 0) \) where the argument itself becomes zero, which of course is not in the domain of the natural logarithm. Similarly, the function \( \bar{g}_2 \) defined in (B.11) has singularities all along the vertical line \( x = 0 \) since the denominator
of the arctangent argument is zero there. However, it turns out that these are all removable
singularities so that these functions can be “patched” and we can define them over all of
\( \mathbb{R}^2 \), where of course \( \mathbb{R} \) denotes the set of real numbers.

Thus we define the following the function \( g_1 : \mathbb{R}^2 \to \mathbb{R} \) by

\[
g_1(x, z) = \begin{cases} 
0 & \text{if } (x, z) = (-a, 0) \text{ or } (x, z) = (a, 0) \\
\bar{g}_1(x, z) & \text{otherwise}
\end{cases}
\]

(B.15)

and \( g_2 : \mathbb{R}^2 \to \mathbb{R} \) by

\[
g_2(x, z) = \begin{cases} 
0 & \text{if } x = 0 \\
\bar{g}_2(x, z) & \text{otherwise}
\end{cases}
\]

(B.16)

We also define \( h_2 : \mathbb{R}^2 \to \mathbb{R} \) by

\[
h_2(x, z) = g_2(x - a, z) - g_2(x + a, z)
\]

(B.17)

for reasons that will become clear later. We can then define the patched versions of \( \bar{g} \) and
\( \bar{f} \) by

\[
g(x, z) = g_1(x, z) + g_2(x - a, z) - g_2(x + a, z) = g_1(x, z) + h_2(x, z)
\]

(B.18)

\[
f(x, z) = g(x, z + b) - g(x, z - b)
\]

(B.19)

which are then of course both defined over all of \( \mathbb{R}^2 \). Of course then \( H_{zn0} \) and therefore
\( H_{zn} \) itself can also be patched by simply setting

\[
H_{zn0}(x, z) = \frac{N_n I_n}{4\pi wh_n} f(x - a, z).
\]

(B.20)
Future references to $H_{zn0}, A_{k,m}, B_{k,m},$ and $T_{k,m}$ use this patched version.

It is strongly suspected that the patched $g_1$ and $g_2$ are continuous functions over their entire domain $\mathbb{R}^2$, from which it follows readily (e.g. by Theorem 4.9 of [60]) that $g, f,$ and $H_{zn0}$ are also continuous over all of $\mathbb{R}^2$. However, as it is not crucial to our ends, we do not formally prove the continuity of $g_1$ and $g_2$. Regarding $g_1$, we do remark that clearly $g_1(x,z) = g_1(x,z) = 0$ when $z = 0$ and $x \notin \{-a, a\}$ so that setting $g_1(x,z) = 0$ when $(x,z) = (-a,0)$ or $(x,z) = (a,0)$ is the natural choice, and is in fact a necessary (but not sufficient) condition for $g_1$ to be continuous. Similarly for $g_2$, we have that

$$
\lim_{x \to 0} g_2(x,z) = \lim_{x \to 0} \overline{g}_2(x,z) = \lim_{x \to 0} 2x \arctan \left( \frac{z}{x} \right) \\
= \left[ \lim_{x \to 0} 2x \right] \left[ \lim_{x \to 0} \arctan \left( \frac{z}{x} \right) \right] \\
= 0 \cdot \lim_{y \to \infty} \arctan y \\
= 0 \cdot \frac{\pi}{2} = 0
$$

for any fixed $z > 0$ since $\arctan$ is continuous. A very similar argument shows the same limit when $z < 0$, and also clearly $g_2(x,z) = g_2(x,z) = 0$ when $z = 0$ and $x \neq 0$ since $\arctan 0 = 0$. Thus $\lim_{x \to 0} g_2(x,z) = 0$ for any fixed $z$ so that the choice of $g_2(x,z) = 0$ when $x = 0$ is natural and again is a necessary condition for overall continuity of $g_2$. We do show limited continuity of $g_2$ later with respect to one of the variables with the other fixed.

In order to be as rigorous as possible, the remainder of the chapter will be in theorem / proof form, culminating in a proof of our final goal. As a convention, we label results dealing with our specific functions as theorems and more general needed results as lemmas.

**Theorem B.1.** The functions $g_1, g_2, h_2, g,$ and $f$ as defined in (B.15) through (B.19) exhibit the following symmetries:
<table>
<thead>
<tr>
<th>Function</th>
<th>Symmetry</th>
<th>Meaning (for all ((x,z) \in \mathbb{R}^2))</th>
</tr>
</thead>
<tbody>
<tr>
<td>(g_1)</td>
<td>Odd symmetry in (x)</td>
<td>(g_1(-x,z) = -g_1(x,z))</td>
</tr>
<tr>
<td></td>
<td>Odd symmetry in (z)</td>
<td>(g_1(x,-z) = -g_1(x,z))</td>
</tr>
<tr>
<td>(g_2)</td>
<td>Even symmetry in (x)</td>
<td>(g_2(-x,z) = g_2(x,z))</td>
</tr>
<tr>
<td></td>
<td>Odd symmetry in (z)</td>
<td>(g_2(x,-z) = -g_2(x,z))</td>
</tr>
<tr>
<td>(h_2)</td>
<td>Odd symmetry in (x)</td>
<td>(h_2(-x,z) = -h_2(x,z))</td>
</tr>
<tr>
<td></td>
<td>Odd symmetry in (z)</td>
<td>(h_2(x,-z) = -h_2(x,z))</td>
</tr>
<tr>
<td>(g)</td>
<td>Odd symmetry in (x)</td>
<td>(g(-x,z) = -g(x,z))</td>
</tr>
<tr>
<td></td>
<td>Odd symmetry in (z)</td>
<td>(g(x,-z) = -g(x,z))</td>
</tr>
<tr>
<td>(f)</td>
<td>Odd symmetry in (x)</td>
<td>(f(-x,z) = -f(x,z))</td>
</tr>
<tr>
<td></td>
<td>Even symmetry in (z)</td>
<td>(f(x,-z) = f(x,z))</td>
</tr>
</tbody>
</table>

**Proof.** First, for the symmetries of \(g_1\), consider any \((x,z) \in \mathbb{R}^2\). If \((x,z) = (-a,0)\) then we have

\[
g_1(-x,z) = g_1(a,0) = 0 = -0 = -g_1(-a,0) = -g_1(x,z)
\]

\[
g_1(x,-z) = g_1(-a,-0) = 0 = -0 = -g_1(-a,0) = -g_1(x,z)
\]  \hspace{1cm} (B.22)

by the definition (B.15). Similarly, if \((x,z) = (a,0)\) then

\[
g_1(-x,z) = g_1(-a,0) = 0 = -0 = -g_1(a,0) = -g_1(x,z)
\]

\[
g_1(x,-z) = g_1(a,-0) = 0 = -0 = -g_1(a,0) = -g_1(x,z)
\]  \hspace{1cm} (B.23)
again by definition (B.15). Otherwise, we note that clearly \((-x, z) \neq (-a, 0)\) and \((-x, z) \neq (a, 0)\) and so

\[
g_1(-x, z) = \overline{g}_1(-x, z) = z \ln \left(\frac{(-x-a)^2 + z^2}{(-x+a)^2 + z^2}\right) = z \ln \left(\frac{(-x+a)^2 + z^2}{(-x-a)^2 + z^2}\right)
\]

\[
= z \ln \left(\frac{(x-a)^2 + z^2}{(x+a)^2 + z^2}\right) = z \ln \left(\frac{(x-a)^2 + z^2}{(x+a)^2 + z^2}\right)^{-1}
\]

\[
= -z \ln \left(\frac{(x-a)^2 + z^2}{(x+a)^2 + z^2}\right) = \overline{g}_1(x, z) = -g_1(x, z). \quad (B.24)
\]

We also clearly have that \((x, -z) \neq (-a, 0)\) and \((x, -z) \neq (a, 0)\) so that

\[
g_1(x, -z) = \overline{g}_1(x, -z) = (-z) \ln \left(\frac{(x-a)^2 + (-z)^2}{(x+a)^2 + (-z)^2}\right)
\]

\[
= -z \ln \left(\frac{(x-a)^2 + z^2}{(x+a)^2 + z^2}\right) = -\overline{g}_1(x, z) = -g_1(x, z). \quad (B.25)
\]

These show the odd symmetry in both \(x\) and \(z\) as desired.

For the symmetries of \(g_2\) we again consider any \((x, z) \in \mathbb{R}^2\). If \(x = 0\) then

\[
g_2(-x, z) = g_2(0, z) = 0 = g_2(0, z) = g_2(x, z)
\]

\[
g_2(x, -z) = g_2(0, -0) = 0 = g_2(0, 0) = g_2(x, z) \quad (B.26)
\]

by definition (B.16). Otherwise we have \(x \neq 0\) so that

\[
g_2(-x, z) = \overline{g}_2(-x, z) = 2(-x) \arctan \left(\frac{z}{-x}\right) = -2x \arctan \left(\frac{-z}{x}\right)
\]

\[
= -2x \left[\arctan \left(\frac{z}{x}\right)\right] = 2x \arctan \left(\frac{z}{x}\right)
\]
\[ g_2(x, z) = g_2(x, -z) = 2x \arctan \left( \frac{-z}{x} \right) = 2x \arctan \left( -\frac{z}{x} \right) = 2x \left[ -\arctan \left( \frac{z}{x} \right) \right] = -2x \arctan \left( \frac{z}{x} \right) = -g_2(x, z) \] (B.27)

since the inverse tangent function itself has odd symmetry. This of course shows the even symmetry in \( x \) and odd symmetry in \( z \).

Regarding the purported symmetries of \( h_2 \), consider again any \( (x, z) \in \mathbb{R}^2 \). Then we have by (B.17) that

\[
h_2(-x, z) = g_2(-x - a, z) - g_2(-x + a, z)
= g_2(-(x + a), z) - g_2(-(x - a), z)
= g_2(x + a, z) - g_2(x - a, z)
= -[g_2(x - a, z) - g_2(x + a, z)]
= -h(x, z)
\] (B.28)

since \( g_2 \) was shown above to have even symmetry in \( x \). Hence \( h_2 \) has odd symmetry in \( x \).

We also have

\[
h_2(x, -z) = g_2(x - a, -z) - g_2(x + a, -z)
= -g_2(x - a, z) + g_2(x + a, z)
= -[g_2(x - a, z) - g_2(x + a, z)]
\]
where we have used the fact that $g_2$ has odd symmetry in $z$. Thus $h_2$ has odd symmetry in $z$ as desired.

Regarding the symmetries of $g$, for any $(x, z) \in \mathbb{R}^2$, we have

\[
g(-x, z) = g_1(-x, z) + h_2(-x, z) = -g_1(x, z) - h_2(x, z) = -[g_1(x, z) + h_2(x, z)] = -g(x, z)
\]

by (B.18) since we showed that both $g_1$ and $h_2$ have odd symmetry in $x$. We also have

\[
g(x, -z) = g_1(x, -z) + h_2(x, -z) = -g_1(x, z) - h_2(x, z) = -[g_1(x, z) + h_2(x, z)] = -g(x, z)
\]

since we showed that $g_1$ and $g_2$ also both have odd symmetry in $z$. This shows the odd symmetry of $g$ in both $x$ and $z$.

Lastly, for the symmetries of $f$, again consider any $(x, z) \in \mathbb{R}^2$ so that

\[
f(-x, z) = g(-x, z + b) - g(-x, z - b) = -g(x, z + b) + g(x, z - b)
\]
for any \((x, z) \in \mathbb{R}^2\) by (B.19) since \(g\) was shown to have odd symmetry in \(x\). Similarly

\[
f(x, -z) = g(x, -z + b) - g(x, -z - b) = g(x, -(z - b)) - g(x, -(z + b))
\]

\[
= -g(x, z - b) + g(x, z + b) = f(x, z),
\]

since \(g\) was also shown to have odd symmetry in \(z\). This completes the proof that \(f\) has odd and even symmetries in \(x\) and \(z\), respectively. We note that these symmetries of \(f\) are expected given the physical situation and the fact that it represents the \(z\) component of the magnetic field. \(\square\)

**Lemma B.2.** Suppose that \(f : E \to \mathbb{R}\) where \(E \subseteq \mathbb{R}\), and that the open interval \((a, b)\) is a subset of \(E\). Also suppose that

\[
\lim_{x \to a^+} f(x) = c \quad \text{(B.34)}
\]

\[
\lim_{x \to b^-} f(x) = d. \quad \text{(B.35)}
\]

If \(f\) is monotonically increasing, then \(c \leq f(x) \leq d\) for all \(x \in (a, b)\). Note that it could be that \(a = -\infty, b = \infty, c = -\infty, \text{ and/or } d = \infty\). Similarly, if \(f\) is monotonically decreasing, then \(c \geq f(x) \geq d\) for all \(x \in (a, b)\), and it could be that \(c = \infty \text{ and/or } d = -\infty\).

**Proof.** First, suppose that \(f\) is increasing and suppose that the conclusion is *not* true so that there is an \(x_0 \in (a, b)\) such that either \(f(x_0) < c\) or \(f(x_0) > d\). In the former case clearly
it cannot be that \( c = -\infty \) since then we could not have \( f(x_0) < c = -\infty \), hence \( c \) must be real. So, let \( \varepsilon = c - f(x_0) \), noting that \( \varepsilon > 0 \) since \( f(x_0) < c \). Then it follows from (B.34) there is a \( a_0 > a \) where \( |f(x) - c| < \varepsilon \) for all \( x \in (a, b) \cap (a, a_0) \). Now let

\[
x_1 = \frac{a + \min(x_0, a_0)}{2}
\] 

(B.36)

so that clearly both \( a < x_1 < \min(x_0, a_0) \leq x_0 < b \) and \( a < x_1 < \min(x_0, a_0) \leq a_0 \).

Therefore \( x_1 \in (a, b) \cap (a, a_0) \) so that \( |f(x_1) - c| < \varepsilon \). However, we also have that \( x_1 < x_0 \) so that \( f(x_1) \leq f(x_0) \) since \( f \) is increasing. Thus

\[
\begin{align*}
f(x_1) & \leq f(x_0) \\
-f(x_1) & \geq -f(x_0) \\
c - f(x_1) & \geq c - f(x_0) = \varepsilon > 0.
\end{align*}
\] 

(B.37)

so that clearly \( |f(x_1) - c| = c - f(x_1) \geq \varepsilon \), which is a contradiction!

We now consider the latter case in which \( f(x_0) > d \), noting that here we cannot have \( d = \infty \) since then \( f(x_0) > d = \infty \) would not be possible. So let \( \varepsilon = f(x_0) - d \), noting that \( \varepsilon > 0 \) since \( f(x_0) > d \). Then, by (B.35), we have that there is a \( b_0 < b \) such that \( |f(x) - d| < \varepsilon \) for all \( x \in (a, b) \cap (b_0, b) \). Let

\[
x_1 = \frac{\max(x_0, b_0) + b}{2}
\] 

(B.38)

so that clearly both \( a < x_0 \leq \max(x_0, b_0) < x_1 < b \) and \( b_0 \leq \max(x_0, b_0) < x_1 < b \).
Hence $x_1 \in (a, b) \cap (b_0, b)$ so that $|f(x_1) - d| < \varepsilon$. We also have that $x_0 < x_1$ so that $f(x_0) \leq f(x_1)$ since $f$ is increasing. Therefore

\[
f(x_0) \leq f(x_1)
\]

\[
0 < \varepsilon = f(x_0) - d \leq f(x_1) - d
\]

so that clearly $|f(x_1) - d| = f(x_1) - d \geq \varepsilon$, which is again a contradiction.

Since both cases lead to a contradiction, we must conclude that in fact $c \leq f(x) \leq d$ for all $x \in (a, b)$ as desired. An obvious analogous proof shows the corresponding result when $f$ is monotonically decreasing. \qed

**Lemma B.3.** Suppose that $f$ is a real function on $\mathbb{R}^2$ and that, for some real $M$,

\[
|f(x, z)| \leq M
\]

for all $(x, z) \in \mathbb{R}^+ \times \mathbb{R}^+$, where $\mathbb{R}^+ = \{ x \in \mathbb{R} \mid x \geq 0 \}$ denotes the set of non-negative real numbers. If $f$ also has even or odd symmetry with respect to $x$ and even or odd symmetry with respect to $z$, then (B.40) holds for all $(x, z) \in \mathbb{R}^2$.

**Proof.** This fact may seem obvious, but we quickly prove it nevertheless. Consider any $(x, z) \in \mathbb{R}^2$. Since $f$ has even or odd symmetry in both $x$ and $z$, it follows that

\[
f(-x, z) = \pm f(x, z) \quad f(x, -z) = \pm f(x, z),
\]

where the signs on the right sides depend on whether the respective symmetries are even or
odd. We then have the following cases:

Case: $x \geq 0$. If also $z \geq 0$ then $(x, z) \in \mathbb{R}^+ \times \mathbb{R}^+$ so that of course $|f(x, z)| \leq M$ by what was given. If $z < 0$ then of course $-z > 0$ so that $(x, -z) \in \mathbb{R}^+ \times \mathbb{R}^+$ and hence

$$|f(x, z)| = |\pm f(x, -z)| = |f(x, -z)| \leq M.$$  \hfill (B.42)

Case: $x < 0$. Then of course $-x > 0$ so that $-x \in \mathbb{R}^+$. If $z \geq 0$ then of course $(-x, z) \in \mathbb{R}^+ \times \mathbb{R}^+$ so that

$$|f(x, z)| = |\pm f(-x, z)| = |f(-x, z)| \leq M.$$  \hfill (B.43)

On the other hand, if $z < 0$ then of course $-z > 0$ so that $(-x, -z) \in \mathbb{R}^+ \times \mathbb{R}^+$. Thus

$$|f(x, z)| = |\pm f(-x, z)| = |f(-x, z)| = |\pm f(-x, -z)| = |f(-x, -z)| \leq M.  \hfill (B.44)$$

Since these cases and sub-cases are exhaustive, this shows the desired result. \qed

**Theorem B.4.** The function $g_1$ as defined in (B.15) is bounded above and below, and in particular

$$|g_1(x, z)| \leq 2a$$  \hfill (B.45)

for all $(x, z) \in \mathbb{R}^2$. 


Proof. To begin, consider the function \( h : \mathbb{R}^+ \to \mathbb{R} \) defined by

\[
h(x) = x \log \left[ \frac{r(x) - a}{r(x) + a} \right], \tag{B.46}
\]

where \( \mathbb{R}^+ = \{ x \in \mathbb{R} \mid x > 0 \} \) is of course the set of positive real numbers. For brevity in what follows, we have also defined \( r : \mathbb{R}^+ \to \mathbb{R} \) in (B.46) by

\[
r(x) = \sqrt{x^2 + a^2}. \tag{B.47}
\]

Now consider any \( x \in \mathbb{R}^+ \) so that \( x > 0 \). From this it follows that \( r(x) = \sqrt{x^2 + a^2} > a > 0 \) so that \( r(x) - a > 0 \), and clearly \( r(x) + a > 0 \) as well since \( r(x) > 0 \). Hence

\[
\frac{r(x) - a}{r(x) + a} > 0 \tag{B.48}
\]

so that the logarithm in (B.46) is defined over the domain of \( h \).

Now, the derivative of \( r \) is clearly

\[
r'(x) = \frac{d}{dx} \sqrt{x^2 + a^2} = \frac{2x}{2\sqrt{x^2 + a^2}} = \frac{x}{\sqrt{x^2 + a^2}} = \frac{x}{r(x)} \tag{B.49}
\]

so that we also have the derivative

\[
\frac{d}{dx} \log \left[ \frac{r(x) - a}{r(x) + a} \right] = \left( \frac{r(x) - a}{r(x) + a} \right)^{-1} \frac{d}{dx} \left( \frac{r(x) - a}{r(x) + a} \right) = \frac{r(x) + a}{r(x) - a} \left( \frac{r'(x)(r(x) + a) - r'(x)(r(x) - a)}{(r(x) + a)^2} \right)
\]
\begin{align*}
  &\left(\frac{r(x) + a}{r(x) - a}\right) \left(\frac{2ar'(x)}{(r(x) + a)^2}\right) \\
  &= \left[\frac{2ar'(x)}{(r(x) + a)(r(x) - a)}\right] \\
  &= \frac{2ar'(x)}{r(x)^2 - a^2} = \frac{2ar'(x)}{x^2 + a^2 - a^2} \\
  &= \frac{2ar'(x)}{x^2} = \frac{2ax}{x^2r(x)} \\
  &= \frac{2a}{xr(x)},
\end{align*}

(B.50)

where we have used (B.49). Thus we have that

\begin{align*}
  h'(x) &= \frac{d}{dx} x \log \left[\frac{r(x) - a}{r(x) + a}\right] \\
  &= \log \left[\frac{r(x) - a}{r(x) + a}\right] + x \frac{d}{dx} \log \left[\frac{r(x) - a}{r(x) + a}\right] \\
  &= \log \left[\frac{r(x) - a}{r(x) + a}\right] + x \frac{2a}{xr(x)} \\
  &= \log \left[\frac{r(x) - a}{r(x) + a}\right] + \frac{2a}{r(x)},
\end{align*}

(B.51)

where of course we have used (B.50). We also have that the second derivative of \( h \) is

\begin{align*}
  h''(x) &= \frac{d}{dx} \log \left[\frac{r(x) - a}{r(x) + a}\right] + \frac{d}{dx} \left(\frac{2a}{r(x)}\right) \\
  &= \frac{2a}{xr(x)} - \frac{2ar'(x)}{r(x)^2} = \frac{2a}{xr(x)} - \frac{2a}{r(x)^2} \left(\frac{x}{r(x)}\right) \\
  &= \frac{2a}{xr(x)} - \frac{2ax}{r(x)^3} = \frac{2a \left[r(x)^2 - x^2\right]}{xr(x)^3} \\
  &= \frac{2a \left[x^2 + a^2 - x^2\right]}{xr(x)^3} = \frac{2a^3}{xr(x)^3},
\end{align*}

(B.52)

where we have of course utilized (B.50) and (B.49). Clearly we have that \( 2a^3 > 0 \) since
\( a > 0, \text{ and } xr(x)^3 > 0 \) since both \( x > 0 \) and \( r(x) > 0 \). It thus follows that always 
\( h''(x) = 2a^3/xr(x)^3 > 0 \), which of course shows that \( h' \) is monotonically increasing over all of \( \mathbb{R}^+ \) since its derivative is positive (refer to Theorem 5.11 of [60]).

Now, clearly we have the limit

\[
\lim_{x \to \infty} r(x) = \lim_{x \to \infty} \sqrt{x^2 + a^2} = \infty \quad (B.53)
\]

so that

\[
\lim_{x \to \infty} \frac{r(x) - a}{r(x) + a} = \lim_{r(x) \to \infty} \frac{1 - a/r(x)}{1 + a/r(x)} = 1. \quad (B.54)
\]

Therefore, since the natural logarithm is continuous,

\[
\lim_{x \to \infty} \log \left[ \frac{r(x) - a}{r(x) + a} \right] = \log \left[ \lim_{x \to \infty} \frac{r(x) - a}{r(x) + a} \right] = \log 1 = 0. \quad (B.55)
\]

Hence

\[
\lim_{x \to \infty} h'(x) = \lim_{x \to \infty} \left[ \log \left( \frac{r(x) - a}{r(x) + a} \right) + \frac{2a}{r(x)} \right]
\]
\[
= \lim_{x \to \infty} \log \left( \frac{r(x) - a}{r(x) + a} \right) + \lim_{x \to \infty} \frac{2a}{r(x)}
\]
\[
= 0 + \lim_{r(x) \to \infty} \frac{2a}{r(x)} = 0 + 0 = 0, \quad (B.56)
\]

where we have used (B.53) and (B.55).

Therefore, since \( \lim_{x \to \infty} h'(x) = 0 \) and \( h' \) is increasing, it follows from Lemma B.2
that $h'(x) \leq 0$ for all $x \in \mathbb{R}^+$. This of course implies that $h$ itself is monotonically decreasing over all of $\mathbb{R}^+$ (again, see Theorem 5.11 of [60]). We also have that

$$\lim_{x \to \infty} h(x) = \lim_{x \to \infty} x \log \left[ \frac{r(x) - a}{r(x) + a} \right] = \lim_{x \to \infty} \frac{\log \left[ \frac{r(x) - a}{r(x) + a} \right]}{1/x}$$

$$= \frac{\lim_{x \to \infty} \log \left[ \frac{r(x) - a}{r(x) + a} \right]}{\lim_{x \to \infty} 1/x} = 0 / 0'$$

(B.57)

which is of course indeterminate. Hence we can use L’Hôpital’s Rule to calculate the limit:

$$\lim_{x \to \infty} h(x) = \lim_{x \to \infty} \frac{\frac{d}{dx} \log \left[ \frac{r(x) - a}{r(x) + a} \right]}{\frac{d}{dx} 1/x}$$

$$= \lim_{x \to \infty} \frac{2a/xr(x)}{-1/x^2} = \lim_{x \to \infty} \frac{-2ax}{r(x)}$$

$$= -2a \lim_{x \to \infty} \frac{x}{r(x)} = -2a \lim_{x \to \infty} \frac{x}{\sqrt{x^2 + a^2}}$$

$$= -2a \lim_{x \to \infty} \sqrt{\frac{x^2}{x^2 + a^2}}$$

$$= -2a \lim_{x \to \infty} \sqrt{\frac{1}{1 + a^2/x^2}} = -2a \sqrt{\frac{1}{1}}$$

$$= -2a$$

(B.58)

where we have of course used (B.50). Therefore, since $\lim_{x \to \infty} h(x) = -2a$ and it has been shown that $h$ is decreasing, it again follows from Lemma B.2 that $h(x) \geq -2a$ for all $x \in \mathbb{R}^+$. This is the primary result of what has been done so far, and we note here that some assistance with a portion of this proof was solicited on the Mathematics Stack Exchange website, see [72].
Next consider any \((x, z) \in \mathbb{R}^+ \times \mathbb{R}^+\). First, we note that we have

\[
2a \left(x^2 - 2xr(z) + z^2 + a^2\right) = 2ax^2 - 4axr(z) + 2az^2 + 2a^3
\]

\[
= r(z)(2x)(-2a) + a \left[2x^2 + 2a^2\right] + 2az^2
\]

\[
= r(z) \left[(x - a) + (x + a)\right] \left[(x - a) - (x + a)\right]
\]

\[
= a \left[(x^2 - 2ax + a^2) + (x^2 + 2ax + a^2)\right] + 2az^2
\]

\[
= r(z) \left[(x - a)^2 - (x + a)^2\right]
\]

\[
+ a \left[(x - a)^2 + (x + a)^2\right] + 2az^2
\]

\[
= r(z)(x - a)^2 - r(z)(x + a)^2
\]

\[
a(x - a)^2 + a(x + a)^2
\]

\[
a^2 + 2az^2 + r(z)z^2 - r(z)z^2
\]

\[
= \left[(x - a)^2 + z^2\right] [r(z) + a]
\]

\[
- \left[(x + a)^2 + z^2\right] [r(z) - a].
\]  \hspace{1cm} (B.59)

Then, as the following is a square, we obviously have

\[
[x - r(z)]^2 \geq 0
\]

\[
x^2 - 2xr(z) + r(z)^2 \geq 0
\]

\[
x^2 - 2xr(z) + z^2 + a^2 \geq 0
\]

\[
2a \left(x^2 - 2xr(z) + z^2 + a^2\right) \geq 0
\]  \hspace{1cm} (since \(2a > 0\))
\[
[(x - a)^2 + z^2] [r(z) + a] \\
- [(x + a)^2 + z^2] [r(z) - a] \geq 0 \quad \text{(by (B.59) above)}
\]

\[
[(x - a)^2 + z^2] [r(z) + a] \geq [(x + a)^2 + z^2] [r(z) - a] \\
\frac{[(x - a)^2 + z^2] [r(z) + a]}{(x + a)^2 + z^2} \geq r(z) - a \quad \text{(since } (x + a)^2 + z^2 > 0) \\
\frac{(x - a)^2 + z^2}{(x + a)^2 + z^2} \geq \frac{r(z) - a}{r(z) + a} \quad \text{(since } r(z) + a > 0) \\
\log \left( \frac{(x - a)^2 + z^2}{(x + a)^2 + z^2} \right) \geq \log \left( \frac{r(z) - a}{r(z) + a} \right) \quad \text{(since } \log \text{ is increasing)} \\
z \log \left( \frac{(x - a)^2 + z^2}{(x + a)^2 + z^2} \right) \geq z \log \left( \frac{r(z) - a}{r(z) + a} \right) \quad \text{(since } z > 0) \\
g_1(x, z) \geq h(z) . \quad \text{(B.60)}
\]

Since it was shown above that \( h(y) \geq -2a \) for all \( y > 0 \), it follows that \( g_1(x, z) \geq h(z) \geq -2a \).

Since \( x > 0 \), we also clearly have

\[-2x < 0 \]
\[-2x - a < -a < 0 < a \]
\[-(x + a) < x - a < x + a \] \quad \text{(B.61)}

so that \( |x - a| < x + a = |x + a| \). From this, it follows that

\[|x - a|^2 < |x + a|^2 \quad \text{(since } x^2 \text{ is strictly increasing when } x \geq 0) \]
\[(x - a)^2 < (x + a)^2 \]

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\[(x-a)^2 + z^2 < (x+a)^2 + z^2\]
\[
\frac{(x-a)^2 + z^2}{(x+a)^2 + z^2} < 1 \quad \text{ (since clearly } (x+a)^2 + z^2 > 0)\]
\[
\log \left[\frac{(x-a)^2 + z^2}{(x+a)^2 + z^2}\right] < \log 1 = 0 \quad \text{ (since log is strictly increasing)}
\]
\[
z \log \left[\frac{(x-a)^2 + z^2}{(x+a)^2 + z^2}\right] < 0 \quad \text{ (since } z > 0)\]
\[
g_1(x, z) < 0. \quad \text{(B.62)}
\]

Hence we have \(-2a \leq g_1(x, z) < 0 \leq 2a\) so that \(|g_1(x, z)| \leq 2a\), this of course being true for any \((x, z) \in \mathbb{R}^+ \times \mathbb{R}^+\) since \(x\) and \(z\) were arbitrary.

Now consider any \((x, z) \in \mathbb{R}^+ \times \mathbb{R}^+\) so that \(x \geq 0\) and \(z \geq 0\). We have the following:

Case: \(z = 0\). Then clearly \(g_1(x, z) = g_1(x, 0) = 0\) by (B.15) since \(\bar{g}_1(x, z) = \bar{g}_1(x, 0) = 0\) when \(x \notin \{-a, a\}\). Hence of course \(|g_1(x, z)| = 0 \leq 2a\).

Case: \(z > 0\). If also \(x > 0\) then \((x, z) \in \mathbb{R}^+ \times \mathbb{R}^+\) so that \(|g_1(x, z)| \leq 2a\) by what was just shown above. If \(x = 0\) then

\[
|g_1(x, z)| = |\bar{g}_1(x, z)| = |\bar{g}_1(0, z)| = |z \log \left[\frac{(-a)^2 + z^2}{a^2 + z^2}\right]|
\]
\[
= |z \log 1| = |z \cdot 0| = 0
\]
\[
\leq 2a \quad \text{(B.63)}
\]

as desired.

Therefore \(|g_1(x, z)| \leq 2a\) for all \((x, z) \in \mathbb{R}^+ \times \mathbb{R}^+\). Since also \(g_1\) has odd symmetry in both \(x\) and \(y\) (as shown in Theorem B.1), it follows that \(|g_1(x, z)| \leq 2a\) over all of \(\mathbb{R}^2\) by Lemma B.3 as desired.
Theorem B.5. The function $h_2$ as defined in (B.17) is bounded above and below, and in particular

$$|h_2(x, z)| \leq 2\pi a$$ \hspace{1cm} (B.64)

for all $(x, z) \in \mathbb{R}^2$.

Proof. First, we note that the function $g_2$ itself as defined in (B.16) is unbounded on $\mathbb{R}^2$ (though we do not prove this as it is not pertinent), and this is why we previously defined $h_2$, which is bounded as we shall now show.

First we consider $h_2$ restricted to $\mathbb{R}^+ \times \mathbb{R}^+$. So consider any fixed $z \in \mathbb{R}^+$ so that $z > 0$ and consider the temporary functions $g : \mathbb{R} \to \mathbb{R}$ and $h : \mathbb{R} \to \mathbb{R}$ defined by

$$g(x) = g_2(x, z)$$

$$h(x) = h_2(x, z) = g_2(x - a, z) - g_2(x + a, z) = g(x - a) - g(x + a).$$ \hspace{1cm} (B.66)

Now, since we have $z$ fixed, we have

$$\frac{d}{dx} \arctan \left( \frac{z}{x} \right) = \left[ \frac{1}{1 + \left( \frac{z}{x} \right)^2} \right] \left[ \frac{-z}{x^2} \right] = \frac{-z}{x^2 + z^2}$$ \hspace{1cm} (B.67)

so long as $x \neq 0$, and therefore clearly

$$g'(x) = \frac{d}{dx} g_2(x, z) = \frac{d}{dx} g_2(x, z) = \frac{d}{dx} 2x \arctan \left( \frac{z}{x} \right)$$
\[= 2 \arctan \left( \frac{z}{x} \right) + 2x \frac{d}{dx} \arctan \left( \frac{z}{x} \right)\]
\[= 2 \arctan \left( \frac{z}{x} \right) - \frac{2xz}{x^2 + z^2}, \quad \text{(B.68)}\]

where again this is only valid only when \(x \neq 0\). We then also have

\[g''(x) = \frac{d}{dx} g'(x) = 2 \frac{d}{dx} \arctan \left( \frac{z}{x} \right) - \frac{d}{dx} \left[ \frac{2xz}{x^2 + z^2} \right]\]
\[= \frac{-2z}{x^2 + z^2} - \frac{2z(x^2 + z^2) - 2xz(2x)}{(x^2 + z^2)^2}\]
\[= \frac{-2z(x^2 + z^2) - 2z(x^2 + z^2) + 4x^2z}{(x^2 + z^2)^2}\]
\[= \frac{-4x^2z - 4z^3 + 4x^2z}{(x^2 + z^2)^2}\]
\[= \frac{-4z^3}{(x^2 + z^2)^2} \quad \text{(B.69)}\]

for \(x \neq 0\), where we have of course used (B.67) and (B.68). Since \(z > 0\) so that \(z^3 > 0\) and the denominator is a square, clearly we have that \(g''(x) < 0\) for all \(x \neq 0\) so that

\[g' \text{ is monotonically decreasing in both } (-\infty, 0) \text{ and } (0, \infty). \quad \text{(B.70)}\]

by Theorem 5.11 of [60].

To momentarily digress, clearly at our fixed \(z > 0\), we have that

\[\lim_{x \to 0^+} \arctan \left( \frac{z}{x} \right) = \frac{\pi}{2}, \quad \text{(B.71)}\]
since \( x > 0 \) for this limit so that \( z/x \to \infty \) as \( x \to 0 \) from the right. Similarly

\[
\lim_{x \to 0^-} \arctan \left( \frac{z}{x} \right) = -\frac{\pi}{2}
\]  
(B.72)

since \( x < 0 \) so that \( z/x \to -\infty \) when \( x \to 0 \) from the left. Hence we have

\[
\lim_{x \to 0^+} g'(x) = 2 \lim_{x \to 0^+} \arctan \left( \frac{z}{x} \right) - \lim_{x \to 0^+} \frac{2xz}{x^2 + z^2}
\]

\[
= 2 \left( \frac{\pi}{2} \right) - \frac{2 \cdot 0 \cdot z}{0 + z^2} = \pi - 0
\]

\[
= \pi
\]  
(B.73)

since \( z > 0 \), where we have utilized (B.71). We also clearly have

\[
\lim_{x \to \infty} \frac{2xz}{x^2 + z^2} = \frac{\infty}{\infty}
\]  
(B.74)

so that we can use L'Hôpital’s Rule to get

\[
\lim_{x \to \infty} \frac{2xz}{x^2 + z^2} = \lim_{x \to \infty} \frac{d}{dx} \frac{2xz}{x^2 + z^2} = \lim_{x \to \infty} \frac{2z}{2x} = 0.
\]  
(B.75)

Therefore

\[
\lim_{x \to \infty} g'(x) = 2 \lim_{x \to \infty} \arctan \left( \frac{z}{x} \right) - \lim_{x \to \infty} \frac{2xz}{x^2 + z^2}
\]

\[
= 2 \arctan 0 - 0 = 0.
\]  
(B.76)

So we have shown that \( \lim_{x \to 0^+} g'(x) = \pi \), \( \lim_{x \to \infty} g'(x) = 0 \), and that \( g' \) is monotonically
decreasing in the open interval \((0, \infty)\) by (B.70). It thus follows from Lemma B.2 that

\[
0 \leq g'(x) \leq \pi \text{ for all } x > 0. \tag{B.77}
\]

Similarly, we also clearly have

\[
\lim_{x \to -\infty} \frac{2xz}{x^2 + z^2} = -\infty \tag{B.78}
\]

so that we can use L’Hôpital’s Rule to evaluate the limit:

\[
\lim_{x \to -\infty} \frac{2xz}{x^2 + z^2} = \lim_{x \to -\infty} \frac{\frac{d}{dx}2xz}{\frac{d}{dx}(x^2 + z^2)} = \lim_{x \to -\infty} \frac{2z}{2x} = 0, \tag{B.79}
\]

and hence

\[
\lim_{x \to -\infty} g'(x) = 2 \lim_{x \to -\infty} \arctan \left( \frac{z}{x} \right) - \lim_{x \to -\infty} \frac{2xz}{x^2 + z^2} = 2 \arctan 0 - 0 = 0. \tag{B.80}
\]

Also

\[
\lim_{x \to 0^-} g'(x) = 2 \lim_{x \to 0^-} \arctan \left( \frac{z}{x} \right) - \lim_{x \to 0^-} \frac{2xz}{x^2 + z^2} = 2 \left( -\frac{\pi}{2} \right) - \frac{2 \cdot 0 \cdot z}{0 + z^2} = -\pi - \frac{0}{z^2} = -\pi, \tag{B.81}
\]

where of course we have used (B.72). Therefore \(\lim_{x \to -\infty} g'(x) = 0\), \(\lim_{x \to 0^-} g'(x) = -\pi\),
and $g'$ is monotonically decreasing in the open interval $(-\infty, 0)$ by (B.70) so that it follows from Lemma B.2 that

$$-\pi \leq g'(x) \leq 0 \text{ for all } x < 0.$$  \hspace{1cm} (B.82)

Here we note that, although $g'$ is decreasing in both $(-\infty, 0)$ and $(0, \infty)$ by (B.70), $g'$ is clearly not decreasing on all of $\mathbb{R} - \{0\}$ since

$$\lim_{x \to 0^-} g'(x) = -\pi < \pi = \lim_{x \to 0^+} g'(x).$$  \hspace{1cm} (B.83)

Now, by the chain rule and (B.66), we have that the derivative of $h$ is

$$h'(x) = \frac{d}{dx} g(x-a) - \frac{d}{dx} g(x+a)$$
$$= g'(x-a) \frac{d}{dx} (x-a) - g'(x+a) \frac{d}{dx} (x+a)$$
$$= g'(x-a) - g'(x+a).$$  \hspace{1cm} (B.84)

Then, for $0 < x < a$ we have that $x-a < 0$ so that $-\pi \leq g'(x-a) \leq 0$ by what was shown above. We also have that $0 < a < x + a$ so that $0 \leq g'(x+a) \leq \pi$ by what was shown before. Thus

$$0 \leq g'(x + a)$$
$$0 \geq -g'(x + a)$$
$$0 \geq g'(x - a) \geq g'(x - a) - g'(x + a)$$
\[ 0 \geq h'(x) \quad (B.85) \]

so that

\[ h \text{ is monotonically decreasing in the open interval } (0, a), \quad (B.86) \]

once again by Theorem 5.11 of [60].

Here we again take a short digression to show that \( h \) as defined in (B.66) is continuous.

First, we have

\[
\lim_{x \to 0^+} g(x) = \lim_{x \to 0^+} g_2(x, z) = \lim_{x \to 0^+} g_2(x, z) = \lim_{x \to 0^+} 2x \arctan \left( \frac{z}{x} \right) \\
= \left[ \lim_{x \to 0^+} 2x \right] \left[ \lim_{x \to 0^+} \arctan \left( \frac{z}{x} \right) \right] = 0 \cdot \frac{\pi}{2} = 0, \quad (B.87)
\]

where we have used (B.71). Also

\[
\lim_{x \to 0^-} g(x) = \lim_{x \to 0^-} g_2(x, z) = \lim_{x \to 0^-} g_2(x, z) = \lim_{x \to 0^-} 2x \arctan \left( \frac{z}{x} \right) \\
= \left[ \lim_{x \to 0^-} 2x \right] \left[ \lim_{x \to 0^-} \arctan \left( \frac{z}{x} \right) \right] = 0 \cdot \left( -\frac{\pi}{2} \right) = 0, \quad (B.88)
\]

where we have used (B.72). Therefore the overall limit exists and of course

\[
\lim_{x \to 0} g(x) = \lim_{x \to 0^+} g(x) = \lim_{x \to 0^-} g(x) = 0. \quad (B.89)
\]

Since we have defined \( g(0) = g_2(0, z) = 0 \), this means that \( g \) is continuous at \( x = 0 \). Also,
clearly \( g \) is also continuous at any other point \( x \neq 0 \) since \( g(x) = g_2(x, z) = g_2(x, z) \) there and \( g_2(x, z) \) is obviously continuous (with respect to \( x \)) since the arctangent function is. Thus we conclude that \( g \) is continuous over all of \( \mathbb{R} \).

It follows from this that \( h \) is also continuous by Theorem 4.9 of [60], being the difference of two (shifted) continuous functions. Therefore, we have

\[
\lim_{x \to 0^+} h(x) = \lim_{x \to 0} h(x) = h(0) = g(0 - a) - g(0 + a) \\
= g(-a) - g(a) = g_2(-a, z) - g_2(a, z) \\
= g_2(a, z) - g_2(a, z) = 0,
\]

(B.90)

where we have exploited the even symmetry of \( g_2 \) in \( x \) as shown in Theorem B.1. We also have

\[
\lim_{x \to a^-} h(x) = \lim_{x \to a} h(x) = h(a) = g(a - a) - g(a + a) \\
= g(0) - g(2a) = g_2(0, z) - g_2(2a, z) = 0 - 2(2a) \arctan \left( \frac{z}{2a} \right) \\
= -4a \arctan \left( \frac{z}{2a} \right),
\]

(B.91)

noting that clearly this is a negative number since \( z > 0 \) and \( a > 0 \) so that \( \arctan(z/2a) > 0 \). Thus it has been shown that \( \lim_{x \to 0^+} h(x) = 0 \), \( \lim_{x \to a^-} = -4a \arctan \left( \frac{z}{2a} \right) \), and \( h \) is monotonically decreasing in \((0, a)\) by (B.86). So it follows that

\[
-4a \arctan \left( \frac{z}{2a} \right) \leq h(x) \leq 0 \text{ for all } x \in (0, a)
\]

(B.92)
by Lemma B.2.

Now consider any $x > a$ so that $x - a > 0$. Then clearly

\[-a < 0 < a\]

\[0 < x - a < x + a\]  (B.93)

so that, since $g'(x)$ is decreasing in $(0, \infty)$ by (B.70),

\[g'(x - a) \geq g'(x + a)\]

\[g'(x - a) - g'(x + a) \geq 0\]

\[h'(x) \geq 0,\]  (B.94)

where we have used (B.84). Hence $h$ is monotonically increasing in the open interval $(a, \infty)$, yet again by Theorem 5.11 of [60]. We also have

\[\lim_{x \to a^+} h(x) = \lim_{x \to a^-} h(x) = -4a \arctan \left(\frac{z}{2a}\right)\]  (B.95)

since $h$ was shown to be continuous, and where we have used (B.91). We also have the limit

\[\lim_{x \to \infty} h(x) = \lim_{x \to \infty} \left[ g(x - a) - g(x + a) \right] = \lim_{x \to \infty} \left[ g_2(x - a, z) - g_2(x + a, z) \right]\]

\[= \lim_{x \to \infty} \left[ \overline{g}_2(x - a, z) - \overline{g}_2(x + a, z) \right]\]

\[= \lim_{x \to \infty} \left[ 2(x - a) \arctan \left(\frac{z}{x - a}\right) - 2(x + a) \arctan \left(\frac{z}{x + a}\right) \right]\]
\begin{align*}
&= 2 \lim_{x \to \infty} \left\{ x \left[ \arctan \left( \frac{z}{x-a} \right) - \arctan \left( \frac{z}{x+a} \right) \right] \\
&\quad - a \left[ \arctan \left( \frac{z}{x-a} \right) + \arctan \left( \frac{z}{x+a} \right) \right] \right\} \\
&= 2 \left\{ \lim_{x \to \infty} x \left[ \arctan \left( \frac{z}{x-a} \right) - \arctan \left( \frac{z}{x+a} \right) \right] \\
&\quad - \lim_{x \to \infty} a \left[ \arctan \left( \frac{z}{x-a} \right) + \arctan \left( \frac{z}{x+a} \right) \right] \right\} \\
&= 2 \left\{ \lim_{x \to \infty} x \left[ \arctan \left( \frac{z}{x-a} \right) - \arctan \left( \frac{z}{x+a} \right) \right] \\
&\quad - a [0 + 0] \right\} \\
&= 2 \lim_{x \to \infty} x \left[ \arctan \left( \frac{z}{x-a} \right) - \arctan \left( \frac{z}{x+a} \right) \right] \\
&= 2 \lim_{x \to \infty} \arctan \left( \frac{z}{x-a} \right) - \arctan \left( \frac{z}{x+a} \right) \\
&= 2 \frac{0 + 0}{0} = 0. \quad (B.96)
\end{align*}

Hence we can use L'Hôpital's Rule to get

\begin{align*}
\lim_{x \to \infty} h(x) &= 2 \lim_{x \to \infty} \frac{\frac{d}{dx} \arctan \left( \frac{z}{x-a} \right) - \frac{d}{dx} \arctan \left( \frac{z}{x+a} \right)}{\frac{d}{dx} 1/x} \\
&= 2 \lim_{x \to \infty} \frac{-\frac{z}{(x-a)^2+z^2} \frac{d}{dx} (x-a) - \frac{-z}{(x+a)^2+z^2} \frac{d}{dx} (x+a)}{-1/x^2} \\
&= 2 \lim_{x \to \infty} \frac{\frac{-z}{(x-a)^2+z^2} - \frac{-z}{(x+a)^2+z^2}}{-1/x^2} \\
&= 2 \lim_{x \to \infty} \frac{1}{x^2} \left[ \frac{z}{(x-a)^2+z^2} - \frac{z}{(x+a)^2+z^2} \right] \\
&= 2 \lim_{x \to \infty} \frac{1}{x^2} \left[ \frac{1}{x^2 - 2ax + a^2 + z^2} - \frac{1}{x^2 + 2ax + a^2 + z^2} \right] \\
&= 2 \lim_{x \to \infty} \frac{1}{1 - 2a/x + (a^2 + z^2)/x^2} - \frac{1}{1 + 2a/x + (a^2 + z^2)/x^2} \\
&= 2 \left[ \frac{1}{1-0+0} - \frac{1}{1+0+0} \right]
\end{align*}
where we have applied the chain rule to (B.67). So, we have shown that \( \lim_{x \to a^+} h(x) = -4a \arctan \left( \frac{z}{2a} \right) \), \( \lim_{x \to \infty} h(x) = 0 \), and that \( h \) is monotonically increasing in \((a, \infty)\). Hence

\[
-4a \arctan \left( \frac{z}{2a} \right) \leq h(x) \leq 0 \text{ for all } x \in (a, \infty) \quad \text{(B.98)}
\]

by Lemma B.2.

So now consider any \( x \in \mathbb{R}^+ \). If \( x \neq a \) then \( x \in (0, a) \) or \( x \in (a, \infty) \) so that

\[
-4a \arctan \left( \frac{z}{2a} \right) \leq h(x) \leq 0 \quad \text{(B.99)}
\]

by (B.92) and (B.98). The only other possibility is that \( x = a \), in which case we have

\[
h(x) = h(a) = \lim_{x \to a} h(x) = \lim_{x \to a^-} h(x) = -4a \arctan \left( \frac{z}{2a} \right) \quad \text{(B.100)}
\]

by (B.91) since \( h \) is continuous. Thus in all cases it is true that

\[
-4a \arctan \left( \frac{z}{2a} \right) \leq h(x) \leq 0 \quad \text{(B.101)}
\]

Now, the arctangent function is bounded so that, in particular,

\[
\arctan \left( \frac{z}{2a} \right) \leq \frac{\pi}{2}
\]
\[-4a \arctan \left( \frac{z}{2a} \right) \geq -4a \frac{\pi}{2} = -2\pi a \]  
(B.102)

since of course $-4a < 0$ as $a > 0$. So, for all $x \in \mathbb{R}^+$, we have

\[-2\pi a \leq -4a \arctan \left( \frac{z}{2a} \right) \leq h(x) \leq 0 \leq 2\pi a \]  
(B.103)

so that $|h(x)| \leq 2\pi a$. Since $z \in \mathbb{R}^+$ was arbitrary, it then follows that

\[ |h_2(x, z)| \leq 2\pi a \]  
(B.104)

Next we briefly show that

\[ g_2(x, 0) = 0 \]  
for any real $x$.  
(B.105)

So consider any $x \in \mathbb{R}$. If $x = 0$ then clearly $g_2(x, 0) = g_2(0, 0) = 0$ by definition (B.16).

If $x \neq 0$ then we have

\[ g_2(x, 0) = \bar{g}_2(x, 0) = 2x \arctan \left( \frac{0}{x} \right) = 2x \cdot 0 = 0 \]  
(B.106)

since $\arctan 0 = 0$. This shows the desired result since $x$ was arbitrary.

To return to our main argument, suppose that $(x, z) \in \mathbb{R}^+ \times \mathbb{R}^+$ so that $x \geq 0$ and $z \geq 0$. We then have the following:
Case: \( x = 0 \). Then

\[
|h_2(x, z)| = |h_2(0, z)| = |g_2(0 - a, z) - g_2(0 + a, z)|
= |g_2(-a, z) - g_2(a, z)| = |g_2(a, z) - g_2(a, z)|
= |0| = 0
\leq 2\pi a,
\]

(B.107)

where we have again used the even symmetry of \( g_2 \) in \( x \) as shown in Theorem B.1.

Case: \( x > 0 \). If also \( z > 0 \) then of course \((x, z) \in \mathbb{R}^+ \times \mathbb{R}^+\) so that \( |h_2(x, z)| \leq 2\pi a \) by (B.104). If \( z = 0 \) then we have that

\[
|h_2(x, z)| = |h_2(x, 0)| = |g_2(x - a, 0) - g_2(x + a, 0)|
= |0 - 0| = |0| = 0
\leq 2\pi a,
\]

(B.108)

where we have used (B.105).

As these cases are exhaustive, this shows that \( |h_2(x, z)| \leq 2\pi a \) for all \((x, z) \in \mathbb{R}^+ \times \mathbb{R}^+\). Since it was proven in Theorem B.1 that \( h_2 \) also has odd symmetry in both \( x \) and \( z \), the main result follows from Lemma B.3.

\[\square\]

**Theorem B.6.** The patched functions \( g, f, \) and \( H_{zn0} \) as defined in (B.18) through (B.20) are all bounded above and below and, in particular,

\[
|g(x, z)| \leq 2(\pi + 1)a
\]

(B.109)
\[ |f(x, z)| \leq 4(\pi + 1)a \quad (B.110) \]
\[ |H_{zn0}(x, z)| \leq \frac{(\pi + 1)N_n I_n a}{\pi w h_n} \quad (B.111) \]

for all \((x, z) \in \mathbb{R}^2\).

**Proof.** Consider any \((x, z) \in \mathbb{R}^2\). It was shown in Theorem B.4 and Theorem B.5 that
\[ |g_1(x, z)| \leq 2a \quad \text{and} \quad |h_2(x, z)| \leq 2\pi a, \]
respectively. Therefore we have by (B.18) that

\[
|g(x, z)| = |g_1(x, z) + h_2(x, z)| \\
\leq |g_1(x, z)| + |h_2(x, z)| \\
\leq 2a + 2\pi a \\
= 2(\pi + 1)a
\]

(B.112)
as desired, where we have of course used the triangle inequality. Hence this is true for any
\((x, z) \in \mathbb{R}^2\) since \((x, z)\) was arbitrary.

We also then have, for any \((x, z) \in \mathbb{R}^2\), that

\[
|f(x, z)| = |g(x, z + b) - g(x, z - b)| \\
\leq |g(x, z + b)| + |g(x, z - b)| \\
\leq 2(\pi + 1)a + 2(\pi + 1)a \\
= 4(\pi + 1)a
\]

(B.113)
by (B.19). Lastly, it then follows from (B.20) that

\[
|H_{zn0}(x, z)| = \left| \frac{N_n I_n}{4\pi w h_n} f(x - a, y) \right| = \left| \frac{N_n I_n}{4\pi w h_n} |f(x - a, y)| \right|
\]

\[
= \frac{N_n I_n}{4\pi w h_n} |f(x - a, y)|
\]

\[
\leq \frac{N_n I_n}{4\pi w h_n} 4(\pi + 1)a
\]

\[
= \frac{(\pi + 1)N_n I_n a}{\pi w h_n}
\]

(B.114)

where we of note that of course $N_n I_n/4\pi w h_n > 0$. These show the remaining desired results.

Lemma B.7. If \( \{a_{k,m}\} \) and \( \{b_{k,m}\} \) are two doubly indexed real sequences such that \( 0 \leq a_{k,m} \leq b_{k,m} \) for all integers \( k, m \geq 0 \) and

\[
\sum_{k=0}^{\infty} \left( \sum_{m=0}^{\infty} b_{k,m} \right)
\]

(B.115)

converges, then

\[
\sum_{k=0}^{\infty} \left( \sum_{m=0}^{\infty} a_{k,m} \right)
\]

(B.116)

also converges. This is the comparison test for iterated series.

Note: Theorem 8.4 of [26] proves the comparison test for proper double series, but this paper does not prove the analogous result for iterated series, so we do this here.
Proof. Since the series of (B.115) converges, it must be that

$$d_k = \sum_{m=0}^{\infty} b_{k,m} \quad (B.117)$$

converges for any $k \geq 0$. Moreover, since $0 \leq a_{k,m} \leq b_{k,m}$ for any $k, m \geq 0$, it follows from the standard comparison test (Theorem 3.25 of [60]) that

$$c_k = \sum_{m=0}^{\infty} a_{k,m} \quad (B.118)$$

also converges for any $k \geq 0$ and that of course $0 \leq c_k \leq d_k$.

Then, again since the iterated series (B.115) converges, we of course have that

$$\sum_{k=0}^{\infty} d_k = \sum_{k=0}^{\infty} \left( \sum_{m=0}^{\infty} b_{k,m} \right) \quad (B.119)$$

converges. Then, since $0 \leq c_k \leq d_k$ for all $k \geq 0$, we have that

$$\sum_{k=0}^{\infty} \left( \sum_{m=0}^{\infty} a_{k,m} \right) = \sum_{k=0}^{\infty} c_k \quad (B.120)$$

also converges, again by the standard comparison theorem. This of course is the desired result. 

\[\Box\]

Theorem B.8. The iterated series

$$\sum_{k=0}^{\infty} \left( \sum_{m=0}^{\infty} |T_{k,m}| \right) \quad (B.121)$$
converges for any point \((\hat{x}, \hat{z}) \in \mathbb{R}^2\), where \(T_{k,m}\) is that defined in (B.8).

**Proof.** First, clearly for \(\alpha_c = 0\) (corresponding with \(\mu_r = 1\)), all of the \(T_{k,m}\) terms are zero with the exception of \(T_{0,0}\), and so the above series clearly converges simply to \(|T_{0,0}|\). This of course corresponds to a situation with effectively no core so that all the image layers disappear and the only contribution is from the original layer. So, in what follows, we can assume \(\mu_r > 1\) so that \(\alpha_c > 0\). It was shown in Appendix A that this results in \(0 < \alpha_c^k < 1\) for any integer \(k > 0\), and that the sequence \(\{\alpha_c^k\}\) strictly decreases. Clearly it is also then the case that \(0 < \sqrt{\alpha_c} < 1\) as well so that the sequence \(\{\sqrt{\alpha_c^k}\}\) also strictly decreases.

Now, clearly \(|2k| = 2k \geq k\) and \(|2k + 1| = 2k + 1 > 2k \geq k\) for all integers \(k \geq 0\). Also, for \(k = 0\), we have that \(|2k - 1| = |-1| = 1 \geq 0 = k\), and, for \(k > 0\) we have \(k \geq 1\) so that

\[
2k \geq 2
\]

\[
2k - 1 \geq 1 > 0 . \tag{B.122}
\]

Hence \(|2k - 1| = 2k - 1\) and

\[
k \geq 1
\]

\[
k + k \geq k + 1
\]

\[
2k \geq k + 1
\]

\[
2k - 1 \geq k
\]

\[
|2k - 1| \geq k . \tag{B.123}
\]
Thus $|2k - 1| \geq k$ for all $k \geq 0$. Since $\{\sqrt{\alpha_c^k}\}$ is a decreasing sequence, it then follows that

$$\sqrt{\alpha_c^{2k}} \leq \sqrt{\alpha_c^k} \quad \sqrt{\alpha_c^{2k+1}} \leq \sqrt{\alpha_c^k} \quad \sqrt{\alpha_c^{2k-1}} \leq \sqrt{\alpha_c^k}$$ (B.124)

for all $k \geq 0$.

Since $0 < \sqrt{\alpha_c} < 1$, we have the following geometric series:

$$\sum_{k=0}^{\infty} \lambda_k \sqrt{\alpha_c^k} = \frac{1}{2} + \sum_{k=1}^{\infty} \sqrt{\alpha_c^k} = \frac{1}{2} + \sum_{k=0}^{\infty} \sqrt{\alpha_c^k} - 1 = \frac{1}{1 - \sqrt{\alpha_c}} - \frac{1}{2} = \frac{2 - (1 - \sqrt{\alpha_c})}{2(1 - \sqrt{\alpha_c})} = \frac{1 + \sqrt{\alpha_c}}{2(1 - \sqrt{\alpha_c})}$$ (B.125)

recalling the definition of $\lambda_k$ in (3.19). Then, for any positive real $M$, we have that iterated series

$$\sum_{k=0}^{\infty} \left( \sum_{m=0}^{\infty} 12M \lambda_k \lambda_m \sqrt{\alpha_c^m} \sqrt{\alpha_c^k} \right) = \sum_{k=0}^{\infty} 12M \lambda_k \sqrt{\alpha_c^k} \left( \sum_{m=0}^{\infty} \lambda_m \sqrt{\alpha_c^{|m|}} \right) = \sum_{k=0}^{\infty} 12M \lambda_k \sqrt{\alpha_c^k} \left( \frac{1 + \sqrt{\alpha_c}}{2(1 - \sqrt{\alpha_c})} \right) = 6M \left( \frac{1 + \sqrt{\alpha_c}}{1 - \sqrt{\alpha_c}} \right) \sum_{k=0}^{\infty} \lambda_k \sqrt{\alpha_c^k} = 6M \left( \frac{1 + \sqrt{\alpha_c}}{1 - \sqrt{\alpha_c}} \right) \left( \frac{1 + \sqrt{\alpha_c}}{2(1 - \sqrt{\alpha_c})} \right) = 3M \left( \frac{1 + \sqrt{\alpha_c}}{1 - \sqrt{\alpha_c}} \right)^2$$ (B.126)
converges.

Here we digress momentarily to note that clearly

$$|k| + |m| \leq \max(|k|, |m|) + \max(|k|, |m|) = 2 \max(|k|, |m|) = 2\xi(k, m)$$

$$\frac{|k| + |m|}{2} \leq \xi(k, m) \tag{B.127}$$

for any $k$ and $m$. Hence

$$\sqrt{\alpha_c} \frac{|k|}{\sqrt{\alpha_c}} \frac{|m|}{\sqrt{\alpha_c}} = \frac{\alpha_c}{\alpha_c} \frac{|k|}{\alpha_c} \frac{|m|}{\alpha_c} = \frac{\alpha_c}{\alpha_c} \geq \alpha_c \xi(k, m) \tag{B.128}$$

since the sequence $\{\alpha_c^k\}$ is strictly decreasing.

Now consider any $(\hat{x}, \hat{z}) \in \mathbb{R}^2$. Theorem B.6 proved that $|H_{zn0}(\hat{x}, \hat{z})| \leq M$ where

$$M = (\pi + 1)N_nI_n\alpha/\pi w h_n.$$ Hence $|A_{k,m}| \leq M$ and $|B_{k,m}| \leq M$ for all indices $k$ and $m$, referring to (B.3) and (B.4) that define $A_{k,m}$ and $B_{k,m}$ in terms of $H_{zn0}$. For any $k \geq 0$ and $m \geq 0$ we then have

$$|T_{k,m}| = |\lambda_k| |\lambda_m| \left| \alpha_c^{\xi(2k,m)} (A_{k,m} + A_{k,-m} + A_{-k,m} + A_{-k,-m}) \right.$$

$$+ \alpha_c^{\xi(2k+1,m)} (B_{k,m} + B_{k,-m}) + \alpha_c^{\xi(2k-1,m)} (B_{-k,m} + B_{-k,-m}) \right|$$

$$\leq \lambda_k \lambda_m \left[ |\alpha_c^{\xi(2k,m)}| |A_{k,m} + A_{k,-m} + A_{-k,m} + A_{-k,-m}| \right.$$

$$+ |\alpha_c^{\xi(2k+1,m)}| |B_{k,m} + B_{k,-m}| + |\alpha_c^{\xi(2k-1,m)}| |B_{-k,m} + B_{-k,-m}| \right]$$

$$\leq \lambda_k \lambda_m \left[ |\alpha_c^{\xi(2k,m)}| (|A_{k,m}| + |A_{k,-m}| + |A_{-k,m}| + |A_{-k,-m}|) \right.$$

$$+ \alpha_c^{\xi(2k+1,m)} (|B_{k,m}| + |B_{k,-m}|) + \alpha_c^{\xi(2k-1,m)} (|B_{-k,m}| + |B_{-k,-m}|) \right]$$

$$\leq \lambda_k \lambda_m \left[ \alpha_c^{\xi(2k,m)} (M + M + M) \right.$$

$$+ \alpha_c^{\xi(2k+1,m)} (M + M) + \alpha_c^{\xi(2k-1,m)} (M + M) \right)$$

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\[ + \alpha_c^{(2k+1,m)} (M + M) + \alpha_c^{(2k-1,m)} (M + M) \]
\[ = \lambda_k \lambda_m \left[ 4M \alpha_c^{(2k,m)} + 2M \alpha_c^{(2k+1,m)} + 2M \alpha_c^{(2k-1,m)} \right] \]
\[ \leq \lambda_k \lambda_m \left[ 4M \alpha_c^{(2k,m)} + 4M \alpha_c^{(2k+1,m)} + 4M \alpha_c^{(2k-1,m)} \right] \]
\[ = 4M \lambda_k \lambda_m \left[ \alpha_c^{(2k,m)} + \alpha_c^{(2k+1,m)} + \alpha_c^{(2k-1,m)} \right] \]
\[ \leq 4M \lambda_k \lambda_m \left[ \sqrt{|\alpha_c^m|} \sqrt{|\alpha_c^{|2k|}} + \sqrt{|\alpha_c^{|2k+1|}} \sqrt{|\alpha_c^{|2k-1|}} \right] \]
\[ = 4M \lambda_k \lambda_m \sqrt{\alpha_c^m} \left[ \sqrt{|\alpha_c^{|2k|}} + \sqrt{|\alpha_c^{|2k+1|}} + \sqrt{|\alpha_c^{|2k-1|}} \right] \]
\[ \leq 4M \lambda_k \lambda_m \sqrt{\alpha_c^m} \left[ \sqrt{\alpha_c^k} + \sqrt{\alpha_c^k} + \sqrt{\alpha_c^k} \right] \]
\[ = 12M \lambda_k \lambda_m \sqrt{\alpha_c^m} \sqrt{\alpha_c^k} \]  \hspace{1cm} (B.129)

so that \( \sum (\sum |T_{k,m}|) \) converges as desired by the comparison test for iterated series (Lemma B.7) since the iterated series (B.126) was shown to converge, noting that this has the same terms as the right side of the inequality (B.129).

**Theorem B.9.** The following three double series all converge absolutely and

\[ \sum_{k=0}^{\infty} \left( \sum_{m=0}^{\infty} T_{k,m} \right) = \sum_{k,m=0}^{\infty} T_{k,m} = \sum_{m=0}^{\infty} \left( \sum_{k=0}^{\infty} T_{k,m} \right) \]  \hspace{1cm} (B.130)

for any point \((\hat{x}, \hat{z}) \in \mathbb{R}^2\), where of course \(T_{k,m}\) is as defined in (B.8).

**Proof.** This readily follows from Theorem B.8 and Theorem 9.6 of [26].
Appendix C

Review of Accuracy Assessment Metrics

Here we review the three metrics used in the accuracy assessment of the analytical solutions. Information on the coefficient of determination comes from [42], though this is common knowledge in statistics. Information on the two relative error metrics is from [29], which is where these validation metrics were first introduced.

C.1 Coefficient of Determination ($R^2$)

Usually used to assess the accuracy of regression models, $R^2$ quantifies how much better of a fit a model of some function is as compared with simply the constant mean of that function over all of the sample points. Suppose that $y$ is some function from a (potentially multidimensional) domain $D$ to the (generally) complex numbers, and also a function $f$ from $D$ to the complex numbers that models $y$. Also consider the finite set of $N$ points \( \{x_i\}_{i \in \{1,2,\ldots,N\}} \) in the domain, and the corresponding $N$-element sets \( \{y_i\} \) and \( \{f_i\} \), where

\[
y_i = y(x_i) \quad \quad \quad f_i = f(x_i). \quad \quad \quad \quad \quad (C.1)
\]
The observed values \( y_i \) could be obtained from experiment or, as in our case, from sophisticated simulations. Of course the \( f_i \) values are obtained by simply evaluating the model at the data points \( x_i \).

The coefficient of determination is then defined as

\[
R^2 = 1 - \frac{\sum_{i=1}^{N} |f_i - y_i|^2}{\sum_{i=1}^{N} |y_i - \bar{y}|^2}
\]  

(C.2)

where

\[
\bar{y} = \frac{1}{N} \sum_{i=1}^{N} y_i
\]  

(C.3)

is of course the mean of the observed values \( y_i \). What makes \( R^2 \) useful for assessing model accuracy are the following properties that are easy to deduce from (C.2):

- \( R^2 \leq 0 \) indicates that the model fits the data only as good as or worse than the constant mean \( \bar{y} \) does. That is, a single constant value over the entire domain \( D \) fits better than the model!

- \( R^2 = 1 \) indicates that the model fits perfectly so that \( f_i = y_i \) at all the data points.

- \( 0 < R^2 < 1 \) indicates that the model fits better than the constant mean value but is not perfect. Obviously values closer to 1 indicate a better fit and closer 0 a worse fit.

We note that \( R^2 > 1 \) is not possible as the second term of (C.2) is always non-negative.

One is often cautioned when using \( R^2 \) for regression model assessment that it can be deceiving. The classic example is that of overfitting \( N \) data points with a polynomial of degree \( N \). As this will always fit the data points perfectly, it will have \( R^2 = 1 \), but these are often very poor models in practice with \( f \) values that wildly differ from a smoother \( y \) for values in between data points. With this in mind it is always a good idea to supplement the usage of \( R^2 \) with qualitative analysis using plots.
C.2 Relative Error Metrics

With the same \( N \) observed values \( y_i \) and modeled values \( f_i \) as in the previous section, both of the relative error metrics are based on calculating the relative error between \( f_i \) and \( y_i \) at each data point. These relative errors are of course

\[
e_{r_i} = \frac{|f_i - y_i|}{|y_i|}, \tag{C.4}
\]

recalling that again each \( y_i \) and \( f_i \) may be generally complex. The metrics then look at this set of \( e_{r_i} \) values from a statistical perspective, noting that of course they are all non-negative and real.

The first metric, called the mean relative error and denoted \( \% \text{RE}^m \), is specified as the mean of the relative errors as a percent as well as the probability of a relative error being no greater than this mean. The mean is of course simply

\[
\bar{e}_r = \frac{1}{N} \sum_{i=1}^{n} e_{r_i}, \tag{C.5}
\]

and the probability of a relative error being no greater than this mean is calculated from the samples as

\[
P = \frac{\text{Number of samples where } e_{r_i} \leq \bar{e}_r}{N}. \tag{C.6}
\]

This metric is useful to assess the accuracy of single model, but makes it difficult to compare models as each will generally have different \( \bar{e}_r \) and \( P \).

The second metric, which is called the specific relative error metric, and is denoted by \( \% \text{RE}^s \), addresses this shortcoming by simply reporting the probability of a relative error being no greater than a specific desired relative error \( e_r \) that is used used for all models.
This is of course calculated from the samples as

\[ P = \frac{\text{Number of samples where } e_{r_i} \leq e_r}{N}. \]  

(C.7)

This enables a more direct comparison between models as a model with a higher specific probability \( P \) is likely to be a better fit than one with a lower \( P \).

In [29] it is emphasized that some care must be taken to avoid the metrics being meaningless or misleading. In particular:

- Samples in which \( f_i = y_i = 0 \) will result in \text{nan} values for \( e_{r_i} \) as \( e_{r_i} = 0/0 \). Though this situation of course represents a perfect fit of the model at these points, the \text{nan} values can corrupt all of the above calculations. It is therefore recommend to replace any \text{nan} values in the set of relative errors by zeros, since this again represents a perfect fit and so \( e_{r_i} = 0 \).

- Samples in which \( f_i \) is nonzero but \( y_i = 0 \) will result in \text{inf} values since \( e_{r_i} = f_i/0 \). Here the infinite relative errors are of course correct, but they make the \( \%\text{RE}^m \) metric meaningless (as this results in \( \overline{e}_r = \infty \) so that \( P \) will always be 100%), noting that these do not adversely affect the \( \%\text{RE}^s \) metric. The recommended solution to this is to bound all \( e_{r_i} \) values prior to calculating \( \overline{e}_r \) and \( P \) in order to get a meaningful \( \%\text{RE}^m \).

Refer to [29] for more details on these metrics and a discussion about how to best bound the \( e_{r_i} \) values in the case of some values being infinite.
Bibliography


[72] Daniel Whitman. *Mathematics Stack Exchange: Showing that $f(x) = x \ln \frac{x-a}{x+a} < -2a$ for all $x > 0$*. Sept. 30, 2020. URL: https://math.stackexchange.com/questions/3846086/showing-that-fx-x-ln-fracx-axa-2a-for-all-x-0 (visited on 10/03/2020).

