7-16-2011

A Statistical Test for the Capacity Coefficient

Joseph W. Houpt  
*Wright State University - Main Campus, joseph.houpt@wright.edu*

James T. Townsend

Follow this and additional works at: [http://corescholar.libraries.wright.edu/psychology](http://corescholar.libraries.wright.edu/psychology)

Part of the [Cognition and Perception Commons](http://corescholar.libraries.wright.edu/psychology), [Cognitive Psychology Commons](http://corescholar.libraries.wright.edu/psychology), [Quantitative Psychology Commons](http://corescholar.libraries.wright.edu/psychology), and the [Statistical Models Commons](http://corescholar.libraries.wright.edu/psychology)

Repository Citation

[http://corescholar.libraries.wright.edu/psychology/8](http://corescholar.libraries.wright.edu/psychology/8)

This Presentation is brought to you for free and open access by the Psychology at CORE Scholar. It has been accepted for inclusion in Psychology Faculty Publications by an authorized administrator of CORE Scholar. For more information, please contact corescholar@www.libraries.wright.edu.
A Statistical Test for the Capacity Coefficient

Joseph W. Houpt  James T. Townsend

INDIANA UNIVERSITY
BLOOMINGTON

Math Psych '11
Medford, MA
July 16, 2011
The workload capacity coefficient \((C(t))\) is a measure of how a person's performance changes with changes in workload.

Until now there have been no non-parametric statistical tests for \(C(t)\).

Develop a statistic test for the capacity coefficient for both OR and AND tasks.

- Null hypothesis: \(C(t) = 1\)

Adapt the Nelson-Aalen Estimator for the cumulative reverse hazard function.

Define unbiased and consistent estimators of OR and AND UCIP performance that are Gaussian Processes in the limit.
The Motivation

- Can we dedicate the same amount of resources to processing each source when there are more sources?
The Motivation

- Can we dedicate the same amount of resources to processing each source when there are more sources?
  - Fewer resources available for each process as the number of sources increases: Limited capacity.
The Motivation

- Can we dedicate the same amount of resources to processing each source when there are more sources?
  - Unchanged amount of resources available for each process as the number of sources increases: Unlimited capacity.

\[ N=1 \quad N=2 \]
The Motivation

- Can we dedicate the same amount of resources to processing each source when there are more sources?
  - More resources available for each process as the number of sources increases: Super capacity.

N=1

N=2
The Measure

- Fix a baseline for performance: Unlimited Capacity, Independent, Parallel (UCIP) model.
The Measure

- Fix a baseline for performance: Unlimited Capacity, Independent, Parallel (UCIP) model.
  - No change due to increased workload
The Measure

- Fix a baseline for performance: Unlimited Capacity, Independent, Parallel (UCIP) model.
  - No change due to increased workload
  - Stochastic independence
The Measure

- Fix a baseline for performance: Unlimited Capacity, Independent, Parallel (UCIP) model.
  - No change due to increased workload
  - Stochastic independence
The Measure

- Fix a baseline for performance: Unlimited Capacity, Independent, Parallel (UCIP) model.
  - No change due to increased workload
  - Stochastic independence

First-Terminating (OR)  Exhaustive (AND)
The Measure

- Fix a baseline for performance: Unlimited Capacity, Independent, Parallel (UCIP) model.
  - No change due to increased workload
  - Stochastic independence

First-Terminating (OR)  
(Townsend & Nozawa, 1995)

$$S_{AB}(t) = S_A(t)S_B(t)$$

Exhaustive (AND)  
(Townsend & Wenger, 2004)

$$F_{AB}(t) = F_A(t)F_B(t)$$
The Measure

- Fix a baseline for performance: Unlimited Capacity, Independent, Parallel (UCIP) model.
  - No change due to increased workload
  - Stochastic independence

First-Terminating (OR)
(Townsend & Nozawa, 1995)

\[ S_{AB}(t) = S_A(t)S_B(t) \]
\[ \log[S_{AB}(t)] = \log[S_A(t)S_B(t)] \]

Exhaustive (AND)
(Townsend & Wenger, 2004)

\[ F_{AB}(t) = F_A(t)F_B(t) \]
\[ \log[F_{AB}(t)] = \log[F_A(t)F_B(t)] \]
The Measure

- Fix a baseline for performance: Unlimited Capacity, Independent, Parallel (UCIP) model.
  - No change due to increased workload
  - Stochastic independence

First-Terminating (OR) (Townsend & Nozawa, 1995)

\[
S_{AB}(t) = S_A(t)S_B(t)
\]
\[
\log[S_{AB}(t)] = \log[S_A(t)S_B(t)]
\]
\[
H_{AB}(t) = H_A(t) + H_B(t)
\]

Exhaustive (AND) (Townsend & Wenger, 2004)

\[
F_{AB}(t) = F_A(t)F_B(t)
\]
\[
\log[F_{AB}(t)] = \log[F_A(t)F_B(t)]
\]
\[
K_{AB}(t) = K_A(t) + K_B(t)
\]
The Measure

- Fix a baseline for performance: Unlimited Capacity, Independent, Parallel (UCIP) model.
  - No change due to increased workload
  - Stochastic independence

First-Terminating (OR)
(Townsend & Nozawa, 1995)

\[
\begin{align*}
S_{AB}(t) &= S_A(t)S_B(t) \\
\log [S_{AB}(t)] &= \log [S_A(t)S_B(t)] \\
H_{AB}(t) &= H_A(t) + H_B(t) \\
C_{or}(t) &= \frac{H_{AB}(t)}{H_A(t) + H_B(t)}
\end{align*}
\]

Exhaustive (AND)
(Townsend & Wenger, 2004)

\[
\begin{align*}
F_{AB}(t) &= F_A(t)F_B(t) \\
\log [F_{AB}(t)] &= \log [F_A(t)F_B(t)] \\
K_{AB}(t) &= K_A(t) + K_B(t) \\
C_{and}(t) &= \frac{K_A(t) + K_B(t)}{K_{AB}(t)}
\end{align*}
\]

Reverse Hazard Functions: Chechile (2011)
Limited Capacity

Worse than predicted by a UCIP model
The Measure

Limited Capacity
Worse than predicted by a UCIP model

Unlimited Capacity
Predicted by a UCIP model
The Measure

Limited Capacity
Worse than predicted by a UCIP model

Unlimited Capacity
Predicted by a UCIP model

Super Capacity
Better than predicted by a UCIP model
The Nelson-Aalen Estimator

Cumulative Hazard Function

\[ H(t) = \int_0^t \frac{f(s)}{1 - F(s)} \, ds = -\log [1 - F(t)] \]

\[ \hat{H}(t) = \sum_{t_i \in \{RT \leq t\}} \frac{1}{Y(t_i)} \]

\[ \text{Var} (\hat{H}(t)) = \sum_{t_i \in \{RT \leq t\}} \frac{1}{Y^2(t_i)} \]

RT = the set of sample response times
\( Y(t) = \# \) of responses that have not occurred as of immediately before \( t \)
The Nelson-Aalen Estimator

Cumulative Hazard Function

- Unbiased: \( \mathbb{E} \left[ H(t) - \hat{H}(t) \right] = 0 \)
- Consistent: \( \lim_{\text{size of } \mathbb{R}^T \to \infty} \hat{H}(t) = H(t) \)
- \( \left[ H(t) - \hat{H}(t) \right] \Rightarrow \text{Gaussian Process} \)

(e.g., Aalen, Borgan, & Gjessing, H. K., 2008)
The Nelson-Aalen Estimator
Cumulative Reverse Hazard Function

\[ K(t) = \int^\infty_t \frac{f(s)}{F(s)} \; ds = \log [F(t)] \]

\[ G(t) = \# \text{ of responses that have occurred up to and including } t \]

\[ \hat{K}(t) = \sum_{t_i \in \{RT > t\}} \frac{1}{G(t_i)} \quad \text{Var} (\hat{K}(t)) = \sum_{t_i \in \{RT > t\}} \frac{1}{G^2(t_i)} \]
The Nelson-Aalen Estimator
Cumulative Reverse Hazard Function

- Unbiased: $E \left[ K(t) - \hat{K}(t) \right] = 0$
- Consistent: $\lim_{\text{size of } R \to \infty} \hat{K}(t) = K(t)$
- $\left[ K(t) - \hat{K}(t) \right] \Rightarrow $ a Gaussian Process
UCIP Performance

OR Process

\[ 1 - F_{\text{UCIP}}(t) = \prod_{i=1}^{m} [1 - F_i(t)] \]

\[ \log (1 - F_{\text{UCIP}}(t)) = \log \left( \prod_{i=1}^{m} [1 - F_i(t)] \right) \]

\[ H_{\text{UCIP}}(t) = \sum_{i=1}^{m} \log (1 - F_i(t)) = \sum_{i=1}^{m} H_i(t). \]

\[ \hat{H}_{\text{UCIP}}(t) \equiv \sum_{i=1}^{m} \hat{H}_i(t) \quad \text{Var} \left( \hat{H}_{\text{UCIP}}(t) \right) = \sum_{i=1}^{m} \sum_{t_j \in RT(i) < t} \frac{1}{Y_i^2(t_j)} \]
UCIP Performance
AND Process

\[ F_{UCIP}(t) = \prod_{i=1}^{m} F_i(t) \]

\[ \log (F_{UCIP}(t)) = \log \left( \prod_{i=1}^{m} F_i(t) \right) \]

\[ K_{UCIP}(t) = \sum_{i=1}^{m} \log (F_i(t)) = \sum_{i=1}^{m} K_i(t). \]

\[ \hat{K}_{UCIP}(t) \equiv \sum_{i=1}^{m} \hat{K}_i(t) \quad \text{Var} \left( \hat{K}_{UCIP}(t) \right) = \sum_{i=1}^{m} \sum_{t_j \in \text{RT}(i) < t} \frac{1}{G_i^2(t_j)} \]
Test Statistic

OR Task

\[ Z_{or}(t) = \sum_{t_j \in \{RT(r) < t\}} \frac{L_{or}(t_j)}{Y_r(t_j)} - \sum_{i=1}^{k} \sum_{t_j \in \{RT(i) < t\}} \frac{L_{or}(t_j)}{Y_i(t_j)} \]

\[ \text{Var}[Z_{or}(t)] = \sum_{t_j \in \{RT(r) < t\}} \frac{L_{or}(t_j)}{Y_r^2(t_j)} + \sum_{i=1}^{k} \sum_{t_j \in \{RT(i) < t\}} \frac{L_{or}(t_j)}{Y_i^2(t_j)} \]
Test Statistic

OR Task

\[
Z_{or}(t) = \sum_{t_j \in \{RT(r) < t\}} \frac{L_{or}(t_j)}{Y_r(t_j)} - \sum_{i=1}^k \sum_{t_j \in \{RT(i) < t\}} \frac{L_{or}(t_j)}{Y_i(t_j)}
\]

\[
\text{Var} [Z_{or}(t)] = \sum_{t_j \in \{RT(r) < t\}} \frac{L_{or}(t_j)}{Y_r^2(t_j)} + \sum_{i=1}^k \sum_{t_j \in \{RT(i) < t\}} \frac{L_{or}(t_j)}{Y_i^2(t_j)}
\]

\[
U_{or} = \frac{Z_{or}(t)}{\text{Var} [Z_{or}(t)]} \sim \mathcal{N}(0, 1)
\]
Test Statistic

AND Task

\[ Z_{\text{and}}(t) = \sum_{t_j \in \{\text{RT}(r) > t\}} \frac{L_{\text{and}}(t_j)}{G_r(t_j)} - \sum_{i=1}^{k} \sum_{t_j \in \{\text{RT}(i) > t\}} \frac{L_{\text{and}}(t_j)}{G_i(t_j)} \]

\[ \text{Var}[Z_{\text{and}}(t)] = \sum_{t_j \in \{\text{RT}(r) > t\}} \frac{L_{\text{and}}(t_j)^2}{G_r^2(t_j)} + \sum_{i=1}^{k} \sum_{t_j \in \{\text{RT}(i) > t\}} \frac{L_{\text{and}}(t_j)^2}{G_i^2(t_j)} \]

\[ U_{\text{and}} = \frac{Z_{\text{and}}(t)}{\text{Var}[Z_{\text{and}}(t)]} \sim \mathcal{N}(0, 1) \]
Weighting Function

\[ L_{or}(t) = \begin{cases} 
\frac{Y_r(t)(\sum_{i=1}^{k} Y_i(t))}{Y_r(t)+\sum_{i=1}^{k} Y_i(t)} & \text{if for all } i, Y_i(t) > 0 \\
0 & \text{otherwise}
\end{cases} \]

\[ L_{and}(t) = \begin{cases} 
\frac{G_r(t)(\sum_{i=1}^{k} G_i(t))}{G_r(t)+\sum_{i=1}^{k} G_i(t)} & \text{if for all } i, G_i(t) > 0 \\
0 & \text{otherwise}
\end{cases} \]
OR: Exponential Race Model

$C_{or}(t) < 1$
OR: Exponential Race Model

$C_{or}(t) > 1$
The Task

PP  PA  AP  AA
The Results

<table>
<thead>
<tr>
<th></th>
<th>OR Task</th>
<th>AND Task</th>
</tr>
</thead>
<tbody>
<tr>
<td>BJ</td>
<td>-10.75 ***</td>
<td>2.88 **</td>
</tr>
<tr>
<td>RS</td>
<td>-4.67 ***</td>
<td>2.93 **</td>
</tr>
<tr>
<td>JS</td>
<td>-8.41 ***</td>
<td>3.16 ***</td>
</tr>
<tr>
<td>MB</td>
<td>-7.12 ***</td>
<td>2.87 **</td>
</tr>
<tr>
<td>RM</td>
<td>-9.69 ***</td>
<td>3.06 **</td>
</tr>
<tr>
<td>LB</td>
<td>-3.02 **</td>
<td>2.70 **</td>
</tr>
<tr>
<td>JG</td>
<td>-5.54 ***</td>
<td>3.01 **</td>
</tr>
<tr>
<td>WY</td>
<td>-6.08 ***</td>
<td>2.99 **</td>
</tr>
<tr>
<td>AW</td>
<td>-5.75 ***</td>
<td>2.88 **</td>
</tr>
</tbody>
</table>

* $p < 0.05$
** $p < 0.01$
*** $p < 0.001$
Adapted the Nelson-Aalen Estimator for the cumulative reverse hazard function.
- Adapted the Nelson-Aalen Estimator for the cumulative reverse hazard function.
- Defined unbiased and consistent estimators of OR and AND UCIP performance that are Gaussian Processes in the limit.
Adapted the Nelson-Aalen Estimator for the cumulative reverse hazard function.

Defined unbiased and consistent estimators of OR and AND UCIP performance that are Gaussian Processes in the limit. Developed a statistical test for the capacity coefficient for both OR and AND tasks.

- Null hypothesis: $C(t) = 1$
- Adapted the Nelson-Aalen Estimator for the cumulative reverse hazard function.
- Defined unbiased and consistent estimators of OR and AND UCIP performance that are Gaussian Processes in the limit.
- Developed a statistical test for the capacity coefficient for both OR and AND tasks.
  - Null hypothesis: $C(t) = 1$
- Yes, we are working on a non-parametric Bayesian $C(t)$ test (with Andrew Heathcote)
Adapted the Nelson-Aalen Estimator for the cumulative reverse hazard function.

Defined unbiased and consistent estimators of OR and AND UCIP performance that are Gaussian Processes in the limit.

Developed a statistical test for the capacity coefficient for both OR and AND tasks.

Null hypothesis: $C(t) = 1$

Yes, we are working on a non-parametric Bayesian $C(t)$ test (with Andrew Heathcote)

Thank you!