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A STRUCTURAL RESULT OF IRREDUCIBLE INCLUSIONS OF TYPE III_λ FACTORS, $\lambda \in (0, 1)$

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ABSTRACT. Given an irreducible inclusion of factors with finite index $N \subset M$, where M is of type $\text{III}_{\lambda^{1/m}}$, N of type $\text{III}_{\lambda^{1/n}}$, $0 < \lambda < 1$, and m, n are relatively prime positive integers, we will prove that if $N \subset M$ satisfies a commuting square condition, then its structure can be characterized by using fixed point algebras and crossed products of automorphisms acting on the middle inclusion of factors associated with $N \subset M$. Relations between $N \subset M$ and a certain G -kernel on subfactors are also discussed.

1. INTRODUCTION

Let $N \overset{E_N^M}{\subset} M$ be an irreducible inclusion of type III factors such that M is of type $\text{III}_{\lambda^{1/m}}$ and N of type $\text{III}_{\lambda^{1/n}}$, where $\lambda \in (0, 1)$, m, n are positive relatively prime integers, and $E_N^M: M \rightarrow N$ is a normal faithful conditional expectation with finite index. We are interested in studying the structure of such an inclusion and its relevance to the classification problem. By the results in [17], such an inclusion can be decomposed into separate sub-inclusions, each of which admits a simple description using automorphisms; more specifically, there exist type III_λ subfactors P and Q with $N \overset{E_N^Q}{\subset} Q \overset{E_Q^P}{\subset} P \overset{E_P^M}{\subset} M$ and such that $\text{Ind}(E_P^M) = m$, $\text{Ind}(E_N^Q) = n$. Moreover, P is the fixed point algebra of the restriction of a modular automorphism of order m on M , whereas Q is the crossed product of N with a modular automorphism of order n on N . As for $Q \subset P$, there exists a joint discrete decomposition in the sense that there exist type II_∞ factors $Q^\infty \subset P^\infty$ and an automorphism θ which acts simultaneously on $Q^\infty \subset P^\infty$ with $\text{mod}(\theta) = \lambda$ and $Q \subset P$ is isomorphic to $Q^\infty \times_\theta \mathbf{Z} \subset P^\infty \times_\theta \mathbf{Z}$. The classification of these sub-inclusions is now well understood. For instance, the top and bottom inclusions $P \subset M$ and $N \subset Q$ are each uniquely determined (cf. [17]) and the middle inclusion $Q \subset P$ is classified by the type II core $Q^\infty \subset P^\infty$, the module of θ and the standard invariant of θ on the tower of higher relative commutants of $Q^\infty \subset P^\infty$ by [23].

Despite the success in classifying these separate sub-inclusions, it remains an open problem to classify the original inclusion $N \subset M$. In this paper, we continue to investigate the feasibility of characterizing the structure of a general inclusion of factors such as $N \subset M$ in terms of automorphisms. First we observe that using Takesaki duality, we can find properly outer, periodic automorphisms α and β

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with non-trivial modules, acting on P and Q respectively, so that M is the crossed product $P \times_{\alpha} \mathbf{Z}_m$ and N is the fixed point algebra $Q^{\mathbf{Z}_n}$ via β . However, it is not possible, in general, to have α and β act on the inclusion $Q \subset P$. In fact we will prove that this is the case if and only if the inclusions $N \subset Q \subset P \subset M$ satisfy certain commuting square conditions which are equivalent to some restriction and extension conditions of the Longo canonical endomorphism of $Q \subset P$ and this property is also equivalent to the existence of two trace-scaling automorphisms θ_1 and θ_2 on $Q^{\infty} \subset P^{\infty}$ such that: $\theta = \theta_1^m = \theta_2^n$, $M = P^{\infty} \times_{\theta_1} \mathbf{Z}$ and $N_1 = Q^{\infty} \times_{\theta_2} \mathbf{Z}$, where N_1 is the basic construction of $N \subset Q$. We can then associate to $N \subset M$ a G -kernel on $Q \subset P$, i.e., a homomorphism of the group G into $\text{Aut}(P, Q)/\text{Int}(Q)$, arising from the subgroup generated by α and β modulo $\text{Int}(Q)$. We can then show that the isomorphism class of $N \subset M$ is determined by the conjugacy class of the corresponding G -kernel.

We are thankful to the referee for bringing the results in [9] to our attention and for making several useful comments and suggestions. We also thank Prof. Hideki Kosaki for pointing out an error in Proposition 1 of the original version of this paper.

2. MAIN RESULTS

We begin by recalling the definition of commuting squares and some basic results about them that we will need.

An inclusion of factors

$$\begin{array}{ccc} P & \subset & M \\ \cup & & \cup \\ Q & \subset & N \end{array}$$

is said to be a *commuting square* with respect to the expectations of minimal indices E_N^M, E_P^M, E_Q^N and E_Q^P if $E_N^M|P = E_Q^P$ or equivalently, if $E_P^M|N = E_Q^N$ (cf. [8]).

The commuting square

$$\begin{array}{ccc} P & \subset & M \\ \cup & & \cup \\ Q & \subset & N \end{array}$$

is called *co-commuting* (cf. [24]) or *non-degenerate* (cf. [22]) if

$$\begin{array}{ccc} N' & \subset & Q' \\ \cup & & \cup \\ M' & \subset & P' \end{array}$$

is also a commuting square with respect to the expectations with minimal indices. We refer the reader to [13, 22, 24] for additional properties of commuting and co-commuting squares. We only mention the following result in [24, 13]: the commuting square

$$\begin{array}{ccc} P & \subset & M \\ \cup & & \cup \\ Q & \subset & N \end{array}$$

is co-commuting if and only if $\text{Ind } E_N^M = \text{Ind } E_Q^P$ or $\text{Ind } E_P^M = \text{Ind } E_Q^N$.

For convenience, unless otherwise stated, E_B^A will denote the conditional expectation of minimal index from A onto B and $[A : B]_0$ the minimal index value.

Lemma 1. *Let*

$$\begin{array}{ccc} P & \subset & M \\ \cup & & \cup \\ Q & \subset & N \end{array}$$

be a commuting square (with respect to the expectations with minimal indices). Let $S \subset R$ be intermediate subfactors such that $P \subset R \subset M, Q \subset S \subset N, E_P^M = E_P^R E_R^M$ and $E_Q^N = E_Q^S E_S^N$. Then

$$\begin{array}{ccc} P & \subset & R \\ \cup & & \cup \\ Q & \subset & S \end{array}$$

is also a commuting square.

Proof. For any $s \in S, E_P^R(s) = E_P^M(s)$, which belongs to Q because

$$\begin{array}{ccc} P & \subset & M \\ \cup & & \cup \\ Q & \subset & N \end{array}$$

is a commuting square by assumption. And so by [8],

$$\begin{array}{ccc} P & \subset & R \\ \cup & & \cup \\ Q & \subset & S \end{array}$$

is a commuting square.

Q.E.D.

Lemma 2. *Let*

$$\begin{array}{ccc} P & \subset & M \\ \cup & & \cup \\ Q & \subset & N \end{array}$$

be a commuting square of factors with respect to the expectations with minimal indices. Then $[M : N]_0 \geq [P : Q]_0$ and $[M : P]_0 \geq [N : Q]_0$.

Proof. By [21], for any $x \in P_+, E_N^M(x) \geq [M : N]_0^{-1}x$. Since $E_Q^P = E_N^M|P, E_Q^P(x) \geq [M : N]_0^{-1}x$. Using [21] again, we have $[P : Q]_0^{-1} \geq [M : N]_0^{-1}$ and so $[M : N]_0 \geq [P : Q]_0$. Similarly, $[M : P]_0 \geq [N : Q]_0$. Q.E.D.

In the following, all factors under consideration are of type III and thus without loss of generality, we may assume that they have a common cyclic and separating vector and are acting standardly on the same Hilbert space by the results of [5]. Let us also recall that for an inclusion of factors $N \subset M$ which act standardly on the same Hilbert space, the canonical endomorphism of $N \subset M$ is defined by $\gamma_{M,N} = \text{Ad } J_N J_M|_M$. For additional properties on the canonical endomorphism in relation to the index theory of $N \subset M$, we refer the reader to [16].

Proposition 1. *Let $N \overset{E_N^Q}{\subset} Q \overset{E_Q^P}{\subset} P$ be an irreducible inclusion of type III factors with finite index. The following are equivalent.*

(i) *There is a subfactor N_0 such that*

$$\begin{array}{ccc} Q & \subset & P \\ \cup & & \cup \\ N & \subset & N_0 \end{array}$$

is a commuting and co-commuting square.

(ii) *There is a choice of the canonical endomorphisms $\gamma_{Q,N}$ and $\gamma_{P,Q}$ such that $\gamma_{Q,N}(P) \subset P$, $\gamma_{P,Q}(N) \subset N$ and*

$$\begin{array}{ccc} \gamma_{P,Q}(Q) & \subset & Q \\ \cup & & \cup \\ \gamma_{P,Q}(N) & \subset & N \end{array}$$

is a commuting and co-commuting square.

Proof. (i) \Rightarrow (ii) By [9], there is a choice of $\gamma_{Q,N}$ and $\gamma_{P,Q}$ of $Q \subset P$ such that $\gamma_{Q,N}(P) \subset P$ and $\gamma_{P,Q}(N) \subset N$. It follows that $\gamma_{P,Q}(Q)$ and $\gamma_{P,Q}(N)$ are the second factors in the downward basic constructions of $Q \subset P$ and $N \subset N_0$, respectively. Since

$$\begin{array}{ccc} Q & \subset & P \\ \cup & & \cup \\ N & \subset & N_0 \end{array}$$

is a co-commuting and commuting square, it is easy to check that

$$\begin{array}{ccc} \gamma_{P,Q}(Q) & \subset & Q \\ \cup & & \cup \\ \gamma_{P,Q}(N) & \subset & N \end{array}$$

is a commuting square which is also co-commuting because $[Q : N]_0 = [\gamma_{P,Q}(Q) : \gamma_{P,Q}(N)]_0$.

(ii) \Rightarrow (i) Let J_P, J_Q and J_N be the modular conjugate operators on their respective factor such that $\gamma_{P,Q} = \text{Ad}(J_Q J_P)$ and $\gamma_{Q,N} = \text{Ad}(J_N J_Q)$. Since $\gamma_{P,N} = \gamma_{Q,N} \cdot \gamma_{P,Q}$, we have the following inclusions of factors:

$$\begin{array}{ccccc} \gamma_{P,Q}(Q) & \subset & \gamma_{P,Q}(P) & \subset & Q \\ \cup & & & & \cup \\ \gamma_{P,Q}(N) & \subset & & & N \\ \cup & & & & \cup \\ \gamma_{P,N}(Q) & \subset & \gamma_{P,N}(P) & \subset & \gamma_{Q,N}(Q) \\ \cup & & & & \\ \gamma_{P,N}(N) & & & & \end{array}$$

Since

$$\begin{array}{ccc} \gamma_{P,Q}(Q) & \subset & Q \\ \cup & & \cup \\ \gamma_{P,Q}(N) & \subset & N \end{array}$$

is a commuting and co-commuting square by assumption, Proposition 2.3 in [9] implies that $\gamma_{Q,N}$ restricts to a canonical endomorphism of $\gamma_{P,Q}(N) \subset \gamma_{P,Q}(Q)$ and thus $\gamma_{P,Q}(N)$ is the basic construction of $\gamma_{P,N}(N) \subset \gamma_{P,N}(Q)$. Let \tilde{N} be the von Neumann algebra generated by $\gamma_{P,Q}(N)$ and $\gamma_{P,N}(P)$. As $\tilde{N}' \cap P = \mathbf{C}$, $\gamma_{P,N}(P)$ is irreducible in \tilde{N} and hence \tilde{N} is a factor. Also since $\gamma_{Q,N}(P) \subset P$, $\gamma_{P,N}(P) \subset \gamma_{P,Q}(P)$ and so $\tilde{N} \subset \gamma_{P,Q}(P)$ as well. By repeated applications of Takesaki's

criterion in [26], we see that there exist conditional expectations on each of the following inclusions: $\gamma_{P,Q}(N) \subset \tilde{N}$, $\gamma_{P,N}(P) \subset \tilde{N}$, $N_0 \subset \gamma_{P,Q}(P)$ and $\tilde{N} \subset N$. Moreover, as these expectations all have finite indices, we may just assume that they have minimal indices.

From the assumption that

$$\begin{array}{ccc} \gamma_{P,Q}(Q) & \subset & Q \\ \cup & & \cup \\ \gamma_{P,Q}(N) & \subset & N \end{array}$$

is a commuting and co-commuting square, it follows that

$$\begin{array}{ccc} \gamma_{P,Q}(N) & \subset & N \\ \cup & & \cup \\ \gamma_{P,N}(Q) & \subset & \gamma_{Q,N}(Q) \end{array}$$

is also a commuting and co-commuting square. Hence by Lemma 1,

$$\begin{array}{ccc} \gamma_{P,Q}(Q) \subset \gamma_{P,Q}(P) & & \gamma_{P,Q}(N) \subset \tilde{N} \\ \cup & \cup & \cup \\ \gamma_{P,Q}(N) \subset \tilde{N} & \text{and} & \gamma_{P,N}(Q) \subset \gamma_{P,N}(P) \end{array}$$

are commuting squares. Thus by Lemma 2,

$$[P : Q]_0 = [\gamma_{P,Q}(P) : \gamma_{P,Q}(Q)]_0 \geq [\tilde{N} : \gamma_{P,Q}(N)]_0 \geq [\gamma_{P,N}(P) : \gamma_{P,N}(Q)]_0 = [P : Q]_0$$

and so $[\tilde{N} : \gamma_{P,Q}(N)]_0 = [P : Q]_0$. Therefore

$$\begin{array}{ccc} \gamma_{P,Q}(Q) & \subset & \gamma_{P,Q}(P) \\ \cup & & \cup \\ \gamma_{P,Q}(N) & \subset & \tilde{N} \end{array}$$

is a commuting and co-commuting square by [24]. Now let $N_0 = \gamma_{P,Q}^{-1}(\tilde{N})$; then

$$\begin{array}{ccc} Q & \subset & P \\ \cup & & \cup \\ N & \subset & N_0 \end{array}$$

is a commuting and co-commuting square.

Q.E.D.

Let us also state and prove the following dual version of Proposition 1.

Proposition 2. Let $Q \overset{E_Q^P}{\subset} P \overset{E_P^M}{\subset} M$ be an irreducible inclusion of type III factors with finite indices. The following are equivalent.

(i) There is a factor M_0 such that

$$\begin{array}{ccc} P & \subset & M \\ \cup & & \cup \\ Q & \subset & M_0 \end{array}$$

is a commuting and co-commuting square.

(ii) There is a choice of the canonical endomorphisms $\gamma_{M,P}$ and $\gamma_{P,Q}$ such that $\gamma_{M,P}(Q) \subset Q$, $\gamma_{P,Q}(M) \subset M$ and

$$\begin{array}{ccc} \gamma_{P,Q}(M) & \subset & M \\ \cup & & \cup \\ \gamma_{P,Q}(P) & \subset & P \end{array}$$

is a commuting and co-commuting square.

Proof. (i) \Rightarrow (ii) By Proposition 2.3 of [9], there exist canonical endomorphisms $\gamma_{M,P}$ and $\gamma_{P,Q}$ that satisfy the stated extension and restriction conditions. Moreover, $\gamma_{P,Q}(P)$ and $\gamma_{P,Q}(M)$ are the basic constructions of $Q \subset P$ and $M_0 \subset M$, respectively, and hence

$$\begin{array}{ccc} \gamma_{P,Q}(M) & \subset & M \\ \cup & & \cup \\ \gamma_{P,Q}(P) & \subset & P \end{array}$$

is a commuting and co-commuting square because

$$\begin{array}{ccc} P & \subset & M \\ \cup & & \cup \\ Q & \subset & M_0 \end{array}$$

is commuting and co-commuting by assumption.

(ii) \Rightarrow (i) By taking the commutants of the factors, the assumptions in (ii) imply that the inclusion $M' \subset P' \subset Q'$ satisfies the hypotheses of (ii) of Proposition 1. Therefore there is a factor M_0 such that

$$\begin{array}{ccc} P' & \subset & Q' \\ \cup & & \cup \\ M' & \subset & M'_0 \end{array}$$

is commuting and co-commuting. Passing to the commutants of these factors will yield the desired commuting and co-commuting square. Q.E.D.

In order to describe the structure of an inclusion of type III factors that satisfies both the commuting and co-commuting square conditions of Propositions 1 and 2, we need the information about the Connes-Takesaki modules (cf. [4]) of the associated automorphisms provided by the next lemma.

Lemma 3. (i) *Let P be a factor of type III_λ , $\lambda \in (0, 1)$, and $m \in \mathbf{N}$. Let φ be a generalized trace on P and $T = [2\pi / \ln \lambda]$. Then $P^{\sigma_{T/m}^\varphi}$ and $P \times_{\sigma_{T/m}^\varphi} \mathbf{Z}_m$ are both of type III_{λ^m} and if α is either the dual or the pre-dual automorphism of $\sigma_{T/m}^\varphi$ on $P \times_{\sigma_{T/m}^\varphi} \mathbf{Z}_m$ or $P^{\sigma_T^\varphi}$, then $\text{mod}(\alpha) \equiv \lambda$.*

(ii) *Let $\lambda \in (0, 1)$ and $Q \overset{E}{\subset} P$ be an inclusion of type III_λ factors with a common discrete decomposition. Then for any $\alpha \in \text{Aut}(P, Q)$ which commutes with E , $\text{mod}(\alpha|P) = \text{mod}(\alpha|Q)$.*

Proof. (i) We will only prove the statement about the module of α as it is well known that $P^{\sigma_{T/m}^\varphi}$ and $P \times_{\sigma_{T/m}^\varphi} \mathbf{Z}_m$ are both of type III_{λ^m} . Suppose that α is the automorphism that is pre-dual to $\sigma_{T/m}^\varphi$ on $P^{\sigma_{T/m}^\varphi}$. Then $P = P^{\sigma_{T/m}^\varphi} \times_\alpha \mathbf{Z}_m$ and $\sigma_{T/m}^\varphi(U) = e^{-2\pi i/m}U$, where U is the unitary in P that implements α . On the other hand, it follows from [17] that the pair $P^{\sigma_T^\varphi} \subset P$ is isomorphic to $P^\infty \times_{\theta^m} \mathbf{Z} \subset P^\infty \times_\theta \mathbf{Z}$, where $\{P^\infty, \theta\}$ is a discrete decomposition of P . Let V be the implementing unitary of θ ; then as φ is a generalized trace on P (cf. [3]), $\sigma_t^\varphi(V) = \lambda^{it}V$, for all $t \in \mathbf{R}$. In particular, $\sigma_{T/m}^\varphi(V) = \lambda^{iT/m}V = e^{-2\pi i/m}V$. Hence $W = UV^* \in P^{\sigma_{T/m}^\varphi}$ and $\theta(x) = \text{Ad } W^* \cdot \alpha(x)$ for every $x \in P^\varphi$. Since $\text{mod}(\theta) = \lambda$, we deduce that $\text{mod}(\alpha) \equiv \lambda$ (cf. [4]).

If α is the dual automorphism to $\sigma_{T/m}^\varphi$, then using the just established result for the pre-dual of $\sigma_{T/m}^\varphi$ and the Takesaki Duality Theorem, we infer that $\text{mod}(\alpha) \equiv \lambda$.

(ii) Let φ be a normal faithful semi-finite weight on Q . Then $\varphi \cdot \alpha \sim \mu^{-1}\varphi$. Since α and E_Q^P commute, $\varphi \cdot E_Q^P \cdot \alpha \sim \mu^{-1}\varphi \cdot E_Q^P$ as well, i.e., $\text{mod}(\alpha) = \text{mod}(\alpha|Q)$. Q.E.D.

We can now prove the following characterization based on automorphisms of an inclusion of the form $N \subset Q \subset P \subset M$ that satisfies the commuting and co-commuting square condition.

Theorem 1. *Let $\lambda \in (0, 1)$, and $m, n \in \mathbf{N}$ be relatively prime. Suppose that $N \overset{E_N^Q}{\subset} Q \overset{E_Q^P}{\subset} P \overset{E_P^M}{\subset} M$ is an irreducible inclusion of factors such that Q and P are both of type III $_{\lambda}$, M of type III $_{\lambda^{1/m}}$ and N of type III $_{\lambda^{1/n}}$, $[M : P]_0 = m$ and $[Q : N]_0 = n$. The following are equivalent.*

(i) *There exist canonical endomorphisms $\gamma_{Q,N}$, $\gamma_{P,Q}$ and $\gamma_{M,P}$ which satisfy: $\gamma_{P,Q}(N) \subset N$, $\gamma_{P,Q}(M) \subset M$, $\gamma_{M,P}(Q) \subset Q$, $\gamma_{Q,N}(N) \subset N$ and such that*

$$\begin{array}{ccc} \gamma_{P,Q}(Q) & \subset & Q \\ \cup & & \cup \\ \gamma_{P,Q}(N) & \subset & N \end{array} \quad \text{and} \quad \begin{array}{ccc} \gamma_{P,Q}(M) & \subset & M \\ \cup & & \cup \\ \gamma_{P,Q}(P) & \subset & P \end{array}$$

are commuting and co-commuting squares.

(ii) *There exist properly outer and periodic automorphisms α, β acting on $Q \subset P$ such that α has order m , β has order n , $\text{mod}(\alpha) \equiv \lambda^{1/m}$, $\text{mod}(\beta) \equiv \lambda^{1/n}$ and such that $N = Q^{\beta}$ and $M = P \times_{\alpha} \mathbf{Z}_m$.*

(iii) *Let $\{Q^{\infty} \subset P^{\infty}, \theta\}$ be a common discrete decomposition of $Q \subset P$. Then θ is both an m and an n power, i.e., there exist trace-scaling automorphisms θ_1 and θ_2 on $Q^{\infty} \subset P^{\infty}$ such that $\theta = \theta_1^m = \theta_2^n$, $M = P^{\infty} \times_{\theta_1} \mathbf{Z}$ and $N_1 = Q^{\infty} \times_{\theta_2} \mathbf{Z}$, where N_1 is the basic construction of $N \subset Q$.*

Proof. The equivalence between (ii) and (iii) was established in [19] and so we only need to prove that (i) and (ii) are equivalent.

(ii) \Rightarrow (i) Assuming that α, β exist, then it is easy to check that

$$\begin{array}{ccc} P & \subset & P \times_{\alpha} \mathbf{Z}_m = M \\ \cup & & \cup \\ Q & \subset & Q \times_{\alpha} \mathbf{Z}_m \end{array} \quad \text{and} \quad \begin{array}{ccc} P^{\beta} & \subset & P \\ \cup & & \cup \\ N = Q^{\beta} & \subset & Q \end{array}$$

are commuting co-commuting squares and (ii) follows from Propositions 1 and 2.

(ii) \Rightarrow (i) By Propositions 1 and 2, there exist factors M_0 and N_0 such that

$$\begin{array}{ccc} P & \subset & M \\ \cup & & \cup \\ Q & \subset & M_0 \end{array} \quad \text{and} \quad \begin{array}{ccc} N_0 & \subset & P \\ \cup & & \cup \\ N & \subset & Q \end{array}$$

are commuting and co-commuting squares.

First let us consider the diagram

$$\begin{array}{ccc} P & \subset & M \\ \cup & & \cup \\ Q & \subset & M_0. \end{array}$$

Let ψ be a generalized trace on Q so that $\sigma_T^{\psi} = \text{Id}$, where $T = \lfloor 2\pi / \ln \lambda \rfloor$. Then

$\sigma_T^{\psi \cdot E_Q^P} = \text{Id}$ on P and, as proved in [17], $P = M^{\sigma_T^{\psi \cdot E_Q^M}}$. It follows that $Q \subset M_0^{\sigma_T^{\psi \cdot E_Q^{M_0}}} \subset M_0$. But because

$$\begin{array}{ccc} P = M^{\sigma_T^{\psi \cdot E_Q^M}} & \subset & M \\ \cup & & \cup \\ M_0^{\sigma_T^{\psi \cdot E_Q^{M_0}}} & \subset & M_0 \end{array}$$

is a commuting and co-commuting square, the subfactor $M_0^{\sigma_T^{\psi \cdot E_Q^{M_0}}}$ has index value m in M_0 , hence $Q = M_0^{\sigma_T^{\psi \cdot E_Q^{M_0}}}$. Now if α is the predual automorphism of $\sigma_T^{\psi \cdot E_Q^M}$ on Q , then $M_0 = Q \times_\alpha \mathbf{Z}_m$ and similarly $M = P \times_\alpha \mathbf{Z}_m$. By Lemma 3, we have $\text{mod}(\alpha) = \text{mod}(\alpha|_Q) \equiv \lambda^{1/m}$.

As for the commuting and co-commuting square

$$\begin{array}{ccc} N_0 & \subset & P \\ \cup & & \cup \\ N & \subset & Q, \end{array}$$

since N is of type $\text{III}_{\lambda^{1/n}}$, there exists a generalized trace φ on N such that $\sigma_{nT}^\varphi = \text{Id}$ on N , where T is as before. Now $\sigma_T^{\varphi \cdot E_N^Q}$ is inner on Q , say $\sigma_T^{\varphi \cdot E_N^Q} = \text{Ad } u$ for some unitary u in Q ; then we may assume that $u^n = 1$ as $N' \cap Q = \mathbf{C}$ and hence $\sigma_{nT}^{\varphi \cdot E_N^Q} = \text{Id}$ on N . As $Q' \cap P = \mathbf{C}$ and P is of type III_λ , we also have $\sigma_T^{\varphi \cdot E_N^Q \cdot E_Q^P} = \text{Ad } u$ on P and because $E_N^Q \cdot E_Q^P E_{N_0}^{N_0} \cdot E_{N_0}^P$, $\sigma_T^{\varphi \cdot E_{N_0}^{N_0} \cdot E_{N_0}^P} = \text{Ad } u$ on P as well. Hence for $x \in N_0$, $\sigma_T^{\varphi \cdot E_{N_0}^{N_0}}(x) = \sigma_T^{\varphi \cdot E_{N_0}^{N_0} \cdot E_{N_0}^P}(x) = u x u^*$ so that $u N_0 u^* = N_0$ and $\text{Ad } u$ defines a properly outer action of \mathbf{Z}_n on N_0 . Moreover, $E_{N_0}^P(u^j) = E_{N_0}^Q(u^j) = 0$ for $0 \leq j \leq n - 1$. Thus $\{N_0, u\}''$ is the crossed product of N_0 by $\sigma_T^{\varphi \cdot E_{N_0}^{N_0}}$ and so it contains P as a subfactor with index value n . Thus $\{N_0, u\}'' = P$. Similarly, Q is the crossed product N by σ_T^φ . Now if we let β be the dual action of $\sigma_T^{\varphi \cdot E_{N_0}^{N_0}}$, then $N = Q^\beta$ and $N_0 = P^\beta$. By Lemma 3, $\text{mod}(\beta) = \text{mod}(\beta|_Q) \equiv \lambda^{1/n}$. Q.E.D.

Remarks. 1) In [2] inclusions of the form $R^H \subset R \times K$, where H and K are finite groups of outer automorphisms on a type II_1 factor R , are studied and thus inclusions satisfying any one of the equivalent conditions of Theorem 1 may be viewed as a subfactor analogue of these group-like inclusions.

2) In view of condition (ii) in Theorem 1, we can define a G -kernel on $Q \subset P$, i.e., a homomorphism of G into $\text{Aut}(P, Q)/\text{Int}(Q)$, where G is the group generated by α and β in $\text{Aut}(P, Q)/\text{Int}(Q)$ and the homomorphism is given by the quotient map. Such a G -kernel may be viewed as a subfactor analogue to those for single factors that were studied in [11, 20, 25].

3) Let $N \overset{E_N^Q}{\subset} Q \overset{E_Q^P}{\subset} P \overset{E_P^M}{\subset} M$ be an irreducible inclusion of factors that satisfy the hypotheses of the equivalent conditions in Theorem 1. It is easy to see that the isomorphism class of $N \subset M$ determines the isomorphism class of the associated

commuting and co-commuting square

$$\begin{array}{ccccc} N_0 & \subset & P & \subset & M \\ \cup & & \cup & & \cup \\ N & \subset & Q & \subset & M_0 \end{array}$$

and vice versa. Indeed, let φ be a generalized trace on N , put $\psi = \varphi \cdot E_N^M$ and let $T = |2\pi/\ln \lambda|$; then by [17] $Q = N \times_{\sigma_T^{\varphi}} \mathbf{Z}_n$ and $P = M^{\sigma_n^{\psi}}$. By the uniqueness of the generalized trace φ , we see that $Q \subset P$ is invariant under isomorphisms of $N \subset M$. Using the spatial uniqueness of the standard form as proved in [10], the extension and restriction properties of the canonical endomorphisms: $\gamma_{Q,N}$, $\gamma_{P,Q}$ and $\gamma_{M,P}$ and the commuting and co-commuting square conditions of

$$\begin{array}{ccc} \gamma_{P,Q}(Q) \subset Q & & \gamma_{P,Q}(M) \subset M \\ \cup & & \cup \\ \gamma_{P,Q}(N) \subset N & \text{and} & \cup \\ & & \cup \end{array}$$

are also preserved under isomorphisms of $N \subset M$. Finally, from the proof of Propositions 1 and 2, the constructions of the subfactors M_0 and N_0 are also preserved under isomorphisms of $N \subset M$.

We now turn to the study of inclusions that satisfy the commuting square conditions explained above by means of the discrete decomposition. We recall the following result proved in [19].

Proposition 3. *Let $\lambda \in (0, 1)$ and $m \in \mathbf{N}$. Suppose that*

$$\begin{array}{ccc} P & \subset & M \\ \cup & & \cup \\ Q & \subset & M_0 \end{array}$$

is a commuting square of factors with finite indices such that: $Q \subset P$ are both of type III $_{\lambda}$ and $M_0 \subset M$ are of type III $_{\lambda^{1/m}}$, $[M_0 : Q]_0 = [M : P]_0 = m$. Then there exist type II $_{\infty}$ factors $Q^{\infty} \subset P^{\infty}$ and a trace-scaling automorphism $\theta \in \text{Aut}(P^{\infty}, Q^{\infty})$ such that $\text{mod}(\theta) = \lambda$ and

$$\begin{array}{ccc} P & \subset & M \\ \cup & & \cup \\ Q & \subset & M_0 \end{array}$$

is isomorphic to

$$\begin{array}{ccc} P^{\infty} \times_{\theta^m} \mathbf{Z} & \subset & P^{\infty} \times_{\theta} \mathbf{Z} \\ \cup & & \cup \\ Q^{\infty} \times_{\theta^m} \mathbf{Z} & \subset & Q^{\infty} \times_{\theta} \mathbf{Z}. \end{array}$$

Using the classification result in [23] for trace-scaling automorphisms on strongly amenable type II $_{\infty}$ inclusions, we obtain the following classification result of inclusions of type III factors satisfying the hypotheses of Proposition 1.

Corollary 1. *Let*

$$\begin{array}{ccc} P & \subset & M \\ \cup & & \cup \\ Q & \subset & M_0 \end{array}$$

be as in Proposition 1 and assume further that $Q \subset P$ is strongly amenable, i.e., its type II core is strongly amenable in the sense defined in [22]. Then the commuting

square is classified by its type II core $Q^\infty \subset P^\infty$ and the standard invariant of θ on $Q^\infty \subset P^\infty$.

Similarly, let

$$\begin{array}{ccc} N_0 & \subset & P \\ \cup & & \cup \\ N & \subset & Q \end{array}$$

be a commuting square of factors with finite indices such that: $Q \subset P$ are both of type III_λ and $N \subset N_0$ are of type $\text{III}_{\lambda^{1/n}}$, $[N_0 : N]_0 = [P : Q]_0 = n$, and let M_0 be the basic construction of $N \subset Q$ and M the basic construction of $N_0 \subset P$. We then obtain a commuting square satisfying the same assumptions as in Proposition 1 and thus

$$\begin{array}{ccc} N_0 & \subset & P \\ \cup & & \cup \\ N & \subset & Q \end{array}$$

is also classified by the type II invariants associated with $Q^\infty \subset P^\infty$ as in Corollary 1 when $Q \subset P$ is strongly amenable.

As an application of Theorem 1, we are going to show that the middle inclusion $Q \subset P$ and the associated G -kernel can be used to study the structure of $N \subset M$. The following proposition is an easy extension to the subfactor case of the results proved in [2] for group-like inclusions.

Proposition 4. *Let $N \xrightarrow{E_N^Q} Q \xrightarrow{E_Q^P} P \xrightarrow{E_P^M} M$ be an irreducible inclusion of type III factors satisfying the hypotheses of the equivalent conditions in Theorem 1. Then $N \subset M$ has finite depth if and only if $Q \subset P$ has finite depth and G is finite.*

Proof. Suppose that $N \subset M$ has finite depth. Then by [1], $Q \subset P$ and $P^\beta \subset P \times_\alpha \mathbf{Z}_m$ both have finite depth, and so $\langle \alpha, \beta \rangle / \langle \alpha, \beta \rangle \cap \text{Int}(P)$ is finite by [2]. On the other hand, as $Q \subset P$ has finite index, $\langle \alpha, \beta \rangle \cap \text{Int}(P) / \langle \alpha, \beta \rangle \cap \text{Int}(Q)$ is finite by [21] and hence $G = \langle \alpha, \beta \rangle / \langle \alpha, \beta \rangle \cap \text{Int}(Q)$ is also finite

Conversely, suppose that $Q \subset P$ has finite depth and G is finite. Then $N = Q^\beta \subset Q \times_\alpha \mathbf{Z}_m$ and $P^\beta \subset M = P \times_\alpha \mathbf{Z}_m$ both have finite depth by [2]. Since

$$\begin{array}{ccc} P^\beta & \subset & P \\ \cup & & \cup \\ N = Q^\beta & \subset & Q \end{array} \quad \text{and} \quad \begin{array}{ccc} P & \subset & P \times_\alpha \mathbf{Z}_m \\ \cup & & \cup \\ Q & \subset & Q \times_\alpha \mathbf{Z}_m \end{array}$$

are commuting squares,

$$\begin{array}{ccc} P^\beta & \subset & M \\ \cup & & \cup \\ N = Q^\beta & \subset & Q \times_\alpha \mathbf{Z}_m \end{array}$$

is also a commuting square, and hence $N \subset M$ has finite depth by [27]. Q.E.D.

Let G be a countable discrete group and let Θ_1 and Θ_2 be two G -kernels on an inclusion of factors $Q \subset P$. As in the single factor case that was studied in [11, 25] we say that Θ_1 and Θ_2 are *conjugate* if there exists $\Phi \in \text{Aut}(P, Q) / \text{Int}(Q)$ such that $\Phi \cdot \Theta_1 \cdot \Phi^{-1} = \Theta_2$. According to [22] an inclusion of factors of the form $R^\beta \subset R \times_\alpha \mathbf{Z}_m$, where α and β are outer automorphisms with order m and n , respectively, on a type II_1 factor R , is classified by the conjugacy class of the G -kernel coming from $\langle \alpha, \beta \rangle / \text{Int}(R)$. It is thus not surprising that a similar

result holds for an inclusion of AFD factors that satisfies any one of the equivalent conditions in Theorem 1.

Theorem 2. *Let $N \subset^{\substack{E_N^Q \\ E_N^Q}} Q \subset^{\substack{E_P^Q \\ E_P^Q}} P \subset^{\substack{E_P^M \\ E_P^M}} M$ be an irreducible inclusion of type III factors that satisfies any one of the equivalent conditions in Theorem 1. Let α and β be the associated automorphisms and $\{G, \Theta\}$ be the kernel on $Q \subset P$ arising from the subgroup generated by α and β modulo $\text{Int}(Q)$. Then the isomorphism class of $N \subset M$ is determined by the conjugacy class of G .*

Proof. Let $N_1 \subset M_1$ be another irreducible inclusion satisfying the same properties as $N \subset M$. Let $Q_1 \subset P_1, \alpha_1, \beta_1$ and Θ_1 be defined accordingly from $N_1 \subset M_1$.

Suppose first that Φ is an isomorphism of $N \subset M$ onto $N_1 \subset M_1$. Then as noted in the Remarks after Theorem 1 above, Φ actually maps $N \subset Q \subset P \subset M$ onto $N_1 \subset Q_1 \subset P_1 \subset M_1$ and so we can identify all the respective factors: $N = N_1, Q = Q_1, P = P_1, M = M_1$. As a result we see that Φ can be extended to an isomorphism mapping

$$\begin{array}{ccc} P \subset P \times_{\alpha} \mathbf{Z}_m & & P \subset P \times_{\beta} \mathbf{Z}_n \\ \cup & \cup & \text{and } \cup & \cup \\ Q \subset Q \times_{\alpha} \mathbf{Z}_m & & Q \subset Q \times_{\beta} \mathbf{Z}_n \end{array}$$

onto

$$\begin{array}{ccc} P \subset P \times_{\alpha_1} \mathbf{Z}_m & & P \subset P \times_{\beta_1} \mathbf{Z}_n \\ \cup & \cup & \text{and } \cup & \cup \\ Q \subset Q \times_{\alpha_1} \mathbf{Z}_m & & Q \subset Q \times_{\beta_1} \mathbf{Z}_n. \end{array}$$

It is then straightforward to prove that there exist unitaries u and v in Q such that $\Phi \cdot \alpha \cdot \Phi^{-1} = \text{Ad } u \cdot \alpha_1$ and $\Phi \cdot \beta \cdot \Phi^{-1} = \text{Ad } v \cdot \beta_1$. Hence Θ and Θ_1 are conjugate.

Conversely, suppose that Θ and Θ_1 are two G -kernels on $Q \subset P$ that are conjugate via an isomorphism Φ of $Q \subset P$ onto $Q_1 \subset P_1$. Then there exist unitaries u and v in Q_1 such that $\Phi \cdot \alpha \cdot \Phi^{-1} = \text{Ad } u \cdot \alpha_1$ and $\Phi \cdot \beta \cdot \Phi^{-1} = \text{Ad } v \cdot \beta_1^l$, where l and n are relatively prime. Then by standard arguments, Φ can be extended to isomorphisms between

$$\begin{array}{ccc} P \subset P \times_{\alpha} \mathbf{Z}_m & & P \subset P \times_{\beta} \mathbf{Z}_n \\ \cup & \cup & \text{and } \cup & \cup \\ Q \subset Q \times_{\alpha} \mathbf{Z}_m & & Q \subset Q \times_{\beta} \mathbf{Z}_n \end{array}$$

and

$$\begin{array}{ccc} P_1 \subset P_1 \times_{\alpha_1} \mathbf{Z}_m & & P_1 \subset P_1 \times_{\beta_1} \mathbf{Z}_n \\ \cup & \cup & \text{and } \cup & \cup \\ Q_1 \subset Q_1 \times_{\alpha_1} \mathbf{Z}_m & & Q_1 \subset Q_1 \times_{\beta_1} \mathbf{Z}_n. \end{array}$$

As $\Phi(N)$ is the downward construction of $Q_1 \subset Q_1 \times_{\beta_1} \mathbf{Z}_n$, we can find a unitary w in Q_1 such that $\text{Ad } w \cdot \Phi(N) = N_1$ and therefore $\text{Ad } w \cdot \Phi$ is an isomorphism between $N \subset M$ and $N_1 \subset M_1$. Q.E.D.

Recall that if R_0 is the hyperfinite type II $_1$ factor, then by [11] in the finite case, and by [20] in the amenable case, G -kernels of R_0 are classified up to conjugacy by their obstructions as defined in [11, 25], which are elements of $H^3(G, \mathbf{T})$: the third cohomology group of G with coefficients in the unit circle \mathbf{T} . It would thus be an interesting problem to classify G -kernels on subfactors by their cohomological invariants, their standard invariants (cf. [18]) and, in the case of type III $_{\lambda}$ factors, their Connes-Takesaki modules.

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