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ANTICOMMUTING DERIVATIONS

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ABSTRACT. We show that there are no non-trivial closable derivations of a C^*-algebra anticommuting with an ergodic action of a compact group, supposing that the set of squares is dense in the group. We also show that there are no non-trivial closable densely defined rank one derivations on any C^*-algebra.

1. INTRODUCTION

This paper is concerned with the theory of unbounded derivations on C^*-algebras, a subject whose study was motivated by questions in quantum physics and statistical mechanics. Derivations are also used in the study of other areas in mathematics, for example differentiation on manifolds, non-commutative geometry, and transformation groups. This paper was motivated in part by a study of representations of the Heisenberg Lie algebra. See, for example, [BR79], [JM84], [Sa91] for surveys of some of these developments.

Let $(\mathfrak{A}, G, \alpha)$ be a C^*-dynamical system and let δ be a (unbounded) derivation of $\mathfrak{A}$. That is, $\mathfrak{A}$ is a C^*-algebra, G a topological group and α is a continuous homomorphism of G into the group of $*$-automorphisms of $\mathfrak{A}$ equipped with the topology of pointwise convergence. There is an extensive literature (see [Br86] and [Sa91] for surveys) showing that under a variety of conditions on α and/or δ commutation of α and δ have consequences for the structure of δ. Versions of many of these results remain valid if we replace the commutation of α and δ by anticommutation (defined below). The purpose of this paper is to show that stronger conclusions are possible in the presence of the anticommutation relation, than are possible in the presence of the commutation relation. Specifically, we show that if $(\mathfrak{A}, G, \alpha)$ is a compact ergodic C^*-dynamical system, $\{g^2 : g \in G\}$ is dense in G and δ is a closable densely defined derivation, then anticommutation of α and δ implies $\delta = 0$. The proof depends on the fact that there are no non-trivial densely defined rank one derivations of any C^*-algebra; we prove this below.

Background material on unbounded derivations, C^*-algebras, and group actions on C^*-algebras can, for example, be found in [BR79], [Br86], and [Sa91].

Let R and S be bounded everywhere defined operators on a Banach space X, and suppose T is a densely defined (possibly unbounded) operator on X. By $RT \subset TS$ we mean that S maps the domain of T into itself and $RTx = TSx$ for any x in the domain of T. Let $(\mathfrak{A}, G, \alpha)$ be a C^*-dynamical system and let δ be a (possibly

unbounded) derivation of $\mathfrak{A}$. We say that α and δ *commute* when $\alpha_g \delta \subset \delta \alpha_g$ for all g in G. We say that α and δ *anticommute* when $\alpha_{g^{-1}} \delta \subset \delta \alpha_g$ for all g in G. Note that if $RS = -SR$, then a formal power series expansion yields $\exp(-tR)S = S \exp(tR)$ for all real numbers t. Hence, if G is the additive group of real numbers, then anticommutation of α and δ corresponds to anticommutation, in the usual sense, of δ and the infinitesimal generator of α.

Before proving the main results, we want to point out that formally an anticommuting pair (α, δ) corresponds to a representation of the Heisenberg commutation relation. The following computations will be formal, in particular, we will ignore the domain problems associated with unbounded operators. Let G be the additive group of real numbers and let γ be the generator of α. Suppose there exist operators H and K so that $\delta(a) = Ka - aK$ and $\gamma(a) = Ha - aH$, and let $C = HK - KH$. It is easy to see that (formally) $\alpha_{-t}\delta = \delta\alpha_t$ implies $\gamma\delta = -\delta\gamma$ which in turn implies $HC = CH$ and $KC = CK$, hence H, K and C is a representation of the Heisenberg Lie algebra, C being the central element. Hence, our main result shows that certain representations of the Heisenberg commutation relations are impossible. It should be noted that commutation of α and δ corresponds to $HK - KH$ being a scalar multiple of the identity.

2. RANK ONE DERIVATIONS

The main result in this section is that C^*-algebras do not admit non-trivial closed rank one *-derivations. This is first proved for a densely defined derivation whose range is a subset of its domain. This is relatively easy, but it illustrates one aspect of the proof of the main result and it is used in the proof of the main result.

Recall, a *derivation* of a C^*-algebra is $\mathfrak{A}$ is a linear map δ from a subalgebra $\mathcal{D}(\delta)$, the *domain* of δ, into $\mathfrak{A}$ that satisfies Leibnitz's rule:

$$\delta(ab) = \delta(a)b + a\delta(b)$$

for all a and b in $\mathfrak{A}$. A *-*derivation* is a derivation δ whose domain is a *-subalgebra of $\mathfrak{A}$ with the property that

$$\delta(a^*) = \delta(a)^*$$

for all a in $\mathfrak{A}$.

Proposition 2.1. *Let δ be a densely defined derivation on a C^*-algebra, whose range is a subset of its domain. If the rank of δ is at most one, then $\delta = 0$.*

Proof. Let δ be a derivation of a C^*-algebra $\mathfrak{A}$. Suppose $\delta(\mathcal{D}(\delta)) \subset D(\delta)$ and the range of δ is one dimensional; then

$$\delta(a) = h(a)R,$$

for any a in $\mathfrak{A}$, where h is a function from $\mathcal{D}(\delta)$ into the complex numbers and R is a non-zero element in $\mathfrak{A}$. Note, R is in the domain of δ, because $\delta(\mathcal{D}(\delta)) \subset D(\delta)$. Fix b in $\mathcal{D}(\delta)$ with $\delta(b) = R$, i.e., $h(b) = 1$. First we will show

(2.1) $Ra + aR$ is a scalar multiple of R

for any a in $\mathfrak{A}$. By density of $\mathcal{D}(\delta)$ in $\mathfrak{A}$ it is enough to show this for a in $\mathcal{D}(\delta)$. Let a be in $\mathcal{D}(\delta)$. If $h(a) \neq 0$, then $h(a)(aR + Ra) = \delta(a^2) = h(a^2)R$. If $h(a) = 0$, then $h(ab + ba)R = \delta(ab + ba) = aR + Ra$. This proves (2.1). Next we will show that

(2.2) $R^2 = 0$.

If $h(R) = 0$, then expanding $\delta(b^2)$ we get $h(b^2)R = Rb + bR$. Applying δ to this equality and using $\delta(R) = 0$ yields $0 = 2R^2$. Next suppose $h(R) \neq 0$. First note that applying δ to R^2 we get

$$(2.3) \qquad h(R^2)R = \delta(R^2) = 2h(R)R^2,$$

so that

$$(2.4) \qquad R^2 = kR,$$

where $k = h(R^2)/2h(R)$. Replacing R^2 by kR in the second equality in (2.3) we see that $kh(R)R = 2kh(R)R$. Using that $h(R)R \neq 0$, it follows that $k = 0$, hence (2.4) implies that $R^2 = 0$ as desired. This proves (2.2). Multiplying (2.1) by R we see that $RaR = 0$ for any a in $\mathfrak{A}$. Taking $a = R^*$ it follows that $RR^*R = 0$; multiplying on the left by R we see that $(R^*R)^2 = 0$, hence that $R = 0$. This contradiction completes the proof of Proposition 2.1. $\qquad\qquad \square$

In the proof of Proposition 2.1 we do not really need that $\mathfrak{A}$ is a C^*-algebra; all we need is that $\mathfrak{A}$ is a topological algebra with the property that the only element R in $\mathfrak{A}$ satisfying $RaR = 0$ for all a in $\mathfrak{A}$ is $R = 0$.

Theorem 2.2. *Let δ be a densely defined closable $*$-derivation of a C^*-algebra. If the rank of δ is at most one, then $\delta = 0$.*

Proof. Suppose $\mathfrak{A}$ is a non-degenerate C^*-algebra of operators on some Hilbert space $\mathcal{H}$, and that δ is a rank one derivation densely defined closed $*$-derivation on $\mathfrak{A}$. Let h, R and b be as in the proof of Proposition 2.1. Then

$$(2.5) \qquad h(b^*)R = \delta(b^*) = \delta(b)^* = R^*.$$

In particular $h(b^*)$ has absolute value one. Let k be a square root of $\overline{h(b^*)}$, and let

$$(2.6) \qquad h'(a) = kh(a) \text{ and } R' = \bar{k}R.$$

Then $\delta(a) = h'(a)R'$ for all a in $\mathfrak{A}$ and

$$R'^* = kR^* = kh(b^*)R = \bar{k}R = R'.$$

So we may assume

$$(2.7) \qquad R^* = R.$$

From the proof of (2.1) above we see that

$$(2.8) \qquad Ra + aR = \lambda(a)R$$

for all a in $\mathfrak{A}$, where λ is a continuous linear functional on $\mathfrak{A}$ and $\lambda(a^*) = \overline{\lambda(a)}$ for all a in $\mathfrak{A}$. From (2.8) we see that both the range and kernel of R are invariant under any a in $\mathfrak{A}$. Let $\mathcal{H}_1$ denote the closure of the range of R and let $\mathcal{H}_0$ denote the kernel of R. The $\mathfrak{A}$ is block diagonal with respect to the decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_0$ of the underlying Hilbert space. Applying (2.8) with $a = R$ we get

$$(2.9) \qquad R^2 = \frac{\lambda(R)}{2}R.$$

Using (2.7) and the definition of $\mathcal{H}_1$ and of $\mathcal{H}_0$ we see that $R = R_1 \oplus 0$, where $R_1^* = R_1$. In particular (2.9) tells us that $R_1^2 = \lambda(R)R_1/2$; it follows that $R_1 = (\lambda(R)/2)\mathbf{1}_1$, where $\mathbf{1}_1$ denotes the identity operator on $\mathcal{H}_1$. Thus replacing δ by $(2/\lambda(R))\delta$, if necessary, we may assume that $R = \mathbf{1}_1 \oplus 0$. If a in $\mathfrak{A}$ has block diagonal form $a = a_1 \oplus a_0$, then it follows from (2.8) that $2a_1 = \lambda(a)\mathbf{1}_1$. In particular each

a in $\mathfrak{A}$ is of the form $(k\mathbf{1}_1) \oplus a_0$, where k is a complex number and a_0 is a bounded linear operator on $\mathcal{H}_0$. Let $\mathfrak{A}_0$ be the subalgebra of $\mathfrak{A}$ consisting of the elements of the form $a = 0 \oplus a_0$.

By adjoining $\mathbf{1} = \mathbf{1}_1 \oplus \mathbf{1}_0$ to $\mathfrak{A}$ and extending δ in the natural way we may assume that $\mathbf{1}$ is in the domain $\mathcal{D}(\delta)$ of δ, see e.g. [BR76] or [BR79]. Let δ_0 denote the linear map on $\mathfrak{A}_0$ determined by $\delta_0(a) = (0 \oplus \mathbf{1}_0)\delta(a)$ for each a in $\mathcal{D}(\delta_0) = \mathcal{D}(\delta) \cap \mathfrak{A}_0$. Then δ_0 is a densely defined closed derivation on $\mathfrak{A}_0$. It follows that $0 \oplus \mathbf{1}_0$ is in $\mathcal{D}(\delta_0)$. But then $R = \mathbf{1} - (0 \oplus \mathbf{1}_0)$ is in $\mathcal{D}(\delta)$. We can now apply Proposition 2.1 to complete the proof.   □

Example 2.3. Let $\mathfrak{A}$ be the algebra of upper 3×3 matrices. Then the derivation $\delta(a) = ba - ab$ will have rank one (or two, or three) for a suitable choice of b in $\mathfrak{A}$. It follows that any positive integer is a candidate for the dimension of the range of some derivation of a (suitable finite dimensional) nest algebra. It is, however, not clear what happens in a general nest, e.g., can the nest algebra of a continuous nest have rank one derivations?

Preliminary calculations suggest that the range of a ∗-derivation can only have even dimension. However, this is not needed below.

3. Derivations anticommuting with a group action

In this section we consider a derivation δ of a C^*-algebra $\mathfrak{A}$ with identity $\mathbf{1}$ and an action α of a group G on $\mathfrak{A}$. We show (given some mild technical condition) that if α acts ergodically, and α and δ anticommute, then $\delta = 0$. That α acts *ergodically* means that the *fixed point algebra*

$$A^\alpha = \{a \in A : \alpha_g(a) = a, \text{ for all } g \in G\}$$

equals $\mathbb{C}\mathbf{1}$, the scalar multiples of $\mathbf{1}$. Recall, α and δ anticommute provided

$$(3.1) \qquad\qquad\qquad\qquad \alpha_g \delta \subset \delta \alpha_{g^{-1}}$$

for all g in G. The situation where G is a d-dimensional torus and α and δ commute, i.e.,

$$(3.2) \qquad\qquad\qquad\qquad \alpha_g \delta \subset \delta \alpha_g$$

for all g in G, and δ is defined on the α-smooth elements in $\mathfrak{A}$, was shown to have far reaching consequences for the structure of δ in [BEJ84], where α is an ergodic (faithful) action. (See [Br86] and [Sa91] for some further developments.) The basis for [BEJ84] was the observation [St74] that there exist unitaries U_γ in $\mathfrak{A}$ such that $\alpha_g(U_\gamma) = (g, \gamma)U_\gamma$ for all g in G and all γ in the dual group $\widehat{G}$. Here (g, γ) denotes the pairing between G and $\widehat{G}$. It then follows from (3.2) that $\delta(U_\gamma) = \beta(\gamma)U_\gamma$ for all γ in $\widehat{G}$, where β is a scalar valued function defined on the dual group $\widehat{G}$. If one assumes (3.1) in place of (3.2), then the corresponding result is $\delta(U_\gamma) = \beta(\gamma)U_{\gamma^{-1}}$ for all γ in $\widehat{G}$. The result proved below does not use this, but it implies $\beta \equiv 0$.

Theorem 3.1. *Let $\mathfrak{A}$ be a C^*-algebra with identity $\mathbf{1}$, δ a closable densely defined ∗-derivation on $\mathfrak{A}$, G a compact group with $\{g^2 : g \in G\}$ dense in G, and let α be an ergodic action of G on $\mathfrak{A}$. If α and δ anticommute, then $\delta = 0$.*

Proof. By replacing δ by its closure, if necessary, we may assume that $\mathbf{1}$ is in the domain $\mathcal{D}(\delta)$ of δ. Suppose δ has rank greater than one. Let a and b be in $\mathcal{D}(\delta)$; then

$$\alpha_g \delta(ab) = (\alpha_g \delta(a))\alpha_{g^{-1}}(b) + \alpha_{g^{-1}}(a)\alpha_g \delta(b)$$

for all g in G. Hence, density of $\{g^2 : g \in G\}$ in G implies

(3.3) $$\delta(ab) = \delta(a)\alpha_g(b) + \alpha_g(a)\delta(b)$$

for all g in G. Now, if a is in $\mathfrak{A}$, then

$$P_0 a = \int_G \alpha_g(a)\,dg$$

determines a projection of $\mathfrak{A}$ onto the fixed point algebra $\mathfrak{A}^\alpha = \mathbb{C}\mathbf{1}$. Hence, integrating (3.3) yields the existence of two scalar valued functions ϕ and ψ defined on $\mathcal{D}(\delta)$, such that

(3.4) $$\delta(ab) = \phi(b)\delta(a) + \psi(a)\delta(b).$$

Expanding $\delta((ka_1 + a_2)b) = k\delta(a_1 b) + \delta(a_2 b)$ using (3.4), where k is a scalar and a_1, a_2 and b are in the domain of δ, it follows that ψ is a linear functional. Similarly, one can show that ϕ is a linear functional. Expanding both sides of $\delta((ab)c) = \delta(a(bc))$ using Leibnitz's rule and (3.4) we get

$$\phi(b)\phi(c)\delta(a) + \phi(c)\psi(a)\delta(b) + \psi(ab)\delta(c)$$
$$= \phi(bc)\delta(a) + \phi(c)\psi(a)\delta(b) + \psi(a)\psi(b)\delta(c)$$

for all a, b and c in $\mathcal{D}(\delta)$. Temporarily, fix b and c in $\mathcal{D}(\delta)$. If $\delta(c) = 0$, pick a in $\mathcal{D}(\delta)$ so that $\delta(a) \neq 0$; it follows that $\phi(b)\phi(c) = \phi(bc)$. If $\delta(c) \neq 0$, pick a in $\mathcal{D}(\delta)$ so that $\delta(a)$ and $\delta(c)$ are linearly independent; then again $\phi(b)\phi(c) = \phi(bc)$. A similar argument can be applied to ψ. We have shown

(3.5) $$\phi(a)\phi(b) = \phi(ab) \text{ and } \psi(a)\psi(b) = \psi(ab)$$

for all g in G and all a and b in $\mathcal{D}(\delta)$. Using (3.4) with either $a = \mathbf{1}$ or $b = \mathbf{1}$ it follows that $\phi(\mathbf{1}) = \psi(\mathbf{1}) = 1$. Now a simple calculation using (3.1) and (3.4) shows that

$$\phi(b)\delta(a) + \psi(a)\delta(b) = \phi(\alpha_g b)\delta(a) + \psi(\alpha_g a)\delta(b)$$

for all g in G and all a and b in $\mathcal{D}(\delta)$. It follows that $\phi(b) = \phi(\alpha_g b)$ and $\psi(a) = \psi(\alpha_g a)$ for all g in G and all a and b in $\mathcal{D}(\delta)$. Expanding $\delta(aP_0 b)$ in two ways, first using (3.4) and then using Leibnitz's rule we get

$$\phi(P_0 b)\delta(a) + \psi(a)\delta(P_0 b) = \delta(a)P_0 b + a\delta(P_0 b)$$

for all a and b in $\mathcal{D}(\delta)$. But $\delta(P_0 b) = 0$, since $P_0 b$ is in $\mathfrak{A}^\alpha = \mathbb{C}\mathbf{1}$, so it follows that $\phi(P_0 b)\mathbf{1} = P_0 b$ for all b in $\mathcal{D}(\delta)$. Hence, (3.5) implies $(P_0 a)(P_0 b) = (P_0 ab)$ for all a and b in $\mathfrak{A}$. It follows that $\mathfrak{A} = \mathbb{C}\mathbf{1}$, contradicting that δ has rank $\geqslant 2$. So δ has rank $\leqslant 1$; now an application of Theorem 2.2 completes the proof. $\qquad\square$

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